

Relative Homological Algebra and Purity in Triangulated Categories

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1. INTRODUCTION

Triangulated categories were introduced by Grothendieck and Verdier in the early sixties as the proper framework for doing homological algebra in an abelian category. Since then triangulated categories have found important applications in algebraic geometry, stable homotopy theory, and representation theory. Our main purpose in this paper is to study a triangulated category, using relative homological algebra which is developed *inside* the triangulated category.

Relative homological algebra has been formulated by Hochschild in categories of modules and later by Heller and Butler and Horrocks in



more general categories with a relative abelian structure. Its main theme consists of a selection of a class of extensions. In triangulated categories there is a natural candidate for extensions, namely the (distinguished) triangles. Let \mathcal{E} be a triangulated category with triangulation Δ . We develop a homological algebra in \mathcal{E} which parallels the homological algebra in an exact category in the sense of Quillen, by specifying a class of triangles $\mathcal{E} \subseteq \Delta$ which is closed under translations and satisfies the analogous formal properties of a proper class of short exact sequences. We call such a class of triangles \mathcal{E} a *proper class of triangles*. Two big differences occur between the homological algebra in a triangulated category and in an exact category. First, the absolute theory in a triangulated category, i.e., if we choose Δ as a proper class, is trivial. Second, a proper class of triangles is uniquely determined by an ideal $\text{Ph}_{\mathcal{E}}(\mathcal{E})$ of \mathcal{E} , which we call the ideal of \mathcal{E} -phantom maps. This ideal is a homological invariant which almost always is non-trivial and in a sense controls the homological behavior of the triangulated category. However, there are non-trivial examples of phantomless categories. To a large extent the relative homological behavior of \mathcal{E} with respect to \mathcal{E} depends on the knowledge of the structure of \mathcal{E} -phantom maps.

The paper is organized as follows. Section 2 contains the basic definitions about proper classes of triangles and phantom maps in a triangulated category \mathcal{E} . We prove that there exists a bijective correspondence between proper classes of triangles and special ideals of \mathcal{E} , and we discuss briefly an analogue of Baer's theory of extensions. In Section 3 we prove that there exists a bijective correspondence between proper classes of triangles in \mathcal{E} and Serre subcategories of the category $\text{mod}(\mathcal{E})$ of finitely presented additive functors $\mathcal{E}^{\text{op}} \rightarrow \mathcal{A}b$, which are closed under the suspension functor. Using this correspondence, we associate to any proper class of triangles \mathcal{E} in \mathcal{E} a uniquely determined abelian category $\mathcal{S}_{\mathcal{E}}(\mathcal{E})$, the \mathcal{E} -Steenrod category of \mathcal{E} . The \mathcal{E} -Steenrod category comes equipped with a homological functor $\mathbf{S}: \mathcal{E} \rightarrow \mathcal{S}_{\mathcal{E}}(\mathcal{E})$, the *projectivization functor*, which is universal for homological functors out of \mathcal{E} which annihilate the ideal of \mathcal{E} -phantom maps. The terminology comes from stable homotopy: the Steenrod category of the stable homotopy category of spectra with respect to the proper class of triangles induced from the Eilenberg–MacLane spectrum corresponding to $\mathbb{Z}/(p)$ is the module category over the mod- p Steenrod algebra. We regard $\mathcal{S}_{\mathcal{E}}(\mathcal{E})$ as an abelian approximation of \mathcal{E} and the idea is to use it as a tool for transferring information between the topological category \mathcal{E} and the algebraic category $\mathcal{S}_{\mathcal{E}}(\mathcal{E})$. This tool is crucial, if the projectivization functor is non-trivial and this happens iff the proper class of triangles \mathcal{E} generates \mathcal{E} in an appropriate sense.

In Section 4, fixing a proper class of triangles \mathcal{E} in a triangulated category \mathcal{E} , we introduce \mathcal{E} -projective objects, \mathcal{E} -projective resolutions,

\mathcal{E} -projective and \mathcal{E} -global dimension and their duals, and we prove the basic tools of homological algebra (Schanuel's lemma, horseshoe lemma, comparison theorem) in this setting. It follows that we can derive additive functors which behave well with respect to triangles in \mathcal{E} . We compare the homological invariants of \mathcal{E} with respect to \mathcal{E} with the homological invariants of its \mathcal{E} -Steenrod category, via the projectivization functor. Finally we study briefly the semisimple and hereditary categories.

In Section 5, we associate with any object A in \mathcal{E} the \mathcal{E} -phantom and the \mathcal{E} -cellular tower of A , which are crucial for the study of the homological invariants of A with respect to \mathcal{E} , e.g., the structure of \mathcal{E} -phantom maps out of A and the \mathcal{E} -projective dimension of A . Using these towers we prove our first main result which asserts that the global dimension of \mathcal{E} with respect to \mathcal{E} is less than equal to 1 iff the projectivization functor $\mathbf{S}: \mathcal{E} \rightarrow \mathcal{S}_{\mathcal{E}}(\mathcal{E})$ is full and reflects isomorphisms. The proper setting for the study of the \mathcal{E} -phantom and the \mathcal{E} -cellular tower is that of a *compactly generated* triangulated category [55]. In this case *homotopy colimits* are defined and we prove that under mild assumptions, any object of \mathcal{E} is a homotopy colimit of its \mathcal{E} -cellular tower and the homotopy colimit of its \mathcal{E} -phantom tower is trivial. Finally we study the *category of extensions* of \mathcal{E} -projective objects, the *phantom topology* and the *phantom filtration* of \mathcal{E} induced by the ideal $\text{Ph}_{\mathcal{E}}(\mathcal{E})$, and we compute the \mathcal{E} -phantom maps which "live forever," i.e. maps in $\bigcap_{n \geq 1} \text{Ph}_{\mathcal{E}}^n(\mathcal{E})$, in terms of the \mathcal{E} -cellular tower.

In Section 6 we associate with any triangulated category \mathcal{E} with enough \mathcal{E} -projectives a new triangulated category $\mathcal{D}_{\mathcal{E}}(\mathcal{E})$ which we call the \mathcal{E} -derived category. Under some mild assumptions $\mathcal{D}_{\mathcal{E}}(\mathcal{E})$ is realized as a full subcategory of \mathcal{E} and is described as the localizing subcategory of \mathcal{E} generated by the \mathcal{E} -projective objects. Moreover, $\mathcal{D}_{\mathcal{E}}(\mathcal{E})$ is compactly generated and any compactly generated category is of this form. This construction, which generalizes the construction of the usual derived category, allows us to prove the existence of total \mathcal{E} -derived functors. Applying these results to the derived category of a Grothendieck category with projectives, we generalize results of Spaltenstein [67], Boekstedt and Neeman [16], Keller [42], and Weibel [71] concerning resolutions of unbounded complexes and the structure of the unbounded derived category, proved in the above papers by using essentially a closed model structure on the category of complexes.

In Section 7 we prove that the stable category of \mathcal{E} modulo \mathcal{E} -projectives admits a natural left triangulated structure, which in many cases it is useful to study.

The results of the first seven sections refer to general proper classes of triangles in unspecified categories. In Section 8 we study proper classes of triangles $\mathcal{E}(\mathcal{A})$ induced by skeletally small subcategories \mathcal{A} in a category \mathcal{E} with coproducts. Under a reasonable condition on \mathcal{A} we show that the

Steenrod category of \mathcal{E} with respect to the proper class $\mathcal{E}(\mathcal{X})$ is the category $\text{Mod}(\mathcal{X})$ of additive functors $\mathcal{X}^{\text{op}} \rightarrow \mathcal{A}b$. We compare the $\mathcal{E}(\mathcal{X})$ -homological properties of objects of \mathcal{E} with the homological properties of projective, flat, and injective functors of the category $\text{Mod}(\mathcal{X})$. Our first main result in this section shows that if \mathcal{X} compactly generates \mathcal{E} , then \mathcal{E} has enough $\mathcal{E}(\mathcal{X})$ -injective objects in a functorial way and admits $\mathcal{E}(\mathcal{X})$ -injective envelopes. This generalizes a result of Krause [50]. Our second main result shows that if \mathcal{X} compactly generates \mathcal{E} or if any object of \mathcal{E} has finite $\mathcal{E}(\mathcal{X})$ -projective dimension, then any coproduct preserving homological functor from \mathcal{E} to a Grothendieck category annihilates the ideal of $\mathcal{E}(\mathcal{X})$ -phantom maps. This generalizes results obtained independently by Christensen and Strickland [21] and Krause [50].

In Section 9 we characterize the $\mathcal{E}(\mathcal{X})$ -semisimple or $\mathcal{E}(\mathcal{X})$ -phantomless categories \mathcal{E} by a host of conditions, generalizing work of Neeman [54]. Perhaps the most characteristic is the condition that \mathcal{E} is a pure-semisimple locally finitely presented category [24]; in particular, \mathcal{E} has filtered colimits. Using this and an old result of Heller [34] (observed also by Keller and Neeman [44]), we give necessary conditions for a skeletally small category \mathcal{D} such that the categories of Pro-objects and Ind-objects over \mathcal{D} admit a triangulated structure. These conditions are sufficient for the existence of a Puppe-triangulated or “pre-triangulated” structure [57] on these categories, in the sense that the octahedral axiom does not necessarily hold.

In Section 10 we study Brown representation theorems, via the concept of a representation equivalence functor, i.e., a functor which is full, surjective on objects, and reflects isomorphisms. We say that a pair $(\mathcal{E}, \mathcal{X})$ consisting of a triangulated category with coproducts \mathcal{E} and a full skeletally small triangulated subcategory $\mathcal{X} \subseteq \mathcal{E}$ satisfies the *Brown representability theorem* or that \mathcal{E} is an $\mathcal{E}(\mathcal{X})$ -Brown category, if the projectivization functor of \mathcal{E} with respect to $\mathcal{E}(\mathcal{X})$ induces a representation equivalence between \mathcal{E} and the category of cohomological functors over \mathcal{X} . Our main result shows that this happens iff \mathcal{X} compactly generates \mathcal{E} and any cohomological functor over \mathcal{X} has projective dimension bounded by 1. This generalizes results of Christensen and Strickland [21] and Neeman [56].

In Section 11 we study the fundamental concept of *purity* in a compactly generated triangulated category which provides a link between representation-theoretic and homological properties. Motivated by the examples of the stable homotopy category of spectra, the derived category of a ring, and the stable module category of a quasi-Frobenius ring, it is natural to regard a compact object as an analogue of a finitely presented object in a triangulated category. In this spirit the proper class of triangles induced by the compact objects are the *pure triangles* and the resulting theory is the *pure homological algebra* of \mathcal{E} . In this case the pure Steenrod category of

\mathcal{E} is the module category $\text{Mod}(\mathcal{E}^b)$, where \mathcal{E}^b denotes the full subcategory of compact objects of \mathcal{E} . We give various formulas for the computation of the pure global dimension and we prove that the pure global dimension bounds the pure global dimension of a smashing or finite localization. Our main result characterizes the pure Brown categories as the pure hereditary ones, and also by a host of equivalent conditions, the most notable being the fullness of the projectivization functor and the conditions concerning the behavior of weak colimits. These results are applied directly to the stable homotopy category, which has pure global dimension 1. By the results of Section 9, a compactly generated triangulated category is pure semisimple iff its pure Steenrod category is locally Noetherian. Motivated by the pure homological theory of module categories, we define a category to be *of finite type* if its pure Steenrod category is locally finite. We prove that the phantomless, Brown, or finite type property is preserved under smashing or finite localization, generalizing results of Hovey et al. [38].

In Section 12 we apply the results of the previous sections (mainly about purity) to derive categories of rings and stable categories of quasi-Frobenius rings. We compute the pure global dimension for many classes of rings and we prove that the derived category $\mathbf{D}(\Lambda)$ of a ring Λ is of finite type iff $\mathbf{D}(\Lambda^{\text{op}})$ is of finite type iff $\mathbf{D}(\Lambda)$ and $\mathbf{D}(\Lambda^{\text{op}})$ are pure semisimple, so finite type is a symmetric condition. However, we show that the Brown property is not symmetric in general. Motivated by the pure homological theory of module categories we formulate the *derived pure semisimple conjecture*, DPSC for short, which asserts that if $\mathbf{D}(\Lambda)$ is pure semisimple, then $\mathbf{D}(\Lambda)$ is of finite type. DPSC implies the still open pure semisimple conjecture asserting that a right pure semisimple ring is of finite representation type. We prove DPSC for many classes of rings, including Artin algebras. Indeed we characterize the Artin algebras with pure semisimple derived categories as the iterated tilted algebras of Dynkin type. We prove that the analogue of DPSC holds for the stable category of a quasi-Frobenius ring Λ , showing that Λ is representation-finite iff the stable category of Λ is pure semisimple (of finite type). The analogue of DPSC in a general compactly generated triangulated category fails. We prove this by answering in the negative a seemingly unrelated problem of Roos [64] concerning the structure of quasi-Frobenius Grothendieck categories. Our results indicate that finite type is the appropriate notion of “representation-finiteness,” at least for stable or derived categories. We close the paper by applying the results of Sections 6 and 7 to derived categories. In particular we obtain structure results for the unbounded derived category and we generalize results of Wheeler [72, 73] concerning the structure of the stable derived category.

A general convention used in the paper is that composition of morphisms in a category is meant diagrammatically: if $f: A \rightarrow B$, $g: B \rightarrow C$ are morphisms, their composition is denoted by $f \circ g$. However, we compose functors in the usual (anti-diagrammatic) order. Our additive categories admit finite direct sums.

2. PROPER CLASSES OF TRIANGLES AND PHANTOM MAPS

2.1. *Triangulated Categories*

Let \mathcal{C} be an additive category and $\Sigma: \mathcal{C} \rightarrow \mathcal{C}$ an additive functor. We define the category $\text{Diag}(\mathcal{C}, \Sigma)$ as follows. An object of $\text{Diag}(\mathcal{C}, \Sigma)$ is a diagram in \mathcal{C} of the form $A \rightarrow B \rightarrow C \rightarrow \Sigma(A)$. A morphism in $\text{Diag}(\mathcal{C}, \Sigma)$ between $A_i \xrightarrow{f_i} B_i \xrightarrow{g_i} C_i \xrightarrow{h_i} \Sigma(A_i)$, $i = 1, 2$, is a triple of morphisms $\alpha: A_1 \rightarrow A_2$, $\beta: B_1 \rightarrow B_2$, $\gamma: C_1 \rightarrow C_2$, such that the diagram (\dagger) commutes:

$$\begin{array}{ccccccc}
 A_1 & \xrightarrow{f_1} & B_1 & \xrightarrow{g_1} & C_1 & \xrightarrow{h_1} & \Sigma(A_1) \\
 \alpha \downarrow & & \beta \downarrow & & \gamma \downarrow & & \Sigma(\alpha) \downarrow \\
 A_2 & \xrightarrow{f_2} & B_2 & \xrightarrow{g_2} & C_2 & \xrightarrow{h_2} & \Sigma(A_2).
 \end{array} \tag{\dagger}$$

A triangulated category [70] is a triple $(\mathcal{C}, \Sigma, \Delta)$, where \mathcal{C} is an additive category, $\Sigma: \mathcal{C} \rightarrow \mathcal{C}$ is an autoequivalence of \mathcal{C} , and Δ is a full subcategory of $\text{Diag}(\mathcal{C}, \Sigma)$ which is closed under isomorphisms and satisfies the following axioms:

(T₁) For any morphism $f: A \rightarrow B$ in \mathcal{C} , there exists an object $A \xrightarrow{f} B \rightarrow^g C \rightarrow^h \Sigma(A)$ in Δ . For any object $A \in \mathcal{C}$, the diagram $0 \rightarrow A \xrightarrow{1_A} A \rightarrow 0$ is in Δ .

(T₂) If $A \xrightarrow{f} B \rightarrow^g C \rightarrow^h \Sigma(A)$ is in Δ , then $B \rightarrow^g C \rightarrow^h \Sigma(A) \rightarrow^{-\Sigma(f)} \Sigma(B)$ is in Δ .

(T₃) If $A_i \xrightarrow{f_i} B_i \xrightarrow{g_i} C_i \xrightarrow{h_i} \Sigma(A_i)$, $i = 1, 2$ are in Δ , and if there are morphisms $\alpha: A_1 \rightarrow A_2$ and $\beta: B_1 \rightarrow B_2$ such that $\alpha \circ f_2 = f_1 \circ \beta$, then there exists a morphism $\gamma: C_1 \rightarrow C_2$ such that the diagram (\dagger) is a morphism in Δ .

(T₄) The Octahedral Axiom. For the formulation of this we refer to Proposition 2.1.

PROPOSITION 2.1 (see also [51]). *Let \mathcal{C} be an additive category equipped with an autoequivalence $\Sigma: \mathcal{C} \rightarrow \mathcal{C}$ and a class of diagrams $\Delta \subseteq \text{Diag}(\mathcal{C}, \Sigma)$. Suppose that the triple $(\mathcal{C}, \Sigma, \Delta)$ satisfies all the axioms of a triangulated category except possibly of the octahedral axiom. Then the following are*

equivalent:

(i) *Base Change.* For any triangle $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} \Sigma(A) \in \Delta$ and any morphism $\varepsilon: E \rightarrow C$, there exists a commutative diagram

$$\begin{array}{ccccccc}
 0 & \longrightarrow & M & \xrightarrow{=} & M & \longrightarrow & 0 \\
 \downarrow & & \alpha \downarrow & & \delta \downarrow & & \downarrow \\
 A & \xrightarrow{f'} & G & \xrightarrow{g'} & E & \xrightarrow{h'} & \Sigma(A) \\
 \parallel \downarrow & & \beta \downarrow & & \varepsilon \downarrow & & \parallel \downarrow \\
 A & \xrightarrow{f} & B & \xrightarrow{g} & C & \xrightarrow{h} & \Sigma(A) \\
 \downarrow & & \gamma \downarrow & & \zeta \downarrow & & \downarrow \\
 0 & \longrightarrow & \Sigma(M) & \xrightarrow{=} & \Sigma(M) & \longrightarrow & 0
 \end{array}$$

in which all horizontal and vertical diagrams are triangles in Δ .

(ii) *Cobase Change.* For any triangle $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} \Sigma(A) \in \Delta$ and any morphism $\alpha: A \rightarrow D$, there exists a commutative diagram

$$\begin{array}{ccccccc}
 0 & \longrightarrow & N & \xrightarrow{=} & N & \longrightarrow & 0 \\
 \downarrow & & \zeta \downarrow & & \delta \downarrow & & \downarrow \\
 \Sigma^{-1}(C) & \xrightarrow{-\Sigma^{-1}(h)} & A & \xrightarrow{f} & B & \xrightarrow{g} & C \\
 \parallel \downarrow & & \alpha \downarrow & & \beta \downarrow & & \parallel \downarrow \\
 \Sigma^{-1}(C) & \xrightarrow{-\Sigma^{-1}(h')} & D & \xrightarrow{f'} & F & \xrightarrow{g'} & C \\
 \downarrow & & \eta \downarrow & & \vartheta \downarrow & & \downarrow \\
 0 & \longrightarrow & \Sigma(N) & \xrightarrow{=} & \Sigma(N) & \longrightarrow & 0
 \end{array}$$

in which all horizontal and vertical diagrams are triangles in Δ .

(iii) *Octahedral Axiom.* For any two morphisms $f_1: A \rightarrow B$, $f_2: B \rightarrow C$ there exists a commutative diagram

$$\begin{array}{ccccccc}
 A & \xrightarrow{f_1} & B & \xrightarrow{g_1} & X & \xrightarrow{h_1} & \Sigma(A) \\
 \parallel \downarrow & & f_2 \downarrow & & \alpha \downarrow & & \parallel \downarrow \\
 A & \xrightarrow{f_1 \circ f_2} & C & \xrightarrow{g_3} & Y & \xrightarrow{h_3} & \Sigma(A) \\
 f_1 \downarrow & & \parallel \downarrow & & \beta \downarrow & & \Sigma(f_1) \downarrow \\
 B & \xrightarrow{f_2} & C & \xrightarrow{g_2} & Z & \xrightarrow{h_2} & \Sigma(B) \\
 \downarrow & & 0 \downarrow & & h_2 \circ \Sigma(g_1) \downarrow & & \downarrow \\
 0 & \longrightarrow & \Sigma(X) & \xrightarrow{=} & \Sigma(X) & \longrightarrow & 0
 \end{array}$$

in which all horizontal and the third vertical diagrams are triangles in Δ .

Proof. (i) \Rightarrow (iii) Let $A \xrightarrow{f_1} B \xrightarrow{f_2} C$ be a diagram in \mathcal{E} . By (T_1) there are triangles $\Sigma^{-1}(Z) \rightarrow^\alpha B \rightarrow^{f_2} C \rightarrow Z$ and $\Sigma^{-1}(X) \rightarrow A \xrightarrow{f_1} B \rightarrow X$ in Δ . Applying base change for the last triangle along α , it is easy to see that if we arrange properly the resulting diagram, then we obtain an octahedral diagram as in (iii). (iii) \Rightarrow (i) If $M \rightarrow E \xrightarrow{\varepsilon} C \xrightarrow{\zeta} \Sigma(M)$ is a triangle, then applying the octahedral axiom to the composition $g \circ \zeta$, we obtain a diagram which if we apply Σ^{-1} and we arrange it properly, we have a diagram as in (i). The proof of (ii) \Leftrightarrow (iii) is similar.



Hence for any triangulated category we may use the above equivalent forms instead of the octahedral axiom, when it is more convenient. Throughout the paper we fix a triangulated category $\mathcal{E} = (\mathcal{E}, \Sigma, \Delta)$; Σ is the *suspension functor*, Δ is the *triangulation*, and the elements of Δ are the *triangles* of \mathcal{E} . A triangle $(T): A \xrightarrow{f} B \rightarrow^g C \rightarrow^h \Sigma(A) \in \Delta$ is called *split* if $h = 0$. Trivially if (T) is split, then the morphisms f, g induce a direct sum decomposition $B \cong A \oplus C$. We denote by Δ_0 the full subcategory of Δ consisting of the split triangles. We call a triangle *semi-split* if it is isomorphic to a coproduct of suspensions of triangles of the form $A \xrightarrow{=} A \rightarrow 0 \rightarrow \Sigma(A)$. Any split triangle is semi-split but the converse is not true.

2.2. Proper Classes of Triangles

A class of triangles \mathcal{E} is *closed under base change* if for any triangle $A \xrightarrow{f} B \rightarrow^g C \rightarrow^h \Sigma(A) \in \mathcal{E}$ and any morphism $\varepsilon: E \rightarrow C$ as in (i) of Proposition 2.1 the triangle $A \xrightarrow{f'} G \rightarrow^{g'} E \rightarrow^{h'} \Sigma(A)$ belongs to \mathcal{E} . Dually a class of triangles \mathcal{E} is *closed under cobase change* if for any triangle $A \xrightarrow{f} B \rightarrow^g C \rightarrow^h \Sigma(A) \in \mathcal{E}$ and any morphism $\alpha: A \rightarrow D$ as in (ii) of Proposition 2.1 the triangle $D \xrightarrow{f'} F \rightarrow^{g'} C \rightarrow^{h'} \Sigma(D)$ belongs to \mathcal{E} . A class of triangles \mathcal{E} is *closed under suspensions* if for any triangle $A \xrightarrow{f} B \rightarrow^g C \rightarrow^h \Sigma(A) \in \mathcal{E}$ and for any $i \in \mathbb{Z}$ the triangle $\Sigma^i(A) \xrightarrow{(-1)^i \Sigma^i f} \Sigma^i(B) \xrightarrow{(-1)^i \Sigma^i g} \Sigma^i(C) \xrightarrow{(-1)^i \Sigma^i h} \Sigma^{i+1}(A) \in \mathcal{E}$. Finally a class of triangles \mathcal{E} is called *saturated* if the following condition holds: if in the situation of base change in Proposition 2.1 the third vertical and the second horizontal triangle is in \mathcal{E} , then the triangle $A \xrightarrow{f} B \rightarrow^g C \rightarrow^h \Sigma(A)$ is in \mathcal{E} . An easy consequence of the octahedral axiom is that \mathcal{E} is saturated iff in the situation of cobase change in Proposition 2.1, if the second vertical and the third horizontal triangle is in \mathcal{E} , then the triangle $A \xrightarrow{f} B \rightarrow^g C \rightarrow^h \Sigma(A)$ is in \mathcal{E} .

The following concept is inspired from the definition of an exact category [61].

DEFINITION 2.2. A full subcategory $\mathcal{E} \subseteq \text{Diag}(\mathcal{C}, \Sigma)$ is called a *proper class of triangles* if the following conditions are true:

- (i) \mathcal{E} is closed under isomorphisms, finite coproducts, and $\Delta_0 \subseteq \mathcal{E} \subseteq \Delta$.
- (ii) \mathcal{E} is closed under suspensions and is saturated.
- (iii) \mathcal{E} is closed under base and cobase change.

EXAMPLE 2.3. (1) The class of split triangles Δ_0 and the class of all triangles Δ in \mathcal{C} are proper classes of triangles. If \mathcal{E} is a proper class of triangles in \mathcal{C} then \mathcal{E}^{op} is a proper class of triangles in \mathcal{C}^{op} . If $\{\mathcal{E}_i; i \in I\}$ is a family of proper classes of triangles, then $\bigcap_{i \in I} \mathcal{E}_i$ is a proper class of triangles. If $\{\mathcal{E}_i; i \in I\}$ is an increasing chain, then the class of triangles $\bigcup_{i \in I} \mathcal{E}_i$ is proper.

(2) Let $U: \mathcal{C} \rightarrow \mathcal{D}$ be an exact functor of triangulated categories and let \mathcal{E} be a proper class of triangles in \mathcal{D} . Let $\mathcal{F} := U^{-1}(\mathcal{E})$ be the class of triangles T in \mathcal{C} such that $U(T) \in \mathcal{E}$. Then \mathcal{F} is a proper class of triangles in \mathcal{C} .

(3) Let $F: \mathcal{C} \rightarrow \mathcal{U}$ be a (co-)homological functor from \mathcal{C} to an abelian category \mathcal{U} . Then we obtain a proper class of triangles $\mathcal{E}(F)$ in \mathcal{C} as follows:

- A triangle $A \rightarrow B \rightarrow C \rightarrow \Sigma(A)$ is in $\mathcal{E}(F)$ iff $\forall i \in \mathbb{Z}$, the induced sequence $0 \rightarrow F^i(A) \rightarrow F^i(B) \rightarrow F^i(C) \rightarrow 0$ is exact in \mathcal{U} , where $F^i = F\Sigma^i$.

(4) If $\mathcal{X} \subseteq \mathcal{C}$ is a class of objects satisfying $\Sigma(\mathcal{X}) = \mathcal{X}$, then we obtain a proper class of triangles $\mathcal{E}(\mathcal{X})$, resp. $\mathcal{E}^{\text{op}}(\mathcal{X})$, in \mathcal{C} as follows:

- A triangle $A \rightarrow B \rightarrow C \rightarrow \Sigma(A)$ is in $\mathcal{E}(\mathcal{X})$, resp. $\mathcal{E}^{\text{op}}(\mathcal{X})$, iff $\forall X \in \mathcal{X}$, the induced sequence $0 \rightarrow \mathcal{E}(X, A) \rightarrow \mathcal{E}(X, B) \rightarrow \mathcal{E}(X, C) \rightarrow 0$, resp. $0 \rightarrow \mathcal{E}(C, X) \rightarrow \mathcal{E}(B, X) \rightarrow \mathcal{E}(A, X) \rightarrow 0$, is exact in $\mathcal{A}b$.

The family of proper classes of triangles in \mathcal{C} is easily seen to be a (big) poset with 0 (the class Δ_0) and 1 (the class Δ), defining $\mathcal{E}_1 \preceq \mathcal{E}_2 \Leftrightarrow \mathcal{E}_1 \subseteq \mathcal{E}_2$.

2.3. Phantom Maps

In general it is difficult to distinguish the morphisms occurring in a triangle by a characteristic property. To solve this problem we proceed as follows.

DEFINITION 2.4. Let \mathcal{E} be a class of triangles in \mathcal{C} (not necessarily proper) and let $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} \Sigma(A)$ be a triangle in \mathcal{E} .

- (i) The morphism $f: A \rightarrow B$ is called an \mathcal{E} -*proper monic*.

- (ii) The morphism $g: B \rightarrow C$ is called an \mathcal{E} -proper epic.
- (iii) The morphism $h: C \rightarrow \Sigma(A)$ is called an \mathcal{E} -phantom map.

The class of \mathcal{E} -phantom maps is denoted by $\text{Ph}_{\mathcal{E}}(\mathcal{E})$.

The concept of phantom maps has homotopy-theoretic origin and the topological terminology is due to A. Heller (see [52]). For a justification of the definition see Section 3 and [52]. Let $\text{Ph}_{\mathcal{E}}(A, B)$ be the set of all \mathcal{E} -phantom maps from A to B and let $\text{Ph}_{\mathcal{E}}^n(A, B)$ be the set of all maps from A to B which can be written as a composition of n \mathcal{E} -phantom maps. We set $\text{Ph}_{\mathcal{E}}^n(\mathcal{E}) = \bigcup_{A, B \in \mathcal{E}} \text{Ph}_{\mathcal{E}}^n(A, B)$.

We recall that an ideal of \mathcal{E} is an additive subfunctor of $\mathcal{E}(-, -)$. An ideal \mathcal{J} of \mathcal{E} is called Σ -stable if $f \in \mathcal{J} \Leftrightarrow \Sigma^n(f) \in \mathcal{J}, \forall n \in \mathbb{Z}$. If \mathcal{J} is a Σ -stable ideal of \mathcal{E} , then define a class of triangles $\mathcal{E}_{\mathcal{J}}$ in \mathcal{E} as follows. The triangle $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} \Sigma(A) \in \mathcal{E}_{\mathcal{J}}$ if $h \in \mathcal{J}(C, \Sigma(A))$. A Σ -stable ideal \mathcal{J} in \mathcal{E} is called *saturated*, if the following condition is true: if f is $\mathcal{E}_{\mathcal{J}}$ -proper epic and $f \circ \sigma \in \mathcal{J}$ then $\sigma \in \mathcal{J}$. By the octahedral axiom this is equivalent to the condition that if g is an $\mathcal{E}_{\mathcal{J}}$ -proper monic and $\rho \circ g \in \mathcal{J}$ then $\rho \in \mathcal{J}$. Trivially if \mathcal{E} is a proper class of triangles, then $\text{Ph}_{\mathcal{E}}^n(\mathcal{E})$ is a Σ -stable saturated ideal of $\mathcal{E}, \forall n \geq 1$. Conversely if \mathcal{J} is a Σ -stable saturated ideal, then the class of triangles $\mathcal{E}_{\mathcal{J}}$ is proper and we have the relations: $\text{Ph}_{\mathcal{E}_{\mathcal{J}}}(\mathcal{E}) = \mathcal{J}$ and $\mathcal{E}_{\text{Ph}_{\mathcal{E}}(\mathcal{E})} = \mathcal{E}$. We collect these observations in the following.

PROPOSITION 2.5. *The map $\Phi: \mathcal{J} \mapsto \mathcal{E}_{\mathcal{J}}$ is a poset isomorphism between the poset of Σ -stable saturated ideals of \mathcal{E} and the poset of proper classes of triangles in \mathcal{E} , with inverse given by $\Psi: \mathcal{E} \mapsto \text{Ph}_{\mathcal{E}}(\mathcal{E})$, e.g., $\Psi(\Delta) = \mathcal{E}(-, -)$ and $\Psi(\Delta_0) = 0$.*

It follows that \mathcal{E} is a proper class of triangles in \mathcal{E} iff $\text{Ph}_{\mathcal{E}}(\mathcal{E})$ is a Σ -stable saturated ideal of \mathcal{E} . This indicates an analogy between the homological theory in \mathcal{E} based on proper classes of triangles and the formulation by Butler and Horrocks [18] of relative homology in an abelian category, based on subfunctors of $\mathcal{E}xt$.

2.4. Baer's Theory

Let \mathcal{E} be a proper class of triangles in \mathcal{E} and let $(T): A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} \Sigma(A)$ be a triangle in \mathcal{E} . We call $h: C \rightarrow \Sigma(A)$ the *characteristic class* of (T) , and usually we denote it by $\text{ch}(T) = h$. Let A, C be two objects of \mathcal{E} and consider the class $\mathcal{E}^*(C, A)$ of all triangles $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} \Sigma(A)$ in \mathcal{E} . We define a relation in $\mathcal{E}^*(C, A)$ as follows. If $(T_i): A \xrightarrow{f_i} B_i \xrightarrow{g_i} C \xrightarrow{h_i} \Sigma(A), i = 1, 2$, are elements of $\mathcal{E}^*(C, A)$, then we define $(T_1) \sim (T_2)$

if there exists a morphism of triangles

$$\begin{array}{ccccccc}
 A & \xrightarrow{f_1} & B_1 & \xrightarrow{g_1} & C & \xrightarrow{h_1} & \Sigma(A) \\
 \parallel \downarrow & & \beta \downarrow & & \parallel \downarrow & & \parallel \downarrow \\
 A & \xrightarrow{f_2} & B_2 & \xrightarrow{g_2} & C & \xrightarrow{h_2} & \Sigma(A).
 \end{array}$$

Obviously β is an isomorphism and \sim is an equivalence relation on the class $\mathcal{E}^*(C, A)$. Using base and cobase change, it is easy to see that we can define (as in the case of the classical Baer's theory in an abelian category) a sum in the class $\mathcal{E}(C, A) := \mathcal{E}^*(C, A)/\sim$ in such a way that $\mathcal{E}(C, A)$ becomes an abelian group and $\mathcal{E}(-, -): \mathcal{E}^{\text{op}} \times \mathcal{E} \rightarrow \mathcal{A}b$ an additive bifunctor. Trivially we have the following.

COROLLARY 2.6. *ch is an isomorphism of bifunctors: $\mathcal{E}(-, -) \rightarrow \text{Ph}_{\mathcal{E}}(-, \Sigma -)$.*

3. THE FREYD AND STEENROD CATEGORY OF A TRIANGULATED CATEGORY

3.1. The Freyd Category

Let \mathcal{E} be an additive category. We recall that an additive functor $F: \mathcal{E}^{\text{op}} \rightarrow \mathcal{A}b$ is called *finitely presented* if there exists an exact sequence $\mathcal{E}(-, A) \rightarrow \mathcal{E}(-, B) \rightarrow F \rightarrow 0$. We denote by $\mathcal{A}(\mathcal{E})$ or $\text{mod}(\mathcal{E})$ the category of finitely presented functors and we call it the *Freyd category* of \mathcal{E} . The Yoneda embedding is denoted by $\mathbb{Y}: \mathcal{E} \hookrightarrow \mathcal{A}(\mathcal{E})$. We define also the Freyd category $\mathcal{B}(\mathcal{E})$ to be $\mathcal{A}(\mathcal{E}^{\text{op}})^{\text{op}}$, so $\mathcal{B}(\mathcal{E}) = \text{mod}(\mathcal{E}^{\text{op}})^{\text{op}}$ is the dual of the category of finitely presented covariant functors over \mathcal{E} . The corresponding Yoneda embedding is denoted by $\mathbb{Y}^{\text{op}}: \mathcal{E} \hookrightarrow \mathcal{B}(\mathcal{E})$. The category $\mathcal{A}(\mathcal{E})$, resp. $\mathcal{B}(\mathcal{E})$, has cokernels, resp. kernels, and the functor \mathbb{Y} , resp. \mathbb{Y}^{op} , has the universal property that any additive functor $F: \mathcal{E} \rightarrow \mathcal{M}$ where \mathcal{M} has cokernels, resp. kernels, admits a unique cokernel, resp. kernel, preserving extension through \mathbb{Y} , resp. \mathbb{Y}^{op} . We recall that a morphism $f: A \rightarrow B$ is a *weak kernel* of $g: B \rightarrow C$, if the sequence of functors $\mathbb{Y}(A) \rightarrow^{\mathbb{Y}(f)} \mathbb{Y}(B) \rightarrow^{\mathbb{Y}(g)} \mathbb{Y}(C)$ is exact. The notion of a *weak cokernel* is defined dually. By results of Freyd [29], the category $\mathcal{A}(\mathcal{E})$, resp. $\mathcal{B}(\mathcal{E})$, is abelian iff any morphism in \mathcal{E} has a weak kernel, resp. weak cokernel. In this case \mathbb{Y} , resp. \mathbb{Y}^{op} , embeds \mathcal{E} as a class of projective, resp. injective, objects in $\mathcal{A}(\mathcal{E})$, resp. $\mathcal{B}(\mathcal{E})$, containing enough projective, resp. injective, objects.

We recall that an abelian category \mathcal{A} is called *Frobenius* if \mathcal{A} has enough projectives, enough injectives, and the projectives coincide with the injectives. By results of Freyd [29], for any triangulated category \mathcal{E} , the category $\mathcal{A}(\mathcal{E})$ is Frobenius abelian and the Yoneda embedding $\mathbb{Y}: \mathcal{E} \hookrightarrow \mathcal{A}(\mathcal{E})$ realizes \mathcal{E} as a class of projective–injective objects in $\mathcal{A}(\mathcal{E})$, containing both enough projectives and injectives. Moreover there exists a unique equivalence $\mathbb{D}: \mathcal{A}(\mathcal{E}) \xrightarrow{\sim} \mathcal{B}(\mathcal{E})$ such that $\mathbb{D}\mathbb{Y} = \mathbb{Y}^{\text{op}}$. It follows that any object F in $\mathcal{A}(\mathcal{E})$ has a description as a kernel, image, or cokernel of a morphism between representable functors. The Freyd category is the *universal homological category* of \mathcal{E} in the sense that \mathbb{Y} is a homological functor and if $F: \mathcal{E} \rightarrow \mathcal{H}$ is a homological functor to an abelian category \mathcal{H} , then there exists a unique exact functor $F^*: \mathcal{A}(\mathcal{E}) \rightarrow \mathcal{H}$ such that $F^*\mathbb{Y} = F$. Note that if idempotents split in \mathcal{E} , then \mathbb{Y} induces an equivalence $\mathcal{E} \approx \text{Proj}(\mathcal{A}(\mathcal{E})) = \text{Inj}(\mathcal{A}(\mathcal{E}))$.

3.2. The Steenrod Category

Let \mathcal{E} be a triangulated category and let \mathcal{F} be a class of morphisms in \mathcal{E} . We denote by $\mathcal{A}(\mathcal{F})$ the full subcategory of $\mathcal{A}(\mathcal{E})$ consisting of all functors of the form $\text{Im } \mathbb{Y}(h)$, for $h \in \mathcal{F}$. Similarly let $\mathcal{B}(\mathcal{F})$ be the full subcategory of $\mathcal{B}(\mathcal{E})$ consisting of all functors of the form $\text{Im } \mathbb{Y}^{\text{op}}(h)$. Trivially the suspension Σ of \mathcal{E} extends to an automorphism of $\mathcal{A}(\mathcal{E})$ and $\mathcal{B}(\mathcal{E})$, denoted also by Σ . A full subcategory \mathcal{U} of $\mathcal{A}(\mathcal{E})$ or $\mathcal{B}(\mathcal{E})$ is called Σ -stable if \mathcal{U} is closed under Σ .

THEOREM 3.1. *The following statements are equivalent.*

- (i) \mathcal{F} is a Σ -stable saturated ideal of \mathcal{E} .
- (ii) $\mathcal{A}(\mathcal{F})$ is a Σ -stable Serre subcategory of $\mathcal{A}(\mathcal{E})$.
- (iii) $\mathcal{B}(\mathcal{F})$ is a Σ -stable Serre subcategory of $\mathcal{B}(\mathcal{E})$.

If one of the above is true, then there is an equivalence $\mathcal{A}(\mathcal{F}) \xrightarrow{\sim} \mathcal{B}(\mathcal{F})$ which induces an equivalence $\mathcal{A}(\mathcal{E})/\mathcal{A}(\mathcal{F}) \xrightarrow{\sim} \mathcal{B}(\mathcal{E})/\mathcal{B}(\mathcal{F})$ in the localizations.

Proof. (ii) \Rightarrow (i) Let $\mathcal{A}(\mathcal{E})/\mathcal{A}(\mathcal{F})$ be the quotient category and let $\mathbf{Q}: \mathcal{A}(\mathcal{E}) \rightarrow \mathcal{A}(\mathcal{E})/\mathcal{A}(\mathcal{F})$ be the exact quotient functor. Then the composition $\mathbf{S}_{\mathcal{F}} = \mathbf{Q}\mathbb{Y}$ is a homological functor and it is easy to see that the kernel ideal $\ker \mathbf{S}_{\mathcal{F}} = \mathcal{F}$. Then obviously \mathcal{F} is a Σ -stable saturated ideal of \mathcal{E} .

(i) \Rightarrow (ii) Assume that \mathcal{F} is a Σ -stable saturated ideal of \mathcal{E} . Then $\mathcal{A}(\mathcal{F})$ is a Σ -stable full subcategory of $\mathcal{A}(\mathcal{E})$. Let $\beta: G \rightarrow H$ be an epimorphism in $\mathcal{A}(\mathcal{E})$ with $G \in \mathcal{A}(\mathcal{F})$. Then there exists a morphism $\mu: B \rightarrow K$ in \mathcal{F} such that $\text{Im } \mathbb{Y}(\mu) = G$ and a morphism $\nu: D \rightarrow L$ such that $\text{Im } \mathbb{Y}(\nu) = H$. Let $\mathbb{Y}(\mu) = \rho \circ \kappa$ and $\mathbb{Y}(\nu) = \sigma \circ \lambda$ be the canonical factor-

izations of $\mathbb{Y}(\mu)$ and $\mathbb{Y}(\nu)$. Since β is epic and $\mathbb{Y}(D)$ is projective, there exists $\omega: \mathbb{Y}(D) \rightarrow G$ such that $\omega \circ \beta = \sigma$. Similarly since ρ is epic, there exists a morphism $\mathbb{Y}(\epsilon): \mathbb{Y}(D) \rightarrow \mathbb{Y}(B)$ such that $\mathbb{Y}(\epsilon) \circ \rho = \omega$. Hence $\mathbb{Y}(\epsilon) \circ \rho \circ \beta = \sigma$. On the other hand, since $\mathbb{Y}(L)$ is injective, there exists a morphism $\mathbb{Y}(\delta): \mathbb{Y}(K) \rightarrow \mathbb{Y}(L)$ such that $\kappa \circ \mathbb{Y}(\delta) = \beta \circ \lambda$. Then $\mathbb{Y}(\epsilon) \circ \mathbb{Y}(\mu) \circ \mathbb{Y}(\delta) = \mathbb{Y}(\epsilon) \circ \rho \circ \beta \circ \lambda = \sigma \circ \lambda = \mathbb{Y}(\nu)$. Hence $\epsilon \circ \mu \circ \delta = \nu$. Since $\mu \in \mathcal{S}$, it follows that $\nu \in \mathcal{S}$. Hence $H \in \mathcal{A}(\mathcal{S})$ and $\mathcal{A}(\mathcal{S})$ is closed under quotient objects. The proof that $\mathcal{A}(\mathcal{S})$ is closed under subobjects is dual and is left to the reader. It remains to prove that $\mathcal{A}(\mathcal{S})$ is closed under extensions. Let $0 \rightarrow F_1 \xrightarrow{\alpha} F_2 \xrightarrow{\beta} F_3 \rightarrow 0$ be a short exact sequence in $\mathcal{A}(\mathcal{E})$ with $F_1, F_3 \in \mathcal{A}(\mathcal{S})$. Let $A_i \xrightarrow{f_i} B_i \xrightarrow{g_i} C_i \rightarrow \Sigma(A_i)$ be triangles in \mathcal{E} such that $F_i = \text{Im } \mathbb{Y}(g_i)$ and $g_1, g_3 \in \mathcal{S}$, $i = 1, 3$. Let $\mathbb{Y}(g_i) = \rho_i \circ \sigma_i$ be the canonical factorizations. Since $\mathbb{Y}(B_3)$ is projective, there exists $\xi: \mathbb{Y}(B_3) \rightarrow F_2$ such that $\xi \circ \beta = \rho_3$. Since $\mathbb{Y}(C_1)$ is injective there exists $\omega: F_2 \rightarrow \mathbb{Y}(C_1)$ such that $\alpha \circ \omega = \sigma_1$. It is easy to see that F_2 is the image of the morphism $\mathbb{Y}(g_2)$, where

$$g_2 = \begin{pmatrix} g_1 & 0 \\ \tau & g_3 \end{pmatrix}: B_1 \oplus B_3 \rightarrow C_1 \oplus C_3$$

and $\tau: B_3 \rightarrow C_1$ is the unique morphism such that $\mathbb{Y}(\tau) = \xi \circ \omega$. Hence it suffices to show that g_2 or equivalently τ is in \mathcal{S} . Since $\mathbb{Y}(f_3) \circ \rho_3 = \mathbb{Y}(f_3) \circ \xi \circ \beta = 0$, there exists a morphism $\kappa: \mathbb{Y}(A_3) \rightarrow F_1$ such that $\mathbb{Y}(f_3) \circ \xi = \kappa \circ \alpha$. Since $\mathbb{Y}(A_3)$ is projective, there exists $\lambda: \mathbb{Y}(A_3) \rightarrow \mathbb{Y}(B_1)$ such that $\lambda \circ \rho_1 = \kappa$. Then $\mathbb{Y}(f_3) \circ \xi = \lambda \circ \rho_1 \circ \alpha$; hence $\mathbb{Y}(f_3) \circ \xi \circ \omega = \lambda \circ \rho_1 \circ \alpha \circ \omega = \lambda \circ \rho_1 \circ \sigma_1 = \lambda \circ \mathbb{Y}(g_1)$. Since λ is of the form $\mathbb{Y}(\mu)$, it follows that $\mathbb{Y}(f_3) \circ \xi \circ \omega = \mathbb{Y}(f_3) \circ \mathbb{Y}(\tau) = \mathbb{Y}(\mu \circ g_1) \Rightarrow f_3 \circ \tau = \mu \circ g_1$. Since g_1 is in \mathcal{S} , it follows that $f_3 \circ \tau \in \mathcal{S}$. But f_3 is $\mathcal{E}_{\mathcal{S}}$ -epic. Since \mathcal{S} is saturated, we have $\tau \in \mathcal{S}$. We conclude that $F_2 \in \mathcal{A}(\mathcal{S})$ and $\mathcal{A}(\mathcal{S})$ is closed under extensions.

The proof of (i) \Leftrightarrow (iii) is similar and is left to the reader. Finally it is easy to see that the equivalence $\mathbb{D}: \mathcal{A}(\mathcal{E}) \rightarrow \mathcal{B}(\mathcal{E})$ defined by $\mathbb{D}(\text{Coker } \mathbb{Y}(f)) = \text{Ker } \mathbb{Y}^{\text{op}}(f)$ sends $\mathcal{A}(\mathcal{S})$ to $\mathcal{B}(\mathcal{S})$, so it induces an equivalence $\mathcal{A}(\mathcal{E})/\mathcal{A}(\mathcal{S}) \approx \mathcal{B}(\mathcal{E})/\mathcal{B}(\mathcal{S})$. ■

COROLLARY 3.2. *The map $\mathcal{E} \mapsto \mathcal{A}(\text{Ph}_{\mathcal{E}}(\mathcal{E}))$ gives a bijection between proper classes of triangles in \mathcal{E} and Σ -stable Serre subcategories of $\mathcal{A}(\mathcal{E})$, with the inverse the map $\mathcal{U} \mapsto \mathcal{E}(F_{\mathcal{U}})$, where $F_{\mathcal{U}}$ is the homological functor $\mathcal{E} \rightarrow \mathcal{A}(\mathcal{E}) \rightarrow \mathcal{A}(\mathcal{E})/\mathcal{U}$.*

DEFINITION 3.3. Let \mathcal{E} be a proper class of triangles in \mathcal{E} . The *Steenrod category*, resp. *dual Steenrod category*, of \mathcal{E} with respect to \mathcal{E} is

defined by

$$\mathcal{S}_{\mathcal{E}}(\mathcal{C}) := \mathcal{A}(\mathcal{C})/\mathcal{A}(\text{Ph}_{\mathcal{E}}(\mathcal{C})), \quad \text{resp. } \mathcal{S}_{\mathcal{E}}^{\text{op}}(\mathcal{C}) := \mathcal{B}(\mathcal{C})/\mathcal{B}(\text{Ph}_{\mathcal{E}}(\mathcal{C}))$$

where $\text{Ph}_{\mathcal{E}}(\mathcal{C})$ is the Σ -stable saturated ideal of \mathcal{E} -phantom maps.

The canonical functor $\mathbf{S}: \mathcal{C} \rightarrow \mathcal{S}_{\mathcal{E}}(\mathcal{C})$, resp. $\mathbf{T}: \mathcal{C} \rightarrow \mathcal{S}_{\mathcal{E}}^{\text{op}}(\mathcal{C})$, given by the composition of the Yoneda embedding $\mathbb{Y}: \mathcal{C} \rightarrow \mathcal{A}(\mathcal{C})$, resp. $\mathbb{Y}^{\text{op}}: \mathcal{C} \rightarrow \mathcal{B}(\mathcal{C})$, and the quotient functor $\mathcal{A}(\mathcal{C}) \rightarrow \mathcal{A}(\mathcal{C})/\mathcal{A}(\text{Ph}_{\mathcal{E}}(\mathcal{C}))$, resp. $\mathcal{B}(\mathcal{C}) \rightarrow \mathcal{B}(\mathcal{C})/\mathcal{B}(\text{Ph}_{\mathcal{E}}(\mathcal{C}))$, is called the *projectivization functor*, resp. *injectivization functor*.

We refer to the next section for the justification of the above definition. The following result summarizes the above observations and explains the terminology for the phantom maps, since these maps are precisely the maps which are invisible in the Steenrod category.

THEOREM 3.4. *Let \mathcal{E} be a proper class of triangles in \mathcal{C} . Then the Steenrod category $\mathcal{S}_{\mathcal{E}}(\mathcal{C})$ is abelian and the projectivization functor $\mathbf{S}: \mathcal{C} \rightarrow \mathcal{S}_{\mathcal{E}}(\mathcal{C})$ is a homological functor having the property that $\mathbf{S}(\phi) = 0$, $\forall \phi \in \text{Ph}_{\mathcal{E}}(\mathcal{C})$. Moreover the pair $(\mathbf{S}, \mathcal{S}_{\mathcal{E}}(\mathcal{C}))$ has the following universal property.*

- *If $H: \mathcal{C} \rightarrow \mathcal{M}$ is a homological functor to an abelian category \mathcal{M} such that $H(\phi) = 0$, $\forall \phi \in \text{Ph}_{\mathcal{E}}(\mathcal{C})$, then there exists a unique exact functor $H^*: \mathcal{S}_{\mathcal{E}}(\mathcal{C}) \rightarrow \mathcal{M}$ such that $H^*\mathbf{S} = H$.*

Proof. It suffices to prove only the universal property. Let $H': \mathcal{A}(\mathcal{C}) \rightarrow \mathcal{M}$ be the unique exact functor which extends uniquely H through \mathbb{Y} . Since H kills \mathcal{E} -phantom maps, H' kills objects of the Serre subcategory $\mathcal{A}(\text{Ph}_{\mathcal{E}}(\mathcal{C}))$, so there exists a unique exact functor $H^*: \mathcal{S}_{\mathcal{E}}(\mathcal{C}) \rightarrow \mathcal{M}$ such that $H^*\mathbf{Q} = H'$. Then $H^*\mathbf{S} = H^*\mathbf{Q}\mathbb{Y} = H'\mathbb{Y} = H$ and H^* is the unique exact functor with this property. ■

Observe that $\mathcal{S}_{\Delta}(\mathcal{C}) = 0$ and $\mathcal{S}_{\Delta_0}(\mathcal{C}) = \mathcal{A}(\mathcal{C})$. By the above theorem, there exists an equivalence $\mathbf{D}: \mathcal{S}_{\mathcal{E}}(\mathcal{C}) \xrightarrow{\sim} \mathcal{S}_{\mathcal{E}}^{\text{op}}(\mathcal{C})$ such that $\mathbf{D}\mathbf{S} = \mathbf{T}$. We leave to the reader to formulate the above result for the dual Steenrod category $\mathcal{S}_{\mathcal{E}}^{\text{op}}(\mathcal{C})$ and the injectivization functor \mathbf{T} .

We regard the Steenrod category as an abelian approximation of \mathcal{C} . So it is useful to know when the projectivization functor $\mathbf{S}: \mathcal{C} \rightarrow \mathcal{S}_{\mathcal{E}}(\mathcal{C})$ is non-trivial.

DEFINITION 3.5. The proper class of triangles \mathcal{E} *generates* \mathcal{C} if the projectivization functor $\mathbf{S}: \mathcal{C} \rightarrow \mathcal{S}_{\mathcal{E}}(\mathcal{C})$ reflects zero objects, i.e., $\mathbf{S}(A) = 0 \Rightarrow A = 0$.

We recall that the *Jacobson radical* $\mathcal{J}ac(\mathcal{E})$ of an additive category \mathcal{E} is the ideal in \mathcal{E} defined by $\mathcal{J}ac(\mathcal{E})(A, B) = \{f: A \rightarrow B; \forall g: B \rightarrow A, \text{ the morphism } 1_A - f \circ g \text{ is invertible}\}$. If F is an additive functor, we denote by $\ker(F)$, resp. $\text{Ker}(F)$, the ideal of morphisms, resp. the full subcategory of objects, annihilated by F . If \mathcal{I} is a Σ -stable ideal of \mathcal{E} , let $\mathbf{S}_{\mathcal{I}}: \mathcal{E} \rightarrow \mathcal{A}(\mathcal{E})/\mathcal{A}(\mathcal{I})$ be the canonical functor, i.e., the composition of $\mathbb{Y}: \mathcal{E} \hookrightarrow \mathcal{A}(\mathcal{E})$ and the quotient functor $\mathcal{A}(\mathcal{E}) \rightarrow \mathcal{A}(\mathcal{E})/\mathcal{A}(\mathcal{I})$.

LEMMA 3.6. $\ker(\mathbf{S}_{\mathcal{I}}) = \mathcal{I}$. Moreover $\mathcal{I} \subseteq \mathcal{J}ac(\mathcal{E})$ iff $\text{Ker}(\mathbf{S}_{\mathcal{I}}) = 0$.

Proof. The first assertion is trivial. If $\mathcal{I} \subseteq \mathcal{J}ac(\mathcal{E})$ and $\mathbf{S}_{\mathcal{I}}(A) = 0$, then obviously $1_A \in \mathcal{I} \subseteq \mathcal{J}ac(\mathcal{E})$; hence $A = 0$. Conversely if $\text{Ker}(\mathbf{S}_{\mathcal{I}}) = 0$ and $f: A \rightarrow B$ is in \mathcal{I} , let $g: B \rightarrow A$ be any morphism and let $\alpha := 1_A - f \circ g$. Then trivially $\mathbf{S}_{\mathcal{I}}(\alpha) = \mathbf{S}_{\mathcal{I}}(1_A)$. If $A \rightarrow^{\alpha} A \rightarrow C \rightarrow \Sigma(A)$ is a triangle in \mathcal{E} , then $\mathbf{S}_{\mathcal{I}}(C) = 0$; hence $C = 0$. It follows that α is invertible and this implies that $f \in \mathcal{J}ac(\mathcal{E})$. ■

COROLLARY 3.7. \mathcal{E} generates \mathcal{E} iff $\text{Ph}_{\mathcal{E}}(\mathcal{E}) \subseteq \mathcal{J}ac(\mathcal{E})$.

4. PROJECTIVE OBJECTS, RESOLUTIONS, AND DERIVED FUNCTORS

We fix throughout a proper class of triangles \mathcal{E} in the triangulated category \mathcal{E} .

4.1. Projective Objects and Global Dimension

DEFINITION 4.1. An object $P \in \mathcal{E}$, resp. $I \in \mathcal{E}$, is called \mathcal{E} -projective, resp. \mathcal{E} -injective, if for any triangle $A \rightarrow B \rightarrow C \rightarrow \Sigma(A)$ in \mathcal{E} , the induced sequence $0 \rightarrow \mathcal{E}(P, A) \rightarrow \mathcal{E}(P, B) \rightarrow \mathcal{E}(P, C) \rightarrow 0$, resp. $0 \rightarrow \mathcal{E}(C, I) \rightarrow \mathcal{E}(B, I) \rightarrow \mathcal{E}(A, I) \rightarrow 0$, is exact in $\mathcal{A}b$.

We denote by $\mathcal{P}(\mathcal{E})$ ($\mathcal{A}(\mathcal{E})$) the full subcategory of \mathcal{E} -projective (\mathcal{E} -injective) objects of \mathcal{E} . As an immediate consequence of the above definition we have that the categories $\mathcal{P}(\mathcal{E})$ and $\mathcal{A}(\mathcal{E})$ are full, additive, closed under isomorphisms, direct summands, and Σ -stable, i.e., $\Sigma(\mathcal{P}(\mathcal{E})) = \mathcal{P}(\mathcal{E})$ and $\Sigma(\mathcal{A}(\mathcal{E})) = \mathcal{A}(\mathcal{E})$.

We say that \mathcal{E} has enough \mathcal{E} -projectives if for any object $A \in \mathcal{E}$ there exists a triangle $K \rightarrow P \rightarrow A \rightarrow^h \Sigma(K)$ in \mathcal{E} with $P \in \mathcal{P}(\mathcal{E})$. Observe that in this case h is a *weakly universal* \mathcal{E} -phantom map out of A in the sense that any \mathcal{E} -phantom map $h': A \rightarrow B$ factors through h in a not necessarily unique way. Dually one defines when \mathcal{E} has enough \mathcal{E} -injectives. We study only the case of \mathcal{E} -projectives since the study of the \mathcal{E} -injectives is dual. However, we use freely the dual results when it is necessary. Note that an

equivalent formulation of the setup of a proper class of triangles in \mathcal{E} such that \mathcal{E} has enough \mathcal{E} -projectives is that of a projective class of morphisms in \mathcal{E} in the sense of Eilenberg and Moore [26]; see [19].

We recall that a full subcategory $\mathcal{L} \subseteq \mathcal{E}$ is called *contravariantly finite* [6] if for any $A \in \mathcal{E}$ there exists a morphism $f: X \rightarrow A$ with $X \in \mathcal{L}$, such that any morphism $g: X' \rightarrow A$ with $X' \in \mathcal{L}$ factors through f . The following result is a direct consequence of the definitions and its proof is left to the reader.

LEMMA 4.2. (i) $\forall P \in \mathcal{P}(\mathcal{E}) : \text{Ph}_{\mathcal{E}}(P, -) = 0$ and $\forall I \in \mathcal{A}(\mathcal{E}) : \text{Ph}_{\mathcal{E}}(-, I) = 0$.

(ii) If \mathcal{E} has enough \mathcal{E} -projectives, then \mathcal{E} generates \mathcal{E} iff $A \in \mathcal{E}$ and $\mathcal{E}(P, A) = 0, \forall P \in \mathcal{P}(\mathcal{E})$, implies $A = 0$.

(iii) \mathcal{E} has enough \mathcal{E} -projectives iff $\mathcal{P}(\mathcal{E})$ is contravariantly finite. In this case, $\mathcal{E} = \mathcal{E}(\mathcal{P}(\mathcal{E}))$; i.e., a triangle $D \rightarrow F \rightarrow C \rightarrow \Sigma(D)$ is in \mathcal{E} iff $\forall P \in \mathcal{P}(\mathcal{E})$ the induced sequence $0 \rightarrow \mathcal{E}(P, D) \rightarrow \mathcal{E}(P, F) \rightarrow \mathcal{E}(P, C) \rightarrow 0$ is exact. Moreover a morphism $f: A \rightarrow B$ is \mathcal{E} -phantom iff $\mathcal{E}(P, f) = 0, \forall P \in \mathcal{P}(\mathcal{E})$. Further $\forall A \in \mathcal{E} : A \in \mathcal{P}(\mathcal{E})$ iff $\text{Ph}_{\mathcal{E}}(A, -) = 0$.

(iv) Let \mathcal{P} be a full additive contravariantly finite subcategory of \mathcal{E} , closed under direct summands such that $\Sigma(\mathcal{P}) = \mathcal{P}$. Then $\mathcal{P} = \mathcal{P}(\mathcal{E}(\mathcal{P}))$.

COROLLARY 4.3. The maps $\mathcal{P} \mapsto \mathcal{E}(\mathcal{P}), \mathcal{E} \mapsto \mathcal{P}(\mathcal{E})$ are inverse bijections between contravariantly finite Σ -stable additive subcategories of \mathcal{E} closed under direct summands and proper classes of triangles \mathcal{E} in \mathcal{E} such that \mathcal{E} has enough \mathcal{E} -projectives.

PROPOSITION 4.4 (Schanuel’s lemma). If $K_i \xrightarrow{f_i} P_i \xrightarrow{g_i} A \xrightarrow{h_i} \Sigma(K_i)$ are triangles in \mathcal{E} with $P_i \in \mathcal{P}(\mathcal{E}), i = 1, 2$, then $K_1 \oplus P_2 \cong K_2 \oplus P_1$.

Proof. Consider the octahedral diagram induced from the composition $g_1 \circ h_2$:

$$\begin{array}{ccccccc}
 P_1 & \xrightarrow{g_1} & A & \xrightarrow{h_1} & \Sigma(K_1) & \xrightarrow{-\Sigma(f_1)} & \Sigma(P_1) \\
 \parallel \downarrow & & h_2 \downarrow & & \downarrow & & \parallel \downarrow \\
 P_1 & \xrightarrow{g_1 \circ h_2} & \Sigma(K_2) & \longrightarrow & X & \longrightarrow & \Sigma(P_1) \\
 g_1 \downarrow & & \parallel \downarrow & & \downarrow & & \downarrow \\
 A & \xrightarrow{h_2} & \Sigma(K_2) & \xrightarrow{-\Sigma(f_2)} & \Sigma(P_2) & \xrightarrow{-\Sigma(g_2)} & \Sigma(A) \\
 \downarrow & & 0 \downarrow & & -\Sigma(g_2) \circ \Sigma(h_1) \downarrow & & \downarrow \\
 0 & \longrightarrow & \Sigma^2(K_1) & \xrightarrow{=} & \Sigma^2(K_1) & \longrightarrow & 0 \quad .
 \end{array}$$

Since h_1, h_2 are \mathcal{E} -phantoms and $P_1, \Sigma(P_2)$ are \mathcal{E} -projectives, we have $g_1 \circ h_2 = 0$ and $g_2 \circ h_1 = 0$. Hence the second horizontal and the third vertical triangles are split. So $X \cong \Sigma(K_2) \oplus \Sigma(P_1) \cong \Sigma(K_1) \oplus \Sigma(P_2) \Rightarrow K_2 \oplus P_1 \cong K_1 \oplus P_2$. ■

If $K \rightarrow P \rightarrow A \rightarrow \Sigma(K)$ is in \mathcal{E} with P in $\mathcal{P}(\mathcal{E})$, then we call the object K a first \mathcal{E} -syzygy of A . An n th \mathcal{E} -syzygy of A is defined as usual by induction. By Schanuel's lemma any two \mathcal{E} -syzygies of A are isomorphic modulo \mathcal{E} -projectives.

We define inductively the \mathcal{E} -projective dimension \mathcal{E} -p.d A of an object $A \in \mathcal{E}$ as follows. If $A \in \mathcal{P}(\mathcal{E})$ then \mathcal{E} -p.d $A = 0$. Next if \mathcal{E} -p.d $A > 0$ define \mathcal{E} -p.d $A \leq n$ if there exists a triangle $K \rightarrow P \rightarrow A \rightarrow \Sigma(K)$ in \mathcal{E} with $P \in \mathcal{P}(\mathcal{E})$ and \mathcal{E} -p.d $K \leq n - 1$. Finally define \mathcal{E} -p.d $A = n$ if \mathcal{E} -p.d $A \leq n$ and \mathcal{E} -p.d $A \not\leq n - 1$. Of course we set \mathcal{E} -p.d $A = \infty$, if \mathcal{E} -p.d $A \neq n, \forall n \geq 0$. The \mathcal{E} -global dimension \mathcal{E} -gl.dim \mathcal{E} of \mathcal{E} is defined by \mathcal{E} -gl.dim $\mathcal{E} = \sup\{\mathcal{E}$ -p.d $A; A \in \mathcal{E}\}$.

EXAMPLE 4.5. $\forall C \neq 0: \Delta$ -p.d $C = \infty$, so if $0 \neq \mathcal{E}: \Delta$ -gl.dim $\mathcal{E} = \infty$. On the other extreme Δ_0 -gl.dim $\mathcal{E} = 0$.

The following is proved using standard arguments.

PROPOSITION 4.6. (i) $\forall A, B \in \mathcal{E}: \mathcal{E}$ -p.d $A \oplus B = \max\{\mathcal{E}$ -p.d A, \mathcal{E} -p.d $B\}$.

(ii) $\forall A \in \mathcal{E}, \forall n \in \mathbb{Z}: \mathcal{E}$ -p.d $A = \mathcal{E}$ -p.d $\Sigma^n(A)$.

(iii) Let $A \rightarrow P \rightarrow B \rightarrow \Sigma(A)$ be a triangle in \mathcal{E} with $P \in \mathcal{P}(\mathcal{E})$. Then either $B \in \mathcal{P}(\mathcal{E})$ or else \mathcal{E} -p.d $A = \mathcal{E}$ -p.d $B - 1$.

4.2. Resolutions and Derived Functors

DEFINITION 4.7. An \mathcal{E} -exact complex $X^\bullet \rightarrow A$ over $A \in \mathcal{E}$ is a diagram $\dots \rightarrow X^{n+1} \xrightarrow{d^{n+1}} X^n \rightarrow \dots \rightarrow X^1 \xrightarrow{d^1} X^0 \xrightarrow{d^0} A \rightarrow 0$ in \mathcal{E} , such that $\forall n \geq 0$:

(i) There are triangles $K^{n+1} \xrightarrow{g^n} X^n \xrightarrow{f^n} K^n \xrightarrow{h^n} \Sigma(K^{n+1})$ in \mathcal{E} , where $K^0 := A$.

(ii) The differential $d^n = f^n \circ g^{n-1}, \forall n \geq 1$ and $d^0 = f^0$.

An \mathcal{E} -projective resolution of $A \in \mathcal{E}$ is an \mathcal{E} -exact complex $P^\bullet \rightarrow A$ as above such that $P^n \in \mathcal{P}(\mathcal{E}), \forall n \geq 0$.

The next result gives a useful characterization of an \mathcal{E} -projective resolution.

PROPOSITION 4.8. Assume that \mathcal{E} has enough \mathcal{E} -projectives and consider a complex $P^\bullet: \dots \rightarrow P^1 \xrightarrow{d^1} P^0 \xrightarrow{d^0} A \rightarrow 0$ in \mathcal{E} with $P^n \in \mathcal{P}(\mathcal{E}), \forall n \geq 0$.

Then P^\bullet is an \mathcal{E} -projective resolution of A iff the induced complex $\mathcal{E}(Q, P^\bullet)$ is exact, $\forall Q \in \mathcal{P}(\mathcal{E})$.

Proof. Assume that the induced complex $\mathcal{E}(Q, P^\bullet)$ is exact in $\mathcal{A}b$, $\forall Q \in \mathcal{P}(\mathcal{E})$. Set $f^0 := d^0$ and let $(T_0): K^1 \rightarrow^{g^0} P^0 \rightarrow^{f^0} A \rightarrow^{h^0} \Sigma(K^1)$ be a triangle in \mathcal{E} . Since $\mathcal{E}(Q, f^0)$ is epic, we have $\mathcal{E}(Q, h^0) = 0$, $\forall Q \in \mathcal{P}(\mathcal{E})$. By Lemma 4.2 it follows that h^0 is \mathcal{E} -phantom; hence (T_0) is in \mathcal{E} . Since $d^1 \circ f^0 = 0$, there exists a morphism $f^1: P^1 \rightarrow K^1$ such that $f^1 \circ g^0 = d^1$. It follows easily that $\mathcal{E}(Q, f^1)$ is epic. Let $(T_1): K^2 \rightarrow^{g^1} P^1 \rightarrow^{f^1} K^1 \rightarrow^{h^1} \Sigma(K^2)$ be a triangle in \mathcal{E} . As above we have that (T_1) is in \mathcal{E} . Since $d^2 \circ d^1 = d^2 \circ f^1 \circ g^0 = 0$, we have $\mathcal{E}(\mathcal{P}(\mathcal{E}), d^2 \circ f^1) \circ \mathcal{E}(\mathcal{P}(\mathcal{E}), g^0) = 0$. Since $\mathcal{E}(\mathcal{P}(\mathcal{E}), g^0)$ is monic, we have $\mathcal{E}(\mathcal{P}(\mathcal{E}), d^2 \circ f^1) = 0$; hence by Lemma 4.2, $d^2 \circ f^1$ is \mathcal{E} -phantom. Since $P^2 \in \mathcal{P}(\mathcal{E})$, by Lemma 4.2, we have $d^2 \circ f^1 = 0$. So there exists a morphism $f^2: P^2 \rightarrow K^2$ such that $d^2 = f^2 \circ g^1$. Continuing inductively in this way we see that P^\bullet is an \mathcal{E} -projective resolution of A . The converse is trivial. \blacksquare

COROLLARY 4.9. *Let $0 \rightarrow P_i^n \rightarrow P_i^{n-1} \rightarrow \dots \rightarrow P_i^0 \rightarrow A_i \rightarrow 0$ be \mathcal{E} -projective resolutions of A_i , $i = 1, 2$. If $A_1 \cong A_2$, then $P_1^0 \oplus P_2^0 \oplus P_1^1 \oplus \dots \cong P_2^0 \oplus P_1^1 \oplus P_2^1 \dots$.*

From now on we assume that \mathcal{E} has enough \mathcal{E} -projectives. For any object $A \in \mathcal{E}$ we fix an \mathcal{E} -projective resolution of A

$$\mathbf{P}_A^\bullet \quad \dots \rightarrow P_A^{n+1} \xrightarrow{\varepsilon_A^{n+1}} P_A^n \rightarrow \dots \rightarrow P_A^1 \xrightarrow{\varepsilon_A^1} P_A^0 \xrightarrow{f_A^0} A \rightarrow 0$$

which, by definition, is the ‘‘Yoneda composition’’ of triangles

$$\mathbf{T}_A^n \quad K_A^{n+1} \xrightarrow{g_A^n} P_A^n \xrightarrow{f_A^n} K_A^n \xrightarrow{h_A^n} \Sigma(K_A^{n+1}) \in \mathcal{E}$$

with $P_A^n \in \mathcal{P}(\mathcal{E})$, $\forall n \geq 0$ and where $K_A^0 := A$. Using standard arguments from relative homological algebra, one can prove a version of the comparison theorem [68] for \mathcal{E} -projective resolutions, the obvious formulation of which is left to the reader. It follows that any two \mathcal{E} -projective resolutions of an object are homotopy equivalent. Using Schanuel’s lemma and the above observations we have the following consequence.

COROLLARY 4.10. *\mathcal{E} -p.d $A \leq n$ iff there exists an \mathcal{E} -projective resolution of A of the form $0 \rightarrow P^n \rightarrow \dots \rightarrow P^1 \rightarrow P^0 \rightarrow A \rightarrow 0$.*

PROPOSITION 4.11 (horseshoe lemma). *Let $(T): A \rightarrow^\alpha B \rightarrow^\beta C \rightarrow^\gamma \Sigma(A)$ be a triangle in \mathcal{E} . Then there are \mathcal{E} -projective resolutions $\mathbf{P}^\bullet(A)$,*

$\mathbf{P}^\bullet(B)$, and $\mathbf{P}^\bullet(C)$ of A , B , and C , respectively, and a commutative diagram

$$\begin{array}{ccccccc} \mathbf{P}^\bullet(A) & \xrightarrow{p^\bullet} & \mathbf{P}^\bullet(B) & \xrightarrow{q^\bullet} & \mathbf{P}^\bullet(C) & \xrightarrow{0} & \Sigma(\mathbf{P}^\bullet(A)) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ A & \xrightarrow{\alpha} & B & \xrightarrow{\beta} & C & \xrightarrow{\gamma} & \Sigma(A) \end{array}$$

such that $\mathbf{P}^n(A) \rightarrow^{p^n} \mathbf{P}^n(B) \rightarrow^{q^n} \mathbf{P}^n(C) \rightarrow^0 \Sigma(\mathbf{P}^n(A))$ is a split triangle, $\forall n \geq 0$. Such a diagram is called an \mathcal{E} -projective resolution of the triangle (T).

Proof. Let $f_A^0: P_A^0 \rightarrow A$ and $f_C^0: P_C^0 \rightarrow C$ be \mathcal{E} -proper epics with $P_A^0, P_C^0 \in \mathcal{P}(\mathcal{E})$. Since γ is \mathcal{E} -phantom, $f_C^0 \circ \gamma = 0$. Using that Σ is an automorphism and a result of Verdier (see [71, p. 378]), the commutative square on the top left corner below is embedded in a diagram

$$\begin{array}{ccccccc} P_C^0 & \xrightarrow{0} & \Sigma(P_A^0) & \xrightarrow{-\Sigma(p)} & \Sigma(P_B^0) & \xrightarrow{\Sigma(q)} & \Sigma(P_C^0) \\ f_C^0 \downarrow & & -\Sigma(f_A^0) \downarrow & & -\Sigma(f_B^0) \downarrow & & \Sigma(f_C^0) \downarrow \\ C & \xrightarrow{\gamma} & \Sigma(A) & \xrightarrow{-\Sigma(\alpha)} & \Sigma(B) & \xrightarrow{-\Sigma(\beta)} & \Sigma(C) \\ h_C^0 \downarrow & & -\Sigma(h_A^0) \downarrow & & -\Sigma(h_B^0) \downarrow & & \Sigma(h_C^0) \downarrow \\ \Sigma(K_C^1) & \xrightarrow{-\Sigma(\phi)} & \Sigma^2(K_A^1) & \xrightarrow{\Sigma^2(\psi)} & \Sigma^2(K_B^1) & \xrightarrow{\Sigma^2(\omega)} & \Sigma^2(K_C^1) \\ -\Sigma(g_C^0) \downarrow & & \Sigma^2(g_A^0) \downarrow & & \Sigma^2(g_B^0) \downarrow & & -\Sigma^2(g_C^0) \downarrow \\ \Sigma(P_C^0) & \xrightarrow{0} & \Sigma^2(P_A^0) & \xrightarrow{-\Sigma^2(p)} & \Sigma^2(P_B^0) & \xrightarrow{\Sigma^2(q)} & \Sigma^2(P_C^0) \end{array}$$

which is commutative except the lower right square which anticommutes and where the rows and columns are triangles. Then we have the following commutative diagram in which the first three vertical and horizontal diagrams are triangles:

$$\begin{array}{ccccccc} K_A^1 & \xrightarrow{-\psi} & K_B^1 & \xrightarrow{\omega} & K_C^1 & \xrightarrow{-\phi} & \Sigma(K_A^1) \\ g_A^0 \downarrow & & g_B^0 \downarrow & & g_C^0 \downarrow & & \Sigma(g_A^0) \downarrow \\ P_A^0 & \xrightarrow{p} & P_B^0 & \xrightarrow{q} & P_C^0 & \xrightarrow{0} & \Sigma(P_A^0) \\ f_A^0 \downarrow & & f_B^0 \downarrow & & f_C^0 \downarrow & & \Sigma(f_A^0) \downarrow \\ A & \xrightarrow{\alpha} & B & \xrightarrow{\beta} & C & \xrightarrow{\gamma} & \Sigma(A) \\ h_A^0 \downarrow & & h_B^0 \downarrow & & h_C^0 \downarrow & & \Sigma(h_A^0) \downarrow \\ \Sigma(K_A^1) & \xrightarrow{-\Sigma(\psi)} & \Sigma(K_B^1) & \xrightarrow{\Sigma(\omega)} & \Sigma(K_C^1) & \xrightarrow{-\Sigma(\phi)} & \Sigma(K_A^1). \end{array}$$

Since the second horizontal triangle is split, we have that P_B^0 is in $\mathcal{P}(\mathcal{E})$. Applying to the above diagram the homological functor $\mathcal{E}(P, -)$, $\forall P \in \mathcal{P}(\mathcal{E})$, a simple diagram chasing argument shows that $0 \rightarrow \mathcal{E}(P, K_A^1) \rightarrow \mathcal{E}(P, K_B^1) \rightarrow \mathcal{E}(P, K_C^1) \rightarrow 0$ is exact. By Lemma 4.2, the first horizontal triangle is in \mathcal{E} . Similarly the second vertical triangle is in \mathcal{E} . Inductively the above procedure completes the proof. ■

Let $F: \mathcal{E} \rightarrow \mathcal{A}$ and $G: \mathcal{E}^{\text{op}} \rightarrow \mathcal{A}$ be additive functors, where \mathcal{A} is an abelian category. Then we define the \mathcal{E} -derived functors

$$\mathcal{L}_n^{\mathcal{E}}F: \mathcal{E} \rightarrow \mathcal{A} \quad \text{and} \quad \mathcal{R}_n^{\mathcal{E}}G: \mathcal{E}^{\text{op}} \rightarrow \mathcal{A}, \quad n \geq 0$$

of F and G with respect to \mathcal{E} , as follows. Let $P^\bullet \rightarrow C$ be an \mathcal{E} -projective resolution of $C \in \mathcal{E}$. Then define $\mathcal{L}_n^{\mathcal{E}}F(C)$ to be the n th-homology of the induced complex $F(P^\bullet)$ and $\mathcal{R}_n^{\mathcal{E}}G(C)$ to be the n th-cohomology of the induced complex $G(P^\bullet)$. In particular $\forall C \in \mathcal{E}$, we define the \mathcal{E} -extension functors $\mathcal{E}xt_{\mathcal{E}}^n(-, C)$ by

$$\mathcal{E}xt_{\mathcal{E}}^n(-, C) := \mathcal{R}_n^{\mathcal{E}}\mathcal{E}(-, C): \mathcal{E}^{\text{op}} \rightarrow \mathcal{A}b, \quad \forall n \geq 0.$$

By the comparison theorem the above \mathcal{E} -derived functors are well defined.

COROLLARY 4.12. *Let $A \rightarrow B \rightarrow C \rightarrow \Sigma(A)$ be in \mathcal{E} . If $F: \mathcal{E} \rightarrow \mathcal{A}$ and $G: \mathcal{E}^{\text{op}} \rightarrow \mathcal{A}$ are additive functors where \mathcal{A} is abelian, then we have exact sequences*

$$\begin{aligned} \cdots \rightarrow \mathcal{L}_1^{\mathcal{E}}F(C) \rightarrow \mathcal{L}_0^{\mathcal{E}}F(A) \rightarrow \mathcal{L}_0^{\mathcal{E}}F(B) \rightarrow \mathcal{L}_0^{\mathcal{E}}F(C) \rightarrow 0 \\ 0 \rightarrow \mathcal{R}_0^{\mathcal{E}}G(C) \rightarrow \mathcal{R}_0^{\mathcal{E}}G(B) \rightarrow \mathcal{R}_0^{\mathcal{E}}G(A) \rightarrow \mathcal{R}_1^{\mathcal{E}}G(C) \rightarrow \cdots \end{aligned}$$

In particular for any $X \in \mathcal{E}$ we have a long exact sequence

$$0 \rightarrow \mathcal{E}xt_{\mathcal{E}}^0(C, X) \rightarrow \mathcal{E}xt_{\mathcal{E}}^0(B, X) \rightarrow \mathcal{E}xt_{\mathcal{E}}^0(A, X) \rightarrow \mathcal{E}xt_{\mathcal{E}}^1(C, X) \rightarrow \cdots .$$

Proof. Applying F and G to the \mathcal{E} -projective resolution of the triangle as in Proposition 4.11 and taking (co-)homology, we get the desired exact sequences. ■

The easy proof of the following is left to the reader.

COROLLARY 4.13. *Let $F: \mathcal{E} \rightarrow \mathcal{A}$, resp. $G: \mathcal{E}^{\text{op}} \rightarrow \mathcal{A}$, be a homological, resp. cohomological, functor where \mathcal{A} is abelian. Then the natural morphism $\mathcal{L}_0^{\mathcal{E}}F \rightarrow F$, resp. $G \rightarrow \mathcal{R}_0^{\mathcal{E}}G$, is an isomorphism iff F , resp. G , kills \mathcal{E} -phantom maps.*

4.3. Projectively Generating Proper Classes

DEFINITION 4.14. A proper class of triangles \mathcal{E} in \mathcal{C} projectively generates \mathcal{C} , if \mathcal{E} generates \mathcal{C} and \mathcal{C} has enough \mathcal{E} -projectives. A full subcategory $\mathcal{F} \subseteq \mathcal{C}$ is called a *generating subcategory* of \mathcal{C} , if \mathcal{F} is Σ -stable and $\forall A \in \mathcal{C}: \mathcal{E}(\mathcal{F}, A) = 0 \Rightarrow A = 0$.

By Lemma 4.2, we have the following.

COROLLARY 4.15. *The map $\mathcal{E} \mapsto \mathcal{P}(\mathcal{E})$ is a bijection between projectively generating proper classes of triangles in \mathcal{C} and contravariantly finite generating subcategories of \mathcal{C} closed under direct summands. The inverse is given by $\mathcal{P} \mapsto \mathcal{E}(\mathcal{P})$.*

Remark 4.16. Obviously \mathcal{C} has enough Δ -projectives and $\mathcal{P}(\Delta) = 0$. Conversely if \mathcal{C} has enough \mathcal{E} -projectives and $\mathcal{P}(\mathcal{E}) = 0$, then $\mathcal{E} = \mathcal{E}(\mathcal{P}(\mathcal{E})) = \mathcal{E}(0) = \Delta$. If $\mathcal{E} \neq 0$, then Δ is not projectively generating. Similarly \mathcal{C} has enough Δ_0 -projectives and $\mathcal{P}(\Delta_0) = \mathcal{C}$. Conversely if \mathcal{C} has enough \mathcal{E} -projectives and $\mathcal{P}(\mathcal{E}) = \mathcal{C}$, then $\mathcal{E} = \mathcal{E}(\mathcal{P}(\mathcal{E})) = \mathcal{E}(\mathcal{C}) = \Delta_0$. Δ_0 is always projectively generating.

The next result, proved independently by Christensen [19], shows that if \mathcal{E} projectively generates \mathcal{C} , then the \mathcal{E} -projective dimension is controlled by the vanishing of the \mathcal{E} -extension functors.

PROPOSITION 4.17. *If \mathcal{E} projectively generates \mathcal{C} , then $\forall A \in \mathcal{C}, \forall n \geq 0$:*

$$\mathcal{E}\text{-p.d } A \leq n \quad \Leftrightarrow \quad \mathcal{E}xt_{\mathcal{E}}^{n+1}(A, -) = 0.$$

Proof. It suffices to prove the (\Leftarrow) direction. Consider the triangles \mathbf{T}_A^n and the resolution \mathbf{P}_A^\bullet after Corollary 4.9. Since the cohomology of the short complex $\mathcal{E}(P_A^n, -) \rightarrow \mathcal{E}(P_A^{n+1}, -) \rightarrow \mathcal{E}(P_A^{n+2}, -)$ is $\mathcal{E}xt_{\mathcal{E}}^{n+1}(A, -) = 0$, the morphism $f_A^{n+1}: P_A^{n+1} \rightarrow K_A^{n+1}$ factors through ε_A^{n+1} . Hence $\varepsilon_A^{n+1} \circ \alpha = f_A^{n+1} \circ g_A^n \circ \alpha = f_A^{n+1}$, for some $\alpha: P_A^n \rightarrow K_A^{n+1}$. Applying the functor $\mathcal{E}(P, -), \forall P \in \mathcal{P}(\mathcal{E})$, to this relation and using that $\mathcal{E}(P, f_A^{n+1})$ is epic, we have that $\mathcal{E}(P, g_A^n) \circ \mathcal{E}(P, \alpha) = 1_{\mathcal{E}(P, K_A^{n+1})}$, so $\mathcal{E}(P, (\alpha, f_A^n))$: $\mathcal{E}(P, P_A^n) \cong \mathcal{E}(P, K_A^{n+1} \oplus K_A^n)$. Since \mathcal{E} generates $\mathcal{C}, P_A^n \cong K_A^{n+1} \oplus K_A^n$. It follows that $K_A^n \in \mathcal{P}(\mathcal{E})$ or equivalently $\mathcal{E}\text{-p.d } A \leq n$. \blacksquare

4.4. The Cartan Morphism

Assume that \mathcal{C} has split idempotents and is skeletally small; i.e., the collection of isoclasses of objects of \mathcal{C} , denoted henceforth by $\text{Iso}(\mathcal{C})$, is a set. The *Grothendieck group* $K_0(\mathcal{C}, \mathcal{E})$ of \mathcal{C} with respect to \mathcal{E} is defined as the quotient of the free abelian group on $\text{Iso}(\mathcal{C})$, modulo the subgroup generated by all elements $(A) - (B) + (C)$, where $A \rightarrow B \rightarrow C \rightarrow \Sigma(A)$

is a triangle in \mathcal{E} . If $\mathcal{E} = \Delta$, then $K_0(\mathcal{E}, \Delta)$ is the usual group defined by Grothendieck. If $\mathcal{E} = \Delta_0$, then $K_0(\mathcal{E}, \Delta_0)$ is the group $K_0(\mathcal{E}, \oplus)$ of the monoidal category (\mathcal{E}, \oplus) . Observe that we have canonical epimorphisms $K_0(\mathcal{E}, \Delta_0) \rightarrow K_0(\mathcal{E}, \mathcal{E})$ and $K_0(\mathcal{E}, \mathcal{E}) \rightarrow K_0(\mathcal{E}, \Delta)$. In case \mathcal{E} has enough \mathcal{E} -projectives, then we define the *Cartan morphism* $c_{\mathcal{E}}: K_0(\mathcal{P}(\mathcal{E}), \oplus) \rightarrow K_0(\mathcal{E}, \mathcal{E})$ by $c_{\mathcal{E}}([P]) = [P]$.

PROPOSITION 4.18. *If $\forall C \in \mathcal{E} : \mathcal{E}\text{-p.d } C < \infty$, then $c_{\mathcal{E}}$ is an isomorphism.*

Proof. We construct an inverse of $c_{\mathcal{E}}$ as follows. Let $0 \rightarrow P_n \rightarrow \dots \rightarrow P^0 \rightarrow A \rightarrow 0$ be an \mathcal{E} -projective resolution of $A \in \mathcal{E}$. Using Schanuel's and horseshoe lemma it follows directly that the assignment $[A] \rightarrow \sum_{i=0}^n (-1)^i [P^i]$ is a well defined morphism $d_{\mathcal{E}}: K_0(\mathcal{E}, \mathcal{E}) \rightarrow K_0(\mathcal{P}(\mathcal{E}), \oplus)$. Clearly $d_{\mathcal{E}}$ is the inverse of $c_{\mathcal{E}}$. ■

4.5. The Steenrod Categories

Let $F: \mathcal{E} \rightarrow \mathcal{A}$ be a homological functor, where \mathcal{A} is abelian. Let $\mathcal{E}(F)$ be the proper class of triangles in \mathcal{E} induced by F . Following Street [68], an object $P \in \mathcal{E}$ is called *F-projective* if $F(P)$ is projective in \mathcal{A} and $\forall A \in \mathcal{E}$, the canonical map $\mathcal{E}(P, A) \rightarrow \mathcal{A}[F(P), F(A)]$ is an isomorphism. Let $\mathcal{P}(F)$ be the full subcategory of *F-projectives*. \mathcal{E} has enough *F-projectives* if $\forall A \in \mathcal{E}$ there exists a triangle $K \rightarrow P \rightarrow A \rightarrow \Sigma(K) \in \mathcal{E}(F)$ with $P \in \mathcal{P}(F)$. It is trivial to see that if \mathcal{E} has enough *F-projectives*, then \mathcal{E} has enough $\mathcal{E}(F)$ -projectives and $\mathcal{P}(\mathcal{E}(F)) = \mathcal{P}(F)$. The next result shows that if \mathcal{E} has enough \mathcal{E} -projectives for a proper class of triangles \mathcal{E} , then $\mathcal{E} = \mathcal{E}(\mathbf{S})$ and $\mathcal{P}(\mathcal{E}(\mathbf{S})) = \mathcal{P}(\mathbf{S})$, where \mathbf{S} is the projectivization functor. It follows that in this case, our formulation of relative homology in \mathcal{E} is equivalent to the theory developed by Street in [68].

PROPOSITION 4.19. *$\forall P \in \mathcal{P}(\mathcal{E})$, $\mathbf{S}(P) \in \text{Proj } \mathcal{S}_{\mathcal{E}}(\mathcal{E})$ and $\forall A \in \mathcal{E}$, the canonical map $\mathbf{S}_{P,A}: \mathcal{E}(P, A) \rightarrow \mathcal{S}_{\mathcal{E}}(\mathcal{E})[\mathbf{S}(P), \mathbf{S}(A)]$ is an isomorphism. Hence \mathbf{S} induces a full embedding $\mathcal{P}(\mathcal{E}) \hookrightarrow \text{Proj } \mathcal{S}_{\mathcal{E}}(\mathcal{E})$. If \mathcal{E} has enough \mathcal{E} -projectives, then:*

(i) $\mathcal{S}_{\mathcal{E}}(\mathcal{E})$ is equivalent to $\mathcal{A}(\mathcal{P}(\mathcal{E}))$ and \mathbf{S} is isomorphic to the restriction $A \mapsto \mathcal{E}(-, A)|_{\mathcal{P}(\mathcal{E})}$. In particular $\mathcal{S}_{\mathcal{E}}(\mathcal{E})$ has enough projectives and $\mathcal{E} = \mathcal{E}(\mathbf{S})$.

(ii) If idempotents split in $\mathcal{P}(\mathcal{E})$, then $\mathbf{S}: \mathcal{P}(\mathcal{E}) \rightarrow \sim \text{Proj } \mathcal{S}_{\mathcal{E}}(\mathcal{E})$.

(iii) A complex $P^\bullet \rightarrow A$ over A is an \mathcal{E} -projective resolution of A iff $\mathbf{S}(P^\bullet) \rightarrow \mathbf{S}(A)$ is a projective resolution of $\mathbf{S}(A)$ in $\mathcal{S}_{\mathcal{E}}(\mathcal{E})$.

Proof. Since the kernel ideal of the functor \mathbf{S} is $\text{Ph}_{\mathcal{E}}(\mathcal{E})$, it follows that $\mathbf{S}_{P,A}$ is injective, since $\text{Ph}_{\mathcal{E}}(P, A) = 0, \forall P \in \mathcal{P}(\mathcal{E})$. Now let $\alpha: \mathbf{S}(P) \rightarrow \mathbf{S}(A)$ be a morphism in $\mathcal{S}_{\mathcal{E}}(\mathcal{E})$. Since $\mathbf{S} = \mathbf{Q}\mathbb{V}$, where $\mathbf{Q}: \mathcal{A}(\mathcal{E}) \rightarrow$

$\mathcal{A}(\mathcal{E})/\mathcal{A}(\text{Ph}_{\mathcal{E}}(\mathcal{E}))$ is the quotient functor, α is represented by a fraction $\mathbb{Y}(P) \leftarrow^s F \rightarrow^f \mathbb{Y}(A)$, where s has kernel and cokernel in $\mathcal{A}(\text{Ph}_{\mathcal{E}}(\mathcal{E}))$. Let $\xi: \mathbb{Y}(P) \rightarrow G$ be the cokernel of s . Then G is the image of a morphism $\mathbb{Y}(h): \mathbb{Y}(C) \rightarrow \mathbb{Y}(D)$, where h is an \mathcal{E} -phantom map; let $\mathbb{Y}(h) = \epsilon \circ \iota$ be the canonical factorization. Then there exists a morphism $\mathbb{Y}(\phi): \mathbb{Y}(P) \rightarrow \mathbb{Y}(C)$ such that $\mathbb{Y}(\phi) \circ \epsilon = \xi$. Then $\xi \circ \iota = \mathbb{Y}(\phi) \circ \epsilon \circ \iota = \mathbb{Y}(\phi \circ h)$. Since h is \mathcal{E} -phantom and P is \mathcal{E} -projective, $\phi \circ h = 0$. Hence $\xi = 0$ and s is split epic. Let $s': \mathbb{Y}(P) \rightarrow F$ be a morphism such that $s' \circ s = 1_{\mathbb{Y}(P)}$. Then $s' \circ f$ is of the form $\mathbb{Y}(\alpha')$ for $\alpha': P \rightarrow A$. Trivially $\mathbf{S}(\alpha') = \mathbf{Q}(s' \circ f) = \alpha$. Hence $\mathbf{S}_{P,A}$ is surjective. The proof that $\mathbf{S}(P)$ is projective is similar and is left to the reader.

(i) One can prove this by using the universal property of $\mathcal{S}_{\mathcal{E}}(\mathcal{E})$. Here is a quick proof. Since \mathcal{E} has enough \mathcal{E} -projectives, the subcategory $\mathcal{P}(\mathcal{E})$ is contravariantly finite in \mathcal{E} . By [13], there exists a short exact sequence of abelian categories $0 \rightarrow \mathcal{A}(\mathcal{E}/\mathcal{P}(\mathcal{E})) \rightarrow \mathcal{A}(\mathcal{E}) \rightarrow^R \mathcal{A}(\mathcal{P}(\mathcal{E})) \rightarrow 0$, where $\mathcal{E}/\mathcal{P}(\mathcal{E})$ is the stable category of \mathcal{E} modulo the full subcategory $\mathcal{P}(\mathcal{E})$ and R is the restriction functor $F \mapsto F|_{\mathcal{P}(\mathcal{E})}$. It is easy to see that $\mathcal{A}(\mathcal{E}/\mathcal{P}(\mathcal{E}))$ is equivalent to $\mathcal{A}(\text{Ph}_{\mathcal{E}}(\mathcal{E}))$ so by Definition 3.3, $\mathcal{S}_{\mathcal{E}}(\mathcal{E})$ is equivalent to $\mathcal{A}(\mathcal{P}(\mathcal{E}))$. In particular $\mathcal{S}_{\mathcal{E}}(\mathcal{E})$ has enough projectives. Part (ii) follows from [28] and part (iii) follows from (i). ■

The above result explains the terminology for the projectivization functor. The next example explains the terminology for the Steenrod category.

EXAMPLE 4.20. Let \mathcal{E} be the stable homotopy category of spectra [51] and consider the full subcategory $\mathcal{X} = \{\Sigma^n(K(\mathbb{Z}/(p)))\}; n \in \mathbb{Z}\}$, where $K(\mathbb{Z}/(p))$ is the Eilenberg–MacLane spectrum corresponding to $\mathbb{Z}/(p)$. Then the Steenrod category $\mathcal{S}_{\mathcal{E}^{\text{op}}(\mathcal{X})}(\mathcal{E})$ of \mathcal{E} with respect to the proper class of triangles $\mathcal{E}^{\text{op}}(\mathcal{X})$ is equivalent to the category of modules over the mod- p Steenrod algebra; see [74] for details.

The description of the Steenrod category in Proposition 4.19 allows us to give a formula for the derived functors of an additive functor $F: \mathcal{E} \rightarrow \mathcal{M}$, where \mathcal{M} is abelian. Let $G := F|_{\mathcal{P}(\mathcal{E})}: \mathcal{P}(\mathcal{E}) \rightarrow \mathcal{M}$. Since $\mathcal{S}_{\mathcal{E}}(\mathcal{E}) = \mathcal{A}(\mathcal{P}(\mathcal{E}))$, G has a unique right exact extension $G^*: \mathcal{S}_{\mathcal{E}}(\mathcal{E}) \rightarrow \mathcal{M}$ through the Steenrod category. The easy proof of the following and its contravariant analogue is left to the reader.

COROLLARY 4.21. $\mathcal{L}_n^{\mathcal{E}}F \cong (\mathcal{L}_n G^*)\mathbf{S}, \forall n \geq 0$.

COROLLARY 4.22. (i) For any $B \in \mathcal{E}$ we have natural isomorphisms

$$\mathcal{E}xt_{\mathcal{E}}^n(-, B) \cong \mathcal{E}xt_{\mathcal{S}_{\mathcal{E}}(\mathcal{E})}^n[\mathbf{S}(-), \mathbf{S}(B)], \quad \forall n \geq 0.$$

The natural map $\mu_{-,B}: \mathcal{C}(-, B) \rightarrow \mathcal{R}_{\mathcal{E}}^0 \mathcal{C}(-, B) = \mathcal{E}xt_{\mathcal{E}}^0(-, B)$ coincides with the map $\mathbf{S}_{-,B}: \mathcal{C}(-, B) \rightarrow \mathcal{S}_{\mathcal{E}}(\mathcal{C})[\mathbf{S}(-), \mathbf{S}(B)]$. Hence $\text{Ker } \mu_{-,B}$ is the right ideal $\text{Ph}_{\mathcal{E}}(-, B)$.

(ii) $\forall A \in \mathcal{E}: \mathcal{E}\text{-p.d } A \geq \text{p.d } \mathbf{S}(A)$; if \mathcal{E} projectively generates \mathcal{C} , then $\mathcal{E}\text{-p.d } A = \text{p.d } \mathbf{S}(A)$. In particular $\mathcal{E}\text{-gl.dim } \mathcal{C} \leq \text{gl.dim } \mathcal{S}_{\mathcal{E}}(\mathcal{C})$.

(iii) Assume that \mathcal{C} is skeletally small with split idempotents. If \mathcal{C} is \mathcal{E} -projectively generated and $\text{gl.dim } \mathcal{S}_{\mathcal{E}}(\mathcal{C}) < \infty$, then the projectivization functor $\mathbf{S}: \mathcal{C} \rightarrow \mathcal{S}_{\mathcal{E}}(\mathcal{C})$ induces an isomorphism $\mathbf{S}_*: K_0(\mathcal{C}, \mathcal{C}) \xrightarrow{\cong} K_0(\mathcal{S}_{\mathcal{E}}(\mathcal{C}))$, and an isomorphism $K_*(\mathcal{A}(\mathcal{C})) \cong K_*(\mathcal{S}_{\mathcal{E}}(\mathcal{C})) \oplus K_*(\mathcal{A}(\mathcal{C}/\mathcal{P}(\mathcal{C})))$ in Quillen's higher K -theory.

Proof. The assertions (i) and (ii) follow trivially from Proposition 4.19. The first part of (iii) follows from Proposition 4.18 and the commutativity of the diagram

$$\begin{CD} K_0(\mathcal{P}(\mathcal{C}), \oplus) @>S>> K_0(\text{Proj}(\mathcal{S}_{\mathcal{E}}(\mathcal{C})), \oplus) \\ @Vc_{\mathcal{E}}VV @VVc_{\mathcal{S}_{\mathcal{E}}(\mathcal{C})}V \\ K_0(\mathcal{C}, \mathcal{C}) @>S_*>> K_0(\mathcal{S}_{\mathcal{E}}(\mathcal{C})). \end{CD}$$

The second part of (iii) follows from a result of Auslander and Reiten [5].



The dual of Proposition 4.19 is also true. We state only the following.

PROPOSITION 4.23. *If \mathcal{C} has enough \mathcal{E} -injectives then the dual Steenrod category of \mathcal{C} with respect to \mathcal{E} is equivalent to $\mathcal{B}(\mathcal{A}(\mathcal{C})) = \text{mod}(\mathcal{A}(\mathcal{C})^{\text{op}})^{\text{op}}$ and the injectivization functor $\mathbf{T}: \mathcal{C} \rightarrow \mathcal{S}_{\mathcal{E}}^{\text{op}}(\mathcal{C})$ is isomorphic to the restriction $A \mapsto \mathcal{C}(A, -)|_{\mathcal{S}(\mathcal{E})}$. If $\mathcal{A}(\mathcal{C})$ has split idempotents, then \mathbf{T} induces an equivalence: $\mathcal{A}(\mathcal{C}) \approx \text{Inj } \mathcal{S}_{\mathcal{E}}^{\text{op}}(\mathcal{C})$.*

If $\mathcal{X} \subseteq \mathcal{C}$ is a class of objects, then $\text{add}(\mathcal{X})$ denotes the full subcategory of \mathcal{C} consisting of all direct summands of finite coproducts of objects of \mathcal{X} .

An \mathcal{E} -injective envelope of $A \in \mathcal{C}$ is an \mathcal{E} -proper monic $\mu: A \rightarrow E$ with $E \in \mathcal{S}(\mathcal{E})$ such that if $\mu \circ \alpha = \mu$, then α is an automorphism of E . Without assuming the existence of enough \mathcal{E} -injectives we have the following.

THEOREM 4.24. *The projectivization functor \mathbf{S} induces a full embedding*

$$\mathbf{S}: \mathcal{S}(\mathcal{C}) \hookrightarrow \text{Inj } \mathcal{S}_{\mathcal{E}}(\mathcal{C}) \quad \text{and} \quad \text{Inj } \mathcal{S}_{\mathcal{E}}(\mathcal{C}) \subseteq \text{add}(\text{Im } \mathbf{S}).$$

In particular $\mathbf{S}: \mathcal{S}(\mathcal{C}) \rightarrow \approx \text{Inj } \mathcal{S}_{\mathcal{E}}(\mathcal{C})$ if $\text{Im } \mathbf{S}$ is closed under direct summands. If this is the case, then \mathcal{C} has enough \mathcal{E} -injectives iff $\mathcal{S}_{\mathcal{E}}(\mathcal{C})$ has

enough injectives. In particular if $\mathcal{S}_{\mathcal{E}}(\mathcal{E})$ has injective envelopes, then \mathcal{E} has \mathcal{E} -injective envelopes.

Proof. Let $P_A^1 \rightarrow \epsilon_A^1 P_A^0 \rightarrow f_A^0 A \rightarrow 0$ be the start of an \mathcal{E} -projective resolution of $A \in \mathcal{E}$. If I is \mathcal{E} -injective then obviously the map $\mathbf{S}_{A,I}: \mathcal{E}(A, I) \rightarrow \mathcal{S}_{\mathcal{E}}(\mathcal{E})[\mathbf{S}(A), \mathbf{S}(I)]$ is injective. Let $\mu: \mathbf{S}(A) \rightarrow \mathbf{S}(I)$ be any morphism. Consider the morphism $\mathbf{S}(f_A^0) \circ \mu: \mathbf{S}(P_A^0) \rightarrow \mathbf{S}(I)$. Since P_A^0 is \mathcal{E} -projective, $\mathbf{S}(f_A^0) \circ \mu = \mathbf{S}(\alpha)$, for some morphism $\alpha: P_A^0 \rightarrow I$. Then $\mathbf{S}(\epsilon_A^1) \circ \mathbf{S}(\alpha) = \mathbf{S}(\epsilon_A^1) \circ \mathbf{S}(f_A^0) \circ \mu = 0$, so $\epsilon_A^1 \circ \alpha: P_A^1 \rightarrow I$ is \mathcal{E} -phantom. Since P_A^1 is \mathcal{E} -projective, $\epsilon_A^1 \circ \alpha = 0$. Since $\epsilon_A^1 = f_A^1 \circ g_A^0$, we have that the morphism $g_A^0 \circ \alpha$ factors through $h_A^1: K_A^1 \rightarrow \Sigma(K_A^2)$. So $g_A^0 \circ \alpha = h_A^1 \circ \beta$, for some morphism $\beta: \Sigma(K_A^2) \rightarrow I$. But since h_A^1 is \mathcal{E} -phantom, it follows that $g_A^0 \circ \alpha: K_A^1 \rightarrow I$ is \mathcal{E} -phantom. Since I is \mathcal{E} -injective, $g_A^0 \circ \alpha = 0$. Hence there exists $\gamma: A \rightarrow I$ such that $f_A^0 \circ \gamma = \beta$. Then $\mathbf{S}(f_A^0) \circ \mathbf{S}(\gamma) = \mathbf{S}(\beta)$. Since $\mathbf{S}(f_A^0)$ is epic, $\mu = \mathbf{S}(\gamma)$; hence $\mathbf{S}_{A,I}$ is an isomorphism. It remains to show that $\mathbf{S}(I) \in \text{Inj } \mathcal{S}_{\mathcal{E}}(\mathcal{E})$. Let $F \in \mathcal{S}_{\mathcal{E}}(\mathcal{E})$ and let $\mathbf{S}(P_1) \xrightarrow{\mathbf{S}(\beta)} \mathbf{S}(P_0) \rightarrow F \rightarrow 0$ be the start of a projective resolution of F . Let $(T): A \xrightarrow{\alpha} P_1 \xrightarrow{\beta} P_0 \rightarrow \Sigma(A)$ be a triangle in \mathcal{E} and let $K \rightarrow P_2 \xrightarrow{f} A \rightarrow \Sigma(K)$ be an \mathcal{E} -projective presentation of A in \mathcal{E} . Since the sequence $\mathbf{S}(A) \xrightarrow{\mathbf{S}(\alpha)} \mathbf{S}(P_1) \xrightarrow{\mathbf{S}(\beta)} \mathbf{S}(P_0) \rightarrow F \rightarrow 0$ is exact, if $G = \text{Ker}(\mathbf{S}(\beta))$, then we have an epimorphism $\mathbf{S}(f) \circ \text{coim } \mathbf{S}(\beta): \mathbf{S}(P_2) \rightarrow G$. Then $\mathbf{S}(P_2) \xrightarrow{\mathbf{S}(f \circ \alpha)} \mathbf{S}(P_1) \rightarrow \mathbf{S}(P_0) \rightarrow F \rightarrow 0$ is part of a projective resolution of F . Applying $\mathcal{S}_{\mathcal{E}}(\mathcal{E})[-, \mathbf{S}(I)]$ to this sequence and using that the map $\mathbf{S}_{C,I}$ is an isomorphism $\forall C \in \mathcal{E}$, it follows easily that the resulting sequence is exact. This means that $\mathcal{E}xt_{\mathcal{S}_{\mathcal{E}}(\mathcal{E})}^1[F, \mathbf{S}(I)] = 0$. Hence $\mathbf{S}(I)$ is injective.

From the triangle (T) it follows that we have an inclusion $\xi: F \hookrightarrow \mathbf{S}(\Sigma(A))$. If F is injective then F is a direct summand of $\mathbf{S}(\Sigma(A))$; hence if in addition $\text{Im } \mathbf{S}$ is closed under direct summands, then $F \cong \mathbf{S}(I)$ for some (obviously \mathcal{E} -injective) I . If \mathcal{E} has enough \mathcal{E} -injectives, let $\Sigma(A) \xrightarrow{\tau} I \rightarrow B \rightarrow \Sigma^2(A)$ be a triangle in \mathcal{E} with $I \in \mathcal{I}(\mathcal{E})$. Then $\xi \circ \mathbf{S}(\tau): F \rightarrow \mathbf{S}(I)$ is monic and $\mathbf{S}(I)$ is injective; hence $\mathcal{S}_{\mathcal{E}}(\mathcal{E})$ has enough injectives. Conversely if $\mathcal{S}_{\mathcal{E}}(\mathcal{E})$ has enough injectives, $\forall C \in \mathcal{E}$, let $\zeta: \mathbf{S}(C) \hookrightarrow E$ be monic with $E \in \text{Inj } \mathcal{S}_{\mathcal{E}}(\mathcal{E})$. Then $E = \mathbf{S}(I)$ for some $I \in \mathcal{I}(\mathcal{E})$ and $\zeta = \mathbf{S}(\nu)$ for a morphism $\nu: C \rightarrow I$ in \mathcal{E} . Since the triangle $C \xrightarrow{\nu} I \rightarrow B \rightarrow \Sigma(C)$ is in \mathcal{E} , it follows that \mathcal{E} has enough \mathcal{E} -injectives. Clearly if $\mathbf{S}(\mu): \mathbf{S}(C) \rightarrow \mathbf{S}(I)$ is an injective envelope, then $\mu: C \rightarrow I$ is an \mathcal{E} -injective envelope. ■

4.6. Semisimple and Hereditary Categories

The following characterization of \mathcal{E} -phantomless categories follows easily from our previous results.

THEOREM 4.25. *The following are equivalent:*

- (i) $\mathcal{E}\text{-gl.dim } \mathcal{C} = 0$.
- (ii) $\mathcal{C} = \mathcal{P}(\mathcal{C})$.
- (iii) $\mathcal{C} = \Delta_0$.
- (iv) $\text{Ph}_{\mathcal{E}}(\mathcal{C}) = 0$.
- (v) *The projectivization functor $\mathbf{S}: \mathcal{C} \rightarrow \mathcal{S}_{\mathcal{E}}(\mathcal{C})$ is faithful.*
- (vi) *The projectivization functor \mathbf{S} induces an equivalence $\mathcal{S}_{\mathcal{E}}(\mathcal{C}) \xrightarrow{\sim} \mathcal{A}(\mathcal{C})$.*

LEMMA 4.26. *Let $G: \mathcal{C} \rightarrow \mathcal{M}$ be a homological and $F: \mathcal{C}^{\text{op}} \rightarrow \mathcal{M}$ a cohomological functor, where \mathcal{M} is abelian. If $\mathcal{E}\text{-p.d } A \leq 1$, there are exact sequences*

$$0 \rightarrow \mathcal{L}_0^{\mathcal{E}} G(A) \rightarrow G(A) \rightarrow \mathcal{L}_1^{\mathcal{E}} G(\Sigma(A)) \rightarrow 0,$$

$$0 \rightarrow \mathcal{R}_{\mathcal{E}}^1 F(\Sigma(A)) \rightarrow F(A) \rightarrow \mathcal{R}_{\mathcal{E}}^0 F(A) \rightarrow 0.$$

Proof. Since $\mathcal{E}\text{-p.d } A \leq 1$, an \mathcal{E} -projective resolution of A is simply a triangle $P_A^1 \rightarrow^{g_A^0} P_A^0 \rightarrow^{f_A^0} A \rightarrow^{h_A^0} \Sigma(P_A^1)$ in \mathcal{C} , with $P_A^1, P_A^0 \in \mathcal{P}(\mathcal{C})$. From the exact sequence $\cdots \rightarrow G(P_A^1) \xrightarrow{G(g_A^0)} G(P_A^0) \rightarrow G(A) \rightarrow G\Sigma(P_A^1) \xrightarrow{G\Sigma(g_A^0)} G\Sigma(P_A^0) \rightarrow \cdots$, by construction we have $\text{Coker}(G(g_A^0)) = \mathcal{L}_0^{\mathcal{E}} G(A)$ and $\text{Ker}(G\Sigma(g_A^0)) = \mathcal{L}_1^{\mathcal{E}} G\Sigma(A)$ and the result follows. The claim for F is similar. ■

The next result computes the ideal $\text{Ph}_{\mathcal{E}}(\mathcal{C})$ in case $\mathcal{E}\text{-gl.dim } \mathcal{C} \leq 1$ and can be regarded as a universal coefficient theorem.

THEOREM 4.27. *Let $A \in \mathcal{C}$ with $\mathcal{E}\text{-p.d } A \leq 1$. Then we have the following.*

- (i) *For any $B \in \mathcal{C}$ there exists a short exact sequence*

$$0 \rightarrow \mathcal{E}xt_{\mathcal{E}}^1(\Sigma(A), B) \rightarrow \mathcal{C}(A, B) \xrightarrow{\mu_{A,B}} \mathcal{E}xt_{\mathcal{E}}^0(A, B) \rightarrow 0.$$

Hence $\text{Ph}_{\mathcal{E}}(A, B) = \mathcal{E}xt_{\mathcal{E}}^1(\Sigma(A), B)$ and A is \mathcal{E} -projective iff $\mathcal{E}xt_{\mathcal{E}}^1(A, -) = 0$.

- (ii) $\text{Ph}_{\mathcal{E}}^2(A, -) = 0$.

If $\mathcal{E}\text{-gl.dim } \mathcal{C} \leq 1$, then the projectivization functor $\mathbf{S}: \mathcal{C} \rightarrow \mathcal{S}_{\mathcal{E}}(\mathcal{C})$ is full and reflects isomorphisms, \mathcal{C} projectively generates \mathcal{C} , and the following relations hold:

$$\text{Ph}_{\mathcal{E}}^2(\mathcal{C}) = 0, \quad \text{Ph}_{\mathcal{E}}(\mathcal{C}) \subseteq \mathcal{F}ac(\mathcal{C}),$$

$$\text{Ph}_{\mathcal{E}}(\Sigma^{-1} -, -) \cong \mathcal{E}xt_{\mathcal{E}}^1(-, -).$$

Proof. Part (i) follows from Lemma 4.26 setting $F = \mathcal{C}(-, B)$. To prove (ii) let $A \rightarrow^{\alpha} B \rightarrow^{\beta} C$ be \mathcal{E} -phantom maps and consider the \mathcal{E} -pro-

jective resolution of A as in the proof of Lemma 4.26. Since α is \mathcal{E} -phantom, the composition $f_A^0 \circ \alpha = 0$. So there exists a map $y: \Sigma(P_A^1) \rightarrow B$ with $h_A^0 \circ y = \alpha$. But with P_A^1 , also $\Sigma(P_A^1)$ is \mathcal{E} -projective. Since β is \mathcal{E} -phantom, the composition $y \circ \beta = 0$. Then $\alpha \circ \beta = h_A^0 \circ y \circ \beta = 0$. Hence $\text{Ph}_{\mathcal{E}}^2(A, -) = 0$. If $\mathcal{E}\text{-gl.dim } \mathcal{E} \leq 1$, then since $\mathcal{E}xt_{\mathcal{E}}^0(A, B) \cong \mathcal{S}_{\mathcal{E}}(\mathcal{E})[\mathbf{S}(A), \mathbf{S}(B)]$, $\forall A, B \in \mathcal{E}$, the functor \mathbf{S} is full. Since $\ker \mathbf{S} = \text{Ph}_{\mathcal{E}}(\mathcal{E})$ is a square zero ideal, it is contained in the Jacobson radical of \mathcal{E} . This implies that \mathbf{S} reflects isomorphisms. The last claim follows from (i) and (ii). ■

Remark 4.28. If \mathcal{E} is skeletally small and \mathcal{A} is abelian, let $\text{Add}(\mathcal{E}, \mathcal{A})$ be the category of additive functors from \mathcal{E} to \mathcal{A} and let $\text{Hom}(\mathcal{E}, \mathcal{A})$ be the full subcategory of homological functors. If $\mathcal{E}\text{-gl.dim } \mathcal{E} \leq 1$, it is easy to see that the kernel-ideal $\ker(\mathcal{L}_0^{\mathcal{E}})$ of the functor $\mathcal{L}_0^{\mathcal{E}}: \text{Hom}(\mathcal{E}, \mathcal{A}) \rightarrow \text{Add}(\mathcal{E}, \mathcal{A})$, satisfies $\ker(\mathcal{L}_0^{\mathcal{E}})^2 = 0$. Hence if $\phi: F \rightarrow G$ is a map of homological functors, then ϕ is invertible iff $\mathcal{L}_0^{\mathcal{E}}(\phi)$ is also.

COROLLARY 4.29. *Assume that $\mathcal{E}\text{-gl.dim } \mathcal{E} \leq 1$ and let $F: \mathcal{E}^{\text{op}} \rightarrow \mathcal{A}b$ be a cohomological functor. Then the following are equivalent.*

(i) F is representable.

(ii) There exists $\Omega \in \mathcal{E}$ and a natural isomorphism $F|_{\mathcal{P}(\mathcal{E})} \rightarrow^{\cong} \mathcal{E}(-, \Omega)|_{\mathcal{P}(\mathcal{E})}$.

Proof. (ii) \Rightarrow (i) Let $\omega: \mathcal{E}(-, \Omega)|_{\mathcal{P}(\mathcal{E})} \rightarrow^{\cong} F|_{\mathcal{P}(\mathcal{E})}$ be a natural isomorphism. Using ω , it follows easily that the short exact sequence of Lemma 4.26 is isomorphic to

$$0 \rightarrow \mathcal{E}xt_{\mathcal{E}}^1[\Sigma -, \Omega] \xrightarrow{\beta} F \xrightarrow{\alpha} \mathcal{E}xt_{\mathcal{E}}^0[-, \Omega] \rightarrow 0.$$

Let $\mu_{-, \Omega}: \mathcal{E}(-, \Omega) \rightarrow \mathcal{E}xt_{\mathcal{E}}^0[-, \Omega]$ be the canonical morphism. Since α is an epimorphism and $\mathcal{E}(-, \Omega)$ is a projective functor, there exists a morphism $\phi: \mathcal{E}(-, \Omega) \rightarrow F$ such that $\phi \circ \alpha = \mu_{-, \Omega}$. Obviously ϕ_P is an isomorphism, $\forall P \in \mathcal{P}(\mathcal{E})$, since $\mu_{P, \Omega}$ and α_P are. Using the natural morphism ϕ and the projective resolution of A , using that F is cohomological and finally using 5-Lemma, we see directly that $\phi_A: \mathcal{E}(A, \Omega) \rightarrow F(A)$ is an isomorphism. Hence $\phi: \mathcal{E}(-, \Omega) \rightarrow^{\cong} F$. ■

5. THE \mathcal{E} -PHANTOM TOWER, THE \mathcal{E} -CELLULAR TOWER, HOMOTOPY COLIMITS, AND COMPACT OBJECTS

Throughout this section we assume that \mathcal{E} is a proper class of triangles in the triangulated category \mathcal{E} and \mathcal{E} has enough \mathcal{E} -projectives. Our

purpose in this section is to construct for any object A of \mathcal{E} the phantom tower and the cellular tower of A , generalizing topological constructions from [51]. These towers are crucial in the study of the homological behavior of A with respect to \mathcal{E} .

5.1. *The Phantom Tower*

We fix an object $A \in \mathcal{E}$ and let

$$\mathbf{P}_A^\bullet: \dots \rightarrow P_A^{n+1} \xrightarrow{\varepsilon_A^{n+1}} P_A^n \rightarrow \dots \rightarrow P_A^1 \xrightarrow{\varepsilon_A^1} P_A^0 \xrightarrow{f_A^0} A \rightarrow 0$$

be the \mathcal{E} -projective resolution of A as in Section 4. We recall that the resolution \mathbf{P}_A^\bullet is obtained by splicing triangles

$$\mathbf{T}_A^n: K_A^{n+1} \rightarrow^{g_A^n} P_A^n \rightarrow^{f_A^n} K_A^n \rightarrow^{h_A^n} \Sigma(K_A^{n+1}) \in \mathcal{E}$$

where $P_A^n \in \mathcal{P}(\mathcal{E})$, $\forall n \geq 0$, $K_A^0 := A$ and $\varepsilon_A^n = f_A^n \circ g_A^{n-1}: P_A^n \rightarrow P_A^{n-1}$, $\forall n \geq 1$.

Consider the morphisms $h_A^{n-1}: K_A^{n-1} \rightarrow \Sigma(K_A^n)$, $\forall n \geq 1$. By construction all these maps are \mathcal{E} -phantoms; i.e., we have $h_A^{n-1} \in \text{Ph}_{\mathcal{E}}(K_A^{n-1}, \Sigma(K_A^n))$. Hence the maps $\Sigma^{n-1}(h_A^{n-1}): \Sigma^{n-1}(K_A^{n-1}) \rightarrow \Sigma^n(K_A^n)$ are also \mathcal{E} -phantom maps. In this way we obtain a tower of objects and \mathcal{E} -phantom maps

$$\begin{aligned} A &\xrightarrow{h_A^0} \Sigma(K_A^1) \xrightarrow{\Sigma(h_A^1)} \Sigma^2(K_A^2) \\ &\rightarrow \dots \rightarrow \Sigma^{n-1}(K_A^{n-1}) \xrightarrow{\Sigma^{n-1}(h_A^{n-1})} \Sigma^n(K_A^n) \rightarrow \dots \end{aligned}$$

which we call the \mathcal{E} -phantom tower of A , with respect to the proper class \mathcal{E} .

PROPOSITION 5.1. *For any morphism $\alpha: A \rightarrow B$, the following are equivalent:*

- (i) $\alpha \in \text{Ph}_{\mathcal{E}}^n(A, B)$.
- (ii) *There exists a morphism $\sigma: \Sigma^n(K_A^n) \rightarrow B$ such that*

$$(-1)^{n+1} h_A^0 \circ \Sigma(h_A^1) \circ \Sigma^2(h_A^2) \circ \dots \circ \Sigma^{n-1}(h_A^{n-1}) \circ \sigma = \alpha.$$

Proof. (i) \Rightarrow (ii) If $\alpha \in \text{Ph}_{\mathcal{E}}(A, B)$, then $f_A^0 \circ \alpha = 0$, since $f_A^0: P_A^0 \rightarrow A$ and α is \mathcal{E} -phantom. Hence there exists $\sigma: \Sigma(K_A^1) \rightarrow B$ with $h_A^0 \circ \sigma = \alpha$. If $\alpha \in \text{Ph}_{\mathcal{E}}^2(A, B)$, then $\alpha = \alpha_1 \circ \alpha_2$ where $\alpha_1: A \rightarrow X$ and $\alpha_2: X \rightarrow B$ are \mathcal{E} -phantom maps. By the above argument, there exists $\sigma_1: \Sigma(K_A^1) \rightarrow X$ with $h_A^0 \circ \sigma_1 = \alpha_1$. Since the composition $\sigma_1 \circ \alpha_2: \Sigma(K_A^1) \rightarrow B$ is \mathcal{E} -phantom and $\Sigma(P_A^1) \in \mathcal{P}(\mathcal{E})$, we have $-\Sigma(f_A^1) \circ \sigma_1 \circ \alpha_2 = 0$. Then as

above there exists $\sigma: \Sigma^2(K_A^2) \rightarrow B$ with $-\Sigma(h_A^1) \circ \sigma = \sigma_1 \circ \alpha_2$. Then $h_A^0 \circ (-\Sigma(h_A^1)) \circ \sigma = h_A^0 \circ \sigma_1 \circ \alpha_2 = \alpha_1 \circ \alpha_2 = \alpha$. A simple induction argument completes the proof. The converse is trivial. \blacksquare

5.2. The Cellular Tower

In this subsection we use the phantom tower to construct the cellular tower of an object. We fix an object A in the triangulated category \mathcal{E} and consider the \mathcal{E} -projective resolution of A as above. For simplicity we set $\omega_A^n := (-1)^n h_A^0 \circ \Sigma(h_A^1) \circ \Sigma^2(h_A^2) \circ \dots \circ \Sigma^n(h_A^n): A \rightarrow \Sigma^{n+1}(K_A^{n+1})$, $A_0 := 0$, and $A_1 := P_A^0$. Consider the triangle $K_A^1 \xrightarrow{s_A^0} P_A^0 \xrightarrow{f_A^0} A \xrightarrow{h_A^0} \Sigma(K_A^1)$ and form the cobase change of this triangle along the phantom map $-h_A^1: K_A^1 \rightarrow \Sigma(K_A^2)$:

$$\begin{array}{ccccccc}
 K_A^1 & \xrightarrow{s_A^0} & P_A^0 & \xrightarrow{f_A^0} & A & \xrightarrow{h_A^0} & \Sigma(K_A^1) \\
 -h_A^1 \downarrow & & \alpha_1 \downarrow & & \parallel \downarrow & & -\Sigma(h_A^1) \downarrow \\
 \Sigma(K_A^2) & \xrightarrow{\beta_2} & A_2 & \xrightarrow{\gamma_2} & A & \xrightarrow{\omega_A^1} & \Sigma^2(K_A^2).
 \end{array} \tag{+}$$

Next consider the second horizontal triangle $\Sigma(K_A^2) \xrightarrow{\beta_2} A_2 \xrightarrow{\gamma_2} A \xrightarrow{\omega_A^1} \Sigma^2(K_A^2)$ in the above diagram, which by construction is in \mathcal{E} , and form the cobase change of this triangle along the phantom map $-\Sigma(h_A^2): \Sigma(K_A^2) \rightarrow \Sigma^2(K_A^3)$:

$$\begin{array}{ccccccc}
 \Sigma(K_A^2) & \xrightarrow{\beta_2} & A_2 & \xrightarrow{\gamma_2} & A & \xrightarrow{\omega_A^1} & \Sigma^2(K_A^2) \\
 -\Sigma(h_A^2) \downarrow & & \alpha_2 \downarrow & & \parallel \downarrow & & -\Sigma^2(h_A^2) \downarrow \\
 \Sigma^2(K_A^3) & \xrightarrow{\beta_3} & A_3 & \xrightarrow{\gamma_3} & A & \xrightarrow{\omega_A^2} & \Sigma^3(K_A^3).
 \end{array}$$

Inductively the above construction produces a tower

$$0 = A_0 \rightarrow A_1 = P_A^0 \xrightarrow{\alpha_1} A_2 \xrightarrow{\alpha_2} A_3 \rightarrow \dots \rightarrow A_n \xrightarrow{\alpha_n} A_{n+1} \rightarrow \dots$$

which we call the \mathcal{E} -cellular tower of A , and commutative diagrams, $\forall n \geq 1$:

$$\begin{array}{ccc}
 A_n & \xrightarrow{\gamma_n} & A \\
 \alpha_n \downarrow & & \parallel \downarrow \\
 A_{n+1} & \xrightarrow{\gamma_{n+1}} & A.
 \end{array}$$

The following result is a direct consequence of the above constructions.

COROLLARY 5.2. For an object $A \in \mathcal{C}$, the following are equivalent:

(i) \mathcal{E} -p.d $A \leq n$.

(ii) The morphisms $\gamma_{n+i}: A_{n+i} \rightarrow A$ of the \mathcal{E} -cellular tower of A constructed above are isomorphisms, $\forall i \geq 1$.

Consequently \mathcal{E} -gl.dim $\mathcal{C} \leq n$ iff the cellular tower of any object of \mathcal{C} stabilizes after the $(n + 1)$ th-step; i.e. the morphisms $\alpha_m: A_m \rightarrow A_{m+1}$ of the cellular tower are isomorphisms, $\forall m \geq n + 1$. If this is the case then $\forall A \in \mathcal{C}: A \cong A_m, \forall m \geq n + 1$.

Using the \mathcal{E} -cellular tower we can prove the main result of this section.

THEOREM 5.3. (α) The following statements are equivalent:

(i) The projectivization functor $\mathbf{S}: \mathcal{C} \rightarrow \mathcal{S}_{\mathcal{E}}(\mathcal{C})$ is full and reflects isomorphisms.

(ii) \mathcal{E} -gl.dim $\mathcal{C} \leq 1$.

(iii) \mathcal{E} generates \mathcal{C} and $\text{Im } \mathbf{S} \subseteq \{F \in \mathcal{S}_{\mathcal{E}}(\mathcal{C}): \text{p.d } F \leq 1\}$.

(β) Assume that idempotents split in $\mathcal{P}(\mathcal{C})$. Then $\{F \in \mathcal{S}_{\mathcal{E}}(\mathcal{C}): \text{p.d } F \leq 1\} \subseteq \text{Im } \mathbf{S}$. If \mathcal{E} -gl.dim $\mathcal{C} \leq 1$, then \mathbf{S} induces an equivalence $\mathbf{S}: \mathcal{F}(\mathcal{C}) \rightarrow \text{Inj } \mathcal{S}_{\mathcal{E}}(\mathcal{C})$; further \mathcal{C} has enough \mathcal{E} -injectives iff $\mathcal{S}_{\mathcal{E}}(\mathcal{C})$ has enough injectives.

(γ) If \mathcal{C} has enough \mathcal{E} -injectives, then the statements in (α) are also equivalent to the following (the inclusion below is an equality if $\mathcal{F}(\mathcal{C})$ has split idempotents):

(iv) $\forall A \in \mathcal{C}: \mathcal{E}$ -i.d $A \leq 1$.

(v) \mathcal{E} generates \mathcal{C} and $\text{Im } \mathbf{S} \subseteq \{F \in \mathcal{S}_{\mathcal{E}}(\mathcal{C}): \text{i.d } F \leq 1\}$.

Proof. (α) (ii) \Rightarrow (i) follows from Theorem 4.27. (i) \Rightarrow (ii) Suppose that (i) is true. Applying to the cobase change diagram (+) the projectivization functor \mathbf{S} , we obtain the following commutative diagram of short exact sequences in $\mathcal{S}_{\mathcal{E}}(\mathcal{C})$

$$\begin{array}{ccccccc}
 0 & \longrightarrow & \mathbf{S}(K_A^1) & \xrightarrow{\mathbf{S}(g_A^0)} & \mathbf{S}(P_A^0) & \xrightarrow{\mathbf{S}(f_A^0)} & \mathbf{S}(A) \longrightarrow 0 \\
 & & \downarrow -\mathbf{S}(h_A^1) & & \downarrow \mathbf{S}(\alpha_1) & & \parallel \downarrow \\
 0 & \longrightarrow & \mathbf{S}(\Sigma(K_A^2)) & \xrightarrow{\mathbf{S}(\beta_2)} & \mathbf{S}(A_2) & \xrightarrow{\mathbf{S}(\gamma_2)} & \mathbf{S}(A) \longrightarrow 0.
 \end{array}$$

Since h_A^1 is an \mathcal{E} -phantom map, $\mathbf{S}(h_A^1) = 0$. Consider the cokernel $\omega: \mathbf{S}(A_2) \rightarrow F$ of $\mathbf{S}(\alpha_1)$. By the above diagram, there exists a unique isomorphism $\zeta: \mathbf{S}(\Sigma(K_A^2)) \rightarrow F$ such that $\mathbf{S}(\beta_2) \circ \omega = \zeta$. Consider the morphism $\omega \circ \zeta^{-1}: \mathbf{S}(A_2) \rightarrow \mathbf{S}(\Sigma(K_A^2))$. Since \mathbf{S} is full, there exists $\phi: A_2 \rightarrow \Sigma(K_A^2)$ such that $\mathbf{S}(\phi) = \omega \circ \zeta^{-1}$. Next consider the morphism $\alpha_1 \circ \phi: P_A^0 \rightarrow$

$\Sigma(K_A^2)$. Then $\mathbf{S}(\alpha_1 \circ \phi) = \mathbf{S}(\alpha_1) \circ \omega \circ \zeta^{-1} = 0$. Hence the morphism $\alpha_1 \circ \phi$ is \mathcal{E} -phantom. Then $\alpha_1 \circ \phi = 0$, since $P_A^0 \in \mathcal{P}(\mathcal{E})$. From the diagram (+), we have that $\Sigma(g_A^1) = \beta_2 \circ \delta_1$, where $P_A^0 \xrightarrow{\alpha_1} A_2 \xrightarrow{\delta_1} \Sigma(P_A^1) \rightarrow \Sigma(P_A^0)$ is a triangle in \mathcal{E} . It follows that ϕ factors through δ_1 ; i.e., there exists $\rho: \Sigma(P_A^1) \rightarrow \Sigma(K_A^2)$, such that $\delta_1 \circ \rho = \phi$. Then $\beta_2 \circ \delta_1 \circ \rho = \beta_2 \circ \phi \Rightarrow \Sigma(g_A^1) \circ \rho = \beta_2 \circ \phi$. But $\mathbf{S}(\beta_2) \circ \omega = \zeta \Rightarrow \mathbf{S}(\beta_2) \circ \omega \circ \zeta^{-1} = 1_{\mathbf{S}(\Sigma(K_A^2))} \Rightarrow \mathbf{S}(\beta_2) \circ \mathbf{S}(\phi) = 1_{\mathbf{S}(\Sigma(K_A^2))} \Rightarrow \mathbf{S}(\beta_2 \circ \phi) = 1_{\mathbf{S}(\Sigma(K_A^2))}$. Since \mathbf{S} reflects isomorphisms, it follows that $\beta_2 \circ \phi = \Sigma(g_A^1) \circ \rho$ is invertible. Then $\Sigma(g_A^1)$ is split monic and consequently $\Sigma(f_A^1): \Sigma(P_A^1) \rightarrow \Sigma(K_A^1)$ is split epic. This implies that K_A^1 is \mathcal{E} -projective as a direct summand of P_A^1 . Hence \mathcal{E} -p.d $A \leq 1$. Part (ii) \Leftrightarrow (iii) follows from (i) and Corollary 4.22, since if \mathcal{E} generates, then \mathcal{E} -p.d $A = \text{p.d } \mathbf{S}(A)$, $\forall A \in \mathcal{E}$.

(β) Let $F \in \mathcal{S}_{\mathcal{E}}(\mathcal{E})$ with $\text{p.d } F \leq 1$. Since $\mathcal{P}(\mathcal{E})$ has split idempotents, by Proposition 4.19, \mathbf{S} induces an equivalence $\mathcal{P}(\mathcal{E}) \approx \text{Proj } \mathcal{S}_{\mathcal{E}}(\mathcal{E})$. Hence F admits a projective resolution of the form $0 \rightarrow \mathbf{S}(P_1) \xrightarrow{\mathbf{S}(\alpha)} \mathbf{S}(P_0) \xrightarrow{f} F \rightarrow 0$ with $P_1, P_0 \in \mathcal{P}(\mathcal{E})$. Then the triangle $P_1 \rightarrow^{\alpha} P_0 \rightarrow A \rightarrow \Sigma(P_1)$ is in \mathcal{E} and obviously $F \cong \mathbf{S}(A)$. Hence $\{F \in \mathcal{S}_{\mathcal{E}}(\mathcal{E}) : \text{p.d } F \leq 1\} \subseteq \text{Im } \mathbf{S}$. If \mathcal{E} -gl.dim $\mathcal{E} \leq 1$, then by (α) we have that $\{F \in \mathcal{S}_{\mathcal{E}}(\mathcal{E}) : \text{p.d } F \leq 1\} = \text{Im } \mathbf{S}$. Hence $\text{Im } \mathbf{S}$ is closed under direct summands and the remaining assertions are consequences of Theorem 4.24.

Part (γ) is similar and is left to the reader. \blacksquare

5.3. The Category of Extensions and the Phantom Filtration

We recall now a well-known construction from [11]. If \mathcal{L} is a class of objects of \mathcal{E} , then the category $\mathcal{L} * \mathcal{L}$ of extensions of \mathcal{L} is defined as follows:

$$\mathcal{L} * \mathcal{L} = \text{add}\{A \in \mathcal{E} : \exists \text{ triangle } X_1 \rightarrow A \rightarrow X_2 \rightarrow \Sigma(X_1) \text{ with } X_1, X_2 \in \mathcal{L}\}.$$

Inductively we define $\mathcal{L}^{*n} := \mathcal{L} * \mathcal{L} * \mathcal{L} * \dots * \mathcal{L}$ (n -factors). It is shown in [11] that the operation $*$ is associative; hence the above definition of \mathcal{L}^{*n} makes sense. Finally define $\mathcal{L}^{*\infty} := \bigcup_{n \geq 1} \mathcal{L}^{*n}$, where we set $\mathcal{L}^{*1} = \mathcal{L}$. The category \mathcal{L}^{*n} is called the category of n th-extensions of \mathcal{L} , for $n = 1, 2, \dots, \infty$.

Remark 5.4. Let A be an object of \mathcal{E} , and consider the \mathcal{E} -cellular tower of A :

$$A_0 \xrightarrow{\alpha_0} A_1 \xrightarrow{\alpha_1} A_2 \xrightarrow{\alpha_2} A_3 \rightarrow \dots \rightarrow A_n \xrightarrow{\alpha_n} A_{n+1} \rightarrow \dots .$$

Then clearly $A_n \in \mathcal{P}(\mathcal{E})^{*n}$, $\forall n \geq 1$.

The next corollary generalizes results of Kelly [46] and Street [68].

COROLLARY 5.5. (α) If $A \in \mathcal{C}$, then $\forall n \geq 0$; consider the following statements:

- (i) \mathcal{E} -p.d $A \leq n$.
- (ii) $A \in \mathcal{P}(\mathcal{E})^{*(n+1)}$.
- (iii) $\text{Ph}_{\mathcal{E}}^{n+1}(A, -) = 0$.

Then (i) \Rightarrow (ii) \Rightarrow (iii). In particular if \mathcal{E} -p.d $A < \infty$, then $\text{Ph}_{\mathcal{E}}(A, -) \subseteq \mathcal{F}ac(A, -)$ and the ideal $\text{Ph}_{\mathcal{E}}(A, A)$ of the ring $\text{End}_{\mathcal{E}}(A)$ is nilpotent. Hence if an object X has a semiprime endomorphism ring, then either $\text{Ph}_{\mathcal{E}}(X, X) = 0$ or else \mathcal{E} -p.d $X = \infty$.

(β) If any object of \mathcal{C} has finite \mathcal{E} -projective dimension, then \mathcal{C} is \mathcal{E} -projectively generated and $\mathcal{C} = \mathcal{P}(\mathcal{E})^{*\infty}$.

(γ) If \mathcal{E} -gl.dim $\mathcal{C} \leq n$, or if $\mathcal{C} = \mathcal{P}(\mathcal{E})^{*(n+1)}$, then $\text{Ph}_{\mathcal{E}}^{n+1}(\mathcal{C}) = 0$.

Proof. (α) The implication (i) \Rightarrow (ii) follows from Corollary 5.2. Let A be in $\mathcal{P}(\mathcal{E}) * \mathcal{P}(\mathcal{E})$, so there exists a triangle $P_1 \xrightarrow{f} A \xrightarrow{g} P_0 \xrightarrow{h} \Sigma(P_1)$ with $P_0, P_1 \in \mathcal{P}(\mathcal{E})$. If $\alpha: A \rightarrow B$, $\beta: B \rightarrow C$ are \mathcal{E} -phantoms, then $\alpha = g \circ \rho$ and $\rho \circ \beta = 0$, for some $\rho: P_0 \rightarrow B$. Then $\alpha \circ \beta = g \circ \rho \circ \beta = 0$. Hence $\text{Ph}_{\mathcal{E}}^2(A, -) = 0$. Now the implication (ii) \Rightarrow (iii) follows by induction. Parts (β) and (γ) are direct consequences of (α). ■

Using the proper class \mathcal{E} , we define a new class of triangles \mathcal{E}^2 as follows:

- A triangle $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} \Sigma(A)$ is in \mathcal{E}^2 if $h \in \text{Ph}_{\mathcal{E}}^2(C, \Sigma(A))$.

PROPOSITION 5.6. The class of triangle \mathcal{E}^2 in \mathcal{C} is proper. If \mathcal{C} has enough \mathcal{E} -projectives then \mathcal{C} has enough \mathcal{E}^2 -projectives and $\mathcal{P}(\mathcal{E}^2) = \mathcal{P}(\mathcal{E}) * \mathcal{P}(\mathcal{E}) = \mathcal{P}(\mathcal{E})^{*2}$. Moreover if \mathcal{E} generates \mathcal{C} then also \mathcal{E}^2 generates \mathcal{C} .

Proof. From Proposition 2.5, it follows that \mathcal{E}^2 is a proper class of triangles. From the construction of the \mathcal{E} -cellular tower we know that $\forall A \in \mathcal{C}$, there exists a triangle $\Sigma(K_A^2) \rightarrow A_2 \rightarrow A \xrightarrow{\omega_A^1} \Sigma^2(K_A^2)$ such that $\omega_A^1 \in \text{Ph}_{\mathcal{E}}^2(\mathcal{C})$ and $A_2 \in \mathcal{P}(\mathcal{E}) * \mathcal{P}(\mathcal{E})$. Hence the above triangle is in \mathcal{E}^2 and it suffices to show that any object in $\mathcal{P}(\mathcal{E}) * \mathcal{P}(\mathcal{E})$ is \mathcal{E}^2 -projective. So let $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} \Sigma(A)$ be a triangle in \mathcal{E}^2 and let $\alpha: X \rightarrow C$ be a morphism with $X \in \mathcal{P}(\mathcal{E}) * \mathcal{P}(\mathcal{E})$. Then the composition $\alpha \circ h$ is in $\text{Ph}_{\mathcal{E}}^2(X, \Sigma(A))$. By Corollary 5.5, we have $\alpha \circ h = 0$, so α factors through g . Hence X is \mathcal{E}^2 -projective and since $\mathcal{P}(\mathcal{E}) * \mathcal{P}(\mathcal{E})$ is closed under direct summands, \mathcal{E}^2 has enough \mathcal{E}^2 -projectives and $\mathcal{P}(\mathcal{E}^2) = \mathcal{P}(\mathcal{E}) * \mathcal{P}(\mathcal{E})$. Since \mathcal{E} generates \mathcal{C} iff $\text{Ph}_{\mathcal{E}}(\mathcal{C}) \subseteq \mathcal{F}ac(\mathcal{C})$ and since $\text{Ph}_{\mathcal{E}^2}(\mathcal{C}) = \text{Ph}_{\mathcal{E}}^2(\mathcal{C})$, the last claim follows trivially. ■

From now on we set $\mathcal{E}^1 := \mathcal{E}$. We note that $\text{Ph}_{\mathcal{E}^2}(\mathcal{E}) = \text{Ph}_{\mathcal{E}^1}^2(\mathcal{E})$. The procedure described in the above proposition produces a filtration of proper classes of triangles

$$\Delta_0 \subseteq \mathcal{E}^n \subseteq \mathcal{E}^{n-1} \subseteq \dots \subseteq \mathcal{E}^2 \subseteq \mathcal{E}^1 = \mathcal{E} \subseteq \Delta,$$

a filtration of phantom ideals

$$\begin{aligned} 0 \subseteq \text{Ph}_{\mathcal{E}^n}(\mathcal{E}) \subseteq \text{Ph}_{\mathcal{E}^{n-1}}(\mathcal{E}) \subseteq \dots \subseteq \text{Ph}_{\mathcal{E}^2}(\mathcal{E}) \subseteq \text{Ph}_{\mathcal{E}^1}(\mathcal{E}) \\ = \text{Ph}_{\mathcal{E}}(\mathcal{E}) \subseteq \mathcal{E}(-, -), \end{aligned}$$

where $\text{Ph}_{\mathcal{E}^n}(\mathcal{E}) = \text{Ph}_{\mathcal{E}}^n(\mathcal{E})$, and a filtration of subcategories

$$0 \subseteq \mathcal{P}(\mathcal{E}) = \mathcal{P}(\mathcal{E}^1) \subseteq \mathcal{P}(\mathcal{E}^2) \subseteq \dots \subseteq \mathcal{P}(\mathcal{E}^{n-1}) \subseteq \mathcal{P}(\mathcal{E}^n) \subseteq \mathcal{E},$$

where $\mathcal{P}(\mathcal{E}^n) = \mathcal{P}(\mathcal{E})^{*n}$. Finally setting $\mathcal{E}^\infty = \bigcap_{n \geq 1} \mathcal{E}^n$ we obtain a proper class of triangles such that $\text{Ph}_{\mathcal{E}^\infty}(\mathcal{E}) = \bigcap_{n \geq 1} \text{Ph}_{\mathcal{E}^n}(\mathcal{E}) = \bigcap_{n \geq 1} \text{Ph}_{\mathcal{E}}^n(\mathcal{E})$ and $\mathcal{P}(\mathcal{E}^\infty) = \bigcup_{n \geq 1} \mathcal{P}(\mathcal{E})^{*n} = \mathcal{P}(\mathcal{E})^{*\infty}$. If \mathcal{E} has enough \mathcal{E} -projectives then for $1 \leq n \leq \infty$, \mathcal{E} has enough \mathcal{E}^n -projectives. Moreover if \mathcal{E} generates \mathcal{E} then the same is true for \mathcal{E}^n . The above data induce a filtration of Steenrod categories

$$\begin{aligned} 0 \hookrightarrow \mathcal{S}_{\mathcal{E}^\infty}(\mathcal{E}) \hookrightarrow \dots \hookrightarrow \mathcal{S}_{\mathcal{E}^n}(\mathcal{E}) \hookrightarrow \mathcal{S}_{\mathcal{E}^{n-1}}(\mathcal{E}) \hookrightarrow \dots \hookrightarrow \mathcal{S}_{\mathcal{E}^2}(\mathcal{E}) \\ \hookrightarrow \mathcal{S}_{\mathcal{E}^1}(\mathcal{E}) \hookrightarrow \mathcal{A}(\mathcal{E}) \end{aligned}$$

and for $1 \leq n \leq m \leq \infty$ each inclusion $\mathcal{S}_{\mathcal{E}^m}(\mathcal{E}) \hookrightarrow \mathcal{S}_{\mathcal{E}^n}(\mathcal{E})$ has an exact right adjoint $\mathbf{S}_{m,n}: \mathcal{S}_{\mathcal{E}^n}(\mathcal{E}) \rightarrow \mathcal{S}_{\mathcal{E}^m}(\mathcal{E})$, with the kernel the Serre subcategory $\mathcal{A}(\text{Ph}_{\mathcal{E}}^n(\mathcal{E}))/\mathcal{A}(\text{Ph}_{\mathcal{E}}^m(\mathcal{E})) \approx \mathcal{A}(\text{Ph}_{\mathcal{E}}^n(\mathcal{E})/\text{Ph}_{\mathcal{E}}^m(\mathcal{E}))$. The following is a consequence of our previous results.

COROLLARY 5.7. *For any n with $1 \leq n \leq \infty$, the following are true.*

- (i) \mathcal{E} is \mathcal{E}^n -phantomless $\Leftrightarrow \text{Ph}_{\mathcal{E}}^n(\mathcal{E}) = 0 \Leftrightarrow \mathcal{E} = \mathcal{P}(\mathcal{E})^{*n} \Leftrightarrow \mathcal{E}^n$ -gl.dim $\mathcal{E} = 0$.
- (ii) \mathcal{E}^n -p.d $A \leq 1 \Rightarrow \forall B \in \mathcal{E} : \text{Ph}_{\mathcal{E}}^n(A, B) = \mathcal{E}xt_{\mathcal{E}^n}^1(\Sigma(A), B)$, $\text{Ph}_{\mathcal{E}}^{2n}(A, B) = 0$.
- (iii) \mathcal{E}^n -gl.dim $\mathcal{E} \leq m \Rightarrow \mathcal{E} = \mathcal{P}(\mathcal{E})^{*n \cdot (m+1)}$ and $\text{Ph}_{\mathcal{E}}^{n \cdot (m+1)}(\mathcal{E}) = 0$.

We define the \mathcal{E} -phantom dimension \mathcal{E} -Ph.dim \mathcal{E} of \mathcal{E} by $\inf\{n \geq 0 : \text{Ph}_{\mathcal{E}}^{n+1}(\mathcal{E}) = 0\}$ and the \mathcal{E} -extension dimension \mathcal{E} -Ext.dim \mathcal{E} of \mathcal{E} by $\inf\{n \geq 0 : \mathcal{P}(\mathcal{E})^{*(n+1)} = \mathcal{E}\}$. Trivially: \mathcal{E} -Ph.dim $\mathcal{E} \leq \mathcal{E}$ -Ext.dim $\mathcal{E} \leq \mathcal{E}$ -gl.dim \mathcal{E} . Using Corollary 5.7(i), the octahedral axiom, and the \mathcal{E} -cellular tower, it is not difficult to see that \mathcal{E} -Ph.dim $\mathcal{E} = \mathcal{E}$ -Ext.dim \mathcal{E} . We thank

J. D. Christensen for this nice observation. It is unclear under what conditions the last inequality is an equality.

5.4. Homotopy Colimits and Compact Objects

In this subsection we study the behavior of the \mathcal{E} -cellular tower under (co-)homological functors. We assume throughout that \mathcal{E} has countable coproducts.

Let $A_0 \rightarrow^{f_0} A_1 \rightarrow^{f_1} A_2 \rightarrow^{f_2} \dots$ be a tower of objects in \mathcal{E} . The *homotopy colimit* $\text{holim}_{\rightarrow} A_n$ of the tower is defined by the triangle

$$\bigoplus_{n \geq 0} A_n \xrightarrow{f} \bigoplus_{n \geq 0} A_n \xrightarrow{g} \text{holim}_{\rightarrow} A_n \xrightarrow{h} \Sigma \left(\bigoplus_{n \geq 0} A_n \right) \tag{T}$$

where f is induced by the morphisms $A_n \xrightarrow{(1, -f_n)} A_n \oplus A_{n+1} \hookrightarrow \bigoplus_{n \geq 0} A_n$. The homotopy colimit is uniquely determined up to a non-unique isomorphism. We include for completeness the proof of the following well-known result.

LEMMA 5.8. *Let $\{A_n; f_n\}_{n \geq 0}$ be a tower of objects in \mathcal{E} .*

(1) *If $F: \mathcal{E} \rightarrow \mathcal{A}b$ is a homological functor which preserves coproducts, then the canonical map $\lim_{\rightarrow} F(A_n) \rightarrow F(\text{holim}_{\rightarrow} A_n)$ is invertible.*

(2) *If $F: \mathcal{E}^{\text{op}} \rightarrow \mathcal{A}b$ is a cohomological functor converting coproducts to products, then the canonical map $F(\text{holim}_{\rightarrow} A_n) \rightarrow \lim_{\leftarrow} F(A_n)$ induces a Milnor short exact sequence: $0 \rightarrow \lim_{\leftarrow}^{(1)} F \Sigma(A_n) \rightarrow F(\text{holim}_{\rightarrow} A_n) \rightarrow \lim_{\leftarrow} F(A_n) \rightarrow 0$.*

Proof. (1) The triangle (T) induces a short exact sequence $0 \rightarrow \bigoplus_{n \geq 0} F(A_n) \rightarrow \bigoplus_{n \geq 0} F(A_n) \rightarrow F(\text{holim}_{\rightarrow} A_n) \rightarrow 0$ in $\mathcal{A}b$ and the result follows. (2) Applying F to the triangle (T), we obtain the long exact sequence $\dots \rightarrow F(\text{holim}_{\rightarrow} A_n) \rightarrow \prod_{n \geq 0} F(A_n) \rightarrow \prod_{n \geq 0} F(A_n) \rightarrow F(\text{holim}_{\rightarrow} \Sigma^{-1}(A_n)) \rightarrow \dots$ in $\mathcal{A}b$, and the result follows from the construction of $\lim_{\leftarrow}^{(i)}$, $i = 0, 1$, of a tower in $\mathcal{A}b$; see [71]. ■

The following concepts are fundamental.

DEFINITION 5.9 [56]. If \mathcal{E} is a triangulated category, then an object $A \in \mathcal{E}$ is called *compact* if the functor $\mathcal{E}(A, -): \mathcal{E} \rightarrow \mathcal{A}b$ preserves all small coproducts. The *compact part* of \mathcal{E} is by definition the full subcategory \mathcal{E}^b of all compact objects. The compact part \mathcal{E}^b of \mathcal{E} is a full triangulated subcategory of \mathcal{E} . The category \mathcal{E} is *compactly generated* if \mathcal{E} has all small coproducts, \mathcal{E}^b is skeletally small, and $\mathcal{E}(X, A) = 0, \forall X \in \mathcal{E}^b$ implies that $A = 0$; i.e., the proper class $\mathcal{E}(\mathcal{E}^b)$ generates \mathcal{E} .

By the construction of the \mathcal{E} -cellular and the \mathcal{E} -phantom tower of an object A , we have a diagram of towers $\{A_n\} \rightarrow \{A\} \rightarrow \{\Sigma^n(K_A^n)\} \rightarrow \{\Sigma(A_n)\}$, such that $A_n \rightarrow A \rightarrow \Sigma^n(K_A^n) \rightarrow \Sigma(A_n)$ is a triangle in \mathcal{E} . It is not known if the induced diagram $\mathcal{E}(A): \text{holim}_{\rightarrow} A_n \rightarrow A \rightarrow \text{holim}_{\rightarrow} \Sigma^n(K_A^n) \rightarrow \Sigma(\text{holim}_{\rightarrow} A_n)$ is a triangle in \mathcal{E} . This happens if \mathcal{E} is a “homotopy” category of a suitable stable closed model category [37], for instance, a derived category or a stable module category. The next result follows from the construction of the cellular and phantom tower.

LEMMA 5.10. (1) *Let T be a compact \mathcal{E} -projective object and let $\text{holim}_{\rightarrow} \Sigma^n(K_A^n)$ be a homotopy colimit of the phantom tower of A . Then $\mathcal{E}(T, \text{holim}_{\rightarrow} \Sigma^n(K_A^n)) = 0$.*

(2) *If $\text{Ph}_{\mathcal{E}}^t(A, -) = 0$, for some $t \geq 0$ or if \mathcal{E} is generated by a class \mathcal{X} of compact \mathcal{E} -projective objects, then $\text{holim}_{\rightarrow} \Sigma^n(K_A^n) = 0$.*

PROPOSITION 5.11. *Let $T \in \mathcal{E}^b$ be a compact object.*

(1) *Let $A \in \mathcal{E}$ and let $\text{holim}_{\rightarrow} A_n$ be a homotopy colimit of the \mathcal{E} -cellular tower $\{A_n; n \geq 0\}$ of A . Then we have an isomorphism $\lim_{\rightarrow} \mathcal{E}(T, A_n) \cong \mathcal{E}(T, \text{holim}_{\rightarrow} A_n)$.*

(2) *If in addition $T \in \mathcal{P}(\mathcal{E})$, then $\mathcal{E}(T, A) \cong \lim_{\rightarrow} \mathcal{E}(T, A_n) \cong \mathcal{E}(T, \text{holim}_{\rightarrow} A_n)$.*

Proof. Part (1) follows from Lemma 5.8. Part (2) follows from (1), applying $\mathcal{E}(T, -)$ to the cobase change diagrams, defining the \mathcal{E} -cellular tower of A . ■

We have the following nice consequence.

COROLLARY 5.12. *If \mathcal{E} is generated by a class \mathcal{X} of compact \mathcal{E} -projective objects, then any object $A \in \mathcal{E}$ is a homotopy colimit of its \mathcal{E} -cellular tower: $A \cong \text{holim}_{\rightarrow} A_n$.*

Proof. Since \mathcal{X} generates \mathcal{E} , the functors $\mathcal{E}(T, -)$, $T \in \mathcal{X}$ collectively reflect isomorphisms and the claim follows from Proposition 5.11. ■

5.5. The Phantom Topology

Our purpose here is to describe the infinite \mathcal{E} -phantom ideal $\text{Ph}_{\mathcal{E}}^{\infty}(\mathcal{E})$ by means of the \mathcal{E} -cellular tower. Fix two objects $A, B \in \mathcal{E}$.

DEFINITION 5.13. The \mathcal{E} -phantom topology of $\mathcal{E}(A, B)$ is the topology with open sets $\phi + \text{Ph}_{\mathcal{E}}^n(A, B)$, $n \geq 0$. The topological group $\mathcal{E}(A, B)$ is complete (in the \mathcal{E} -phantom topology) if it is Hausdorff and any Cauchy sequence has a limit.

Consider the \mathcal{E} -phantom tower of A and let $\omega_A^n := (-1)^n h_A^0 \circ \Sigma(h_A^1) \circ \Sigma^2(h_A^2) \circ \dots \circ \Sigma^n(h_A^n): A \rightarrow \Sigma^{n+1}(K_A^{n+1})$ as in Section 5.1. Then by Proposition 5.1, we have clearly $\text{Im } \mathcal{E}(\omega_A^n, B) = \text{Ph}_{\mathcal{E}}^{n+1}(A, B)$. Now applying the functor $\mathcal{E}(-, B)$ to the \mathcal{E} -cellular tower of A , we obtain a short exact sequence of inverse systems in $\mathcal{A}b$:

$$0 \rightarrow \{\mathcal{E}(A, B)/\text{Ph}_{\mathcal{E}}^n(A, B)\} \rightarrow \{\mathcal{E}(A_n, B)\} \rightarrow \{\text{Im } \mathcal{E}(\beta_A^n, B)\} \rightarrow 0.$$

Hence we have the exact sequence

$$\begin{aligned} 0 &\rightarrow \lim_{\leftarrow} \mathcal{E}(A, B)/\text{Ph}_{\mathcal{E}}^n(A, B) \rightarrow \lim_{\leftarrow} \mathcal{E}(A_n, B) \rightarrow \lim_{\leftarrow} \text{Im } \mathcal{E}(\beta_A^n, B) \\ &\rightarrow \lim_{\leftarrow}^{(1)} \mathcal{E}(A, B)/\text{Ph}_{\mathcal{E}}^n(A, B) \rightarrow \lim_{\leftarrow}^{(1)} \mathcal{E}(A_n, B) \\ &\rightarrow \lim_{\leftarrow}^{(1)} \text{Im } \mathcal{E}(\beta_A^n, B) \rightarrow 0. \end{aligned} \tag{1}$$

Assume now that \mathcal{E} has countable coproducts and is generated by a class of objects \mathcal{X} such that $\Sigma(\mathcal{X}) = \mathcal{X} \subseteq \mathcal{P}(\mathcal{E}) \cap \mathcal{E}^b$. Then since $\text{holim}_{\rightarrow} A_n \cong A$, we have the following exact sequence (2): $0 \rightarrow \lim_{\leftarrow}^{(1)} \mathcal{E}(\Sigma(A_n), B) \rightarrow \mathcal{E}(A, B) \rightarrow \lim_{\leftarrow} \mathcal{E}(A_n, B) \rightarrow 0$. From the short exact sequence of the inverse systems

$$0 \rightarrow \{\text{Ker } \mathcal{E}(\omega_A^{n-1}, B)\} \rightarrow \{\mathcal{E}(\Sigma^n(K_A^n), B)\} \rightarrow \{\text{Ph}_{\mathcal{E}}^n(A, B)\} \rightarrow 0$$

using that $\{\text{Ker } \mathcal{E}(\omega_A^{n-1}, B)\} = \{\text{Im } \mathcal{E}(\Sigma(\beta_A^n), B)\}$, we obtain the exact sequence

$$\begin{aligned} 0 &\rightarrow \lim_{\leftarrow} \text{Im } \mathcal{E}(\Sigma(\beta_n), B) \rightarrow \lim_{\leftarrow} \mathcal{E}(\Sigma^n(K_A^n), B) \rightarrow \lim_{\leftarrow} \text{Ph}_{\mathcal{E}}^n(A, B) \\ &\rightarrow \lim_{\leftarrow}^{(1)} \text{Im } \mathcal{E}(\Sigma(\beta^n), B) \rightarrow \lim_{\leftarrow}^{(1)} \mathcal{E}(\Sigma^n(K_A^n), B) \\ &\rightarrow \lim_{\leftarrow}^{(1)} \text{Ph}_{\mathcal{E}}^n(A, B) \rightarrow 0. \end{aligned} \tag{3}$$

But $\forall B \in \mathcal{E}$, the \mathcal{E} -phantom tower of A induces a short exact sequence

$$\begin{aligned} 0 &\rightarrow \lim_{\leftarrow}^{(1)} \mathcal{E}(\Sigma^{n+1}(K_A^n), B) \rightarrow \mathcal{E}\left(\text{holim}_{\rightarrow} \Sigma^n(K_A^n), B\right) \\ &\rightarrow \lim_{\leftarrow} \mathcal{E}(\Sigma^n(K_A^n), B) \rightarrow 0. \end{aligned}$$

Since $\text{holim}_{\rightarrow} \Sigma^n(K_A^n) = 0$, we have $\lim_{\leftarrow}^{(1)} \mathcal{E}(\Sigma^{n+1}(K_A^n), B) = \lim_{\leftarrow} \mathcal{E}(\Sigma^n(K_A^n), B) = 0$. Since this holds $\forall B \in \mathcal{E}$, by (3), $\lim_{\leftarrow} \text{Im } \mathcal{E}(\beta_n, B) = \lim_{\leftarrow}^{(1)} \text{Ph}_{\mathcal{E}}^n(A, B) = 0$, $\lim_{\leftarrow} \text{Ph}_{\mathcal{E}}^n(A, B) \cong \lim_{\leftarrow}^{(1)} \text{Im } \mathcal{E}(\Sigma(\beta_n), B)$. Since $\lim_{\leftarrow}^{(1)} \mathcal{E}(A, B)/\text{Ph}_{\mathcal{E}}^n(A, B) = 0$, using (1) and collecting all the above information we deduce the following.

PROPOSITION 5.14. (1) $\lim_{\leftarrow} \mathcal{E}(A_n, B) \cong \lim_{\leftarrow} \mathcal{E}(A, B) / \text{Ph}_{\mathcal{E}}^n(A, B)$.

(2) $\lim_{\leftarrow}^{(1)} \text{Ph}_{\mathcal{E}}^n(A, B) = 0$.

(3) $\text{Ph}_{\mathcal{E}}^{\infty}(A, B) = \bigcap_{n \geq 1} \text{Ph}_{\mathcal{E}}^n(A_n, B) = \lim_{\leftarrow} \text{Ph}_{\mathcal{E}}^n(A, B) \cong \lim_{\leftarrow}^{(1)} \mathcal{E}(\Sigma(A_n), B)$.

(4) $\mathcal{E}(A, B)$ is complete in the \mathcal{E} -phantom topology iff $\lim_{\leftarrow}^{(1)} \mathcal{E}(\Sigma(A_n), B) = 0$.

It is well known that $\lim_{\leftarrow}^{(1)} G_n = 0$ for an inverse system $\{G_n\}$ of Artinian groups. Hence we have the following consequence.

COROLLARY 5.15. If \mathcal{E} -p.d $A < \infty$ or $A \in \mathcal{P}(\mathcal{E})^{*\infty}$ or $\mathcal{E}(\Sigma(A_n), B)$ is Artinian, $\forall n \geq 0$, then $\mathcal{E}(A, B)$ is complete in the \mathcal{E} -phantom topology, i.e., $\text{Ph}_{\mathcal{E}}^{\infty}(A, B) = 0$.

Note that the above computations are useful for the construction and the study of convergence of the Adams spectral sequence [19, 68] for the proper class \mathcal{E} .

6. LOCALIZATION AND THE RELATIVE DERIVED CATEGORY

We consider in this section a triangulated category \mathcal{E} equipped with a proper class of triangles \mathcal{E} and we assume always that \mathcal{E} has enough \mathcal{E} -projectives.

We have seen that the class of \mathcal{E} -phantom maps $\text{Ph}_{\mathcal{E}}(\mathcal{E}) = \{f: \forall P \in \mathcal{P}(\mathcal{E}); \mathcal{E}(P, f) = 0\}$ is an ideal in \mathcal{E} , and the condition $\text{Ph}_{\mathcal{E}}(\mathcal{E}) = 0$ has nice consequences for the triangulated category \mathcal{E} . So it is useful to find a way to “kill” the \mathcal{E} -phantom maps. Define a class of morphisms by $\mathcal{R} := \{f \mid \mathbf{S}(f) \text{ is invertible}\}$, i.e., $\forall A, B \in \mathcal{E}$:

$$\mathcal{R}_{A, B} = \left\{ f \in \mathcal{E}(A, B) \mid \forall P \in \mathcal{P}(\mathcal{E}); \mathcal{E}(P, f): \mathcal{E}(P, A) \xrightarrow{\cong} \mathcal{E}(P, B) \right\}.$$

The easy proof of the following is left to the reader.

LEMMA 6.1. The class of morphisms \mathcal{R} is a multiplicative system of morphisms in \mathcal{E} compatible with its triangulation.

We call the elements of \mathcal{R} \mathcal{E} -quasi-isomorphisms. By the above lemma, the localized category $\mathcal{E}[\mathcal{R}^{-1}]$ can be calculated by left and right fractions. We recall that $\mathcal{E}[\mathcal{R}^{-1}]$ is a triangulated category and there exists an exact functor of triangulated categories $\mathbf{R}: \mathcal{E} \rightarrow \mathcal{E}[\mathcal{R}^{-1}]$, which sends the elements of \mathcal{R} to isomorphisms in $\mathcal{E}[\mathcal{R}^{-1}]$ and has the following universal property. If $F: \mathcal{E} \rightarrow \mathcal{F}$ is an exact functor to a triangulated category \mathcal{F}

which sends \mathcal{R} to isomorphisms, then there exists a unique up to isomorphism exact functor $G: \mathcal{E}[\mathcal{R}^{-1}] \rightarrow \mathcal{T}$ with $GR = F$. From now on we denote by $\mathcal{D}_{\mathcal{E}}(\mathcal{E}) := \mathcal{E}[\mathcal{R}^{-1}]$ and by $\mathbf{R}: \mathcal{E} \rightarrow \mathcal{D}_{\mathcal{E}}(\mathcal{E})$ the canonical functor. We call the triangulated category $\mathcal{D}_{\mathcal{E}}(\mathcal{E})$ the \mathcal{E} -derived category of \mathcal{E} with respect to \mathcal{E} . In general $\mathcal{D}_{\mathcal{E}}(\mathcal{E})$ lives in a higher universe than \mathcal{E} . If $\mathcal{A}_{\mathcal{E}}(\mathcal{E}) = \text{Ker } \mathbf{R} = \{A \in \mathcal{E} : \mathbf{R}(A) = 0\}$ is the full subcategory of \mathcal{E} -acyclic objects, then we have a short exact sequence of triangulated categories:

$$0 \rightarrow \mathcal{A}_{\mathcal{E}}(\mathcal{E}) \rightarrow^{\mathbf{j}} \mathcal{E} \rightarrow^{\mathbf{R}} \mathcal{D}_{\mathcal{E}}(\mathcal{E}) \rightarrow 0.$$

LEMMA 6.2. *The kernel-ideal $\text{ker } \mathbf{R}$ is contained in the ideal of \mathcal{E} -phantom maps.*

Proof. If $\mathbf{R}(f) = 0$, then there exists a morphism $s \in \mathcal{R}$ with $f \circ s = 0$. By the definition of \mathcal{R} , we have $\mathcal{E}(\mathcal{P}(\mathcal{E}), f) = 0$. Then by Lemma 4.2, $f \in \text{Ph}_{\mathcal{E}}(\mathcal{E})$. ■

LEMMA 6.3. *$\forall P \in \mathcal{P}(\mathcal{E})$ and $\forall B \in \mathcal{E}$, the morphism*

$$\mathbf{R}_{P,B}: \mathcal{E}(P, B) \rightarrow \mathcal{D}_{\mathcal{E}}(\mathcal{E})[\mathbf{R}(P), \mathbf{R}(B)]$$

is an isomorphism.

Proof. By Lemma 6.2, $\mathbf{R}_{P,B}$ is injective. Let $\tilde{a}: \mathbf{R}(P) \rightarrow \mathbf{R}(B)$ be a morphism in $\mathcal{D}_{\mathcal{E}}(\mathcal{E})$. It is well known that \tilde{a} can be represented as a diagram $P \xleftarrow{s} X \xrightarrow{f} B$ with $s \in \mathcal{R}$. Then $\mathcal{E}(\mathcal{P}(\mathcal{E}), s): \mathcal{E}(\mathcal{P}(\mathcal{E}), X) \xrightarrow{\cong} \mathcal{E}(\mathcal{P}(\mathcal{E}), P)$. Let $g \in \mathcal{E}(P, X)$ be the unique morphism such that $g \circ s = 1_P$. By the definition of the morphisms in the category of fractions $\mathcal{D}_{\mathcal{E}}(\mathcal{E})$, we have $\mathbf{R}(g \circ f) = \tilde{a}$, so $\mathbf{R}_{P,B}$ is surjective. ■

By Lemma 6.3 and induction we have the following direct consequence.

LEMMA 6.4. *$\forall P \in \mathcal{P}(\mathcal{E})^{*n}$, $n = 0, 1, 2, \dots, \infty$ and $\forall B \in \mathcal{E}$, the morphism*

$$\mathbf{R}_{P,B}: \mathcal{E}(P, B) \rightarrow \mathcal{D}_{\mathcal{E}}(\mathcal{E})[\mathbf{R}(P), \mathbf{R}(B)]$$

is an isomorphism. In particular the induced functor $\mathbf{R}: \mathcal{P}(\mathcal{E})^{\infty} \hookrightarrow \mathcal{D}_{\mathcal{E}}(\mathcal{E})$ is fully faithful.*

6.1. Relative Derived Categories

We seek conditions such that the functor \mathbf{R} induces an equivalence between $\mathcal{D}_{\mathcal{E}}(\mathcal{E})$ and a full subcategory of \mathcal{E} . We fix an object $A \in \mathcal{E}$, and consider the \mathcal{E} -cellular tower of A :

$$(T) \quad A_0 \xrightarrow{\alpha_0} A_1 \xrightarrow{\alpha_1} A_2 \rightarrow \dots \rightarrow A_n \xrightarrow{\alpha_n} A_{n+1} \rightarrow \dots, \\ A_n \in \mathcal{P}(\mathcal{E})^{*n}, \quad \forall n \geq 0.$$

From now on we assume that the triangulated category \mathcal{E} has (countable) coproducts. Consider a homotopy colimit $\text{holim}_{\rightarrow} A_n$.

$$\bigoplus_{n \geq 0} A_n \xrightarrow{\zeta} \bigoplus_{n \geq 0} A_n \xrightarrow{\theta} \text{holim}_{\rightarrow} A_n \xrightarrow{\eta} \Sigma \left(\bigoplus_{n \geq 0} A_n \right) \quad (!)$$

and the following triangle, where the morphism ψ is naturally induced from (T):

$$\mathcal{E}(A): \text{holim}_{\rightarrow} A_n \xrightarrow{\psi} A \xrightarrow{\phi} \hat{A} \xrightarrow{\chi} \Sigma \left(\text{holim}_{\rightarrow} A_n \right). \quad (!!)$$

Let $\widehat{\mathcal{P}(\mathcal{E})}^{*\infty}$ be the full additive subcategory generated by all objects which are isomorphic to a homotopy colimit of a tower (T) as above with $A_n \in \mathcal{P}(\mathcal{E})^{*n}$, $n \geq 0$:

$$\widehat{\mathcal{P}(\mathcal{E})}^{*\infty} := \text{add} \left\{ A \in \mathcal{E} \mid A \cong \text{holim}_{\rightarrow} A_n, \text{ where } A_n \in \mathcal{P}(\mathcal{E})^{*n}, \forall n \geq 0 \right\}.$$

Now we can prove the main result of this section. First we recall that a *localizing* subcategory of \mathcal{E} is a full triangulated subcategory of \mathcal{E} closed under coproducts. If $\mathcal{U} \subseteq \mathcal{E}$, then the localizing subcategory $\text{Loc}\langle \mathcal{U} \rangle$ generated by \mathcal{U} is the smallest localizing subcategory of \mathcal{E} which contains \mathcal{U} . Finally we set ${}^{\perp} \mathcal{U} := \{A \in \mathcal{E} \mid \mathcal{E}(A, U) = 0, \forall U \in \mathcal{U}\}$ and $\text{Add}(\mathcal{U})$ to be the full subcategory of \mathcal{E} consisting of all direct summands of arbitrary coproducts of objects of \mathcal{U} .

THEOREM 6.5. *Assume that there exists a set of compact \mathcal{E} -projective objects \mathcal{X} in \mathcal{E} such that $\Sigma(\mathcal{X}) = \mathcal{X}$ and a morphism $f \in \mathcal{R}$ iff the morphism $\mathcal{E}(\mathcal{X}, f)$ is an isomorphism. Then we have the following:*

- (i) $\widehat{\mathcal{P}(\mathcal{E})}^{*\infty} = {}^{\perp} \mathcal{A}_{\mathcal{E}}(\mathcal{E}) = \text{Loc}\langle \widehat{\mathcal{P}(\mathcal{E})}^{*\infty} \rangle$.
- (ii) The functor \mathbf{R} induces a triangle equivalence: $\mathbf{R}: \widehat{\mathcal{P}(\mathcal{E})}^{*\infty} \rightarrow \approx \mathcal{D}_{\mathcal{E}}(\mathcal{E})$.
- (iii) The category $\mathcal{D}_{\mathcal{E}}(\mathcal{E})$ is compactly generated and any compactly generated triangulated category arises in this way.

Proof. (i) Let $P \in \widehat{\mathcal{P}(\mathcal{E})}^{*\infty}$ and $L \in \mathcal{A}_{\mathcal{E}}(\mathcal{E})$. If $\alpha: P \rightarrow L$ is a morphism, then $\mathbf{R}(\alpha) = 0$, since L is \mathcal{E} -acyclic. Lemma 6.3 implies that $\alpha = 0$. Hence $\widehat{\mathcal{P}(\mathcal{E})}^{*\infty} \subseteq {}^{\perp} \mathcal{A}_{\mathcal{E}}(\mathcal{E})$. By induction and using Lemma 6.4 it follows that $\mathcal{P}(\mathcal{E})^{*n} \subseteq {}^{\perp} \mathcal{A}_{\mathcal{E}}(\mathcal{E})$. Now let $\text{holim}_{\rightarrow} A_n$ be a homotopy colimit of a tower of objects $A_n \in \mathcal{P}(\mathcal{E})^{*n}$. By Lemma 5.8, we have a short exact sequence $0 \rightarrow \lim_{\leftarrow}^{(1)} \mathcal{E}(\Sigma(A_n), L) \rightarrow \mathcal{E}(\text{holim}_{\rightarrow} A_n, L) \rightarrow \lim_{\leftarrow} \mathcal{E}(A_n, L) \rightarrow 0$. Since L is an \mathcal{E} -acyclic object, we have $\mathcal{E}(\text{holim}_{\rightarrow} A_n, L) = 0$. Hence $\widehat{\mathcal{P}(\mathcal{E})}^{*\infty} \subseteq {}^{\perp} \mathcal{A}_{\mathcal{E}}(\mathcal{E})$. If A is in ${}^{\perp} \mathcal{A}_{\mathcal{E}}(\mathcal{E})$ and $\{A_n; n \geq 0\}$ is the \mathcal{E} -cellular tower of A , then in the triangle (!!) above we have $\hat{A} \in \mathcal{A}_{\mathcal{E}}(\mathcal{E})$. Indeed

applying $\mathcal{E}(\mathcal{X}, -)$ to the triangle (!) and using Proposition 5.11, it follows that $\mathcal{E}(\mathcal{X}, \psi): \mathcal{E}(\mathcal{X}, \text{holim}_{\rightarrow} A_n) \rightarrow \mathcal{E}(\mathcal{X}, A)$ is an isomorphism. Hence by hypothesis, $\psi \in \mathcal{R}$ and this implies that $\mathbf{R}(\hat{A}) = 0$, i.e., $\hat{A} \in \mathcal{A}_{\mathcal{E}}(\mathcal{E})$. Since $A \in {}^{\perp}\mathcal{A}_{\mathcal{E}}(\mathcal{E})$, we have $\phi = 0$. It follows that A is a direct summand of $\text{holim}_{\rightarrow} A_n$; hence by definition $A \in \overline{\mathcal{P}(\mathcal{E})}^{*\infty}$. We conclude that $\overline{\mathcal{P}(\mathcal{E})}^{*\infty} = {}^{\perp}\mathcal{A}_{\mathcal{E}}(\mathcal{E})$, so it is a localizing subcategory of \mathcal{E} . By construction $\overline{\mathcal{P}(\mathcal{E})}^{*\infty}$ is the localizing subcategory of \mathcal{E} generated by $\mathcal{P}(\mathcal{E})$.

(ii) It is well known and easy to see that \mathbf{R} induces a full embedding ${}^{\perp}\mathcal{A}_{\mathcal{E}}(\mathcal{E}) \hookrightarrow \mathcal{D}_{\mathcal{E}}(\mathcal{E}) = \mathcal{E}/\mathcal{A}_{\mathcal{E}}(\mathcal{E})$. Hence \mathbf{R} induces a full embedding $\overline{\mathcal{P}(\mathcal{E})}^{*\infty} \hookrightarrow \mathcal{D}_{\mathcal{E}}(\mathcal{E})$. If A is an object in $\mathcal{D}_{\mathcal{E}}(\mathcal{E})$, then by the above argument the morphism $\psi: \text{holim}_{\rightarrow} A_n \rightarrow A$ is in \mathcal{R} . Since by construction $\text{holim}_{\rightarrow} A_n \in \overline{\mathcal{P}(\mathcal{E})}^{*\infty}$, it follows that the functor $\mathbf{R}: \overline{\mathcal{P}(\mathcal{E})}^{*\infty} \rightarrow \mathcal{D}_{\mathcal{E}}(\mathcal{E})$ is surjective on objects, so it is a triangle equivalence.

(iii) If $A \in \overline{\mathcal{P}(\mathcal{E})}^{*\infty}$, then from the triangle $\mathcal{E}(A)$ we deduce trivially that the morphism $\psi: \text{holim}_{\rightarrow} A_n \rightarrow A$ is an isomorphism. If $\mathcal{E}(\mathcal{X}, A) = 0$, then $\mathcal{E}(\mathcal{X}, \text{holim}_{\rightarrow} A_n) = 0$. This implies that the morphism $\mathcal{E}(\mathcal{X}, \zeta)$ in the triangle (!) is an isomorphism. Hence by hypothesis, ζ is in \mathcal{R} . It follows that $\mathbf{R}(\text{holim}_{\rightarrow} A_n) = 0$, so $A \cong \text{holim}_{\rightarrow} A_n \in \mathcal{A}_{\mathcal{E}}(\mathcal{E})$. Hence $A = 0$. Since \mathcal{X} consists of compact objects and it is contained $\overline{\mathcal{P}(\mathcal{E})}^{*\infty}$, it follows that $\overline{\mathcal{P}(\mathcal{E})}^{*\infty}$ is compactly generated.

It remains to prove that any compactly generated category \mathcal{E} arises in this way. Consider the proper class of triangles $\mathcal{E} = \mathcal{E}(\mathcal{E}^b)$ in \mathcal{E} . Since \mathcal{E}^b is skeletally small it follows easily (see Section 8) that \mathcal{E} has enough \mathcal{E} -projectives and $\mathcal{P}(\mathcal{E}) = \text{Add}(\mathcal{E}^b)$. Since \mathcal{E}^b generates \mathcal{E} , the projectivization functor \mathbf{S} reflects isomorphisms; hence \mathcal{R} consists of isomorphisms. Choosing $\mathcal{X} = \text{Iso}(\mathcal{E}^b)$ to be the set of isoclasses of compact objects above, we have $\mathcal{E} = \mathcal{E}[\mathcal{R}^{-1}] = \mathcal{D}_{\mathcal{E}}(\mathcal{E}) = \overline{\text{Add}(\mathcal{E}^b)}^{*\infty}$. ■

In some cases the compactness condition in Theorem 6.5 can be avoided:

Remark 6.6. Let $H: \mathcal{E} \rightarrow \mathcal{M}$ be a coproduct preserving homological functor to an AB4 category \mathcal{M} . Assume that H kills \mathcal{E} -phantom maps and satisfies the property: $H(f)$ is an isomorphism $\Rightarrow f \in \mathcal{R}$. Then parts (i) and (ii) in Theorem 6.5 are true. Indeed it suffices to show that $H(\psi): H(\text{holim}_{\rightarrow} A_n) \rightarrow H(A)$ is an isomorphism. Observe first that $H(\text{holim}_{\rightarrow} A_n) \cong \lim_{\rightarrow} H(A_n)$ and $H(\text{holim}_{\rightarrow} \Sigma^n(K_A^n)) \cong \lim_{\rightarrow} H(\Sigma^n(K_A^n)) = 0$. Hence applying H to the diagrams defining the \mathcal{E} -cellular tower of A , we have $H(\psi): H(\text{holim}_{\rightarrow} A_n) \rightarrow \cong H(A)$. Note that under the above assumptions, if $f \in \mathcal{R}$, then $H(f)$ is an isomorphism. Indeed since H kills \mathcal{E} -phantom maps, $H = H^* \mathbf{S}$ where $H^*: \mathcal{S}_{\mathcal{E}}(\mathcal{E}) \rightarrow \mathcal{M}$ is exact. Hence if $f \in \mathcal{R}$, i.e., $\mathbf{S}(f)$ is invertible, then so is $H(f)$. It follows that \mathcal{E} and H define the same phantom maps.

We call the objects of $\widehat{\mathcal{P}(\mathcal{E})}^{*\infty}$, \mathcal{E} -cofibrant. Hence any object of \mathcal{E} has an \mathcal{E} -cofibrant resolution, i.e., admits an \mathcal{E} -quasi-isomorphism to an \mathcal{E} -cofibrant object. The following are direct consequences of Theorem 6.5 or Remark 6.6.

COROLLARY 6.7. *Keep the hypothesis of Theorem 6.5 or Remark 6.6. Then the inclusion $\mathbf{i}: \widehat{\mathcal{P}(\mathcal{E})}^{*\infty} \hookrightarrow \mathcal{E}$ has a right adjoint $\mathbf{i}^!$ and the inclusion $\mathbf{j}: \mathcal{A}_{\mathcal{E}}(\mathcal{E}) \hookrightarrow \mathcal{E}$ has a left adjoint \mathbf{j}^* . The quotient functor $\mathbf{R}: \mathcal{E} \rightarrow \mathcal{D}_{\mathcal{E}}(\mathcal{E})$ has a fully faithful left adjoint. For any object $A \in \mathcal{E}$, there exists a functorial triangle*

$$\mathbf{i}^!(A) \rightarrow A \rightarrow \mathbf{j}^*(A) \rightarrow \Sigma(\mathbf{i}^!(A)),$$

where the involved morphisms are the adjunction morphisms. Any triangle $P \rightarrow A \rightarrow L \rightarrow \Sigma(P)$ with $P \in \widehat{\mathcal{P}(\mathcal{E})}^{*\infty}$ and $L \in \mathcal{A}_{\mathcal{E}}(\mathcal{E})$ is isomorphic to the triangle above by a unique isomorphism of triangles which extends the identity of A .

COROLLARY 6.8. *A full triangulated subcategory of $\mathcal{D}_{\mathcal{E}}(\mathcal{E})$ equals $\mathcal{D}_{\mathcal{E}}(\mathcal{E})$ if it contains \mathcal{X} and is closed under coproducts.*

COROLLARY 6.9. *If \mathcal{E} is compactly generated then any object A of \mathcal{E} is the homotopy colimit $\text{holim}_{\rightarrow} A_n$ of a tower $\{A_n; A_n \rightarrow^{\alpha_n} A_{n+1}\}$, where $A_n \in \text{Add}(\mathcal{E}^b)^{*n}$, $\forall n \geq 0$.*

6.2. Total Derived Functors

Let \mathcal{E} and \mathcal{B} be triangulated categories with proper classes of triangles \mathcal{E} and \mathcal{F} , respectively. We assume that the pairs $(\mathcal{E}, \mathcal{E}), (\mathcal{B}, \mathcal{F})$ satisfy the conditions of Theorem 6.5 or Remark 6.6.

Let $F: \mathcal{E} \rightarrow \mathcal{A}$ be an additive functor to an additive category \mathcal{A} . The total left \mathcal{E} -derived functor of F is an additive functor $\underline{\underline{L}}F: \mathcal{D}_{\mathcal{E}}(\mathcal{E}) \rightarrow \mathcal{A}$ together with a morphism $\zeta: \underline{\underline{L}}F\mathbf{R} \rightarrow F$ such that for any additive functor $G: \mathcal{D}_{\mathcal{E}}(\mathcal{E}) \rightarrow \mathcal{A}$ and morphism $\alpha: \mathbf{G}\mathbf{R} \rightarrow F$, there exists a unique natural morphism $\xi: G \rightarrow \underline{\underline{L}}F$ such that $\xi\mathbf{R} \circ \zeta = \alpha$. Similarly if $F: \mathcal{E} \rightarrow \mathcal{B}$ is an additive functor, then the total left $(\mathcal{E}, \mathcal{F})$ -derived functor of F is an additive functor $\underline{\underline{L}}F: \mathcal{D}_{\mathcal{E}}(\mathcal{E}) \rightarrow \mathcal{D}_{\mathcal{F}}(\mathcal{B})$ which is the total left \mathcal{E} -derived functor for the composite $\mathbf{R}_{\mathcal{B}}F: \mathcal{E} \rightarrow \mathcal{D}_{\mathcal{F}}(\mathcal{B})$.

THEOREM 6.10. (1) *Any additive functor $F: \mathcal{E} \rightarrow \mathcal{A}$ to an additive category \mathcal{A} has a total \mathcal{E} -derived functor $\underline{\underline{L}}F: \mathcal{D}_{\mathcal{E}}(\mathcal{E}) \rightarrow \mathcal{A}$. If F is homological then so is $\underline{\underline{L}}F$.*

(2) *Any additive functor $F: \mathcal{E} \rightarrow \mathcal{B}$ has a total left $(\mathcal{E}, \mathcal{F})$ -derived functor $\underline{\underline{L}}F: \mathcal{D}_{\mathcal{E}}(\mathcal{E}) \rightarrow \mathcal{D}_{\mathcal{F}}(\mathcal{B})$. If F is exact then so is $\underline{\underline{L}}F$.*

Proof. In case (2) just set $\underline{\underline{L}}F = \mathbf{R}_{\mathcal{B}}\mathbf{F}i: \mathcal{D}_{\mathcal{E}}(\mathcal{C}) \rightarrow \mathcal{D}_{\mathcal{E}}(\mathcal{B})$. Case (1) is similar. ■

Remark 6.11. Let \mathcal{U} be a Serre subcategory of $\mathcal{A}b$. Define a class of morphisms $\mathcal{R}(\mathcal{U})$ in \mathcal{C} as follows: $\mathcal{R}(\mathcal{U}) = \bigcup_{A, B \in \mathcal{C}} \mathcal{R}_{A, B}(\mathcal{U})$, where a morphism $f: A \rightarrow B$ belongs to $\mathcal{R}_{A, B}(\mathcal{U})$ if $\forall P \in \mathcal{P}(\mathcal{C})$, the morphism $\mathcal{E}(P, f)$ is an isomorphism in the Gabriel-quotient $\mathcal{A}b/\mathcal{U}$, i.e., $\ker \mathcal{E}(P, f)$, $\text{coker } \mathcal{E}(P, f) \in \mathcal{U}$. As above the class $\mathcal{R}(\mathcal{U})$ is a multiplicative system of morphisms compatible with the triangulation of \mathcal{C} . The class $\text{Ph}_{\mathcal{E}}^{\mathcal{U}}(\mathcal{C})$ of \mathcal{U} -, \mathcal{E} -phantom maps defined as the class of morphisms $f: A \rightarrow B$ such that $\mathcal{E}(P, f) = 0$ in $\mathcal{A}b/\mathcal{U}$, i.e., $\text{Im } \mathcal{E}(P, f) \in \mathcal{U}$, $\forall P \in \mathcal{P}(\mathcal{C})$, is then an ideal of \mathcal{C} . One can develop the theory of the present section to this situation without important changes. The above theory then corresponds to the case $\mathcal{U} = 0$. Note that this applies to the stable homotopy category of spectra, choosing as \mathcal{U} suitable Serre subcategories of $\mathcal{A}b$ associated with various sets of primes.

7. THE STABLE TRIANGULATED CATEGORY

Let \mathcal{C} be a triangulated category and let \mathcal{E} be a proper class of triangles in \mathcal{C} . We assume that \mathcal{C} has enough \mathcal{E} -projectives. We denote by $\mathcal{C}/\mathcal{P}(\mathcal{E})$ the stable category of \mathcal{C} modulo the full subcategory $\mathcal{P}(\mathcal{E})$ of \mathcal{E} -projective objects. If $\varpi: \mathcal{C} \rightarrow \mathcal{C}/\mathcal{P}(\mathcal{E})$ is the canonical functor, then we set $\varpi(A) = \underline{A}$ and $\varpi(f) = \underline{f}$.

Our aim is to show that $\mathcal{C}/\mathcal{P}(\mathcal{E})$ carries in a natural way a left triangulated structure, which in some cases it is useful to study. For the notion of left triangulated categories we refer to [12], or to [43] in which the dual notion is treated.

For each object $A \in \mathcal{C}$ choose an \mathcal{E} -projective presentation of A , i.e., a triangle $K_A^1 \xrightarrow{g_A^0} P_A^0 \xrightarrow{f_A^0} A \xrightarrow{h_A^0} \Sigma(K_A^1)$ in \mathcal{C} with $P_A^0 \in \mathcal{P}(\mathcal{E})$. By Schanuel’s lemma, we see that if we choose another \mathcal{E} -projective presentation $K \rightarrow P \rightarrow A \rightarrow \Sigma(K)$ of A , then $K_A^1 \oplus P \cong K \oplus P_A^0$. Hence the object K_A^1 is uniquely determined in $\mathcal{C}/\mathcal{P}(\mathcal{E})$. We denote it by $\Omega_{\mathcal{E}}(\underline{A}) = \underline{K}_A^1$. Let $f: A \rightarrow B$ be a morphism in \mathcal{C} . Then there are morphisms $p_f^0: P_A^0 \rightarrow P_B^0$, $k_f^1: K_A^1 \rightarrow K_B^1$ and a morphism of triangles

$$\begin{array}{ccccccc} K_A^1 & \xrightarrow{g_A^0} & P_A^0 & \xrightarrow{f_A^0} & A & \xrightarrow{h_A^0} & \Sigma(K_A^1) \\ k_f^1 \downarrow & & p_f^0 \downarrow & & f \downarrow & & \Sigma(k_f^1) \downarrow \\ K_B^1 & \xrightarrow{g_B^0} & P_B^0 & \xrightarrow{f_B^0} & B & \xrightarrow{h_B^0} & \Sigma(K_B^1). \end{array}$$

If $q_f^0: P_A^0 \rightarrow P_B^0$ is another lifting of f which induces a morphism $l_f^1: K_A^1 \rightarrow K_B^1$ in the above diagram, then $h_A^0 \circ [\Sigma(k_f^1) - \Sigma(l_f^1)] = 0$, so there

exists a morphism $\alpha: \Sigma(P_A^0) \rightarrow \Sigma(K_B^1)$ such that $\Sigma(k_f^1) - \Sigma(l_f^1) = \Sigma(g_A^0) \circ \alpha$. Then the morphism $\Sigma(k_f^1) - \Sigma(l_f^1)$ or equivalently the morphism $k_f^1 - l_f^1$ is zero in $\mathcal{E}/\mathcal{P}(\mathcal{E})$, i.e., $\underline{k}_f^1 = \underline{l}_f^1$. Hence setting $\Omega_{\mathcal{E}}(f) = \underline{k}_f^1$, we obtain an additive functor $\Omega_{\mathcal{E}}: \mathcal{E}/\mathcal{P}(\mathcal{E}) \rightarrow \mathcal{E}/\mathcal{P}(\mathcal{E})$, which we call the \mathcal{E} -loop functor of \mathcal{E} with respect to \mathcal{E} . If $T: A \xrightarrow{\alpha} B \xrightarrow{\beta} C \xrightarrow{\gamma} \Sigma(A)$ is a triangle in \mathcal{E} , then T induces a morphism of triangles

$$\begin{array}{ccccccc} K_C^1 & \xrightarrow{g_C^0} & P_C^0 & \xrightarrow{f_C^0} & C & \xrightarrow{h_C^0} & \Sigma(K_C^1) \\ c \downarrow & & d \downarrow & & \parallel \downarrow & & \Sigma(c) \downarrow \\ A & \xrightarrow{\alpha} & B & \xrightarrow{\beta} & C & \xrightarrow{\gamma} & \Sigma(A). \end{array}$$

It is easy to see as above that the morphism $c: K_C^1 \rightarrow A$ is independent in $\mathcal{E}/\mathcal{P}(\mathcal{E})$ of the chosen \mathcal{E} -projective presentation of \mathcal{E} . We call the image of c in $\mathcal{E}/\mathcal{P}(\mathcal{E})$ the *characteristic class* of the triangle T and we denote it by $\underline{c} := \text{ch}(T)$. Hence the triangle T in \mathcal{E} induces a diagram

$$\underline{T}: \Omega_{\mathcal{E}}(\underline{C}) \xrightarrow{\text{ch}(T)} \underline{A} \xrightarrow{\alpha} \underline{B} \xrightarrow{\beta} \underline{C} \quad \text{in } \mathcal{E}/\mathcal{P}(\mathcal{E}).$$

We define a triangulation $\Delta_{\mathcal{E}}$ of the pair $(\mathcal{E}/\mathcal{P}(\mathcal{E}), \Omega_{\mathcal{E}})$ as follows:

- A diagram $\Omega_{\mathcal{E}}(\underline{C}') \rightarrow \underline{A}' \rightarrow \underline{B}' \rightarrow \underline{C}'$ in $\mathcal{E}/\mathcal{P}(\mathcal{E})$ belongs to $\Delta_{\mathcal{E}}$ iff it is isomorphic to a diagram of the form \underline{T} , where T is a triangle in \mathcal{E} .

THEOREM 7.1. *The triple $(\mathcal{E}/\mathcal{P}(\mathcal{E}), \Omega_{\mathcal{E}}, \Delta_{\mathcal{E}})$ is a left triangulated category.*

Proof. The proof is similar to the proof of the main theorem of [12]. We prove only how any morphism $f: \underline{B} \rightarrow \underline{C}$ is embedded in a triangle in $\mathcal{E}/\mathcal{P}(\mathcal{E})$. Consider the base-change diagram of the \mathcal{E} -projective presentation of C along the morphism f :

$$\begin{array}{ccccccc} K_C^1 & \xrightarrow{\alpha} & A & \xrightarrow{\beta} & B & \xrightarrow{\gamma} & \Sigma(K_C^1) \\ \parallel \downarrow & & \delta \downarrow & & f \downarrow & & \parallel \downarrow \\ K_C^1 & \xrightarrow{g_C^0} & P_C^0 & \xrightarrow{f_C^0} & C & \xrightarrow{h_C^0} & \Sigma(K_C^1). \end{array}$$

Then consider the cobase-change diagram of the \mathcal{E} -projective presentation of C along the morphism $\alpha: K_C^1 \rightarrow A$

$$\begin{array}{ccccccc} K_C^1 & \xrightarrow{g_C^0} & P_C^0 & \xrightarrow{f_C^0} & C & \xrightarrow{h_C^0} & \Sigma(K_C^1) \\ \alpha \downarrow & & \varepsilon \downarrow & & \parallel \downarrow & & \Sigma(\alpha) \downarrow \\ A & \xrightarrow{\zeta} & D & \xrightarrow{\theta} & C & \xrightarrow{\eta} & \Sigma(A). \end{array}$$

We know that the lower triangle in the above diagram is in \mathcal{E} . Hence by the definition of $\Delta_{\mathcal{E}}$ we have a triangle $\Omega_{\mathcal{E}}(\underline{C}) \rightarrow \underline{A} \rightarrow \underline{D} \rightarrow \underline{C}$ in $\mathcal{E}/\mathcal{P}(\mathcal{E})$. It is not difficult to see that the above two diagrams produce an isomorphism $D \cong B \oplus P_C^0$. Hence $\underline{D} \cong \underline{B}$, such that the morphism $\underline{D} \rightarrow \underline{C}$ is isomorphic to f . Hence the diagram $\Omega_{\mathcal{E}}(\underline{C}) \rightarrow \underline{A} \rightarrow \underline{B} \xrightarrow{f} \underline{C}$ belongs to the triangulation $\Delta_{\mathcal{E}}$ of $\mathcal{E}/\mathcal{P}(\mathcal{E})$. ■

The stable category $\mathcal{E}/\mathcal{P}(\mathcal{E})$ is not necessarily triangulated. The next result describes when this happens. Its proof is left to the reader (compare [12]).

THEOREM 7.2. *The stable category $\mathcal{E}/\mathcal{P}(\mathcal{E})$ is triangulated iff \mathcal{E} has enough \mathcal{E} -injectives and $\mathcal{P}(\mathcal{E}) = \mathcal{I}(\mathcal{E})$. If $\mathcal{E}/\mathcal{P}(\mathcal{E})$ is triangulated, then $\mathcal{E}\text{-gl.dim } \mathcal{E} = 0$ or ∞ . Moreover $\mathcal{E}\text{-gl.dim } \mathcal{E} = 0$ iff $\mathcal{E} = \Delta_0$.*

8. PROJECTIVITY, INJECTIVITY, AND FLATNESS

Throughout this section we fix a triangulated category \mathcal{E} with coproducts and a full skeletally small additive subcategory $0 \neq \mathcal{X} \subseteq \mathcal{E}$, closed under isomorphisms, Σ, Σ^{-1} , and direct summands. Our aim is to study the homological theory of \mathcal{E} based on the proper class of triangles $\mathcal{E}(\mathcal{X})$:

- A triangle $A \rightarrow B \rightarrow C \rightarrow \Sigma(A)$ is in $\mathcal{E}(\mathcal{X})$ iff $\forall X \in \mathcal{X}$, the induced sequence

$$0 \rightarrow \mathcal{E}(X, A) \rightarrow \mathcal{E}(X, B) \rightarrow \mathcal{E}(X, C) \rightarrow 0$$

is exact in $\mathcal{A}b$.

LEMMA 8.1. *\mathcal{E} has enough $\mathcal{E}(\mathcal{X})$ -projectives and $\mathcal{P}(\mathcal{E}(\mathcal{X})) = \text{Add}(\mathcal{X})$.*

Proof. For any $C \in \mathcal{E}$, let $I_C := \{X \rightarrow C \mid X \in \text{Iso}(\mathcal{X})\}$. If $X_C := \bigoplus_{\lambda \in I_C} X_\lambda$, then $X_C \in \mathcal{P}(\mathcal{E}(\mathcal{X}))$ since $\mathcal{X} \subseteq \mathcal{P}(\mathcal{E}(\mathcal{X}))$ and $\mathcal{P}(\mathcal{E}(\mathcal{X}))$ is closed under coproducts. The set I_C induces a morphism $f: X_C \rightarrow C$, which by construction is an $\mathcal{E}(\mathcal{X})$ -proper epic. Then the triangle $A \rightarrow X_C \xrightarrow{f} C \rightarrow \Sigma(A) \in \mathcal{E}(\mathcal{X})$ and \mathcal{E} has enough $\mathcal{E}(\mathcal{X})$ -projectives. Further $\text{Add}(\mathcal{X}) \subseteq \mathcal{P}(\mathcal{E}(\mathcal{X}))$, since $\mathcal{P}(\mathcal{E}(\mathcal{X}))$ is closed under coproducts and direct summands and contains \mathcal{X} . If $P \in \mathcal{P}(\mathcal{E}(\mathcal{X}))$, let $A \rightarrow X_p \rightarrow \alpha P \rightarrow \Sigma(A)$ be a triangle in $\mathcal{E}(\mathcal{X})$ with X_p a coproduct of objects of \mathcal{X} . Since P is $\mathcal{E}(\mathcal{X})$ -projective, α splits and P is a direct summand of X_p , so $P \in \text{Add}(\mathcal{X})$. ■

Since \mathcal{E} has enough $\mathcal{E}(\mathcal{X})$ -projectives, we can apply the results of the previous sections to the pair $(\mathcal{E}, \mathcal{X})$. An important example of the above situation occurs in the stable homotopy category of spectra [51], choosing \mathcal{X} to be the category of finite spectra. For other examples we refer to Section 12.

LEMMA 8.2. (1) $f: A \rightarrow B$ is an $\mathcal{E}(\mathcal{X})$ -phantom map iff $\forall X \in \mathcal{X}: \mathcal{E}(X, f) = 0$.

(2) The ideal $\text{Ph}_{\mathcal{E}(\mathcal{X})}(\mathcal{E}) \subseteq \text{Fac}(\mathcal{E})$ iff $\forall X \in \mathcal{X}: \mathcal{E}(X, A) = 0 \Rightarrow A = 0$.

8.1. Projective and Flat Functors

Since \mathcal{X} is skeletally small, we can consider the category $\text{Mod}(\mathcal{X})$ of right \mathcal{X} -modules, i.e., the category of contravariant additive functors from \mathcal{X} to the category $\mathcal{A}b$ of abelian groups. Define a functor

$$\mathbf{S}': \mathcal{E} \rightarrow \text{Mod}(\mathcal{X}) \quad \text{by} \quad \mathbf{S}'(A) = \mathcal{E}(-, A)|_{\mathcal{X}}$$

and

$$\mathbf{S}'(f) = \mathcal{E}(-, f)|_{\mathcal{X}},$$

where $\mathcal{E}(-, A)|_{\mathcal{X}}$ denotes the restriction of the representable functor to \mathcal{X} .

There is a well-defined tensor product functor $- \otimes_{\mathcal{X}} -: \text{Mod}(\mathcal{X}) \times \text{Mod}(\mathcal{X}^{\text{op}}) \rightarrow \mathcal{A}b$, which satisfies all the usual properties [53]. A functor F in $\text{Mod}(\mathcal{X})$ is called *flat*, if $F \otimes_{\mathcal{X}} -: \text{Mod}(\mathcal{X}^{\text{op}}) \rightarrow \mathcal{A}b$ is exact. The full subcategory of $\text{Mod}(\mathcal{X})$ consisting of flat functors is denoted by $\text{Flat}(\text{Mod}(\mathcal{X}))$. Also we denote by $\text{Proj}(\text{Mod}(\mathcal{X}))$, resp. $\text{Inj}(\text{Mod}(\mathcal{X}))$, the full subcategory of $\text{Mod}(\mathcal{X})$ consisting of all projective, resp. injective, objects. Observe that $\forall X \in \mathcal{X}$, $\mathbf{S}'(X)$ is the representable functor $\mathcal{X}(-, X)$, so $\mathbf{S}'(X) \in \text{Proj}(\text{Mod}(\mathcal{X}))$.

Remark 8.3. The functor $\mathbf{S}': \mathcal{E} \rightarrow \text{Mod}(\mathcal{X})$ is homological and a triangle T is in $\mathcal{E}(\mathcal{X})$ iff $\mathbf{S}'(T)$ is a short exact sequence in $\text{Mod}(\mathcal{X})$. Moreover \mathbf{S}' preserves products, and \mathbf{S}' preserves coproducts iff $\mathcal{X} \subseteq \mathcal{E}^b$. Obviously \mathbf{S}' kills $\mathcal{E}(\mathcal{X})$ -phantom maps, so there exists a unique exact functor $\mathbf{H}: \mathcal{S}_{\mathcal{E}(\mathcal{X})}(\mathcal{E}) \rightarrow \text{Mod}(\mathcal{X})$ with $\mathbf{H}\mathbf{S} = \mathbf{S}'$, where $\mathcal{S}_{\mathcal{E}(\mathcal{X})}(\mathcal{E})$ is the Steenrod category and \mathbf{S} is the projectivization functor.

PROPOSITION 8.4. *The following conditions are equivalent:*

(i) *The functor $\mathbf{H}: \mathcal{S}_{\mathcal{E}(\mathcal{X})}(\mathcal{E}) \rightarrow \text{Mod}(\mathcal{X})$ is an equivalence of categories.*

(ii) *The functor \mathbf{S}' induces an equivalence: $\mathcal{P}(\mathcal{E}(\mathcal{X})) = \text{Add}(\mathcal{X}) \approx \text{Proj}(\text{Mod}(\mathcal{X}))$.*

(iii) *For any object $X \in \mathcal{X}$ and any family $\{X_i\}_{i \in I} \subseteq \mathcal{X}$, the natural morphism $\bigoplus_{i \in I} \mathcal{E}(X, X_i) \rightarrow \mathcal{E}(X, \bigoplus_{i \in I} X_i)$ is an isomorphism.*

If (i) holds, then identifying $\mathcal{S}_{\mathcal{E}(\mathcal{X})}(\mathcal{E}) = \text{Mod}(\mathcal{X})$ and $\mathbf{S}' = \mathbf{S}$, we have that \mathbf{S} induces natural isomorphisms:

$$\mathcal{E}xt_{\mathcal{E}(\mathcal{X})}^n(-, C) \cong \mathcal{E}xt_{\text{Mod}(\mathcal{X})}^n[\mathbf{S}(-), \mathbf{S}(C)]: \mathcal{E}^{\text{op}} \rightarrow \mathcal{A}b,$$

$$\forall C \in \mathcal{E}, \quad \forall n \geq 0.$$

Further the ideal $\ker \mathbf{S} = \text{Ph}_{\mathcal{E}(\mathcal{X})}(\mathcal{E})$, and if \mathcal{X} generates \mathcal{E} , then $\forall A \in \mathcal{E} : \mathcal{E}(\mathcal{X})\text{-p.d } A = \text{p.d } \mathbf{S}(A)$. In particular: $\mathcal{E}(\mathcal{X})\text{-gl.dim } \mathcal{E} \leq \text{gl.dim Mod}(\mathcal{X})$.

Proof. (ii) \Rightarrow (iii) Let $\{X_i\}_{i \in I}$ be any family of objects in \mathcal{X} . Since $\bigoplus_{i \in I} X_i$ is in $\mathcal{P}(\mathcal{E}(\mathcal{X}))$, by hypothesis we have $\bigoplus_{i \in I} \mathbf{S}'(X_i) \cong \mathbf{S}'(\bigoplus_{i \in I} X_i)$. It follows that for any $X \in \mathcal{X}$, $\bigoplus_{i \in I} \mathcal{E}(X, X_i) = \bigoplus_{i \in I} \mathbf{S}'(X_i)(X) \cong \mathbf{S}'(\bigoplus_{i \in I} X_i)(X) = \mathcal{E}(X, \bigoplus_{i \in I} X_i)$.

(iii) \Rightarrow (ii) Since \mathbf{S}' restricted to \mathcal{X} is the Yoneda embedding, it follows that $\mathbf{S}'|_{\mathcal{X}}: \mathcal{X} \hookrightarrow \mathbf{P}$ is fully faithful where \mathbf{P} is the category of finitely generated projective functors in $\text{Mod}(\mathcal{X})$. Since $\text{Proj}(\text{Mod}(\mathcal{X})) = \text{Add}(\mathbf{P})$, the hypothesis implies easily that $\mathbf{S}'|_{\mathcal{X}}$ can be extended to a full embedding $\mathcal{P}(\mathcal{E}(\mathcal{X})) = \text{Add}(\mathcal{X}) \hookrightarrow \text{Proj}(\text{Mod}(\mathcal{X}))$. Let F be a projective functor. Then there exists a family $\{X_i\}_{i \in I} \subseteq \mathcal{X}$ such that $F \oplus G = \bigoplus_{i \in I} \mathbf{S}'(X_i) = \mathbf{S}'(\bigoplus_{i \in I} X_i)$, for some functor G . This decomposition gives us an idempotent $f: \mathbf{S}'(\bigoplus_{i \in I} X_i) \rightarrow \mathbf{S}'(\bigoplus_{i \in I} X_i)$ with $\text{Im}(f) = F$. Then there exists an idempotent $e: \bigoplus_{i \in I} X_i \rightarrow \bigoplus_{i \in I} X_i$ such that $\mathbf{S}'(e) = f$. Since \mathcal{E} has coproducts, by [28] the idempotent e splits producing a direct sum decomposition $\bigoplus_{i \in I} X_i = K \oplus L$ with $\text{Im}(e) = K$. Then obviously $\mathbf{S}'(K) \cong F$ and consequently $\mathbf{S}': \mathcal{P}(\mathcal{E}(\mathcal{X})) \rightarrow \text{Proj}(\text{Mod}(\mathcal{X}))$ is an equivalence.

(i) \Leftrightarrow (ii) If condition (i) holds, then \mathbf{H} restricts to an equivalence between $\text{Proj } \mathcal{S}_{\mathcal{E}(\mathcal{X})}(\mathcal{E})$ and $\text{Proj}(\text{Mod}(\mathcal{X}))$. Since $\mathcal{P}(\mathcal{E}(\mathcal{X})) = \text{Add}(\mathcal{X})$ has split idempotents, by Proposition 4.19 we have an equivalence $\mathcal{P}(\mathcal{E}(\mathcal{X})) \approx \text{Proj } \mathcal{S}_{\mathcal{E}(\mathcal{X})}(\mathcal{E})$, induced by \mathbf{S} . It follows that $\mathbf{S}' = \mathbf{H}\mathbf{S}$ induces an equivalence $\mathcal{P}(\mathcal{E}(\mathcal{X})) \approx \text{Proj}(\text{Mod}(\mathcal{X}))$. Conversely if (ii) holds, then the functor \mathbf{S}' induces the desired equivalence $\mathbf{H}: \mathcal{S}_{\mathcal{E}(\mathcal{X})}(\mathcal{E}) = \mathcal{A}(\mathcal{P}(\mathcal{E}(\mathcal{X}))) \rightarrow \approx \mathcal{A}(\text{Proj}(\text{Mod}(\mathcal{X}))) = \text{Mod}(\mathcal{X})$.

The last assertions follow from Section 4. ■

We call \mathcal{X} *self-compact* if \mathcal{X} satisfies condition (iii) of Proposition 8.4. From now on we write $\mathbf{S} = \mathbf{S}'$, so that if \mathcal{X} is self-compact, \mathbf{S}' is the projectivization functor and $\mathcal{S}_{\mathcal{E}(\mathcal{X})}(\mathcal{E}) = \text{Mod}(\mathcal{X})$. We would like to know when \mathbf{S} has its image in the subcategory $\text{Flat}(\text{Mod}(\mathcal{X}))$ of flat functors. The motivation for this is that in case \mathcal{X} is triangulated, the category $\mathcal{E}oh(\mathcal{X}^{\text{op}}, \mathcal{A}b)$, resp. $\mathcal{H}om(\mathcal{X}, \mathcal{A}b)$, of cohomological, resp. homological, functors over \mathcal{X} and the category $\text{Flat}(\text{Mod}(\mathcal{X}))$, resp. $\text{Flat}(\text{Mod}(\mathcal{X}^{\text{op}}))$, of

flat right, resp. left, \mathcal{L} -modules, coincide [13]. It is natural to compare \mathcal{E} with the cohomological functors over \mathcal{L} , via the homological functor \mathbf{S} .

PROPOSITION 8.5. *The following conditions are equivalent:*

- (i) $\text{Im } \mathbf{S} \subseteq \text{Flat}(\text{Mod}(\mathcal{L}))$.
- (ii) \mathcal{L} has weak cokernels and the inclusion $i: \mathcal{L} \hookrightarrow \mathcal{E}$ preserves them.
- (iii) \mathcal{L} has weak cokernels and any morphism in \mathcal{L} is a weak kernel.
- (iv) \mathcal{L} has weak cokernels and any injective right \mathcal{L} -module is flat.

Proof. (ii) \Rightarrow (i) Since \mathcal{L} has weak cokernels, it follows from [13] that a functor $F \in \text{Mod}(\mathcal{L})$ is flat iff F sends diagrams of the form $(*)$: $X \xrightarrow{f} Y \xrightarrow{g} Z$, where g is a weak cokernel of f in \mathcal{L} , to exact sequences in $\mathcal{A}b$. Hence we must show that if $(*)$ is a diagram as above, then $\mathbf{S}(A)(Z) \rightarrow^{\mathbf{S}(A)(g)} \mathbf{S}(A)(Y) \rightarrow^{\mathbf{S}(A)(f)} \mathbf{S}(A)(X)$ is exact, $\forall A \in \mathcal{E}$. Obviously this diagram is isomorphic to the complex (\dagger) : $\mathcal{E}(Z, A) \xrightarrow{\mathcal{E}(g, A)} \mathcal{E}(Y, A) \xrightarrow{\mathcal{E}(f, A)} \mathcal{E}(X, A)$. Let $X \xrightarrow{f} Y \xrightarrow{g'} A' \xrightarrow{h} \Sigma(X)$ be a triangle in \mathcal{E} . Since g is a weak cokernel of f in \mathcal{L} and the inclusion $i: \mathcal{L} \hookrightarrow \mathcal{E}$ preserves weak cokernels, there are morphisms $\tau: Z \rightarrow A'$ and $\rho: A' \rightarrow Z$ such that $g' = g \circ \tau$ and $g = g' \circ \rho$. Then let $\omega \in \text{Ker } \mathcal{E}(f, A)$ be such that $f \circ \omega = 0$. Since g' is a weak cokernel of f , there exists a morphism $\phi: A' \rightarrow A$ with $g' \circ \phi = \omega$. Then $g \circ \tau \circ \phi = \omega \Rightarrow \omega \in \text{Im } \mathcal{E}(g, A)$. Hence (\dagger) is exact in $\mathcal{A}b$, and consequently $\mathbf{S}(A)$ is a flat functor in $\text{Mod}(\mathcal{L})$.

(i) \Rightarrow (ii) Let $f: X \rightarrow Y$ be a morphism in \mathcal{L} and let $X \xrightarrow{f} Y \xrightarrow{g} A \xrightarrow{h} \Sigma(X)$ be a triangle in \mathcal{E} . Since \mathbf{S} is homological, we have an exact sequence in $\text{Mod}(\mathcal{L})$: $\mathbf{S}(X) \rightarrow^{\mathbf{S}(f)} \mathbf{S}(Y) \rightarrow^{\mathbf{S}(g)} \mathbf{S}(A) \rightarrow^{\mathbf{S}(h)} \mathbf{S}(\Sigma(X))$. Let $\kappa: \mathbf{S}(Y) \rightarrow F$ be the cokernel of $\mathbf{S}(f)$, and let $\lambda: F \hookrightarrow \mathbf{S}(A)$ be the canonical inclusion, such that $\mathbf{S}(g) = \kappa \circ \lambda$. Since $\mathbf{S}(X), \mathbf{S}(Y)$ are representable, F is finitely presented. By hypothesis $\mathbf{S}(A)$ is a flat functor. Hence the morphism λ factors through some representable functor $\mathcal{X}(-, Z) = \mathbf{S}(Z)$ with $Z \in \mathcal{L}$. So there are morphisms $m: F \rightarrow \mathbf{S}(Z)$ and $\nu: \mathbf{S}(Z) \rightarrow \mathbf{S}(A)$ such that $\lambda = m \circ \nu$. Since $Z \in \mathcal{L}$ and $\mathbf{S}|_{\mathcal{L}}$ is fully faithful, there exists a unique morphism $\gamma: Z \rightarrow A$ such that $\nu = \mathbf{S}(\gamma)$. Similarly for the morphism $\kappa \circ m: \mathbf{S}(Y) \rightarrow \mathbf{S}(Z)$, there exists a unique morphism $\alpha: Y \rightarrow Z$ with $\kappa \circ m = \mathbf{S}\alpha$. Then $\mathbf{S}(g) = \kappa \circ \lambda = \kappa \circ m \circ \nu = \mathbf{S}(\alpha) \circ \mathbf{S}(\gamma) = \mathbf{S}(\alpha \circ \gamma)$. Since $\mathbf{S}|_{\mathcal{L}}$ is fully faithful, $g = \alpha \circ \gamma$. We show that $\alpha: Y \rightarrow Z$ is a weak cokernel of f in \mathcal{L} . First $\mathbf{S}(f) \circ \mathbf{S}(g) = 0 \Rightarrow \mathbf{S}(f) \circ \kappa \circ \lambda = 0 \Rightarrow \mathbf{S}(f) \circ \kappa = 0 \Rightarrow \mathbf{S}(f) \circ \kappa \circ m = 0 \Rightarrow \mathbf{S}(f) \circ \mathbf{S}(\alpha) = 0 \Rightarrow f \circ \alpha = 0$. Now let $\rho: Y \rightarrow W$ be a morphism in \mathcal{L} such that $f \circ \rho = 0$. Since g is a weak cokernel of f in \mathcal{E} , there exists a morphism $\tau: A \rightarrow W$ with $\rho = g \circ \tau$. But then ρ factors through α as $\rho = \alpha \circ \gamma \circ \tau$. Hence \mathcal{L} has weak cokernels. To show that the inclusion $i: \mathcal{L} \hookrightarrow \mathcal{E}$ preserves them, it suffices to show that if ω :

$Y \rightarrow B$ is a morphism in \mathcal{E} with $f \circ \omega = 0$, then ω factors through α . If $f \circ \omega = 0$, there exists $\phi: A \rightarrow B$ with $g \circ \phi = \omega$. Then $\alpha \circ \gamma \circ \phi = \omega$.

(iii) \Leftrightarrow (iv) Follows from [13]. (ii) \Leftrightarrow (iii) The easy proof is left to the reader. ■

8.2. Injective Objects and Functors

Assume, besides the default assumptions concerning the pair $(\mathcal{E}, \mathcal{Z})$, that \mathcal{Z} generates \mathcal{E} and consists of compact objects.

Let $F \in \text{Flat}(\text{Mod}(\mathcal{Z}^{\text{op}}))$; i.e., $F: \mathcal{Z} \rightarrow \mathcal{A}b$ is a flat functor. Consider the functor

$$\mathbb{D}[\mathbf{S}(-) \otimes_{\mathcal{Z}} F]: \mathcal{E}^{\text{op}} \rightarrow \mathcal{A}b, \tag{*}$$

where $\mathbb{D} = \text{Hom}_{\mathbb{Z}}(-, \mathbb{Q}/\mathbb{Z})$ and $\mathbb{D}[\mathbf{S}(-) \otimes_{\mathcal{Z}} F](A) = \text{Hom}_{\mathbb{Z}}[\mathbf{S}(A) \otimes_{\mathcal{Z}} F, \mathbb{Q}/\mathbb{Z}]$. Since F is a flat, the above functor is cohomological and converts coproducts to products. Since \mathcal{E} is compactly generated by \mathcal{Z} , by Brown representability theorem [55] there exists an object $\mathbf{D}(F) \in \mathcal{E}$, unique up to isomorphism, and a natural isomorphism

$$\omega: \mathbb{D}[\mathbf{S}(-) \otimes_{\mathcal{Z}} F] \xrightarrow{\cong} \mathcal{E}[-, \mathbf{D}(F)]. \tag{+}$$

Since \mathbf{S} sends triangles in $\mathcal{E}(\mathcal{Z})$ to short exact sequences in $\text{Mod}(\mathcal{Z})$, it follows that $\mathbb{D}[\mathbf{S}(-) \otimes_{\mathcal{Z}} F]$, hence $\mathcal{E}[-, \mathbf{D}(F)]$, sends triangles in $\mathcal{E}(\mathcal{Z})$ to short exact sequences in $\mathcal{A}b$. We infer that $\mathbf{D}(F)$ is an $\mathcal{E}(\mathcal{Z})$ -injective object in \mathcal{E} .

Choose $F = \mathcal{Z}(X, -)$ in $(*)$, where $X \in \mathcal{Z}$. Then $\mathbb{D}[\mathbf{S}(-) \otimes_{\mathcal{Z}} \mathcal{Z}(X, -)] = \mathbb{D}\mathcal{E}(X, -)$. In this case we denote the object $\mathbf{D}(\mathcal{Z}(X, -))$ by $\mathbb{D}(X)$. Observe that if $\mathcal{E}(C, \mathbb{D}(X)) = 0, \forall X \in \mathcal{Z}$, then $\mathbb{D}\mathcal{E}(X, C) = 0 \Rightarrow \mathcal{E}(X, C) = 0, \forall X \in \mathcal{Z}$. Since \mathcal{Z} generates $\mathcal{E}, C = 0$. Hence the set $\{\mathbb{D}(X); X \in \text{Iso}(\mathcal{Z})\}$ cogenerates \mathcal{E} .

For a class of objects $\mathcal{Y} \subseteq \mathcal{E}$, let $\text{Prod}(\mathcal{Y})$ be the full subcategory of retracts of products of objects of \mathcal{Y} . The following generalizes results of [21, 50].

THEOREM 8.6. *\mathcal{E} has $\mathcal{E}(\mathcal{Z})$ -injective envelopes and $\mathcal{A}(\mathcal{E}(\mathcal{Z})) = \text{Prod}(\mathbb{D}(\mathcal{Z}))$. The functor \mathbf{S} induces an equivalence $\mathbf{S}: \mathcal{A}(\mathcal{E}(\mathcal{Z})) \rightarrow \widetilde{\text{Inj}}(\text{Mod}(\mathcal{Z}))$. Finally for any $\mathcal{E}(\mathcal{Z})$ -injective object E , the ring $\Lambda := \text{End}_{\mathcal{E}}(E)$ is F -semiperfect; i.e., $\Lambda/\mathcal{F}ac(\Lambda)$ is Von Neumann regular and idempotents can be lifted modulo $\mathcal{F}ac(\Lambda)$. In particular an indecomposable $\mathcal{E}(\mathcal{Z})$ -injective object in \mathcal{E} has a local endomorphism ring.*

Proof. For $C \in \mathcal{E}$, let $J_C := \{C \rightarrow \mathbb{D}(X) | X \in \text{Iso}(\mathcal{Z})\}$. Set $E_C = \prod_{\mu \in J_C} \mathbb{D}(X_{\mu})$. The set J_C induces a morphism $g_C: C \rightarrow E_C$. Let $(T): C \xrightarrow{g_C} E_C \rightarrow A \rightarrow \Sigma(C)$ be a triangle in \mathcal{E} . Applying, $\forall X \in \mathcal{Z}$, the functor

$\mathcal{E}(-, \mathbb{D}(X))$ to the triangle it follows by construction that $0 \rightarrow \mathcal{E}(A, \mathbb{D}(X)) \rightarrow \mathcal{E}(E_C, \mathbb{D}(X)) \rightarrow \mathcal{E}(C, \mathbb{D}(X)) \rightarrow 0$ is exact. But this sequence is isomorphic to the sequence $0 \rightarrow \mathbb{D}\mathcal{E}(X, A) \rightarrow \mathbb{D}\mathcal{E}(X, E_C) \rightarrow \mathbb{D}\mathcal{E}(X, C) \rightarrow 0$. Since \mathbb{Q}/\mathbb{Z} cogenerates $\mathcal{A}b$, the sequence $0 \rightarrow \mathcal{E}(X, C) \rightarrow \mathcal{E}(X, E_C) \rightarrow \mathcal{E}(X, A) \rightarrow 0$ is exact, i.e., $(T) \in \mathcal{E}(\mathcal{X})$. Since E_C is $\mathcal{E}(\mathcal{X})$ -injective, we have that \mathcal{E} has enough $\mathcal{E}(\mathcal{X})$ -injectives and the characterization of $\mathcal{E}(\mathcal{X})$ -injective objects follows. By Theorem 4.24, $\mathbf{S}: \mathcal{S}(\mathcal{E}(\mathcal{X})) \hookrightarrow \text{Inj}(\text{Mod}(\mathcal{X}))$ is fully faithful. Let F be an injective functor in $\text{Mod}(\mathcal{X})$. Applying Brown representability to the functor $\mathcal{E}(\mathbf{S}(-), F)$, we have an isomorphism $\omega: \mathcal{E}(\mathbf{S}(-), F) \rightarrow \mathcal{E}(-, E_F)$ and plainly E_F is in $\mathcal{S}(\mathcal{E}(\mathcal{X}))$. Then $\omega|_{\mathcal{X}}: \mathcal{E}(\mathbf{S}(-), F)|_{\mathcal{X}} = F \cong \mathbf{S}(E_F)$, so \mathbf{S} is surjective. By Theorem 4.24, \mathcal{E} has $\mathcal{E}(\mathcal{X})$ -injective envelopes. The last assertions are true since they are true in $\text{Mod}(\mathcal{X})$; see [39]. ■

We close this subsection showing that any object of \mathcal{E} admits a functorial “embedding” into an $\mathcal{E}(\mathcal{X})$ -injective object. We assume from now on that \mathcal{X} is a triangulated subcategory of \mathcal{E} . By Proposition 8.5, for any object $A \in \mathcal{E}$, $\mathbf{S}(A)$ is a flat right \mathcal{X} -module. It is well known that $\mathbb{D}\mathbf{S}(A)$ is an injective left \mathcal{X} -module. Since \mathcal{X} is a triangulated subcategory of \mathcal{E} , by the dual of Proposition 8.5, we infer that $\mathbb{D}\mathbf{S}(A)$ is a flat left \mathcal{X} -module. Hence choosing $F = \mathbb{D}\mathbf{S}(A)$ in $(*)$, we obtain an $\mathcal{E}(\mathcal{X})$ -injective object $\mathbf{D}(\mathbb{D}\mathbf{S}(A))$ in \mathcal{E} , which we denote by $\mathbf{D}(A)$, equipped with the natural isomorphism $\omega: \mathbb{D}[\mathbf{S}(-) \otimes_{\mathcal{X}} \mathbb{D}\mathbf{S}(A)] \xrightarrow{\cong} \mathcal{E}[-, \mathbf{D}(A)]$.

THEOREM 8.7. *The assignment $A \mapsto \mathbf{D}(A)$ defines an additive functor $\mathbf{D}: \mathcal{E} \rightarrow \mathcal{E}$ and $\forall A \in \mathcal{E}$, the object $\mathbf{D}(A)$ is $\mathcal{E}(\mathcal{X})$ -injective. Further there exists a natural morphism $\delta: \text{Id}_{\mathcal{E}} \rightarrow \mathbf{D}$, such that $\delta_A: A \rightarrow \mathbf{D}(A)$ is an $\mathcal{E}(\mathcal{X})$ -proper monic.*

Proof. It is well known [65] that there is a canonical morphism $F \otimes_{\mathcal{X}} \mathbb{D}(G) \rightarrow \mathbb{D}(F, G)$ which is invertible if F is finitely presented. Hence we have a natural morphism $\phi: \mathbf{S}(-) \otimes_{\mathcal{X}} \mathbb{D}\mathbf{S}(A) \rightarrow \mathbb{D}[\mathbf{S}(-), \mathbf{S}(A)]$ such that $\phi|_{\mathcal{X}}$ is invertible. We denote by $\psi: \mathcal{E}(-, A) \rightarrow \mathcal{E}[-, \mathbf{D}(A)]$ the composition

$$\begin{aligned} \mathcal{E}(-, A) &\xrightarrow{\mathbf{S}_{-, A}} [\mathbf{S}(-), \mathbf{S}(A)] \xrightarrow{\mu} \mathbb{D}^2[\mathbf{S}(-), \mathbf{S}(A)] \\ &\xrightarrow{\mathbb{D}(\phi)} \mathbb{D}[\mathbf{S}(-) \otimes_{\mathcal{X}} \mathbb{D}\mathbf{S}(A)] \xrightarrow{\omega} \mathcal{E}[-, \mathbf{D}(A)], \end{aligned}$$

where μ is the canonical monomorphism. Observe that $\psi|_{\mathcal{X}}: \mathbf{S}(A) \rightarrow \mathbf{S}(\mathbf{D}(A))$ is a monomorphism. By Yoneda’s lemma there exists a unique morphism $\delta_A: A \rightarrow \mathbf{D}(A)$ such that $\psi = \mathcal{E}(-, \delta_A)$. Then $\psi|_{\mathcal{X}} = \mathcal{E}(-, \delta_A)|_{\mathcal{X}} = \mathbf{S}(\delta_A)$ and since $\psi|_{\mathcal{X}}$ is a monomorphism, the same is true for $\mathbf{S}(\delta_A)$; i.e., δ_A is an $\mathcal{E}(\mathcal{X})$ -proper monic. We leave to the reader to prove that $A \mapsto \mathbf{D}(A)$ defines an additive functor $\mathbf{D}: \mathcal{E} \rightarrow \mathcal{E}$ and $\{\delta_A\}_{A \in \mathcal{E}}$ are the components of a natural morphism $\delta: \text{Id}_{\mathcal{E}} \rightarrow \mathbf{D}$. ■

Note that we have the formula $\mathbf{SD} \cong \mathbb{D}^2\mathbf{S}$, so \mathbf{D} can be regarded as a “double dual” functor in \mathcal{C} . The next result is a direct consequence of the above theorems.

COROLLARY 8.8. $\forall A, B \in \mathcal{C}, \forall X \in \mathcal{X}, \text{Ph}_{\mathcal{E}(\mathcal{X})}(A, \mathbb{D}(X)) = 0 = \text{Ph}_{\mathcal{E}(\mathcal{X})}(A, \mathbf{D}(B))$. Moreover $f \in \text{Ph}_{\mathcal{E}(\mathcal{X})}(\mathcal{C})$ iff $\mathbf{D}(f) = 0$ iff $\mathcal{E}(f, \mathbb{D}(X)) = 0, \forall X \in \mathcal{X}$.

8.3. Homological Functors Preserving Coproducts

Assume from now on that \mathcal{X} is self-compact and satisfies the following condition

($\dagger\dagger$) \mathcal{X} has weak kernels and the inclusion $\mathcal{X} \hookrightarrow \mathcal{C}$ preserves them, which is dual to condition (ii) of Proposition 8.5.

Fix a coproduct preserving homological functor $H: \mathcal{C} \rightarrow \mathcal{A}b$. Set $H' := H|_{\mathcal{X}}: \mathcal{X} \rightarrow \mathcal{A}b$ and let $H^* := - \otimes_{\mathcal{X}} H': \text{Mod}(\mathcal{X}) \rightarrow \mathcal{A}b$ be the unique colimit preserving functor extending H' through the Yoneda embedding $\mathcal{X} \hookrightarrow \text{Mod}(\mathcal{X})$. Using Corollary 4.21, it is not difficult to see that $H^*\mathbf{S} \cong \mathcal{L}_0^{\mathcal{E}(\mathcal{X})}H$. Let $\omega: \mathcal{L}_0^{\mathcal{E}(\mathcal{X})}H \rightarrow H$ be the canonical morphism; in particular the restriction $\omega|_{\mathcal{P}(\mathcal{E}(\mathcal{X}))}$ is invertible. Since H is homological and \mathcal{X} satisfies condition ($\dagger\dagger$), by [13] it follows easily that H' is flat, so H^* is exact (see also [50]). We infer that $H^*\mathbf{S} = \mathcal{L}_0^{\mathcal{E}(\mathcal{X})}H$ is homological.

LEMMA 8.9. Assume that (α): any object of \mathcal{C} has finite $\mathcal{E}(\mathcal{X})$ -projective dimension or (β): $\mathcal{X} \subseteq \mathcal{C}^b$ and \mathcal{X} generates \mathcal{C} . Then $\omega: \mathcal{L}_0^{\mathcal{E}(\mathcal{X})}H \rightarrow^{\cong} H$.

Proof. (α) Assume first that $\mathcal{E}(\mathcal{X})$ -p.d $A \leq 1$ and let $P_1 \rightarrow P_0 \rightarrow A \rightarrow \Sigma(P_1)$ be an $\mathcal{E}(\mathcal{X})$ -projective resolution of A . Since $\omega|_{\mathcal{P}(\mathcal{E}(\mathcal{X}))}$ is invertible, applying the homological functors $\mathcal{L}_0^{\mathcal{E}(\mathcal{X})}H, H$ to the above triangle and using 5-Lemma, it follows that $\omega_A: \mathcal{L}_0^{\mathcal{E}(\mathcal{X})}H(A) \rightarrow H(A)$ is invertible. Since any object of \mathcal{C} has finite $\mathcal{E}(\mathcal{X})$ -projective dimension, by induction we have that ω is an isomorphism.

(β) Set $\mathcal{U} := \{A \in \mathcal{C} \mid \omega_A: \mathcal{L}_0^{\mathcal{E}(\mathcal{X})}H(A) \rightarrow^{\cong} H(A)\}$. Since $\mathcal{L}_0^{\mathcal{E}(\mathcal{X})}H, H$ are coproduct preserving homological functors, \mathcal{U} is a full triangulated subcategory of \mathcal{C} , closed under coproducts, and contains the compact generating subcategory \mathcal{X} of \mathcal{C} . It follows that $\mathcal{U} = \mathcal{C}$ and consequently ω is invertible. ■

THEOREM 8.10. Under the assumptions of Lemma 8.9, we have the following.

(i) There exists an exact colimit preserving functor $H^*: \text{Mod}(\mathcal{X}) \rightarrow \mathcal{A}b$, unique up to isomorphism, such that $H^*\mathbf{S} = H$. The functor H^* admits a right adjoint which preserves injectives and we can identify $\mathbf{S}(-) \otimes_{\mathcal{X}} H|_{\mathcal{X}} = H$.

(ii) $H(f) = 0, \forall f \in \text{Ph}_{\mathcal{E}(\mathcal{X})}(\mathcal{E})$.

(iii) If $\text{Flat}^b(\text{Mod}(\mathcal{E}^{\text{op}}))$ is the category of coproduct preserving homological functors $\mathcal{E} \rightarrow \mathcal{A}b$, then the assignment $F \mapsto F^* \mathbf{S}$ induces an equivalence $\text{Flat}(\text{Mod}(\mathcal{E}^{\text{op}})) \rightarrow \approx \text{Flat}^b(\text{Mod}(\mathcal{E}^{\text{op}}))$, with the inverse given by $H \mapsto H|_{\mathcal{E}}$.

Proof. (i) Follows from the above analysis and Lemma 8.9. (ii) Follows from (i), since \mathbf{S} kills $\mathcal{E}(\mathcal{X})$ -phantom maps. (iii) The easy proof is left to the reader. ■

Remark 8.11. In Theorem 8.10, $\mathcal{A}b$ can be replaced by any abelian AB5 category.

8.4. The Roos Spectral Sequence

Let $\{F_i\}$ be a filtered system of functors in $\text{Mod}(\mathcal{X})$ and let G be another functor. Then by [65] we have a Roos's spectral sequence: $E_2^{p,q} = \lim_{\leftarrow}^{(p)} \mathcal{E}xt^q[F_i, G] \Rightarrow \mathcal{E}xt^n[\lim_{\rightarrow} F_i, G]$. Assume now that \mathcal{X} compactly generates \mathcal{E} and satisfies condition (ii) of Proposition 8.5. Let $A, B \in \mathcal{E}$. Since $\mathbf{S}(A)$ is flat, there exists a functor $\mathcal{J}: I \rightarrow \mathcal{X}$ from a small filtered category I such that $\mathbf{S}(A) = \lim_{\rightarrow} \mathbf{S}(X_i)$, where $X_i = \mathcal{J}(i)$. Trivially the Roos spectral sequence for the functors $\mathbf{S}(A), \mathbf{S}(B)$ collapses, giving isomorphisms

$$\lim_{\leftarrow}^{(n)} \mathcal{E}(X_i, B) \cong \mathcal{E}xt^n[\mathbf{S}(A), \mathbf{S}(B)] \cong \mathcal{E}xt_{\mathcal{E}(\mathcal{X})}^n[A, B], \quad \forall n \geq 0.$$

COROLLARY 8.12. $\mathcal{E}(\mathcal{X})$ -i.d $B \leq n$ iff $\lim_{\leftarrow}^{(t)} \mathcal{E}(X_i, B) = 0$, for all $t \geq n + 1$ and for all filtered direct systems $\{X_i\}$ of objects of \mathcal{X} .

Let $A_0 \rightarrow A_1 \rightarrow \dots$ be a tower of objects of \mathcal{E} . Since \mathcal{X} is compact, the functor \mathbf{S} preserves coproducts. Hence by Lemma 5.8 we have $\mathbf{S}(\text{holim}_{\rightarrow} A_i) \cong \lim_{\rightarrow} \mathbf{S}(A_i)$. Since for a tower $\{G_n\}$ in $\mathcal{A}b$, $\lim_{\leftarrow}^{(p)} G_n$ vanishes $\forall p \geq 2$, the next result follows from Roos's spectral sequence by standard arguments.

COROLLARY 8.13. $\forall B \in \mathcal{E}$ and $\forall n \geq 0$, there exists a short exact sequence

$$\begin{aligned} 0 \rightarrow \lim_{\leftarrow}^{(1)} \mathcal{E}xt_{\mathcal{E}(\mathcal{X})}^{n-1}[A_i, B] &\rightarrow \mathcal{E}xt_{\mathcal{E}(\mathcal{X})}^n \left[\text{holim}_{\rightarrow} A_i, B \right] \\ &\rightarrow \lim_{\leftarrow} \mathcal{E}xt_{\mathcal{E}(\mathcal{X})}^n[A_i, B] \rightarrow 0. \end{aligned}$$

If the tower above is the \mathcal{E} -cellular tower of A , then since $A \cong \text{holim}_{\rightarrow} A_i$, the above short exact sequence gives a method to compute the extension functors of A by means of the extension functors of its cells A_i .

9. PHANTOMLESS TRIANGULATED CATEGORIES

Let \mathcal{E} be a triangulated category with coproducts and let $\mathcal{X} \subseteq \mathcal{E}$ be a skeletally small full additive subcategory, closed under Σ, Σ^{-1} , direct summands, and isomorphisms. Our main result in this section characterizes the $\mathcal{E}(\mathcal{X})$ -phantomless triangulated categories and generalizes results of Neeman [54]. First we need a simple lemma and some definitions.

LEMMA 9.1. *If \mathcal{E} has products and $\mathcal{X} \subseteq \mathcal{E}^b$, then for any set of objects $\{A_i\}_{i \in I} \subseteq \mathcal{E}$, the canonically defined triangle $\bigoplus_{i \in I} A_i \rightarrow \prod_{i \in I} A_i \rightarrow A \rightarrow \Sigma(\bigoplus_{i \in I} A_i)$ is in $\mathcal{E}(\mathcal{X})$.*

Proof. Since \mathbf{S} preserves products and also coproducts (since $\mathcal{X} \subseteq \mathcal{E}^b$), the exact sequences $\cdots \rightarrow \mathbf{S}(\bigoplus_{i \in I} A_i) \rightarrow \mathbf{S}(\prod_{i \in I} A_i) \rightarrow \mathbf{S}(A) \rightarrow \cdots$ and $\cdots \rightarrow \bigoplus_{i \in I} \mathbf{S}(A_i) \rightarrow \prod_{i \in I} \mathbf{S}(A_i) \rightarrow \mathbf{S}(A) \rightarrow \cdots$ are isomorphic. Since in module categories the morphism from a coproduct to the product is a monomorphism, we see that $0 \rightarrow \mathbf{S}(\bigoplus_{i \in I} A_i) \rightarrow \mathbf{S}(\prod_{i \in I} A_i) \rightarrow \mathbf{S}(A) \rightarrow 0$ is exact and the assertion follows. ■

We recall [13] that an additive category \mathcal{E} is called *weak abelian* if \mathcal{E} has weak kernels and weak cokernels, and any morphism is a weak kernel and a weak cokernel. Trivially triangulated categories are weak abelian. We recall [30] that a Grothendieck category \mathcal{G} is called *locally Noetherian*, resp. *locally Artinian*, resp. *locally finite*, if \mathcal{G} has a set of generators consisting of Noetherian, resp. Artinian, resp. finite length, objects. A functor category is called *perfect* [39] if any flat functor is projective. We need the following result from [13].

PROPOSITION 9.2 [13]. *Let \mathcal{E} be a skeletally small additive category with split idempotents. Then the following are equivalent:*

- (i) $\text{Mod}(\mathcal{E})$ is a Frobenius category.
- (ii) \mathcal{E} is weak abelian and $\text{Mod}(\mathcal{E})$ is locally Noetherian.
- (iii) \mathcal{E} is weak abelian and $\text{Mod}(\mathcal{E})$ is perfect.
- (iv) \mathcal{E} is weak abelian and $\text{Proj}(\text{Mod}(\mathcal{E}))$, resp. $\text{Inj}(\text{Mod}(\mathcal{E}))$, is closed under products, resp. coproducts.

In this case the projective, injective, and flat functors coincide. Moreover the module category $\text{Mod}(\mathcal{E})$ is Frobenius and locally finite iff $\text{Mod}(\mathcal{E}^{\text{op}})$ is Frobenius and locally finite iff \mathcal{E} is weak abelian and $\text{Mod}(\mathcal{E}), \text{Mod}(\mathcal{E}^{\text{op}})$ are locally Noetherian (perfect).

Let \mathcal{F} be an additive category. We recall that an object $A \in \mathcal{F}$ is called *finitely presented* if the functor $\mathcal{F}(A, -)$ commutes with filtered colimits. Following [24] we say that \mathcal{F} is *locally finitely presented* if \mathcal{F} has filtered

colimits, any object of \mathcal{F} is a filtered colimit of finitely presented objects, and the full subcategory of finitely presented objects of \mathcal{F} is skeletally small. Finally we recall that an additive category is a *Krull–Schmidt category*, if any of its objects is a finite coproduct of indecomposable objects and any indecomposable object has a local endomorphism ring. In case \mathcal{C} is compactly generated and $\mathcal{X} = \mathcal{C}^b$, then parts (1), (12), (14), (15) of the next result have been observed independently by Krause [50].

THEOREM 9.3. *The following statements are equivalent:*

- (1) *The functor $\mathbf{S}: \mathcal{C} \rightarrow \text{Mod}(\mathcal{X})$ induces an equivalence: $\mathcal{C} \approx \text{Flat}(\text{Mod}(\mathcal{X}))$.*
- (2) *$\mathcal{X} \subseteq \mathcal{C}^b$ and $\mathcal{E}(\mathcal{X})\text{-gl.dim } \mathcal{C} = 0$.*
- (3) *$\mathcal{X} \subseteq \mathcal{C}^b$ and $\text{Ph}_{\mathcal{E}(\mathcal{X})}(\mathcal{C}) = 0$.*
- (4) *$\mathcal{X} \subseteq \mathcal{C}^b$ and $\mathcal{C} = \text{Add}(\mathcal{X})$, resp. $\mathcal{X} \subseteq \mathcal{C}^b$ generates \mathcal{C} and $\mathcal{C} = \text{Prod}(\mathbb{D}(\mathcal{X}))$.*
- (5) *\mathcal{X} generates \mathcal{C} , $\mathcal{X} \subseteq \mathcal{C}^b$, and the module category $\text{Mod}(\mathcal{X})$ is Frobenius.*
- (6) *\mathcal{X} generates \mathcal{C} , $\mathcal{X} \subseteq \mathcal{C}^b$ is weak abelian, and $\text{Add}(\mathcal{X})$, resp. $\mathcal{S}(\mathcal{E}(\mathcal{X}))$, is closed under products, resp. coproducts.*
- (7) *\mathcal{X} generates \mathcal{C} , $\mathcal{X} \subseteq \mathcal{C}^b$ is weak abelian, and \mathcal{C} is $\mathcal{E}(\mathcal{X})$ -Frobenius; i.e., \mathcal{C} has enough $\mathcal{E}(\mathcal{X})$ -injectives and $\mathcal{P}(\mathcal{E}(\mathcal{X})) = \mathcal{S}(\mathcal{E}(\mathcal{X}))$.*
- (8) *The unique (exact) extension $\mathbf{S}^*: \mathcal{A}(\mathcal{C}) \rightarrow \text{Mod}(\mathcal{X})$ of \mathbf{S} through the Freyd category $\mathcal{A}(\mathcal{C})$ of \mathcal{C} is an equivalence of categories.*
- (9) *For any abelian category \mathcal{M} and for any homological functor $H: \mathcal{C} \rightarrow \mathcal{M}$, there exists a unique exact functor $H^*: \text{Mod}(\mathcal{X}) \rightarrow \mathcal{M}$ with $H^*\mathbf{S} = H$.*
- (10) *\mathcal{X} generates \mathcal{C} , $\mathcal{X} \subseteq \mathcal{C}^b$ is weak abelian (resp. and \mathcal{C} has products), and the following conditions are true:*

(α) *For any family $\{X_i\}_{i \in I} \subseteq \mathcal{X}$ the canonical (mono-)morphism*

$$\mu: \bigoplus_{i \in I} \mathcal{C}(-, X_i) \rightarrow \prod_{i \in I} \mathcal{C}(-, X_i) \quad \left(\text{resp. } \nu: \bigoplus_{i \in I} X_i \rightarrow \prod_{i \in I} X_i \right)$$

splits.

(β) *For any set I and directed family $\{X_i\}_{i \in I} \subseteq \mathcal{X}$ and for any $Y \in \mathcal{X}$*

$$\lim_{\leftarrow}^{(n)} \mathcal{C}(X_i, Y) = 0, \quad \forall n \geq 1.$$

(11) \mathcal{X} generates \mathcal{E} , $\mathcal{X} \subseteq \mathcal{E}^b$ is weak abelian, and if $\{X_i; i \in I\}$ and $\{Y_j; j \in J\}$ are filtered direct systems of objects of \mathcal{X} , then

$$\lim_{\leftarrow}^{(n)} \lim_{\rightarrow} \mathcal{E}(X_i, Y_j) = 0, \quad \forall n \geq 1.$$

(12) $\mathcal{X} \subseteq \mathcal{E}^b$ is a Krull–Schmidt weak abelian category which generates \mathcal{E} and for any sequence $X_1 \rightarrow^{f_1} X_2 \rightarrow^{f_2} X_3 \rightarrow \dots$ of non-isomorphisms between indecomposable objects of \mathcal{X} , there exists $N \in \mathbb{N}$, such that $f_1 \circ f_2 \circ \dots \circ f_N = 0$.

(13) $\mathcal{X} \subseteq \mathcal{E}^b$ generates \mathcal{E} and $\mathcal{E}(\mathcal{X}) = \lim_{\rightarrow} \Delta_0$; i.e., a triangle is in $\mathcal{E}(\mathcal{X})$ iff it is a filtered colimit of split triangles.

(14) $\mathcal{X} \subseteq \mathcal{E}^b$ and if $A \in \mathcal{E}$ then $A = \bigoplus_{i \in I} X_i$, where $X_i \in \mathcal{X}$ and $\text{End}_{\mathcal{E}}(X_i)$ is a local ring. Moreover any two such decompositions are isomorphic.

(15) $\mathcal{X} = \mathcal{E}^b$ and \mathcal{E} is a locally finitely presented category.

If one of the above conditions is true, then \mathcal{E} is a pure semisimple locally finitely presented category with products with $\mathcal{X} = \mathcal{E}^b$ as its full subcategory of finitely presented objects. Further the projectivization functor \mathbf{S} induces equivalences

$$\mathcal{E} = \text{Add}(\mathcal{E}^b) \approx \text{Proj}(\text{Mod}(\mathcal{E}^b)) = \text{Flat}(\text{Mod}(\mathcal{E}^b)) = \text{Inj}(\text{Mod}(\mathcal{E}^b))$$

and a triangle equivalence $\Delta/\Delta_0 \approx \underline{\text{Mod}}(\mathcal{E}^b)$, where Δ/Δ_0 is the stable triangulated category of all triangles modulo the split triangles [13] and $\underline{\text{Mod}}(\mathcal{E}^b)$ is the stable triangulated category of the module category $\text{Mod}(\mathcal{E}^b)$ modulo projectives.

Proof. In the first part of the proof we show that the first nine statements are equivalent by the scheme: (1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (4) \Rightarrow (5) \Rightarrow (1), (5) \Leftrightarrow (6), (7), (8), and (8) \Leftrightarrow (9). In the second part we show that (5) \Leftrightarrow (10), (11), (12), that (3) \Leftrightarrow (13), that (4) \Leftrightarrow (14) and finally that (1) \Leftrightarrow (15).

If (1) is true, then since \mathbf{S} is faithful, we have $\ker \mathbf{S} = \text{Ph}_{\mathcal{E}(\mathcal{X})}(\mathcal{E}) = 0$. Trivially $\mathcal{X} \subseteq \mathcal{E}^b$ generates \mathcal{E} . Hence (1) \Rightarrow (2). Trivially (2) \Rightarrow (3), (3) \Rightarrow (4) and by Theorem 4.25 we have (4) \Rightarrow (5). If (5) is true then since $\mathcal{X} \subseteq \mathcal{E}^b$, by Proposition 8.4, the functor $\mathbf{S}: \mathcal{P}(\mathcal{E}(\mathcal{X})) \rightarrow \text{Proj}(\text{Mod}(\mathcal{X})) = \text{Flat}(\text{Mod}(\mathcal{X}))$ is an equivalence. Since by Proposition 9.2, \mathcal{X} is weak abelian, it follows by Proposition 8.5 that $\text{Im } \mathbf{S}$ consists of flat = projective modules. Now since \mathcal{X} generates \mathcal{E} , $\forall A \in \mathcal{E}$, $\mathcal{E}(\mathcal{X})\text{-p.d } A = \text{p.d } \mathbf{S}(A) = 0$. This implies that $\mathcal{E}(\mathcal{X})\text{-gl.dim } \mathcal{E} = 0$; hence $\mathcal{E} = \mathcal{P}(\mathcal{E}(\mathcal{X})) \approx \text{Flat}(\text{Mod}(\mathcal{X}))$, so (1) is true. (5) \Leftrightarrow (6) Since $\text{Add}(\mathcal{X}) \approx \text{Proj}(\text{Mod}(\mathcal{X}))$ and $\mathcal{I}(\mathcal{E}(\mathcal{X})) \approx \text{Inj}(\text{Mod}(\mathcal{X}))$, the category $\text{Add}(\mathcal{X})$, resp. $\mathcal{I}(\mathcal{E}(\mathcal{X}))$, is closed

under products, resp. coproducts, iff $\text{Proj}(\text{Mod}(\mathcal{Z}))$, resp. $\text{Inj}(\text{Mod}(\mathcal{Z}))$, is closed under products, resp. coproducts. Then the assertion follows from Proposition 9.2. (5) \Leftrightarrow (7) If \mathcal{Z} compactly generates \mathcal{E} and \mathcal{Z} is weak abelian, then trivially $\text{Mod}(\mathcal{Z})$ is Frobenius iff \mathcal{E} is $\mathcal{E}(\mathcal{Z})$ -Frobenius and the claim follows. (5) \Rightarrow (8) Since (5) \Rightarrow (2) and $\text{Mod}(\mathcal{Z})$ is the Steenrod category of \mathcal{E} , (8) follows from Theorem 4.25. That (8) \Rightarrow (5) is a consequence of the fact that $\mathcal{A}(\mathcal{E})$ is Frobenius. Finally (8) \Leftrightarrow (9) follows from the fact that $\mathcal{A}(\mathcal{E})$ is the universal homological category of \mathcal{E} .

So far we proved that the conditions (1), \dots , (9) are equivalent. Observe that the conditions (1) and (4) imply that $\mathcal{Z} = \mathcal{E}^b$. In particular \mathcal{Z} is a triangulated subcategory of \mathcal{E} and \mathcal{E} is compactly generated.

(5) \Leftrightarrow (10) If (5) is true then $\bigoplus_{i \in I} \mathcal{E}(-, X_i)$ is injective, since $\text{Mod}(\mathcal{Z})$ is Frobenius. Hence the canonical monomorphism μ splits. Let $\{X_i\}_{i \in I} \subseteq \mathcal{Z}$, with I a filtered set and let $Y \in \mathcal{Z}$. Consider the (flat) functor $F = \lim_{\rightarrow, I} \mathcal{Z}(-, X_i)$. From the Roos spectral sequence of Section 8.4 for the functors $F, \mathcal{Z}(-, Y)$, it follows that $\lim_{\leftarrow, I}^{(n)} \mathcal{E}(X_i, Y) \cong \mathcal{E}xt^n[F, \mathcal{Z}(-, Y)]$, $\forall n \geq 1$. Since any flat is projective, we have $\mathcal{E}xt^n(F, \mathcal{Z}(-, Y)) = 0$, $\forall n \geq 1$; hence condition (10)(β) holds. For the parenthetical case observe that since $\text{Flat}(\text{Mod}(\mathcal{Z}))$ has products, by (1) we have that \mathcal{E} has products; hence the canonical morphism ν is defined. By Lemma 9.1, ν is an $\mathcal{E}(\mathcal{Z})$ -proper monic, which splits since $\text{Ph}_{\mathcal{E}(\mathcal{Z})}(\mathcal{E}) = 0$ by (2). Conversely if (10) is true, then by (β) we have $\mathcal{E}xt^n[F, \mathcal{Z}(-, Y)] = 0$, $\forall n \geq 1$, for any flat functor F and any object $Y \in \mathcal{Z}$, again from Roos's spectral sequence. Since \mathcal{Z} is weak abelian, \mathcal{Z} has weak cokernels so by [40], we have that $\mathcal{Z}(-, Y)$ is pure injective. Then by condition (α), any free functor is pure injective. It follows that any projective is pure injective and then by [40], the category $\text{Mod}(\mathcal{Z})$ is perfect; i.e., any flat functor is projective. By Proposition 9.2, $\text{Mod}(\mathcal{Z})$ is Frobenius. For the parenthetical case observe that μ is isomorphic to $\mathbf{S}(\nu)$, since $\mathcal{Z} \subseteq \mathcal{E}^b$.

(5) \Leftrightarrow (11) As in the proof of (5) \Leftrightarrow (10), using Roos's spectral sequence.

(5) \Leftrightarrow (12) Suppose that (5) is true. Then $\text{Mod}(\mathcal{Z})$ is perfect and the assertion follows from the characterization of perfect functor categories in [39]. If (12) is true, by [39] we have that $\text{Mod}(\mathcal{Z})$ is perfect and hence Frobenius since \mathcal{Z} is weak abelian.

(4) \Leftrightarrow (14) If (4) is true then by (5), $\text{Mod}(\mathcal{Z})$ is perfect and \mathcal{E} is identified with $\text{Proj}(\text{Mod}(\mathcal{Z}))$. Then the assertion follows from [39]. The converse is trivial.

(3) \Leftrightarrow (13) If any triangle in $\mathcal{E}(\mathcal{Z})$ is a filtered colimit of split triangles, then obviously \mathcal{E} is $\mathcal{E}(\mathcal{Z})$ -phantomless. Conversely if \mathcal{E} is

$\mathcal{E}(\mathcal{X})$ -phantomless, then the assertion follows from the equivalence of (3) with the condition (1).

(1) \Leftrightarrow (15) If (1) is true, then $\mathcal{E} = \text{Flat}(\text{Mod}(\mathcal{X}))$ which obviously is locally finitely presented. Moreover by (4), the full subcategory of finitely presented objects of \mathcal{E} coincides with $\mathcal{X} = \mathcal{E}^b$. By [24] and condition (11), we have that \mathcal{E} is pure semisimple. Conversely if (15) is true, and if \mathcal{Y} is the category of finitely presented objects of \mathcal{E} , then obviously $\mathcal{Y} \subseteq \mathcal{E}^b$ and we know by [24] that the functor $\mathbf{S}: \mathcal{E} \rightarrow \text{Flat}(\text{Mod}(\mathcal{Y}))$ is an equivalence. By condition (1) with \mathcal{X} replaced by \mathcal{Y} it follows that \mathcal{E} is $\mathcal{E}(\mathcal{Y})$ -phantomless. This implies as above that $\mathcal{Y} = \mathcal{E}^b$. Since by hypothesis $\mathcal{E}^b = \mathcal{X}$, it follows that $\mathcal{X} = \mathcal{Y}$ and (15) implies (1).

The last part follows from the above proof, noting that by [13], we have always an equivalence $\Delta/\Delta_0 \approx \underline{\mathcal{A}}(\mathcal{E})$, where the latter is the stable category of the Frobenius category $\mathcal{A}(\mathcal{E})$ modulo projectives. ■

EXAMPLE 9.4. Any semisimple algebraic stable homotopy category in the sense of Hovey et al. [38] satisfies any one of the conditions of Theorem 9.3. For other examples we refer to Section 12.

We recall that an additive category \mathcal{D} is called (*von Neumann*) *regular* if for any morphism $f: X \rightarrow Y$ in \mathcal{D} , there exists $g: Y \rightarrow X$ such that $f \circ g \circ f = f$. It is not difficult to see that a triangulated category \mathcal{D} is abelian iff \mathcal{D} is semisimple abelian iff \mathcal{D} is regular with split idempotents. It is well known that if \mathcal{D} is regular and skeletally small, then all left or right modules over \mathcal{D} are flat.

COROLLARY 9.5. *If \mathcal{E} is compactly generated, then the following are equivalent:*

- (i) *All triangles in \mathcal{E} are semi-split.*
- (ii) *\mathcal{E} is $\mathcal{E}(\mathcal{E}^b)$ -phantomless and \mathcal{E}^b is von Neumann regular.*
- (iii) *\mathcal{E} is $\mathcal{E}(\mathcal{E}^b)$ -phantomless and \mathcal{E}^b is (semisimple) abelian.*
- (iv) *The Steenrod category $\text{Mod}(\mathcal{E}^b)$ is semisimple abelian.*
- (v) *The projectivization functor $\mathbf{S}: \mathcal{E} \rightarrow \text{Mod}(\mathcal{E}^b)$ is an equivalence.*

9.1. Puppe Triangulations

Let \mathcal{F} be an abelian category with enough injectives. By [43], the stable category $\underline{\mathcal{F}}$ modulo injectives is right triangulated with suspension functor $\Omega^{-1}: \underline{\mathcal{F}} \rightarrow \underline{\mathcal{F}}$.

DEFINITION 9.6. A *stable structure* on \mathcal{F} is a pair (Σ, δ) , where $\Sigma: \mathcal{F} \rightarrow \mathcal{F}$ is an automorphism of \mathcal{F} and $\delta: \Sigma \rightarrow \Omega^{-3}$ is a natural isomorphism in $\underline{\mathcal{F}}$, such that $\delta\Omega^{-1} + \Omega^{-1}\delta = 0$. The last condition makes sense

since we can identify $\Sigma\Omega^{-1} \cong \Omega^{-1}\Sigma$. \mathcal{F} is called Σ -stable if \mathcal{F} admits a stable structure (Σ, δ) .

Observe that if \mathcal{F} is Σ -stable, then \mathcal{F} is Frobenius. Now let \mathcal{P} be an additive category equipped with an automorphism $\Sigma: \mathcal{P} \rightarrow \mathcal{P}$. A class \mathbf{P} of diagrams $A \rightarrow B \rightarrow C \rightarrow \Sigma(A)$ in \mathcal{P} is called a *Puppe-triangulation* of the pair (\mathcal{P}, Σ) , if \mathbf{P} satisfies all the axioms of a triangulation, except possibly of the octahedral axiom. In this case \mathcal{P} is called *Puppe-triangulated*; see [57] where the terminology *pre-triangulated* is used. However, the last term has been used by many authors to mean different things and we shall avoid it. Our terminology is justified by the fact that according to Heller [34], D. Puppe was the first who introduced the axioms (T_1) , (T_2) , and (T_3) of Section 2; see for instance [60].

EXAMPLE 9.7. Let $(\mathcal{F}, \Sigma, \Delta)$ be a Puppe-triangulated category. Then the Freyd category $\mathcal{A}(\mathcal{F})$ is Frobenius and the suspension Σ of \mathcal{F} induces an auto-equivalence $\Sigma_*: \mathcal{A}(\mathcal{F}) \rightarrow \mathcal{A}(\mathcal{F})$, defined by $\Sigma_*(F) = \text{Coker } \mathcal{A}(-, \Sigma(f))$, where $\mathcal{A}(-, A) \rightarrow^{\mathcal{A}(-, f)} \mathcal{A}(-, B) \rightarrow F \rightarrow 0$ is a finite presentation of $F \in \mathcal{A}(\mathcal{F})$. We leave to the reader to check that in fact $\mathcal{A}(\mathcal{F})$ is Σ_* -stable.

The following is due to Heller and was observed independently by Keller and Neeman.

THEOREM 9.8 [34, 44]. *Let \mathcal{F} be a Frobenius category and $\Sigma: \mathcal{F} \rightarrow \mathcal{F}$ an automorphism of \mathcal{F} . If \mathcal{P} is the full subcategory of projective–injective objects of \mathcal{F} , then there exists a bijective correspondence between Puppe-triangulations \mathbf{P} of the pair (\mathcal{P}, Σ) and Σ -stable structures (Σ, δ) on \mathcal{F} .*

In the next two paragraphs and for reasons of comparison, we rename triangulated categories, and we call them *Verdier-triangulated* categories. Hence a Verdier-triangulated category is a Puppe-triangulated category which satisfies the octahedral axiom. Theorem 9.8 suggests the following conjecture, which we believe is due to Keller and Neeman.

Conjecture 9.9. There exists a Σ -stable Frobenius category \mathcal{E} , such that the Puppe-triangulated category $(\mathcal{P}, \Sigma, \mathbf{P})$ of Theorem 9.8 is not Verdier-triangulated.

The following provides support for the above conjecture to be true.

Remark 9.10. By Example 9.7, a Puppe-triangulated category \mathcal{F} is embedded in a Σ -stable Frobenius category as a full subcategory of projective–injective objects. It follows that the above conjecture fails iff any Puppe-triangulated category is Verdier-triangulated. If the latter holds, then it would follow that for any Verdier-triangulated category \mathcal{F} , the octahedral axiom (T_4) of Verdier is a formal consequence of the axioms

(T_1) , (T_2) , and (T_3) . This is very unlikely to be true, but it is not known if there exists a counterexample.

9.2. *Triangulations and Categories of Pro-objects and Ind-objects*

Let \mathcal{Z} be a skeletally small triangulated category. It is very natural to ask if the categories $\text{Proj}(\text{Mod}(\mathcal{Z}))$, $\text{Flat}(\text{Mod}(\mathcal{Z}))$ admit a triangulated structure in such a way that the inclusions $\mathcal{Z} \hookrightarrow \text{Proj}(\text{Mod}(\mathcal{Z}))$, $\mathcal{Z} \hookrightarrow \text{Flat}(\text{Mod}(\mathcal{Z}))$ are exact. We devote this subsection to a discussion of this question.

By Theorem 9.3, if a triangulated category \mathcal{E} is $\mathcal{E}(\mathcal{Z})$ -phantomless with \mathcal{Z} as in Theorem 9.3, then $\mathcal{Z} = \mathcal{E}^b$ is triangulated, the Steenrod category $\text{Mod}(\mathcal{Z})$ is Frobenius, and its category of projectives (equivalently the category of cohomological functors over \mathcal{Z}) has a triangulated structure which extends the triangulated structure of \mathcal{Z} . The following result provides a partial converse.

THEOREM 9.11. *Let \mathcal{D} be a skeletally small additive category with split idempotents. Then for the statements*

- (i) *the category $\text{Flat}(\text{Mod}(\mathcal{D}))$ is triangulated,*
- (ii) *$\text{Mod}(\mathcal{D})$ is Σ -stable,*

we have (i) \Rightarrow (ii) and (ii) implies that $\text{Flat}(\text{Mod}(\mathcal{D}))$ is Puppe-triangulated.

If (i) holds, then \mathcal{D} is triangulated, the inclusion $i: \mathcal{D} \hookrightarrow \text{Flat}(\text{Mod}(\mathcal{D}))$ is exact, $\text{Mod}(\mathcal{D})$ is Frobenius, and there are identifications:

$$\text{Add}(\mathcal{D}) = \text{Proj}(\text{Mod}(\mathcal{D})) = \text{Flat}(\text{Mod}(\mathcal{D})) = \text{Inj}(\text{Mod}(\mathcal{D})).$$

Proof. If $\mathcal{E} := \text{Flat}(\text{Mod}(\mathcal{D}))$ is triangulated with suspension Σ , then \mathcal{E} has coproducts and \mathcal{E}^b is a full triangulated subcategory of \mathcal{E} . By [24], the category of finitely presented objects of $\text{Flat}(\text{Mod}(\mathcal{D}))$ can be identified with \mathcal{D} . Trivially $\mathcal{D} \subseteq \mathcal{E}^b$ generates \mathcal{E} . Let $\alpha: F \rightarrow G$ be an $\mathcal{E}(\mathcal{E}^b)$ -phantom map in \mathcal{E} , i.e., $(\mathcal{E}^b, \alpha) = 0$. Then $(\mathcal{D}, \alpha) = 0$, which trivially by Yoneda implies that $\alpha = 0$. By Theorem 9.3, it follows that $\mathcal{D} = \mathcal{E}^b$; hence \mathcal{D} is a triangulated subcategory of \mathcal{E} and $\text{Mod}(\mathcal{D})$ is Frobenius, which trivially is Σ -stable. Hence (i) \Rightarrow (ii). Conversely by Theorem 9.8, (ii) implies that $\text{Mod}(\mathcal{D})$ is Frobenius and $\text{Flat}(\text{Mod}(\mathcal{D}))$ is Puppe-triangulated. ■

For a skeletally small additive category \mathcal{D} , let $\text{Pro}(\mathcal{D}), \text{Ind}(\mathcal{D})$ be the induced Pro-, Ind-categories; see [31]. In [13], we gave necessary and sufficient conditions for $\text{Pro}(\mathcal{D}), \text{Ind}(\mathcal{D})$ to be abelian. It is useful to know when the Pro-, Ind-categories can be endowed with a triangulated struc-

ture, since these categories are usually the codomain of derived functors defined on triangulated categories; see [25].

COROLLARY 9.12. (a) *For the statements*

- (i) *the category $\text{Ind}(\mathcal{D})$, resp. $\text{Pro}(\mathcal{D})$, is triangulated,*
- (ii) *$\text{Mod}(\mathcal{D})$, resp. $\text{Mod}(\mathcal{D}^{\text{op}})$, is a Σ -stable category,*

we have (i) \Rightarrow (ii) and (ii) implies that $\text{Ind}(\mathcal{D})$, resp. $\text{Pro}(\mathcal{D})$, is Puppe-triangulated.

(b) *For the statements*

- (i) *the categories $\text{Pro}(\mathcal{D})$, $\text{Ind}(\mathcal{D})$ are triangulated,*
- (ii) *$\text{Mod}(\mathcal{D})$, or equivalently $\text{Mod}(\mathcal{D}^{\text{op}})$, is a locally finite Σ -stable category,*

we have (i) \Rightarrow (ii) and (ii) implies that $\text{Ind}(\mathcal{D})$ and $\text{Pro}(\mathcal{D})$ are Puppe-triangulated. If (i) holds, then $\text{Mod}(\mathcal{D})$, $\text{Mod}(\mathcal{D}^{\text{op}})$ are locally finite Frobenius categories, \mathcal{D} is triangulated, and the inclusions $\mathcal{D} \hookrightarrow \text{Ind}(\mathcal{D})$, $\mathcal{D} \hookrightarrow \text{Pro}(\mathcal{D})$ are exact.

Proof. It is well known that we can identify $\text{Ind}(\mathcal{D}) = \text{Flat}(\text{Mod}(\mathcal{D}))$ and $\text{Pro}(\mathcal{D}) = \text{Flat}(\text{Mod}(\mathcal{D}^{\text{op}}))^{\text{op}}$, and we have an equivalence: $\text{Pro}(\mathcal{D})^{\text{op}} \approx \text{Ind}(\mathcal{D}^{\text{op}})$. Then the assertions are consequences of Theorem 9.11 and Proposition 9.2. ■

Note that if Conjecture 9.9 is not true, then the map $\mathcal{E} \mapsto \text{Mod}(\mathcal{E}^b)$ gives a bijective correspondence between $\mathcal{E}(\mathcal{E}^b)$ -phantomless compactly generated triangulated categories and Σ -stable Frobenius module categories over skeletally small triangulated categories. The inverse map is given by $\text{Mod}(\mathcal{D}) \mapsto \text{Ind}(\mathcal{D})$.

10. BROWN REPRESENTATION THEOREMS

Throughout this section we assume that the triangulated category \mathcal{E} has coproducts. We fix a skeletally small full additive subcategory $\mathcal{X} \subseteq \mathcal{E}$ which is closed under isomorphisms, direct summands, and Σ, Σ^{-1} . Consider the homological functor

$$\mathbf{S}': \mathcal{E} \rightarrow \text{Mod}(\mathcal{X}), \quad \mathbf{S}'(A) = \mathcal{E}(-, A)|_{\mathcal{X}}$$

and

$$\mathbf{S}'(f) = \mathcal{E}(-, f)|_{\mathcal{X}},$$

where $\mathcal{E}(-, A)|_{\mathcal{X}}$ denotes the restriction of the representable functor to \mathcal{X} . We recall that an additive functor $F: \mathcal{A} \rightarrow \mathcal{B}$ is called *complete* [45] or a

representation equivalence [1], if F is full, surjective on objects and reflects isomorphisms.

DEFINITION 10.1. The pair $(\mathcal{C}, \mathcal{X})$ satisfies the Brown representability theorem (BRT for short) if the canonical functor $\mathbf{S}' : \mathcal{C} \rightarrow \text{Mod}(\mathcal{X})$ induces a representation equivalence between \mathcal{C} and the flat functors $\text{Flat}(\text{Mod}(\mathcal{X}))$ over \mathcal{X} .

Our aim is to characterize when the pair $(\mathcal{C}, \mathcal{X})$ satisfies BRT. By Section 8, the functor \mathbf{S}' satisfies $\text{Im } \mathbf{S}' \subseteq \text{Flat}(\text{Mod}(\mathcal{X}))$ iff the following condition is true:

(\dagger) \mathcal{X} has weak cokernels and the inclusion functor $\mathcal{X} \hookrightarrow \mathcal{C}$ preserves them.

Further \mathbf{S}' induces an equivalence between $\mathcal{P}(\mathcal{E}(\mathcal{X}))$ and $\text{Proj}(\text{Mod}(\mathcal{X}))$ iff \mathcal{X} is self-compact iff $\text{Mod}(\mathcal{X})$ is the Steenrod category of \mathcal{C} with respect to $\mathcal{E}(\mathcal{X})$ and $\mathbf{S}' = \mathbf{S}$ is the projectivization functor. Our main result in this section is the following:

THEOREM 10.2. The following statements are equivalent:

- (i) The pair $(\mathcal{C}, \mathcal{X})$ satisfies BRT.
- (ii) (α) \mathcal{X} is self-compact, generates \mathcal{C} , and satisfies condition (\dagger).
 (β) $\text{Flat}(\text{Mod}(\mathcal{X})) = \{F \in \text{Mod}(\mathcal{X}) \mid \text{p.d } F \leq 1\}$.
- (iii) (α) \mathcal{X} is self-compact and satisfies condition (\dagger).
 (β) $\mathcal{E}(\mathcal{X})\text{-gl.dim } \mathcal{C} \leq 1$.
 (γ) Any $F \in \text{Flat}(\text{Mod}(\mathcal{X}))$ has finite projective dimension.

Proof. (i) \Rightarrow (ii) By BRT, $\text{Im } \mathbf{S}' = \text{Flat}(\text{Mod}(\mathcal{X}))$; hence condition (\dagger) is true. Since $\mathbf{S}' : \mathcal{C}/\text{Ph}_{\mathcal{E}(\mathcal{X})}(\mathcal{C}) \approx \text{Flat}(\text{Mod}(\mathcal{X}))$, it follows directly that \mathcal{X} is self-compact. Since idempotents split in $\mathcal{P}(\mathcal{E}(\mathcal{X})) = \text{Add}(\mathcal{X})$, by Theorem 5.3, $\text{Flat}(\text{Mod}(\mathcal{X})) = \text{Im } \mathbf{S}' = \{F \in \text{Mod}(\mathcal{X}) \mid \text{p.d } F \leq 1\}$. Finally Theorem 5.3 ensures that \mathcal{X} generates \mathcal{C} and $\mathcal{E}(\mathcal{X})\text{-gl.dim } \mathcal{C} \leq 1$. Part (ii) \Rightarrow (iii) is trivial since if \mathcal{X} generates \mathcal{C} and condition (\dagger) is true, then $\mathcal{E}(\mathcal{X})\text{-gl.dim } \mathcal{C} \leq \sup\{\text{p.d } F \mid F \in \text{Flat}(\text{Mod}(\mathcal{X}))\}$.

(iii) \Rightarrow (i) Since $\mathcal{E}(\mathcal{X})\text{-gl.dim } \mathcal{C} \leq 1$, by Theorem 5.3 the functor $\mathbf{S}' = \mathbf{S} : \mathcal{C} \rightarrow \text{Flat}(\text{Mod}(\mathcal{X}))$ is full and reflects isomorphisms. It remains to show that \mathbf{S} is surjective on objects. Let F be a flat functor, let $0 \rightarrow \mathbf{S}(P_n) \xrightarrow{a} \mathbf{S}(P_{n-1}) \xrightarrow{b} \mathbf{S}(P_{n-2}) \rightarrow \dots \rightarrow \mathbf{S}(P_0) \rightarrow F \rightarrow 0$ be a projective resolution, and let $b = d \circ e$ be the canonical factorization, where $d : \mathbf{S}(P_{n-1}) \rightarrow G$ and $e : G \hookrightarrow \mathbf{S}(P_{n-2})$. Then there exists $f : P_n \rightarrow P_{n-1}$ such that $\mathbf{S}(f) = a$, since $\mathbf{S} : \mathcal{P}(\mathcal{E}(\mathcal{X})) \approx \text{Proj}(\text{Mod}(\mathcal{X}))$. Since $\mathbf{S}(f)$ is monic, the triangle $P_n \xrightarrow{f} P_{n-1} \rightarrow L_{n-1} \rightarrow \Sigma(P_n)$ is in $\mathcal{E}(\mathcal{X})$, and then $\mathbf{S}(L_{n-1}) \cong G$. Since \mathbf{S} is full, there exists a morphism $g : L_{n-1} \rightarrow P_{n-2}$ with $\mathbf{S}(g) = e$. Since $\mathbf{S}(g)$

is monic, the triangle $L_{n-1} \xrightarrow{g} P_{n-2} \rightarrow L_{n-2} \rightarrow \Sigma(L_{n-1})$ is in $\mathcal{E}(\mathcal{X})$, and then $\mathbf{S}(L_{n-2}) \cong \text{Im}(P_{n-2} \rightarrow P_{n-3})$. Continuing in this way, it follows that F is in $\text{Im } \mathbf{S}$, so \mathbf{S} is surjective on objects. ■

If $(\mathcal{E}, \mathcal{X})$ satisfies BRT, then we have the classical version of the Brown representability theorem [51, 55].

COROLLARY 10.3. *If the pair $(\mathcal{E}, \mathcal{X})$ satisfies BRT and if $F: \mathcal{E}^{\text{op}} \rightarrow \mathcal{A}b$ is a cohomological functor converting coproducts to products, then F is representable.*

Proof. Since the restriction functor $\tilde{F} := F|_{\mathcal{X}} \in \text{Mod}(\mathcal{X})$ is flat, by BRT there exists $A \in \mathcal{E}$, such that $\alpha: \mathbf{S}(A) = \mathcal{E}(-, A)|_{\mathcal{X}} \rightarrow \tilde{F}|_{\mathcal{X}} = \tilde{F}$. Since $\mathcal{P}(\mathcal{E}(\mathcal{X})) = \text{Add}(\mathcal{X})$ and F sends coproducts to products, α can be extended to an isomorphism $\beta: F|_{\mathcal{P}(\mathcal{E}(\mathcal{X}))} \cong \mathcal{E}(-, A)|_{\mathcal{P}(\mathcal{E}(\mathcal{X}))}$. Then the result follows from Corollary 4.29. ■

The author is indebted to the referee for the following nice remark.

Remark 10.4. If the pair $(\mathcal{E}, \mathcal{X})$ satisfies BRT, then \mathcal{E} is compactly generated. Indeed it suffices to show that $\mathcal{X} \subseteq \mathcal{E}^b$. Let $\{A_i; i \in I\}$ be a set of objects in \mathcal{E} and choose an $\mathcal{E}(\mathcal{X})$ -projective resolution $P_{i1} \rightarrow P_{i0} \rightarrow A_i \rightarrow \Sigma(P_{i1})$ of A_i , $\forall i \in I$. Then the sum $\oplus P_{i1} \rightarrow \oplus P_{i0} \rightarrow \oplus A_i \rightarrow \Sigma(\oplus P_{i1})$ of these resolutions is a triangle which is an $\mathcal{E}(\mathcal{X})$ -projective resolution of $\oplus A_i$. If $X \in \mathcal{X}$, then applying $\mathcal{E}(X, -)$ to the triangle and using 5-Lemma and the self-compactness of \mathcal{X} , it follows easily that the canonical map $\oplus \mathcal{E}(X, A_i) \rightarrow \mathcal{E}(X, \oplus A_i)$ is invertible. Hence X is compact.

A direct consequence of the above representability result is the following.

COROLLARY 10.5. *If the pair $(\mathcal{E}, \mathcal{X})$ satisfies BRT, then \mathcal{E} has products.*

The following is a consequence of Theorems 4.24 and 5.3.

COROLLARY 10.6. *If BRT holds for the pair $(\mathcal{E}, \mathcal{X})$, then \mathcal{E} has $\mathcal{E}(\mathcal{X})$ -injective envelopes and the functor \mathbf{S} induces an equivalence $\mathbf{S}: \mathcal{A}(\mathcal{E}(\mathcal{X})) \rightarrow \tilde{\text{Inj}}(\text{Mod}(\mathcal{X}))$.*

COROLLARY 10.7. *Assume that \mathcal{X} is self-compact, generates \mathcal{E} , and satisfies condition (\dagger) . If (α) any flat right \mathcal{X} -module is a direct union of countably presented pure submodules, (β) if any flat right \mathcal{X} -module is countably presented, or (γ) if the module category $\text{Mod}(\mathcal{X})$ satisfies one of the following conditions: $\text{p.gl.dim Mod}(\mathcal{X}) \leq 1$ or $\text{Kdim Mod}(\mathcal{X}) \leq 1$, where $\text{p.gl.dim Mod}(\mathcal{X})$ denotes pure global dimension and $\text{Kdim Mod}(\mathcal{X})$ denotes Krull dimension [30] of the Grothendieck category $\text{Mod}(\mathcal{X})$, then the pair $(\mathcal{E}, \mathcal{X})$ satisfies BRT.*

Proof. By [39, 65], we have $\sup\{\text{p.d } F; F \in \text{Flat}(\text{Mod}(\mathcal{Z}))\} \leq \text{p.gl.dim Mod}(\mathcal{Z}) \leq \text{Kdim Mod}(\mathcal{Z})$. So the result follows from Theorem 10.2 in case (γ) . In the first two cases it follows from [40, 65], since then for any flat F , $\text{p.d } F \leq 1$. ■

COROLLARY 10.8. *Let $\mathcal{E}/\text{Add}(\mathcal{Z}), \mathcal{E}/\mathcal{I}(\mathcal{E}(\mathcal{Z}))$ be the stable categories of \mathcal{E} modulo $\text{Add}(\mathcal{Z}), \mathcal{I}(\mathcal{E}(\mathcal{Z}))$ and let $\text{Flat}(\text{Mod}(\mathcal{Z})), \overline{\text{Flat}}(\text{Mod}(\mathcal{Z}))$ be the stable categories of flat modules modulo projectives, injectives. If the pair $(\mathcal{E}, \mathcal{Z})$ satisfies BRT, then \mathbf{S} induces equivalences $\mathcal{E}/\text{Add}(\mathcal{Z}) \cong \text{Flat}(\text{Mod}(\mathcal{Z}))$ and $\mathcal{E}/\mathcal{I}(\mathcal{E}(\mathcal{Z})) \cong \overline{\text{Flat}}(\text{Mod}(\mathcal{Z}))$. $\mathcal{E}/\text{Add}(\mathcal{Z})$ has kernels and coproducts and $\mathcal{E}/\mathcal{I}(\mathcal{E}(\mathcal{Z}))$ has cokernels and products.*

Proof. By BRT the functor \mathbf{S} induces a full and surjective on objects functor $\mathbf{S}: \mathcal{E}/\text{Add}(\mathcal{Z}) \rightarrow \text{Flat}(\text{Mod}(\mathcal{Z}))$. Let $f: A \rightarrow B$ be a morphism in \mathcal{E} such that $\mathbf{S}(f)$ factors through a projective $G = \mathbf{S}(P), P \in \text{Add}(\mathcal{Z})$, via the morphisms $\alpha = \mathbf{S}(g): \mathbf{S}(A) \rightarrow \mathbf{S}(P)$ and $\beta = \mathbf{S}(h): \mathbf{S}(P) \rightarrow \mathbf{S}(B)$. Then $f - g \circ h: A \rightarrow B$ is a phantom map. Let $P_1 \rightarrow^\sigma P_0 \rightarrow^\rho A \rightarrow^\tau \Sigma(P_1)$ be an $\mathcal{E}(\mathcal{Z})$ -projective resolution of A . Since $f - g \circ h$ is phantom, we have $\rho \circ (f - g \circ h) = 0$; hence there exists $\omega: \Sigma(P_1) \rightarrow B$ such that $\tau \circ \omega = f - g \circ h$ or $f = \tau \circ \omega + g \circ h$. Passing to the stable category we have $f = 0$ in $\mathcal{E}/\text{Add}(\mathcal{Z})$. This means that $\mathbf{S}|_{\mathcal{E}/\text{Add}(\mathcal{Z})}$ is faithful, hence an equivalence. Finally the category $\text{Flat}(\text{Mod}(\mathcal{Z}))$ is a left triangulated category [12], and since for any flat F we have $\text{p.d } F \leq 1$, it is trivial to see that $\text{Flat}(\text{Mod}(\mathcal{Z}))$ has kernels. Since $\text{Flat}(\text{Mod}(\mathcal{Z}))$ has coproducts it is easy to see that the same is true for $\overline{\text{Flat}}(\text{Mod}(\mathcal{Z}))$. The case of injectives is treated similarly. ■

There are pairs for which BRT fails (see Section 12). Let \mathcal{E}^{\aleph_0} be the full subcategory of \mathcal{E} consisting of objects A for which there exists a triangle $P_1 \rightarrow P_0 \rightarrow A \rightarrow \Sigma(P_1)$ in $\mathcal{E}(\mathcal{Z})$ with P_1, P_0 direct summands of countable coproducts of objects of \mathcal{Z} . Let $\text{Ph}_{\mathcal{E}(\mathcal{Z})}(\mathcal{E}^{\aleph_0})$ be the restriction of $\text{Ph}_{\mathcal{E}(\mathcal{Z})}(\mathcal{E})$ to \mathcal{E}^{\aleph_0} and let $\text{Flat}^{\aleph_0}(\text{Mod}(\mathcal{Z}))$ be the full subcategory of countably presented flat functors [39, 65].

PROPOSITION 10.9. *The functor \mathbf{S} induces a representation equivalence $\mathbf{S}: \mathcal{E}^{\aleph_0} \rightarrow \text{Flat}^{\aleph_0}(\text{Mod}(\mathcal{Z}))$. Hence $\mathcal{E}^{\aleph_0}/\text{Ph}_{\mathcal{E}(\mathcal{Z})}(\mathcal{E}^{\aleph_0}) \simeq \text{Flat}^{\aleph_0}(\text{Mod}(\mathcal{Z}))$.*

Proof. By construction we have that \mathbf{S} has image in $\text{Flat}^{\aleph_0}(\text{Mod}(\mathcal{Z}))$, and any object $A \in \mathcal{E}^{\aleph_0}$ has $\mathcal{E}(\mathcal{Z})$ -p.d $A \leq 1$. Since [65] any countably presented flat functor F has $\text{p.d } F \leq 1$, we see that the functor \mathbf{S} is surjective on objects. ■

Recall from [65] that the *weight* $w(\mathcal{Z})$ of a skeletally small category \mathcal{Z} is defined by $w(\mathcal{Z}) := \text{card}\{\bigsqcup \mathcal{X}(X, Y): X, Y \in \text{Iso}(\mathcal{Z})\}$.

PROPOSITION 10.10 [65]. *If \mathcal{Z} is a skeletally small additive category with weight $w(\mathcal{Z}) \leq \aleph_t$, for some $t \geq 0$, then any flat module \mathcal{Z} -module F has $\text{p.d } F \leq t + 1$.*

The following consequence is a generalization of the main result of [56].

COROLLARY 10.11. *If for the pair $(\mathcal{E}, \mathcal{X})$, the subcategory \mathcal{X} has weight $w(\mathcal{X}) \leq \aleph_0$ and satisfies condition (\dagger) , then the following are equivalent.*

- (i) *The pair $(\mathcal{E}, \mathcal{X})$ satisfies BRT.*
- (ii) *\mathcal{X} is a compact generating subcategory of \mathcal{E} .*

For examples of pairs $(\mathcal{E}, \mathcal{X})$ satisfying BRT we refer to Sections 11 and 12.

11. PURITY

Throughout this section we fix a compactly generated triangulated category \mathcal{E} . Our aim here is to apply the theory of Section 8, in case $\mathcal{X} = \mathcal{E}^b$ is the full subcategory of compact objects. The homological algebra in \mathcal{E} based on the proper class of triangles $\mathcal{E}(\mathcal{E}^b)$ is called the *pure homological algebra* of \mathcal{E} . This kind of purity was also considered independently by Krause [50].

TERMINOLOGY 11.1. A *pure triangle* in \mathcal{E} is a triangle in $\mathcal{E}(\mathcal{E}^b)$. The *pure projective*, resp. *pure injective*, objects $\text{PProj}(\mathcal{E})$, resp. $\text{PInj}(\mathcal{E})$, are the $\mathcal{E}(\mathcal{E}^b)$ -projective, resp. $\mathcal{E}(\mathcal{E}^b)$ -injective, objects. The *pure projective dimension* $\text{p.p.d } A$ of $A \in \mathcal{E}$ is defined as $\mathcal{E}(\mathcal{E}^b)$ -p.d A . The *pure global dimension* $\text{p.gl.dim } \mathcal{E}$ is $\mathcal{E}(\mathcal{E}^b)$ -gl.dim \mathcal{E} and the *pure extension functors* $\mathcal{P}\mathcal{E}xt^*$ are the functors $\mathcal{E}xt_{\mathcal{E}(\mathcal{E}^b)}^*$. The ideal $\text{Ph}(\mathcal{E})$ of *pure phantom maps* is the ideal $\text{Ph}_{\mathcal{E}(\mathcal{E}^b)}(\mathcal{E})$, and so on.

By Section 8 we know that \mathcal{E} has enough pure-injectives and enough pure-projectives with $\text{PProj}(\mathcal{E}) = \text{Add}(\mathcal{E}^b)$ and $\text{PInj}(\mathcal{E}) = \text{Prod}(\mathbb{D}(\mathcal{E}^b))$. The pure Steenrod category of \mathcal{E} is the module category $\text{Mod}(\mathcal{E}^b)$ and the projectivization functor is the restricted Yoneda functor $\mathbf{S}: \mathcal{E} \rightarrow \text{Mod}(\mathcal{E}^b)$, which induces an equivalence between $\text{PProj}(\mathcal{E})$ and $\text{Proj}(\text{Mod}(\mathcal{E}^b))$ and between $\text{PInj}(\mathcal{E})$ and $\text{Inj}(\text{Mod}(\mathcal{E}^b))$. The motivating source of the above definitions and terminology is explained below. Its main theme is to try to describe objects of a larger category by filtered (homotopy) colimits of objects of a smaller and better behaved full subcategory.

11.1. Motivation

It is well known that the proper framework for the study of purity in a module category is that of a locally finitely presented Grothendieck category [65]. If \mathcal{E} is a locally finitely presented Grothendieck category, and $\text{f.p}(\mathcal{E})$ is the full subcategory of finitely presented objects, then the restricted Yoneda functor $\mathcal{Y}: \mathcal{E} \rightarrow \text{Mod}(\text{f.p}(\mathcal{E}))$ is fully faithful and identi-

fies \mathcal{E} with the full subcategory $\text{Flat}(\text{Mod}(\text{f.p}(\mathcal{E})))$ of flat functors and the pure-projective objects of \mathcal{E} with the projective objects of $\text{Mod}(\text{f.p}(\mathcal{E}))$. A short exact sequence (E) in \mathcal{E} is pure iff $\mathcal{H}(E)$ is exact. The pure global dimension of the Grothendieck category \mathcal{E} is computed as $\text{p.gl.dim } \mathcal{E} = \sup\{\text{p.d } F \mid F \in \text{Flat}(\text{Mod}(\text{f.p}(\mathcal{E})))\}$. In particular \mathcal{E} is pure semisimple iff the module category $\text{Mod}(\text{f.p}(\mathcal{E}))$ is perfect; i.e., any flat functor is projective [65]. Hence the pure homological theory of \mathcal{E} is strongly connected with the behavior of $\text{Mod}(\text{f.p}(\mathcal{E}))$ via the functor \mathcal{H} .

Now if \mathcal{E} is a triangulated category with coproducts such that \mathcal{E}^b is skeletally small, it is natural to consider \mathcal{E}^b as the full subcategory of “finitely presented” objects, since $\forall X \in \mathcal{E}^b$, the functor $\mathcal{E}(X, -)$ preserves coproducts. This analogy is justified by the fact that in the stable homotopy category the compact objects are the finite spectra and also in the derived category where the compact objects are the perfect complexes which behave by [69], like finitely presented objects. See also Section 12, where the example of the stable module category provides further justification. If \mathcal{E} has a set of compact generators, i.e., \mathcal{E} is compactly generated, then \mathcal{E} can be considered as the triangulated analogue of a locally finitely presented Grothendieck category. Then the Steenrod category $\text{Mod}(\mathcal{E}^b)$ is the analogue of $\text{Mod}(\text{f.p}(\mathcal{E}))$ and the functor $\mathbf{S}: \mathcal{E} \rightarrow \text{Mod}(\mathcal{E}^b)$ is the analogue of the functor \mathcal{H} above. A triangle (T) is in $\mathcal{E}(\mathcal{E}^b)$ iff $\mathbf{S}(T)$ is short exact. Hence the triangles in $\mathcal{E}(\mathcal{E}^b)$ can be considered as the “pure” triangles. The functor \mathbf{S} is not fully faithful in general, due to the presence of phantom maps, and this is the major difference between the two theories. Indeed by Theorem 9.3, \mathbf{S} is fully faithful iff \mathcal{E} is phantomless iff \mathbf{S} induces an equivalence between \mathcal{E} and $\text{Flat}(\text{Mod}(\mathcal{E}^b))$. Also Theorem 9.3 characterizes the phantomless categories as the categories \mathcal{E} such that $\text{Mod}(\mathcal{E}^b)$ is perfect or equivalently all pure triangles are split. Hence the phantomless categories are the pure semisimple categories and the global dimension of \mathcal{E} with respect to the pure triangles is the pure global dimension of \mathcal{E} .

We recall that a right \mathcal{E}^b -module G is called *FP-injective* if $\mathcal{E}xt^1(F, G) = 0$ for any finitely presented module F . Since \mathcal{E}^b is triangulated, by [13] or [50], the category $\text{Flat}(\text{Mod}(\mathcal{E}^b))$ of flat right \mathcal{E}^b -modules, the category $\text{Coh}((\mathcal{E}^b)^{\text{op}}, \mathcal{A}b)$ of cohomological functors over \mathcal{E}^b , and the category $\text{FPInj}(\text{Mod}(\mathcal{E}^b))$ of FP-injective right \mathcal{E}^b -modules are identical.

PROPOSITION 11.2.

$$\begin{aligned} \text{p.gl.dim } \mathcal{E} &= \sup\{\text{p.p.d } A; A \in \mathcal{E} \mid \text{p.p.d } A < \infty\} \\ &= \sup\{\text{p.d } F; F \in \text{Flat}(\text{Mod}(\mathcal{E}^b)) \mid \text{p.d } F < \infty\} \\ &= \sup\{\text{p.d } F; F \in \text{Flat}(\text{Mod}(\mathcal{E}^b))\} \end{aligned}$$

$$\begin{aligned}
&= \sup\{\text{i.d } F; F \in \text{FPInj}(\text{Mod}(\mathcal{E}^b))\} \\
&= \sup\{\text{i.d } F; F \in \text{Flat}(\text{Mod}(\mathcal{E}^b))\} \\
&= \sup\{\text{p.d } F; F \in \text{FPInj}(\text{Mod}(\mathcal{E}^b))\}.
\end{aligned}$$

Proof. Since $\forall A \in \mathcal{E}$, $\text{p.p.d } A = \text{p.d } \mathbf{S}(A)$, and since $\text{Im } \mathbf{S} \subseteq \text{Flat}(\text{Mod}(\mathcal{E}^b))$, we have $\text{p.gl.dim } \mathcal{E} \leq \sup\{\text{p.d } F; F \in \text{Flat}(\text{Mod}(\mathcal{E}^b))\}$. By a result of Jensen [65], $\sup\{\text{p.d } F; F \in \text{Flat}(\text{Mod}(\mathcal{E}^b))\} = \sup\{\text{p.d } F; F \in \text{Flat}(\text{Mod}(\mathcal{E}^b)) \mid \text{p.d } F < \infty\}$. We show that if F is a flat right \mathcal{E}^b -module of finite projective dimension, then there exists $A \in \mathcal{E}$ such that $\text{p.p.d } A = \text{p.d } F$. If $\text{p.d } F \leq 1$, then we know that there exists $A \in \mathcal{E}$ such that $F = \mathbf{S}(A)$. Suppose that $\infty > \text{p.d } F \geq 2$. There exists a small filtered category I and a functor $\mathcal{F}: I \rightarrow \mathcal{E}^b$ such that $F = \lim_{\rightarrow, \mathcal{F}} \mathbf{S}(X_i)$. Consider the triangle $\oplus_{\mathcal{F}} X_i \rightarrow \oplus_{\mathcal{F}} X_i \rightarrow A \rightarrow \Sigma(\oplus_{\mathcal{F}} X_i)$ in \mathcal{E} . Then we have a long exact sequence $\cdots \rightarrow \oplus_{\mathcal{F}} \mathbf{S}(\Sigma^{-1}(X_i)) \xrightarrow{\alpha} \mathbf{S}(\Sigma^{-1}(A)) \xrightarrow{\beta} \oplus_{\mathcal{F}} \mathbf{S}(X_i) \xrightarrow{\gamma} \oplus_{\mathcal{F}} \mathbf{S}(X_i) \xrightarrow{\delta} \mathbf{S}(A) \rightarrow \cdots$. The equivalence Σ induces an autoequivalence Σ_* of $\text{Flat}(\text{Mod}(\mathcal{E}^b))$ given by $\Sigma_*(F) = F\Sigma$ with inverse Σ_*^{-1} defined by $\Sigma_*^{-1}(F) = F\Sigma^{-1}$. Then $F = \text{Im}(\delta)$ and $\text{Im}(\alpha) = F\Sigma^{-1}$. Let $H = \text{Im}(\beta)$ and $G = \text{Im}(\gamma)$. Obviously $\text{p.d } \Sigma_*(F) = \text{p.d } F = \text{p.d } \Sigma_*^{-1}(F)$ and we have an exact sequence $0 \rightarrow \Sigma_*^{-1}(F) \rightarrow \mathbf{S}(\Sigma^{-1}(A)) \xrightarrow{\beta} \oplus_{\mathcal{F}} \mathbf{S}(X_i) \xrightarrow{\gamma} \oplus_{\mathcal{F}} \mathbf{S}(X_i) \rightarrow F \rightarrow 0$. From this exact sequence, since $\text{p.d } F \geq 2$ and $\oplus_{\mathcal{F}} \mathbf{S}(X_i)$ is a projective module, we have $\text{p.d } H = \text{p.d } F - 2 = \text{p.d } \Sigma_*^{-1}(F) - 2$. But then from the short exact sequence $0 \rightarrow \Sigma_*^{-1}(F) \rightarrow \mathbf{S}(\Sigma^{-1}(A)) \rightarrow H \rightarrow 0$, and the fact that F , hence $\Sigma_*^{-1}(F)$, has finite projective dimension, we deduce that $\text{p.d } F = \text{p.d } \Sigma_*^{-1}(F) = \text{p.d } \mathbf{S}(\Sigma^{-1}(A)) = \text{p.d } \mathbf{S}(A) = \text{p.p.d } A$. Hence for any flat right \mathcal{E}^b -module F with $\text{p.d } F < \infty$, there exists an object $A \in \mathcal{E}$, such that $\text{p.p.d } A = \text{p.d } F$. This implies trivially that $\sup\{\text{p.d } F; F \in \text{Flat}(\text{Mod}(\mathcal{E}^b)) \mid \text{p.d } F < \infty\} \leq \text{p.gl.dim } \mathcal{E}$. As a consequence we have that the first three equalities are true. Using that the flat and the FP-injective modules coincide, the proof of the remaining equalities is left to the reader. ■

By Proposition 10.10, we have the following consequence.

COROLLARY 11.3. *If $w(\mathcal{E}^b) \leq \aleph_t$, $t \geq 0$, then $\text{p.gl.dim } \mathcal{E} \leq t + 1$.*

If \mathcal{E} is the stable homotopy category of spectra, then it is well known that \mathcal{E} is compactly generated and the full subcategory \mathcal{E}^b of finite spectra has a countable skeleton. Hence by Corollary 11.3, $\text{p.gl.dim } \mathcal{E} \leq 1$.

COROLLARY 11.4. *If \mathcal{E} is the stable homotopy category, then $\text{p.gl.dim } \mathcal{E} = 1$.*

Proof. If $\text{p.gl.dim } \mathcal{E} = 0$, then by Theorem 9.3, $\text{Mod}(\mathcal{E}^b)$ is perfect. It is easy to see [65] that this implies that any compact object X has a perfect

endomorphism ring. In \mathcal{E} this is not true; e.g., choose X to be the sphere spectrum. ■

Remark 11.5. The pure global dimension is invariant under triangle equivalence. Indeed any triangle equivalence $F: \mathcal{E} \rightarrow \mathcal{D}$ restricts to an equivalence $F^b: \mathcal{E}^b \rightarrow \mathcal{D}^b$. Hence $\text{p.gl.dim } \mathcal{E} = \text{p.gl.dim } \mathcal{D}$, by Proposition 11.2.

We denote the full subcategories of $\text{Mod}(\mathcal{E}^b)$ consisting of the modules of finite projective, resp. injective, dimension by $\text{Proj}^{<\infty}(\text{Mod}(\mathcal{E}^b))$, resp. $\text{Inj}^{<\infty}(\text{Mod}(\mathcal{E}^b))$. Since the flat and the FP-injective modules coincide, it is trivial to see that

$$\begin{aligned} \text{Proj}^{<\infty}(\text{Mod}(\mathcal{E}^b)) &\subseteq \text{Flat}(\text{Mod}(\mathcal{E}^b)) \\ &= \text{FPInj}(\text{Mod}(\mathcal{E}^b)) \supseteq \text{Inj}^{<\infty}(\text{Mod}(\mathcal{E}^b)). \end{aligned}$$

Finally the *finitistic projective dimension* of $\text{Mod}(\mathcal{E}^b)$ is defined as $\text{f.p.d Mod}(\mathcal{E}^b) = \sup\{\text{p.d } F \in \text{Proj}^{<\infty}(\text{Mod}(\mathcal{E}^b))\}$. The definition of the finitistic injective dimension $\text{f.i.d Mod}(\mathcal{E}^b)$ of $\text{Mod}(\mathcal{E}^b)$ is similar. Note that since \mathcal{E}^b is triangulated, by [13] the flat or FP-injective dimension of any left or right \mathcal{E}^b -module is 0 or ∞ .

11.2. Pure Hereditary and Brown Categories

Let $\mathcal{I}: I \rightarrow \mathcal{E}$ be a functor from a small filtered category I . We use the notations: $A_i = \mathcal{I}(i)$ for $i \in \mathcal{I}$ and $\alpha_{ij} = \mathcal{I}(i \rightarrow j): A_i \rightarrow A_j$ for an arrow $i \rightarrow j$ in I . A *weak colimit* of the functor \mathcal{I} is an object A in \mathcal{E} together with morphisms $f_i: A_i \rightarrow A$, which are compatible with the system $\{A_i, \alpha_{ij}\}$ in the sense that for any arrow $i \rightarrow j$ in I we have $f_i = \alpha_{ij} \circ f_j$, and if $g_i: A_i \rightarrow B$ is another compatible family, then there exists a (not necessarily unique) morphism $\omega: A \rightarrow B$ such that $f_i \circ \omega = g_i$, for any object $i \in \mathcal{I}$. For example a homotopy colimit of a tower is a weak colimit. A weak colimit of the functor \mathcal{I} is denoted by $\text{w.lim}_{\rightarrow, \mathcal{I}} A_i$. By [51] a weak colimit of the functor \mathcal{I} can be constructed as a weak cokernel $\theta_{\mathcal{I}}: \bigoplus_{i \in I} A_i \rightarrow \text{w.lim}_{\rightarrow, \mathcal{I}} A_i$ of the canonical morphism $\zeta_{\mathcal{I}}: \bigoplus_{\alpha_{ij} \in I^2} A_i \rightarrow \bigoplus_{i \in I} A_i$. Hence it can be computed from the triangle $\bigoplus_{\alpha_{ij} \in I^2} A_i \xrightarrow{\zeta_{\mathcal{I}}} \bigoplus_{i \in I} A_i \xrightarrow{\theta_{\mathcal{I}}} \text{w.lim}_{\rightarrow, \mathcal{I}} A_i \rightarrow \Sigma(\bigoplus_{\alpha_{ij} \in I^2} A_i)$.

DEFINITION 11.6 [51]. A weak colimit $\text{w.lim}_{\rightarrow, \mathcal{I}} A_i$ of a functor $\mathcal{I}: I \rightarrow \mathcal{E}$ is called *minimal* if the canonical morphism

$$\lim_{\rightarrow, \mathcal{I}} \mathbf{S}(A_i) \rightarrow \mathbf{S}\left(\text{w.lim}_{\rightarrow, \mathcal{I}} A_i\right)$$

is an isomorphism. We denote a minimal weak colimit by $\text{m.w.lim}_{\rightarrow, \mathcal{I}} A_i$.

By Lemma 5.8 the homotopy colimit of a tower is a minimal weak colimit.

DEFINITION 11.7. A triangulated category \mathcal{E} is called a *Brown category*, if \mathcal{E} has coproducts, \mathcal{E}^b is skeletally small, and the pair $(\mathcal{E}, \mathcal{E}^b)$ satisfies BRT; i.e., the projectivization functor $\mathbf{S}: \mathcal{E} \rightarrow \text{Flat}(\text{Mod}(\mathcal{E}^b))$ is a representation equivalence.

Part (1) \Leftrightarrow (4) in the next result is due to Neeman [56] (see also [19, 21]) and parts (1) \Leftrightarrow (9) \Rightarrow (11) were proved independently by Hovey et al. [38]; see also [51]. We believe that our proofs are simpler. Note that the characterizations (4), (5) below are identical with the characterizations of flat modules over the Steenrod algebra; see [51].

THEOREM 11.8. *Let \mathcal{E} be a compactly generated triangulated category. Then the following are equivalent.*

- (1) \mathcal{E} is a Brown category.
- (2) The projectivization functor $\mathbf{S}: \mathcal{E} \rightarrow \text{Mod}(\mathcal{E}^b)$ is full.
- (3) $\text{p.gl.dim } \mathcal{E} \leq 1$.
- (4) Let $F \in \text{Mod}(\mathcal{E}^b)$. Then $F \in \text{Flat}(\text{Mod}(\mathcal{E}^b))$ iff $\text{p.d } F \leq 1$.
- (5) Let $F \in \text{Mod}(\mathcal{E}^b)$. Then $F \in \text{Flat}(\text{Mod}(\mathcal{E}^b))$ iff $\text{i.d } F \leq 1$.
- (6) $\text{f.p.d Mod}(\mathcal{E}^b) \leq 1$.
- (7) $\text{f.i.d Mod}(\mathcal{E}^b) \leq 1$.
- (8) If $\mathcal{I}: I \rightarrow \mathcal{E}^b$ and $\mathcal{J}: J \rightarrow \mathcal{E}^b$ are functors from small filtered categories, with $\mathcal{I}(i) = X_i$, $\mathcal{J}(j) = Y_j$, then

$$\lim_{\leftarrow \mathcal{I}}^{(n)} \lim_{\rightarrow \mathcal{J}} \mathcal{E}(X_i, Y_j) = 0, \quad \forall n \geq 2.$$

(9) Any functor $\mathcal{I}: I \rightarrow \mathcal{E}^b$ from a small filtered category I has a minimal weak colimit $\text{m.w.lim}_{\rightarrow \mathcal{I}} X_i$ in \mathcal{E} .

(10) Any functor $\mathcal{I}: I \rightarrow \mathcal{E}^b$ from a small filtered category I has a weak colimit $\text{w.lim}_{\rightarrow \mathcal{I}} X_i$ in \mathcal{E} with the following property. For any coproduct preserving homological functor $F: \mathcal{E} \rightarrow \mathcal{A}b$, the canonical map ϕ below is invertible:

$$\phi: \lim_{\rightarrow \mathcal{I}} F(X_i) \xrightarrow{\cong} F\left(\text{w.lim}_{\rightarrow \mathcal{I}} X_i\right).$$

(11) Any object of \mathcal{E} is a minimal weak colimit of a functor $\mathcal{I}: I \rightarrow \mathcal{E}^b$ from a small filtered category.

If one of the above equivalent conditions is true, then any two minimal weak colimits are isomorphic and for any weak colimit $\text{w.lim}_{\rightarrow \mathcal{I}} X_i$ with $X_i \in \mathcal{E}^b$,

we have

$$\text{w.lim}_{\mathcal{F}} X_i \cong \text{m.w.lim}_{\mathcal{F}} X_i \oplus P, \quad \text{where } P \in \text{Add}(\mathcal{E}^b).$$

Proof. Using Theorems 5.3, 10.2 and Proposition 11.2 it follows trivially that the first seven conditions are equivalent. We include only a proof that (2) \Rightarrow (1). If \mathbf{S} is full, then by Theorem 5.3 we have that $\text{p.gl.dim } \mathcal{E} \leq 1$. Then Proposition 11.2 ensures that any flat functor $F \in \text{Mod}(\mathcal{E}^b)$ has $\text{p.d } F \leq 1$. Finally by Theorem 10.2 it follows that \mathcal{E} is a Brown category.

(1) \Leftrightarrow (8) Let F, G be flat functors. Then F is a filtered colimit of representables $\mathbf{S}(X_i) = \mathcal{E}^b(-, X_i)$ and G is a filtered colimit of representables $\mathbf{S}(Y_j) = \mathcal{E}^b(-, Y_j)$, where I, J are small filtered categories and $\mathcal{F}: I \rightarrow \mathcal{E}^b, \mathcal{G}: J \rightarrow \mathcal{E}^b$ are functors with $\mathcal{F}(i) = X_i$ and $\mathcal{G}(j) = Y_j$. By Roos’s spectral sequence of Section 8.4 for the functors F, G , we have $\lim_{\leftarrow \mathcal{F}}^{(n)} \mathcal{E}(X_i, G) \cong \lim_{\leftarrow \mathcal{F}}^{(n)} \lim_{\rightarrow \mathcal{G}} \mathcal{E}(X_i, Y_j) \cong \mathcal{E}xt^n(F, G), \forall n \geq 0$. If (1) holds, then the above isomorphism implies (8), by (3). Conversely if (8) holds, then $\mathcal{E}xt^n(F, G) = 0, \forall n \geq 2$, for any two flat functors F, G . By [40], this implies that any flat G has pure injective dimension bounded by one. Since flat and FP-injective functors coincide and the pure injective dimension coincides with the injective dimension for FP-injective functors, (3) follows.

(9) \Rightarrow (1) Let $F = \lim_{\rightarrow \mathcal{F}} \mathbf{S}(X_i)$ be a flat functor, where $\mathcal{F}: I \rightarrow \mathcal{E}^b$ is a functor and I is a small filtered category. Consider a minimal weak colimit $\text{m.w.lim}_{\rightarrow \mathcal{F}} X_i$ of \mathcal{F} in \mathcal{E} . By hypothesis: $F = \lim_{\rightarrow \mathcal{F}} \mathbf{S}(X_i) \cong \mathbf{S}(\text{m.w.lim}_{\rightarrow \mathcal{F}} X_i)$, so \mathbf{S} is essentially surjective. Since \mathcal{E} is compactly generated, \mathbf{S} reflects isomorphisms and it remains to prove that \mathbf{S} is full. It suffices to show that if $\mathcal{F}: I \rightarrow \mathcal{E}^b$ and $\mathcal{G}: J \rightarrow \mathcal{E}^b$ are functors from small filtered categories, then any map $\alpha': \mathbf{S}(\text{m.w.lim}_{\rightarrow \mathcal{F}} X_i) \rightarrow \mathbf{S}(\text{m.w.lim}_{\rightarrow \mathcal{G}} Y_j)$ is of the form $\mathbf{S}(\alpha)$. By construction we have projective presentations $\mathbf{S}(\oplus X_i) \xrightarrow{\mathbf{S}(\zeta_{\mathcal{F}})} \mathbf{S}(\oplus X_i) \xrightarrow{\xi_{\mathcal{F}}} \mathbf{S}(\text{m.w.lim}_{\rightarrow \mathcal{F}} X_i) \rightarrow 0$ and $\mathbf{S}(\oplus Y_j) \xrightarrow{\mathbf{S}(\zeta_{\mathcal{G}})} \mathbf{S}(\oplus Y_j) \xrightarrow{\xi_{\mathcal{G}}} \mathbf{S}(\text{m.w.lim}_{\rightarrow \mathcal{G}} Y_j) \rightarrow 0$, and there exists a commutative diagram

$$\begin{CD} \mathbf{S}(\oplus X_i) @>\mathbf{S}(\zeta_{\mathcal{F}})>> \mathbf{S}(\oplus X_i) @>\xi_{\mathcal{F}}>> \mathbf{S}(\text{m.w.lim}_{\rightarrow \mathcal{F}} X_i) @>> 0 \\ @V\exists\beta'VV @V\exists\gamma'VV @V\alpha'VV \\ \mathbf{S}(\oplus Y_j) @>\mathbf{S}(\zeta_{\mathcal{G}})>> \mathbf{S}(\oplus Y_j) @>\xi_{\mathcal{G}}>> \mathbf{S}(\text{m.w.lim}_{\rightarrow \mathcal{G}} Y_j) @>> 0 \end{CD}$$

Since \mathbf{S} induces an equivalence between $\text{Add}(\mathcal{E}^b)$ and $\text{Proj}(\text{Mod}(\mathcal{E}^b))$, there exist morphisms $\beta: \oplus X_i \rightarrow \oplus Y_j, \gamma: \oplus X_i \rightarrow \oplus Y_j$ such that $\mathbf{S}(\beta) = \beta', \mathbf{S}(\gamma) = \gamma'$, and $\beta \circ \zeta_{\mathcal{F}} = \zeta_{\mathcal{G}} \circ \gamma$. Since $\text{m.w.lim}_{\rightarrow \mathcal{F}} X_i$ and $\text{m.w.lim}_{\rightarrow \mathcal{G}} Y_j$

are weak cokernels of $\zeta_{\mathcal{F}}$ and $\zeta'_{\mathcal{F}}$, respectively, there exists a morphism $\alpha: \text{m.w.lim}_{\rightarrow \mathcal{F}} X_i \rightarrow \text{m.w.lim}_{\rightarrow \mathcal{F}} Y_j$ such that $\gamma \circ \theta_{\mathcal{F}} = \theta_{\mathcal{F}} \circ \alpha$. But then obviously $\mathbf{S}(\alpha) = \alpha'$. Hence \mathbf{S} is full.

(1) \Rightarrow (9) By the equivalence of (1) with (3), for any flat right \mathcal{E}^b -module F , we have $\text{p.d } F \leq 1$. Let $\mathcal{F}: I \rightarrow \mathcal{E}^b$ be a functor from a small filtered category I and consider the flat modules $F = \lim_{\rightarrow \mathcal{F}} \mathbf{S}(X_i)$ and $G = \mathbf{S}(\text{w.lim}_{\rightarrow \mathcal{F}} X_i)$. Then identifying $\mathbf{S}(\oplus X_i)$ with $\oplus \mathbf{S}(X_i)$, we have the following exact commutative diagram

$$\begin{array}{ccccc} \oplus \mathbf{S}(X_i) & \xrightarrow{\mathbf{S}(\zeta_{\mathcal{F}})} & \oplus \mathbf{S}(X_i) & \xrightarrow{\alpha} & F \longrightarrow 0 \\ \parallel \downarrow & & \parallel \downarrow & & \exists! \omega \downarrow \\ \mathbf{S}(\oplus X_i) & \xrightarrow{\mathbf{S}(\zeta_{\mathcal{F}})} & \mathbf{S}(\oplus X_i) & \xrightarrow{\mathbf{S}(\theta_{\mathcal{F}})} & G \end{array}$$

By BRT there exists $A \in \mathcal{E}$ and a morphism $f: A \rightarrow \text{w.lim}_{\rightarrow \mathcal{F}} X_i$, such that $\mathbf{S}(A) = F$ and $\mathbf{S}(f) = \omega$. Similarly there exists a (unique) morphism $\mu: \oplus X_i \rightarrow A$, such that $\mathbf{S}(\mu) = \alpha$. Let $\mathbf{S}(\zeta_{\mathcal{F}}) = \beta \circ \gamma$ be the canonical factorization of $\mathbf{S}(\zeta_{\mathcal{F}})$ in $\text{Mod}(\mathcal{E}^b)$, where $\beta: \mathbf{S}(\oplus X_i) \rightarrow H$ and $\gamma: H \rightarrow \mathbf{S}(\oplus X_i)$. By BRT there exists $Q \in \mathcal{E}$ and (unique) morphisms $\kappa: \oplus X_i \rightarrow Q$ and $\lambda: Q \rightarrow \oplus X_i$ such that $\mathbf{S}(\kappa) = \beta$ and $\mathbf{S}(\lambda) = \gamma$. Since $\mathbf{S}(\mu)$ is epic the triangle $Q \xrightarrow{\lambda} \oplus X_i \xrightarrow{\mu} A \xrightarrow{f} \Sigma(Q)$ is pure. Since $\text{p.d } F \leq 1$, the functor $H = \mathbf{S}(Q)$ is projective; hence the morphism $\mathbf{S}(\kappa)$ is split epic. Since \mathbf{S} reflects isomorphisms, κ is split epic so there exists $\rho: Q \rightarrow \oplus X_i$ with $\rho \circ \kappa = 1_Q$ inducing a direct sum decomposition $\oplus X_i \cong Q \oplus P$. Then we have the following morphisms of triangles:

$$\begin{array}{ccccccc} Q & \xrightarrow{\lambda} & \oplus X_i & \xrightarrow{\mu} & A & \xrightarrow{f} & \Sigma(Q) \\ \rho \downarrow & & \parallel \downarrow & & f \downarrow & & \Sigma(\rho) \downarrow \\ \oplus X_i & \xrightarrow{\zeta_{\mathcal{F}}} & \oplus X_i & \xrightarrow{\theta_{\mathcal{F}}} & \text{w.lim}_{\rightarrow \mathcal{F}} X_i & \xrightarrow{\eta_{\mathcal{F}}} & \Sigma(\oplus X_i) \\ \kappa \downarrow & & \parallel \downarrow & & \exists g \downarrow & & \Sigma(\kappa) \downarrow \\ Q & \xrightarrow{\lambda} & \oplus X_i & \xrightarrow{\mu} & A & \xrightarrow{f} & \Sigma(\oplus X_i). \end{array}$$

Since $\rho \circ \kappa = 1_Q$, the morphism $f \circ g$ is invertible; in particular f is split monic. It is easy to see that $\text{Coker}(f) = \Sigma(P)$; hence $\text{w.lim}_{\rightarrow \mathcal{F}} X_i \cong A \oplus \Sigma(P)$. Observe that $\zeta_{\mathcal{F}} \circ \mu = \kappa \circ \lambda \circ \mu = 0$. Let $\phi: \oplus X_i \rightarrow E$ be a morphism such that $\zeta_{\mathcal{F}} \circ \phi = 0$. Since by construction $\theta_{\mathcal{F}}$ is a weak cokernel of $\zeta_{\mathcal{F}}$, we have $\phi = \theta_{\mathcal{F}} \circ \psi$ for a morphism $\psi: \text{w.lim}_{\rightarrow \mathcal{F}} X_i \rightarrow E$. Hence $\phi = \mu \circ f \circ \psi$ and this implies that the morphism $\mu: \oplus X_i \rightarrow A$ is a weak cokernel of $\zeta_{\mathcal{F}}$. Hence A is a weak colimit of the functor $\mathcal{F}: I \rightarrow \mathcal{E}^b$. Since by the construction $\mathbf{S}(A) = \lim_{\rightarrow \mathcal{F}} \mathbf{S}(X_i)$, we conclude that A is a minimal

weak colimit of the functor \mathcal{F} . If A' is another minimal weak colimit, then $\mathbf{S}(A) \cong \mathbf{S}(A')$ and by BRT we have $A \cong A'$. The above argument shows that if \hat{A} is any other weak colimit, then $\hat{A} \cong A \oplus P$ where $P \in \text{Add}(\mathcal{E}^b)$.

(9) \Leftrightarrow (10) Note that by Theorem 8.10, a homological functor $F: \mathcal{E} \rightarrow \mathcal{A}b$ preserves coproducts iff $F \cong G^* \mathbf{S}$, where G^* is exact and preserves colimits. From this it follows directly that (9) implies (10). Conversely part (9) follows from (10), choosing $F = \mathcal{E}(X, -): \mathcal{E} \rightarrow \mathcal{A}b$, for any $X \in \mathcal{E}^b$.

(1) \Leftrightarrow (11) If \mathcal{E} is Brown and $A \in \mathcal{E}$, then $\mathbf{S}(A) = \lim_{\rightarrow \mathcal{F}} \mathbf{S}(X_i)$, where $\mathcal{F}: I \rightarrow \mathcal{E}^b$ is a functor from a small filtered category I . By (9), the functor \mathcal{F} has a minimal weak colimit $A' = \text{m.w.}\lim_{\rightarrow \mathcal{F}} X_i$. Then $\mathbf{S}(A) \cong \mathbf{S}(A')$ and since by (1), \mathbf{S} is a representation equivalence, we have $A \cong A'$. Conversely if (11) holds, let $A \in \mathcal{E}$. Then as above $A = \text{m.w.}\lim_{\rightarrow \mathcal{F}} X_i$. By construction of the weak colimit of \mathcal{F} , there exists a triangle $\oplus X_j \xrightarrow{\alpha} \oplus X_i \xrightarrow{\beta} A \xrightarrow{\gamma} \Sigma(\oplus X_j)$. By the construction of the colimit of the functor $\mathbf{S}\mathcal{F}: I \rightarrow \text{Mod}(\mathcal{E}^b)$, it follows directly that γ is a pure phantom map, so the above triangle is pure. This implies that $\text{p.p.d } A \leq 1$. ■

COROLLARY 11.9. *Suppose that $\text{p.gl.dim } \mathcal{E} \leq 1$. If $A = \text{m.w.}\lim_{\rightarrow \mathcal{F}} X_i$ with $X_i \in \mathcal{E}^b$, then for any $B \in \mathcal{E}$ there exists a short exact sequence*

$$0 \rightarrow \lim_{\leftarrow \mathcal{F}}^{(1)} \mathcal{E}(\Sigma(X_i), B) \rightarrow \mathcal{E}(A, B) \rightarrow \lim_{\leftarrow \mathcal{F}} \mathcal{E}(X_i, B) \rightarrow 0.$$

Hence $\mathcal{P}\mathcal{E}xt^1(\Sigma(A), B) = \text{Ph}(A, B) = \lim_{\leftarrow \mathcal{F}}^{(1)} \mathcal{E}(\Sigma(X_i), B)$ and $\text{Ph}^2(\mathcal{E}) = 0$.

If $A = \text{m.w.}\lim_{\rightarrow \mathcal{F}} X_i$, $B = \text{m.w.}\lim_{\rightarrow \mathcal{G}} Y_j$ are representations of A, B as minimal weak colimits of functors $\mathcal{F}: I \rightarrow \mathcal{E}^b$ and $\mathcal{G}: J \rightarrow \mathcal{E}^b$, then $\text{Ph}(A, B) = \lim_{\leftarrow \mathcal{F}}^{(1)} \lim_{\rightarrow \mathcal{G}} \mathcal{E}(\Sigma(X_i), Y_j)$ and there exists a short exact sequence

$$0 \rightarrow \lim_{\leftarrow \mathcal{F}}^{(1)} \lim_{\rightarrow \mathcal{G}} \mathcal{E}(\Sigma(X_i), Y_j) \rightarrow \mathcal{E}(A, B) \rightarrow \lim_{\leftarrow \mathcal{F}} \lim_{\rightarrow \mathcal{G}} \mathcal{E}(X_i, Y_j) \rightarrow 0.$$

Proof. By Theorem 4.27, there exists a short exact sequence $0 \rightarrow \mathcal{P}\mathcal{E}xt^1(\Sigma(A), B) \rightarrow \mathcal{E}(A, B) \rightarrow \mathcal{P}\mathcal{E}xt^0(A, B) \rightarrow 0$. Since $\mathcal{E}xt^n[\mathbf{S}(A), \mathbf{S}(B)] \cong \mathcal{P}\mathcal{E}xt^n(A, B)$, $\forall n \geq 0$, we have $\mathcal{E}xt^1[\mathbf{S}(\Sigma(A)), \mathbf{S}(B)] \cong \mathcal{E}xt^1[\lim_{\rightarrow \mathcal{F}} \mathbf{S}(\Sigma(X_i)), \mathbf{S}(B)] \cong \mathcal{P}\mathcal{E}xt^1(\Sigma(A), B)$. Using Roos's spectral sequence as in the proof of Theorem 11.8 we see directly that

$$\begin{aligned} \mathcal{E}xt^1 \left[\lim_{\rightarrow \mathcal{F}} \mathbf{S}(\Sigma(X_i)), \mathbf{S}(B) \right] &\cong \lim_{\leftarrow \mathcal{F}}^{(1)} [\mathbf{S}(\Sigma(X_i), \mathbf{S}(B))] \\ &\cong \lim_{\leftarrow \mathcal{F}}^{(1)} \mathcal{E}(\Sigma(X_i), B). \end{aligned}$$

In the same way $\mathcal{P}\mathcal{E}xt^0(A, B) \cong \lim_{\leftarrow \mathcal{F}} \mathcal{E}(X_i, B)$ and the assertion follows. ■

COROLLARY 11.10. *If \mathcal{E}, \mathcal{D} are Brown categories, then*

$$\mathcal{E}^b \approx \mathcal{D}^b \Leftrightarrow \mathcal{E}/\text{Ph}(\mathcal{E}) \approx \mathcal{D}/\text{Ph}(\mathcal{D}) \Leftrightarrow \text{PInj}(\mathcal{E}) \approx \text{PInj}(\mathcal{D}).$$

Proof. Trivial since $\mathcal{E}/\text{Ph}(\mathcal{E}) \approx \text{Flat}(\text{Mod}(\mathcal{E}^b))$, $\mathcal{D}/\text{Ph}(\mathcal{D}) \approx \text{Flat}(\text{Mod}(\mathcal{D}^b))$. ■

Remark 11.11 [21]. If $\text{p.gl.dim } \mathcal{E} \leq 1$, then we have a $+$ -linear extension of categories

$$0 \rightarrow \text{Ph}(\mathcal{E}) \rightarrow \mathcal{E} \xrightarrow{\mathbf{S}} \text{Flat}(\text{Mod}(\mathcal{E}^b)) \rightarrow 0 \quad (\text{E})$$

in the sense of [9], which represents an element of the second Hochschild–Mitchell–Baues–Wirsching cohomology group

$$H^2[\text{Flat}(\text{Mod}(\mathcal{E}^b)), \text{Ph}(\mathcal{E})].$$

This element is zero if the above extension splits and this happens if there exists a functor $\mathbf{T}: \text{Flat}(\text{Mod}(\mathcal{E}^b)) \rightarrow \mathcal{E}$ such that $\mathbf{ST} = \text{Id}$. If this is true, then $\forall B \in \mathcal{E}$, the short exact sequence $0 \rightarrow \mathcal{P}\mathcal{E}xt^1[\Sigma -, B] \rightarrow \mathcal{E}(-, B) \rightarrow \mathcal{P}\mathcal{E}xt^0[-, B] \rightarrow 0$ splits. This implies that $\mathcal{P}\mathcal{E}xt^1[-, B]$ is a cohomological functor as a direct summand of $\mathcal{E}(-, \Sigma(B))$. Hence if $P_1 \rightarrow P_0 \rightarrow A \rightarrow \Sigma(P_1)$ is a pure triangle with $P_1, P_0 \in \text{PProj}(\mathcal{E})$, then we have a long exact sequence $\cdots \rightarrow \mathcal{P}\mathcal{E}xt^1[\Sigma^2(P_1), B] \rightarrow \mathcal{P}\mathcal{E}xt^1[\Sigma(A), B] \rightarrow \mathcal{P}\mathcal{E}xt^1[\Sigma(P_0), B] \rightarrow \cdots$ and this shows that $\mathcal{P}\mathcal{E}xt^1[\Sigma(A), B] = 0$. Hence B is pure injective. It follows that $\text{p.gl.dim } \mathcal{E} = 0$. Hence if \mathcal{E} is pure hereditary, then \mathcal{E} is pure semisimple iff the extension (E) splits. Since the stable homotopy category of spectra \mathcal{E} has $\text{p.gl.dim } \mathcal{E} = 1$, we have $H^2[\text{Flat}(\text{Mod}(\mathcal{E}^b)), \text{Ph}(\mathcal{E})] \neq 0$. In this case all the above results are applied to this situation. In particular we recover the various (topological) versions of the Brown representability theorem.

Remark 11.12. If $\text{p.gl.dim } \mathcal{E} \leq 2$, then the projectivization functor $\mathbf{S}: \mathcal{E} \rightarrow \text{Flat}(\text{Mod}(\mathcal{E}^b))$ is surjective on objects. Indeed let $0 \rightarrow \mathbf{S}(P_2) \xrightarrow{\mathbf{S}(f)} \mathbf{S}(P_1) \xrightarrow{\mathbf{S}(g)} \mathbf{S}(P_0) \xrightarrow{\epsilon} F \rightarrow 0$ be a projective resolution of the flat functor F . If $P_2 \xrightarrow{f} P_1 \xrightarrow{\beta} B \rightarrow \Sigma(P_2)$ is a triangle in \mathcal{E} , then obviously $\text{Im } \mathbf{S}(g) \cong \mathbf{S}(B)$ and $\text{coim } \mathbf{S}(g) = \mathbf{S}(\beta)$. Hence $\text{im } \mathbf{S}(g): \mathbf{S}(B) \hookrightarrow \mathbf{S}(P_0)$. Since $\text{p.p.d } B \leq 1$, by Theorem 4.27 it follows that $\text{im } \mathbf{S}(g) = \mathbf{S}(\alpha)$ for some morphism $\alpha: B \rightarrow P_0$. If $B \rightarrow \alpha P_0 \rightarrow A \rightarrow \Sigma(B)$ is a triangle in \mathcal{E} , then obviously $F \cong \mathbf{S}(A)$. In case \mathcal{E} is the unbounded derived category $\mathbf{D}(\Lambda)$, where Λ is a right hereditary ring, then by [22] the converse is true.

The interpretation of the pure global dimension in terms of properties of the projectivization functor \mathbf{S} shows that we have, in general non-reversible, implications:

$$\mathbf{S} \text{ is faithful} \quad \Rightarrow \quad \mathbf{S} \text{ is full} \quad \Rightarrow \quad \mathbf{S} \text{ is surjective on objects.}$$

11.3. Homological Functors

If $\text{p.gl.dim } \mathcal{C} = 0$, then by Theorem 9.3, any homological functor $H: \mathcal{C} \rightarrow \mathcal{G}$ to an abelian category \mathcal{G} has a unique exact extension $H^*: \text{Mod}(\mathcal{C}^b) \rightarrow \mathcal{G}$, such that $H^* \mathbf{S} = H$. If the homological functor H preserves coproducts and \mathcal{G} is Grothendieck, then without any restriction on $\text{p.gl.dim } \mathcal{C}$ we have the following consequence of Theorem 8.10, part (i) of which was observed also independently by Christensen and Strickland [21] in the Brown case, and by Krause [50] in general. Part (ii) is due to Krause [50] with a different proof. Note that Neeman proved in [57] that any product preserving homological functor $\mathcal{C} \rightarrow \mathcal{A}b$ is representable. However, the proof of this general result is much more difficult.

COROLLARY 11.13. *Let $H: \mathcal{C} \rightarrow \mathcal{G}$ be a homological functor which preserves coproducts to a Grothendieck category \mathcal{G} . Then:*

(i) *$H(\phi) = 0$, for any pure-phantom map ϕ and there exists a unique coproduct preserving functor $H^*: \text{Mod}(\mathcal{C}^b) \rightarrow \mathcal{G}$, such that $H^* \mathbf{S} = H$. Hence \mathbf{S} is the universal coproduct preserving homological functor out of \mathcal{C} to Grothendieck categories. The functor H^* is exact and admits a right adjoint which preserves injectives, and we can identify $H^* = - \otimes_{\mathcal{C}^b} H|_{\mathcal{C}^b}$ and $H = \mathbf{S}(-) \otimes_{\mathcal{C}^b} H|_{\mathcal{C}^b}$.*

(ii) *If $\mathcal{G} = \text{Mod}(\Lambda)$ for a ring Λ , then H preserves products iff $H \cong \mathcal{C}(T, -)$, where $T \in \mathcal{C}^b$. In this case H^* has a left adjoint which preserves projectives.*

Proof. Part (i) follows from Theorem 8.10. If $H: \mathcal{C} \rightarrow \text{Mod}(\Lambda)$ preserves products, then H^* preserves limits, so it is representable: $H^* \cong (M, -)$. Since by (i), H^* is exact and preserves colimits, M is a finitely generated projective functor, hence of the form $M = \mathbf{S}(T)$, for $T \in \mathcal{C}^b$. Then $H \cong H^* \mathbf{S} \cong (\mathbf{S}(T), \mathbf{S}(-)) \cong \mathcal{C}(T, -)$. ■

If $H: \mathcal{C} \rightarrow \mathcal{A}b$ is a homological functor, then $\hat{H} := \mathbf{S}(-) \otimes_{\mathcal{C}^b} H|_{\mathcal{C}^b}: \mathcal{C} \rightarrow \mathcal{A}b$ is a homological functor preserving coproducts and there is a natural morphism $\phi: \hat{H} \rightarrow H$. It follows by the above result that ϕ is invertible iff H preserves coproducts. Consider now the dual situation. Let $F: \mathcal{C}^{\text{op}} \rightarrow \mathcal{A}b$ be a cohomological functor, and consider the functor $\tilde{F} := (\mathbf{S}(-), F|_{\mathcal{C}^b}): \mathcal{C}^{\text{op}} \rightarrow \mathcal{A}b$. It is not difficult to see that \tilde{F} sends coproducts to products and there exists a natural morphism $\psi: F \rightarrow \tilde{F}$. However, \tilde{F} is

not cohomological in general. Indeed \tilde{F} is cohomological iff ψ is invertible iff $F = \mathcal{E}(-, E)$, where E is pure injective. If for $A \in \mathcal{E}$, $\mathcal{D}(A)$ denotes the category of morphisms $X_a \rightarrow A$ with $X_a \in \mathcal{E}^b$, then $\hat{H}(A) = \lim_{\rightarrow \mathcal{D}(A)} H(X_a)$ and $\tilde{F}(A) = \lim_{\leftarrow \mathcal{D}(A)} F(X_a)$ are the functors considered by Margolis [51]. If BRT holds, then the next corollary, which gives a trivial proof to a result of Adams [51], shows that \tilde{F} satisfies a weak exact condition.

COROLLARY 11.14. *If $\text{p.gl.dim } \mathcal{E} \leq 1$ and $P \rightarrow^f A \rightarrow^g B \rightarrow \Sigma(P)$ is a triangle in \mathcal{E} , where P is pure projective, then the sequence $\tilde{F}(B) \rightarrow \tilde{F}(A) \rightarrow \tilde{F}(P)$ is exact.*

Proof. By BRT $F|_{\mathcal{E}^b} = \mathbf{S}(C)$. Hence $\tilde{F} = [\mathbf{S}(-), \mathbf{S}(C)] = \mathcal{P}\mathcal{E}xt^0(-, C)$. Consider the complex $(*)$: $[\mathbf{S}(B), \mathbf{S}(C)] \rightarrow [\mathbf{S}(A), \mathbf{S}(C)] \rightarrow [\mathbf{S}(P), \mathbf{S}(C)]$. Let $\alpha: \mathbf{S}(A) \rightarrow \mathbf{S}(C)$ be such that $\mathbf{S}(f) \circ \alpha = 0$. By BRT, $\alpha = \mathbf{S}(\beta)$. Then $f \circ \beta$ is pure phantom. Since P is pure projective, $f \circ \beta = 0$. Hence $\beta = g \circ \gamma$, where $\gamma: B \rightarrow C$. Then α factors through $(\mathbf{S}(g), \mathbf{S}(C))$ and this means that the complex $(*)$ is exact. ■

Remark 11.15. Suppose \mathcal{E} is monogenic [38]; i.e., \mathcal{E} admits a compact object T such that \mathcal{E} coincides with the localizing subcategory generated by T , for example a tilting complex [62] in the derived category of a ring or the sphere spectrum [51] in the stable homotopy category. The relative homological algebra based on the proper class $\mathcal{E}(\mathcal{L})$, where $\mathcal{L} = \text{add}\{\Sigma^n(T); n \in \mathbb{Z}\}$, equivalently on the proper class $\mathcal{E}(H)$ where $H = \mathcal{E}(T, -)$, can be considered as the “absolute homological algebra” of \mathcal{E} and is studied in [14]; see also Sections 12.4 and 12.5. The pure and the absolute theory are related by a Butler–Horrocks spectral sequence [18]; see [14].

Remark 11.16. Assume that \mathcal{E} has all small coproducts, and let \mathcal{L} be an α -localizing subcategory of \mathcal{E} in the sense of [57], where α is an infinite cardinal. Then the Steenrod category of \mathcal{E} with respect to $\mathcal{E}(\mathcal{L})$ is the category $\mathcal{E}x(\mathcal{L}^{\text{op}}, \mathcal{A}b)$ of all functors $\mathcal{L}^{\text{op}} \rightarrow \mathcal{A}b$ converting coproducts of fewer than α objects of \mathcal{L} to products in $\mathcal{A}b$. See [57] for a detailed analysis of this situation which can be regarded as a higher analogue of the present theory, which in turn corresponds to the case $\alpha = \aleph_0$.

11.4. Pure Global Dimension under Smashing and Finite Localization

Fix a compactly generated triangulated category \mathcal{E} . We recall that a localizing subcategory \mathcal{L} of \mathcal{E} is called *smashing* if the inclusion functor $i: \mathcal{L} \hookrightarrow \mathcal{E}$ has a right adjoint which preserves coproducts; equivalently the quotient functor $\pi: \mathcal{E} \rightarrow \mathcal{E}/\mathcal{L}$ has a (fully faithful) right adjoint $\tau: \mathcal{E}/\mathcal{L} \rightarrow \mathcal{E}$ which preserves coproducts. It is easy to see [50] that if $\mathcal{L} \subseteq \mathcal{E}$ is smashing, then the Verdier quotient \mathcal{E}/\mathcal{L} is compactly generated and

the functor π sends \mathcal{E}^b to $(\mathcal{E}/\mathcal{L})^b$, so π induces an exact functor $\pi^b: \mathcal{E}^b \rightarrow (\mathcal{E}/\mathcal{L})^b$. Since π is exact and preserves coproducts, by Corollary 11.13 or [50], there exists a unique exact colimit preserving functor $\pi_b^*: \text{Mod}(\mathcal{E}^b) \rightarrow \text{Mod}((\mathcal{E}/\mathcal{L})^b)$ such that $\pi_b^* \mathbf{S}_{\mathcal{E}} = \mathbf{S}_{\mathcal{E}/\mathcal{L}} \pi$, where $\mathbf{S}_{\mathcal{E}}, \mathbf{S}_{\mathcal{E}/\mathcal{L}}$ are the projectivization functors. Observe that by the above relation, π_b^* preserves flat functors and moreover projective functors, since π preserves compact objects. Similarly since τ is exact and preserves coproducts, there exists a unique exact colimit preserving functor $\tau_b^*: \text{Mod}((\mathcal{E}/\mathcal{L})^b) \rightarrow \text{Mod}(\mathcal{E}^b)$ such that $\mathbf{S}_{\mathcal{E}} \tau = \tau_b^* \mathbf{S}_{\mathcal{E}/\mathcal{L}}$. Since $\pi \tau = \text{Id}_{\mathcal{E}/\mathcal{L}}$, the above properties imply that $\pi_b^* \tau_b^* \cong \text{Id}_{\text{Mod}((\mathcal{E}/\mathcal{L})^b)}$. If $\lim_{\rightarrow} \mathbf{S}_{\mathcal{E}/\mathcal{L}}(Y_j)$ is a flat functor over $(\mathcal{E}/\mathcal{L})^b$, then since τ_b^* preserves filtered colimits, $\tau_b^*(\lim_{\rightarrow} \mathbf{S}_{\mathcal{E}/\mathcal{L}}(Y_j)) = \lim_{\rightarrow} \tau_b^* \mathbf{S}_{\mathcal{E}/\mathcal{L}}(Y_j) \cong \lim_{\rightarrow} \mathbf{S}_{\mathcal{E}} \tau(Y_j)$. Since $\mathbf{S}_{\mathcal{E}} \tau(Y_j)$ are flat functors and a filtered colimit of flat functors is flat, it follows that τ_b^* preserves flatness.

THEOREM 11.17. (1) *If \mathcal{L} is a smashing subcategory of \mathcal{E} , then*

$$\text{p.gl.dim } \mathcal{E}/\mathcal{L} \leq \text{p.gl.dim } \mathcal{E}.$$

(2) *If \mathcal{L} is a localizing subcategory of \mathcal{E} which is generated by a set of compact objects from \mathcal{E} , so \mathcal{L} is compactly generated and smashing, then*

$$\max\{\text{p.gl.dim } \mathcal{L}, \text{p.gl.dim } \mathcal{E}/\mathcal{L}\} \leq \text{p.gl.dim } \mathcal{E}.$$

Proof. (1) Let F be a flat functor over $(\mathcal{E}/\mathcal{L})^b$. Let $P^\bullet \rightarrow \tau_b^*(F)$ be a projective resolution of the flat functor $\tau_b^*(F)$ in $\text{Mod}(\mathcal{E}^b)$. Since π_b^* is exact and preserves projectives, $\pi_b^*(P^\bullet) \rightarrow \pi_b^* \tau_b^*(F) \cong F$ is a projective resolution of F in $\text{Mod}((\mathcal{E}/\mathcal{L})^b)$ and the assertion follows. The second part is proved in a similar way. ■

The following consequence is due to Hovey et al. [38].

COROLLARY 11.18. *Smashing or finite localizations of Brown, resp. phantomless categories are Brown, resp. phantomless.*

11.5. Compactly Generated Triangulated Categories of Finite Type

By Theorem 9.3 a compactly generated triangulated category \mathcal{E} is pure semisimple iff its pure Steenrod category $\text{Mod}(\mathcal{E}^b)$ is locally Noetherian or perfect. The pure homological theory of module categories [39, 65] suggests the following definition.

DEFINITION 11.19. A compactly generated triangulated category \mathcal{E} is said to be *of finite type* if its pure Steenrod category $\text{Mod}(\mathcal{E}^b)$ is locally finite.

Categories of finite type can be regarded as the “representation-finite” triangulated categories. For a justification of this statement we refer to the next section.

COROLLARY 11.20. *If \mathcal{L} is a smashing subcategory of \mathcal{E} and \mathcal{E} is of finite type, then so is \mathcal{E}/\mathcal{L} . If \mathcal{L} is a localizing subcategory of \mathcal{E} generated by a set of compact objects from \mathcal{E} and \mathcal{E} is of finite type, then so are $\mathcal{L}, \mathcal{E}/\mathcal{L}$.*

Proof. If \mathcal{L} is smashing then by [50], $\text{Mod}((\mathcal{E}/\mathcal{L})^b)$ is a localized quotient of $\text{Mod}(\mathcal{E}^b)$. Since \mathcal{E} is of finite type, $\text{Mod}(\mathcal{E}^b)$ is locally finite. Hence by [59], $\text{Mod}((\mathcal{E}/\mathcal{L})^b)$ is locally finite; i.e., \mathcal{E}/\mathcal{L} is of finite type. The second part is left to the reader. ■

The pure homological theory of module categories suggests the following

QUESTION 11.21. *Let \mathcal{E} be a pure semisimple compactly generated triangulated category. Is it true that \mathcal{E} is of finite type?*

A positive answer to Question 11.21 would follow from an affirmative answer to the following problem about Grothendieck categories posed by Roos [64] almost 30 years ago.

Roos's Problem 11.22. *Let \mathcal{E} be a locally Noetherian Grothendieck category in which any injective object is projective. Is it true that \mathcal{E} is locally finite?*

Indeed if \mathcal{E} is pure semisimple, then the module category $\text{Mod}(\mathcal{E}^b)$ is a locally Noetherian Frobenius Grothendieck category. If Roos's problem has an affirmative answer, then $\text{Mod}(\mathcal{E}^b)$ is locally finite, so \mathcal{E} is of finite type. In Section 12 we shall see that in general both Question 11.21 and Roos's problem have a negative answer.

11.6. Krull–Gabriel Dimension and Dual Pure Global Dimension

We close this section by presenting a characterization of categories of finite type, using two new dimensions which are invariant under triangle equivalence.

Define the *dual pure global dimension* $\text{p.gl.dim } \mathcal{E}^{\text{op}}$ of \mathcal{E} as the supremum of the projective dimensions in $\text{Mod}((\mathcal{E}^b)^{\text{op}})$ of the homological functors $\mathcal{E}^b \rightarrow \mathcal{A}b$:

$$\text{p.gl.dim } \mathcal{E}^{\text{op}} := \sup \left\{ \text{p.d } F \mid F \in \text{Flat} \left(\text{Mod} \left((\mathcal{E}^b)^{\text{op}} \right) \right) \right\}.$$

Note that $\text{p.gl.dim } \mathcal{E}^{\text{op}}$ is *not* the pure global dimension of \mathcal{E}^{op} , since \mathcal{E}^{op} is never compactly generated [57]. However, the above definition is reasonable, since if $\mathbf{D}(\Lambda)$ is the unbounded derived category of a ring Λ , then from the results of the next section it follows that $\text{p.gl.dim } \mathbf{D}(\Lambda)^{\text{op}} = \text{p.gl.dim } \mathbf{D}(\Lambda^{\text{op}})$. Similarly if $\underline{\text{Mod}}(\mathcal{D})$ is the stable category of a locally finite Frobenius module category $\text{Mod}(\mathcal{D})$, then $\text{p.gl.dim } \underline{\text{Mod}}(\mathcal{D})^{\text{op}} = \text{p.gl.dim } \underline{\text{Mod}}(\mathcal{D}^{\text{op}})$. Finally if \mathcal{E} is the stable homotopy category, then

using the Spanier–Whitehead duality functor $D: \mathcal{E}^b \rightarrow \approx (\mathcal{E}^b)^{\text{op}}$ [51], it follows that $\text{p.gl.dim } \mathcal{E}^{\text{op}} = \text{p.gl.dim } \mathcal{E} = 1$. More generally if \mathcal{T}, \mathcal{S} are compactly generated and there exists an equivalence $(\mathcal{T}^b)^{\text{op}} \approx \mathcal{S}^b$, then $\text{p.gl.dim } \mathcal{T}^{\text{op}} = \text{p.gl.dim } \mathcal{S}$ and $\text{p.gl.dim } \mathcal{S}^{\text{op}} = \text{p.gl.dim } \mathcal{T}$.

Finally let $\text{KGdim } \mathcal{E}^b$ be the *Krull–Gabriel dimension* [32, 49], of the abelian category $\mathcal{A}(\mathcal{E}^b) = \text{mod}(\mathcal{E}^b)$. Note that this dimension is the finitely presented version of the Krull dimension $\text{Kdim Mod}(\mathcal{E}^b)$ of $\text{Mod}(\mathcal{E}^b)$ defined by Gabriel [30]. By [49], $\text{Kdim Mod}(\mathcal{E}^b) \leq \text{KGdim } \mathcal{E}^b$ with equality if \mathcal{E} is pure semisimple. It seems that the Krull–Gabriel dimension is more stable than the pure global dimensions and deserves further study. For instance it is symmetric: $\text{KGdim } \mathcal{E}^b = \text{KGdim}(\mathcal{E}^b)^{\text{op}}$; see [32]. We denote this common value by $\text{KGdim } \mathcal{E}$ and by abuse of language we call it the *Krull–Gabriel dimension* of \mathcal{E} . Note that by [32] it follows easily that $\max\{\text{p.gl.dim } \mathcal{E}^{\text{op}}, \text{p.gl.dim } \mathcal{E}\} \leq \text{KGdim } \mathcal{E}$.

The next result shows that Question 11.21 has an affirmative answer if the implication $\text{p.gl.dim } \mathcal{E} = 0 \Rightarrow \text{p.gl.dim } \mathcal{E}^{\text{op}} = 0$ is true.

PROPOSITION 11.23. (1) *p.gl.dim $\mathcal{E}^{\text{op}} = 0$ iff the Steenrod category $\text{Mod}(\mathcal{E}^b)$ is locally Artinian. More generally assume that for any compact object X , any decreasing filtered family of subobjects of $\mathcal{E}^b(-, X)$ in $\mathcal{A}(\mathcal{E}^b)$ contains a cofinal subfamily of cardinality $\aleph_t, t \geq -1$. Then $\text{p.gl.dim } \mathcal{E}^{\text{op}} \leq t + 1$.*

(2) *The following statements are equivalent:*

- (i) *\mathcal{E} is of finite type.*
- (ii) *$\text{p.gl.dim } \mathcal{E}^{\text{op}} = 0 = \text{p.gl.dim } \mathcal{E}$.*
- (iii) *$\text{KGdim } \mathcal{E} = 0$.*

Proof. Part (1) follows easily from results of Gruson and Jensen [32] and Simson [65]. Part (2) follows from part (1) and Proposition 9.2, using the well-known duality $\text{mod}(\mathcal{E}^b)^{\text{op}} \approx \text{mod}((\mathcal{E}^b)^{\text{op}})$ given by Coker $\mathcal{E}^b(-, f) \mapsto \text{Ker } \mathcal{E}^b(f, -)$. ■

QUESTION 11.24. *What is the Krull–Gabriel dimension of a Brown category? In particular, what is $\text{KGdim } \mathcal{E}$, if \mathcal{E} is the stable homotopy category? Certainly $\text{KGdim } \mathcal{E} \geq 1$ if \mathcal{E} is Brown. It seems reasonable to conjecture that $\text{KGdim } \mathcal{E} = 1$.*

12. APPLICATIONS TO DERIVED AND STABLE CATEGORIES

In this section we apply our previous results (mainly about purity) to derived categories of rings and to stable module categories of quasi-Frobenius rings.

12.1. *Derived Categories*

Let Λ be an associative ring (with 1 always). We denote by $\mathbf{D}(\Lambda)$ the unbounded derived category of the abelian category $\text{Mod}(\Lambda)$ of all right Λ -modules. We denote also by $\text{Proj}(\Lambda)$ the full subcategory of all projective modules and by \mathcal{P}_Λ the full subcategory of finitely generated projective modules in $\text{Mod}(\Lambda)$. Finally we denote by $\mathcal{H}^b(\mathcal{P}_\Lambda)$ the bounded homotopy category of \mathcal{P}_Λ , and by $\mathbf{D}^b(\text{mod}(\Lambda))$ the bounded derived category of finitely presented modules. It is well known [62] that $\mathbf{D}(\Lambda)$ is a compactly generated triangulated category and $\mathcal{H}^b(\mathcal{P}_\Lambda)$ is, up to equivalence, the full subcategory of compact objects of $\mathbf{D}(\Lambda)$, i.e., $\mathbf{D}(\Lambda)^b \approx \mathcal{H}^b(\mathcal{P}_\Lambda)$. For the notion of pure semisimplicity in module categories, we refer to [39, 65]. Our first result is a consequence of Corollary 11.3.

PROPOSITION 12.1. *If $|\Lambda| \leq \aleph_t$, for some $t \geq 0$, then $\text{p.gl.dim } \mathbf{D}(\Lambda) \leq t + 1$.*

EXAMPLE 12.2. Let $\mathbf{D}(\mathcal{O}_{\mathbb{P}^n}(k))$ be the derived category of quasi-coherent sheaves on the projective n -space over the field k . By Beilinson's description [10] of the category $\mathbf{D}(\mathcal{O}_{\mathbb{P}^n}(k))$, it follows that if $|k| \leq \aleph_t$, then $\text{p.gl.dim } \mathbf{D}(\mathcal{O}_{\mathbb{P}^n}(k)) \leq t + 1$.

PROPOSITION 12.3. (1) *If $\text{p.gl.dim } \mathbf{D}(\Lambda) = 0$, then $\text{r.p.gl.dim } \Lambda = 0$; i.e., Λ is a right pure semisimple ring. Moreover $\text{r.gl.dim } \Lambda < \infty$.*

(2) *If Λ is derived equivalent to a right pure semisimple right hereditary ring, then $\mathbf{D}(\Lambda)$ is pure semisimple.*

(3) *Λ is a semisimple ring iff $\mathbf{D}(\Lambda)$ is (semisimple) abelian iff all triangles in $\mathbf{D}(\Lambda)$ are semi-split iff there exists a (uniquely determined) set of division rings $\{\Gamma_i; i \in I\}$ and an equivalence $\mathbf{D}(\Lambda) \approx \prod_{i \in I} \text{Mod}(\Gamma_i)$.*

Proof. (1) By Theorem 9.3, the module category $\text{Mod}(\mathcal{H}^b(\mathcal{P}_\Lambda))$ is Frobenius; in particular it is locally Noetherian and perfect. Consider the inclusion $i: \mathcal{P}_\Lambda \hookrightarrow \mathcal{H}^b(\mathcal{P}_\Lambda)$, and let $i^*: \text{Mod}(\mathcal{H}^b(\mathcal{P}_\Lambda)) \rightarrow \text{Mod}(\mathcal{P}_\Lambda) = \text{Mod}(\Lambda)$ be the restriction functor. It is well known [59] that i^* has a fully faithful right adjoint $i^!$ given by the Kan construction. Hence $\text{Mod}(\Lambda)$ is a localized quotient of $\text{Mod}(\mathcal{H}^b(\mathcal{P}_\Lambda))$ and since the latter is locally Noetherian, we have that $\text{Mod}(\Lambda)$ is also locally Noetherian; i.e., Λ is right Noetherian. Let $A \in \text{Mod}(\Lambda)$ and let $A^\bullet[0] \in \mathbf{D}(\Lambda)$ be the module A considered as a stalk complex concentrated in degree zero. Since $\mathbf{D}(\Lambda)$ is pure semisimple, we have $\mathbf{D}(\Lambda) = \text{Add}(\mathcal{H}^b(\mathcal{P}_\Lambda))$. Hence there exists a family $\{X_i^\bullet; i \in I\} \subseteq \mathcal{H}^b(\mathcal{P}_\Lambda)$ such that $A^\bullet[0] \oplus Y^\bullet = \bigoplus_{i \in I} X_i^\bullet$, for some complex Y^\bullet . Applying the cohomology functor H^0 , we have $A \oplus H^0(Y^\bullet) = \bigoplus_{i \in I} H^0(X_i^\bullet)$. Since Λ is right Noetherian, $H^0(X_i^\bullet)$ is a finitely presented module, $\forall i \in I$. Hence A is a direct summand of a coproduct of finitely presented modules, so A is pure projective. Hence any right module is pure projective and Λ is a right pure semisimple ring. It remains

to show that $\text{r.gl.dim } \Lambda < \infty$. If $A \in \text{Mod}(\Lambda)$ is an indecomposable module, then by Theorem 9.3, $A^\bullet[0] = \bigoplus_{i \in I} P_i^\bullet$ where each P_i^\bullet is compact with a local endomorphism ring. Since $A^\bullet[0]$ is indecomposable, we have that $A^\bullet[0]$ is isomorphic to some $P_i^\bullet \in \mathcal{H}^b(\mathcal{P}_\Lambda)$. Of course this implies that A has finite projective dimension. Since A was an arbitrary indecomposable, and since Λ is right Artinian (as a right pure semisimple ring), we have that Λ has finite (right) global dimension.

(2) Let Λ be a right pure semisimple right hereditary ring. Since $\text{r.gl.dim } \Lambda \leq 1$, for any complex $X^\bullet \in \mathbf{D}(\Lambda)$ we have $X^\bullet \cong \bigoplus_{i \in \mathbb{Z}} \Sigma^{-i}(H^i(X^\bullet))$; see [54]. Since Λ is right hereditary, $\forall i \in \mathbb{Z}$, the module $H^{-i}(\Sigma^i(X^\bullet))$ is isomorphic to an object in $\mathcal{H}^b(\text{Proj}(\Lambda))$. Since Λ is right pure semisimple, $H^{-i}(\Sigma^i(X^\bullet))$ is a direct summand of a coproduct of finitely presented modules, and since Λ is right coherent as a right hereditary ring, any finitely presented module is isomorphic to an object in $\mathcal{H}^b(\mathcal{P}_\Lambda)$. Putting these things together we have that $X^\bullet \in \text{Add}(\mathcal{H}^b(\mathcal{P}_\Lambda))$. This shows that $\mathbf{D}(\Lambda) = \text{Add}(\mathcal{H}^b(\mathcal{P}_\Lambda))$. Then by Theorem 9.3, $\mathbf{D}(\Lambda)$ is pure semisimple. This completes the proof since by Remark 11.5, the pure global dimension of a derived category is invariant under derived equivalence.

(3) Using Corollary 9.5, the easy proof is left to the reader. ■

COROLLARY 12.4. *Let Λ be a countable ring. If Λ is not right Artinian or right perfect, or $\text{r.gl.dim } \Lambda = \infty$, then $\text{p.gl.dim } \mathbf{D}(\Lambda) = 1$. In particular $\text{p.gl.dim } \mathbf{D}(\mathbb{Z}) = 1$.*

We recall that a ring Λ is called *representation-finite* if Λ is right Artinian and the set of isoclasses of indecomposable finitely presented right modules is finite. It is well known that the representation-finite commutative rings are exactly the Artinian principal ideal rings. Proposition 12.3 implies easily the following.

COROLLARY 12.5. *Let Λ be a commutative ring. Then $\text{p.gl.dim } \mathbf{D}(\Lambda) = 0$ iff Λ is a finite product of fields.*

EXAMPLE 12.6. Let \mathcal{H} be the category of commutative cocommutative connected graded Hopf algebras over a field k with $\text{char}(k) = p > 0$, generated by elements of degrees $2p^i, i \geq 0$. By [66], \mathcal{H} is a Grothendieck (functor) category which is hereditary and pure semisimple. As in Proposition 12.3, $\mathbf{D}(\mathcal{H})$ is pure semisimple.

Remark 12.7. If $\text{p.gl.dim } \mathbf{D}(\Lambda) = 0$, then it is not true that Λ is right hereditary.

Indeed let Γ be a representation-finite piecewise hereditary algebra over an algebraically closed field, with $\text{gl.dim } \Gamma \geq 2$ [33]. Then there exists a representation-finite hereditary algebra Λ and a derived equivalence

$\mathbf{D}(\Lambda) \approx \mathbf{D}(\Gamma)$. By Proposition 12.3, Γ is right pure semisimple but not hereditary.

The author is indebted to the referee for the nice proof of the following result.

PROPOSITION 12.8. *If Λ is a right hereditary ring, then*

$$\text{r.p.gl.dim } \Lambda = \text{p.gl.dim } \mathbf{D}(\Lambda).$$

Proof. By a result of Neeman [54], any complex A^\bullet in $\mathbf{D}(\Lambda)$ is isomorphic to the coproduct of its cohomology groups: $A^\bullet = \bigoplus_{n \in \mathbb{Z}} \Sigma^{-n}(H^n(A^\bullet))$. Hence it suffices to show that $\text{p.p.d } A = \text{p.p.d } A^\bullet[0]$, $\forall A \in \text{Mod}(\Lambda)$.

Let $Y: \text{Mod}(\Lambda) \rightarrow \text{Mod}(\mathcal{H}^b(\mathcal{P}_\Lambda))$ be the composition of the embedding $\text{Mod}(\Lambda) \hookrightarrow \mathbf{D}(\Lambda)$ given by $A \mapsto A^\bullet[0]$, with the projectivization functor $S: \mathbf{D}(\Lambda) \rightarrow \text{Mod}(\mathcal{H}^b(\mathcal{P}_\Lambda))$. The cohomology functor $H^0: \mathbf{D}(\Lambda) \rightarrow \text{Mod}(\Lambda)$ preserves coproducts and kills pure phantom maps. Hence it can be extended uniquely to an exact colimit preserving functor $\tilde{H}^0: \text{Mod}(\mathcal{H}^b(\mathcal{P}_\Lambda)) \rightarrow \text{Mod}(\Lambda)$, which is given by $F \mapsto F(\Lambda)$ and satisfies $\tilde{H}^0 Y = \text{Id}_{\text{Mod}(\Lambda)}$. It is not difficult to see that Y commutes with filtered colimits. Using the well-known fact that a short exact sequence in a module category is pure iff it is a filtered colimit of split short exact sequences, it follows that Y preserves pure exactness. Since Λ is right hereditary, hence right coherent, it follows trivially that Y takes pure projective modules to projective functors. Hence the image under Y of a pure projective resolution of A is a projective resolution of $Y(A)$ in $\text{Mod}(\mathcal{H}^b(\mathcal{P}_\Lambda))$. The latter can be lifted to a pure projective resolution of $A^\bullet[0]$ in $\mathbf{D}(\Lambda)$. Conversely \tilde{H}^0 preserves pure exactness, since it is exact and colimit preserving. Since Λ is right coherent as a right hereditary ring, it follows that the cohomology functor H^0 induces a functor $H^0: \mathcal{H}^b(\mathcal{P}_\Lambda) \rightarrow \text{mod}(\Lambda)$. Since \tilde{H}^0 is exact and preserves coproducts and $\tilde{H}^0 S = H^0$, it follows easily that \tilde{H}^0 preserves pure projectivity. Now any pure projective resolution of $A^\bullet[0]$ in $\mathbf{D}(\Lambda)$ induces via S a (pure) projective resolution of $Y(A)$ in $\text{Mod}(\mathcal{H}^b(\mathcal{P}_\Lambda))$, which in turn induces via \tilde{H}^0 , a pure projective resolution of A in $\text{Mod}(\Lambda)$.

The above observations about pure projective resolutions in $\text{Mod}(\Lambda)$ and in $\mathbf{D}(\Lambda)$ imply trivially that $\text{p.p.d } A = \text{p.p.d } A^\bullet[0]$. ■

PROPOSITION 12.9 (see also [56]). *Let Λ be a right coherent ring and assume that $\mathbf{D}(\Lambda)$ is a Brown category. Then for any flat right Λ -module F , we have $\text{p.d } F \leq 1$. If moreover Λ has finite weak global dimension, then $\text{r.p.gl.dim } \Lambda \leq 1$.*

Proof. Let A be a right Λ -module, and consider the stalk complex $A^\bullet[0]$. By BRT there exists a pure triangle $(\dagger): P_1^\bullet \rightarrow^f P_0^\bullet \rightarrow^s A^\bullet[0]$

$\rightarrow^{h^\bullet} \Sigma(P_1^\bullet)$, such that P_1^\bullet, P_0^\bullet are in $\text{Add}(\mathcal{H}^b(\mathcal{P}_\Lambda))$. Applying the cohomology functor H^0 , we deduce a short exact sequence $(\dagger\dagger): 0 \rightarrow K \rightarrow L \rightarrow A \rightarrow 0$, where K, L are direct summands of coproducts of cohomology modules of complexes in $\mathcal{H}^b(\mathcal{P}_\Lambda)$. Since Λ is right coherent, these cohomologies are sums of finitely presented modules. This implies that K, L are pure projective modules. If A is flat, then $(\dagger\dagger)$ is pure; hence $\text{p.p.d } A \leq 1$. But then $\text{p.d } A \leq 1$, since $\text{p.d } A = \text{p.p.d } A$ for A flat.

If $\text{w.gl.dim } \Lambda < \infty$, then it suffices to prove that $(\dagger\dagger)$ is pure. Let M be a finitely presented right Λ -module and $\alpha: M \rightarrow A$ a morphism. Since Λ is right coherent, $\text{w.gl.dim } \Lambda < \infty$, and M is finitely presented, it follows that $M^\bullet[0]$ is compact. Since h^\bullet is pure phantom, the composition $M^\bullet[0] \rightarrow^{\alpha^\bullet[0]} A^\bullet[0] \rightarrow^{h^\bullet} \Sigma(P_1^\bullet)$ is zero. So there exists a map $\beta^\bullet: M^\bullet[0] \rightarrow P_0^\bullet$ such that $\beta^\bullet \circ g^\bullet = \alpha^\bullet[0]$. Applying H^0 , we have $H^0(\beta^\bullet) \circ H^0(g^\bullet) = \alpha$. Hence α factors through L , so $(\dagger\dagger)$ is pure. ■

Remark 12.10. It is not difficult to see that if Λ is a right coherent ring with $\text{w.gl.dim } \Lambda < \infty$, then $\text{r.p.gl.dim } \Lambda \leq \text{p.gl.dim } \mathbf{D}(\Lambda)$. We refer to the work of Christensen et al. [22] for details and a thorough discussion of related topics. In particular in [22] it is proved that the above inequality can be strict.

Since the pure global dimension coincides with the global dimension for a von Neumann regular ring (obviously coherent) [39], we have the following consequence.

COROLLARY 12.11. *Let Λ be a von Neumann regular ring. Then $\text{p.gl.dim } \mathbf{D}(\Lambda) \leq 1$ iff Λ is right hereditary. Moreover $\text{p.gl.dim } \mathbf{D}(\Lambda) = 0$ iff Λ is semisimple Artinian.*

By Theorem 3.2 of [7], we have the following.

COROLLARY 12.12. *Let $\Lambda = k\langle X \rangle$ be the free algebra over a set of variables X , with $|X| \leq \aleph_0$. If $|\Lambda| = \max\{\aleph_0, |k|\} = \aleph_t$, then $\text{p.gl.dim } \mathbf{D}(\Lambda) \leq 1 \Leftrightarrow t = 0$. In case $X = \{x\}$, then for the polynomial ring $k[x]$, $\text{p.gl.dim } \mathbf{D}(k[x]) = 1$.*

Other examples of rings Λ with a Brown derived category are Dedekind domains and free Boolean rings on countably many generators; see [39].

EXAMPLE 12.13. There exists a ring Λ such that $\text{p.gl.dim } \mathbf{D}(\Lambda) \geq 2$ or equivalently $\mathbf{D}(\Lambda)$ is not a Brown category. Indeed take Λ to be a von Neumann regular ring with $\text{r.gl.dim } \Lambda \geq 2$ (for instance $\prod_1^\infty \mathbb{C}$ or a von Neumann regular ring with $|\Lambda| = \aleph_n, n \geq 2$; see [39]). Then by Corollary 12.11, $\mathbf{D}(\Lambda)$ is not Brown.

That BRT does not hold in general was observed first by Keller (see [56]) and is a consequence of the following results of Osofsky [58]. Let Λ

be a ring with cardinality \aleph_n and $\text{r.gl.dim } \Lambda = \kappa$. Suppose that Λ is a regular local ring with $|\Lambda| = |\Lambda/\mathcal{I}ac(\Lambda)|$ or a complete regular local ring or $\Lambda = k[\mathbf{x}_1, \dots, \mathbf{x}_\kappa]$ and k is a field. Then if Q is the quotient field, we have $\text{p.d.}_\Lambda Q = \min(\kappa, n + 1)$. In particular assuming the continuum hypothesis $2^{\aleph_0} = \aleph_1$, we have $\text{p.d.}_{\mathbb{R}[\mathbf{x}, \mathbf{y}, \mathbf{z}]} \mathbb{R}(\mathbf{x}, \mathbf{y}, \mathbf{z}) = 2$. Hence there are coherent rings Λ having flat modules with arbitrary large finite projective dimension. For example let k be a field and let t be determined by the equation $\aleph_t = \max(\aleph_0, |k|)$. Then the flat module $k(\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n)$ over the coherent ring $k[\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n]$ has projective dimension $t + 1$. By Proposition 12.9 for these rings, $\mathbf{D}(\Lambda)$ is not Brown, if $t \geq 1$. By Remark 12.10, it follows that for any $n \in \mathbb{N} \cup \{\infty\}$, there exists a (von Neumann regular) ring Λ with $\text{p.gl.dim } \mathbf{D}(\Lambda) = n$.

The next example shows that the left and right pure global dimensions of the derived category of a ring can be different, if they are positive. In particular it shows that the Brown property is not left–right hand symmetric.

EXAMPLE 12.14. There exists a ring Λ with the following properties:

- (i) $\text{p.gl.dim } \mathbf{D}(\Lambda) \neq \text{p.gl.dim } \mathbf{D}(\Lambda^{\text{op}})$.
- (ii) $\mathbf{D}(\Lambda)$ is a Brown category, but $\mathbf{D}(\Lambda^{\text{op}})$ is not.

Indeed Kaplansky in [41] constructed a von Neumann regular ring Λ of cardinality $|\Lambda| = 2^{\aleph_0}$ with $\text{r.gl.dim } \Lambda = 1$ and $\text{l.gl.dim } \Lambda \geq 2$. By Corollary 12.11, $\text{p.gl.dim } \mathbf{D}(\Lambda) = 1$ and by Remark 12.10, $\text{p.gl.dim } \mathbf{D}(\Lambda^{\text{op}}) \geq 2$. Hence by Theorem 11.8, $\mathbf{D}(\Lambda)$ is Brown and $\mathbf{D}(\Lambda^{\text{op}})$ is not. Assuming continuum hypothesis $2^{\aleph_0} = \aleph_1$, it follows from Proposition 12.1 that $\text{p.gl.dim } \mathbf{D}(\Lambda^{\text{op}}) = 2$.

12.2. The Derived Pure Semisimple Conjecture, Categories of Finite Type, and Roos's Problem

The next result shows that finite type is a symmetric condition for derived categories.

PROPOSITION 12.15. *The following are equivalent:*

- (i) $\mathbf{D}(\Lambda)$ is of finite type.
- (ii) $\mathbf{D}(\Lambda^{\text{op}})$ is of finite type.
- (iii) $\mathbf{D}(\Lambda)$ and $\mathbf{D}(\Lambda^{\text{op}})$ are pure-semisimple.
- (iv) $\text{KGdim } \mathbf{D}(\Lambda) = 0$.

Proof. The natural duality $\mathcal{P}_\Lambda^{\text{op}} \approx \mathcal{P}_{\Lambda^{\text{op}}}$ extends to a duality $\mathcal{H}^b(\mathcal{P}_\Lambda)^{\text{op}} \approx \mathcal{H}^b(\mathcal{P}_{\Lambda^{\text{op}}})$, so $(\mathbf{D}(\Lambda)^b)^{\text{op}} \approx \mathbf{D}(\Lambda^{\text{op}})^b$. Using the notation of Section 11, we have $\text{p.gl.dim } \mathbf{D}(\Lambda)^{\text{op}} = \text{p.gl.dim } \mathbf{D}(\Lambda^{\text{op}})$ and the assertion follows from Proposition 11.23. ■

COROLLARY 12.16. (1) *If $\mathbf{D}(\Lambda)$ is of finite type, then Λ is representation-finite of finite global dimension.*

(2) *If Λ is (derived equivalent to) a representation-finite right hereditary ring, then $\mathbf{D}(\Lambda)$ is of finite type.*

(3) *Let Λ be a PI-ring or a ring with a Morita duality, for instance, an Artin algebra or a quasi-Frobenius ring. If $\mathbf{D}(\Lambda)$ is pure semisimple, then Λ is of finite representation type and of finite global dimension.*

(4) *If Λ is a representation-infinite Artin algebra of finite global dimension, then $\text{KGdim } \mathbf{D}(\Lambda) \neq 1$. If Λ is derived equivalent to a hereditary algebra, then $\text{KGdim } \mathbf{D}(\Lambda) \neq 1$, independently of the representation type.*

Proof. By a well-known result of Auslander [39], a ring is representation-finite iff it is left and right pure semisimple. Then (1) and (2) follow from Propositions 12.3 and 12.15. (3) follows from results of Simson, Herzog [35] claiming that right pure semisimple rings with polynomial identity or with Morita duality are representation-finite.

(4) Since Λ has finite global dimension, the canonical full embedding $\text{mod}(\Lambda) \hookrightarrow \mathbf{D}(\Lambda)$ has its image in $\mathcal{H}^b(\mathcal{P}_\Lambda)$. As in the proof of Proposition 12.3, it follows that $\text{Mod}(\text{mod}(\Lambda))$ is a localized quotient of $\text{Mod}(\mathcal{H}^b(\mathcal{P}_\Lambda))$. It follows from results of Gabriel [30] that $\text{Kdim } \text{Mod}(\text{mod}(\Lambda)) \leq \text{Kdim } \text{Mod}(\mathcal{H}^b(\mathcal{P}_\Lambda)) \leq \text{GKdim } \mathbf{D}(\Lambda)$. Assume that $\text{GKdim } \mathbf{D}(\Lambda) = 1$, so $\text{Kdim } \text{Mod}(\text{mod}(\Lambda)) \leq 1$. Since $\text{Kdim } \text{Mod}(\text{mod}(\Lambda)) \neq 1$ for any Artin algebra (see [36, 47]), it follows that $\text{Kdim } \text{Mod}(\text{mod}(\Lambda)) = 0$. Then by a result of Auslander [2], it follows that Λ is representation-finite and this is not the case. Hence $\text{GKdim } \mathbf{D}(\Lambda) \neq 1$. If Λ is derived equivalent to a hereditary algebra Γ and $\text{GKdim } \mathbf{D}(\Lambda) = 1$, then $\text{GKdim } \mathbf{D}(\Gamma) = 1$ and as above we deduce that Γ is representation finite. Hence by (2), $\mathbf{D}(\Lambda)$ is of finite type. Then by Proposition 12.15, $\text{GKdim } \mathbf{D}(\Lambda) = 0$ and this is not true. Hence $\text{GKdim } \mathbf{D}(\Lambda) \neq 1$. ■

Part (4) of the above corollary is a partial analogue of a module-theoretic result of Krause [47] and Herzog [36]. We do not know if for any Artin algebra Λ it holds that $\text{GKdim } \mathbf{D}(\Lambda) \neq 1$.

We recall that the still open *pure semisimple conjecture* asserts that a right pure semisimple ring Λ is left pure semisimple; equivalently Λ is representation-finite. We formulate an analogue of this conjecture in the derived category.

Derived Pure Semisimple Conjecture 12.17. Pure semisimplicity in $\mathbf{D}(\Lambda)$ is left–right hand symmetric: $\text{p.gl.dim } \mathbf{D}(\Lambda) = 0 \Leftrightarrow \text{p.gl.dim } \mathbf{D}(\Lambda^{\text{op}}) = 0$. Equivalently if $\mathbf{D}(\Lambda)$ is pure semisimple then $\mathbf{D}(\Lambda)$ is of finite type.

Remark 12.18. The truth of the derived pure semisimple conjecture (DPSC) implies the truth of the pure semisimple conjecture (PSC). Indeed

if DPSC is true and PSC is not, then by [35], there exists a right hereditary right pure semisimple ring which is not left pure semisimple. By Proposition 12.3, $\mathbf{D}(\Lambda)$ is pure semisimple; hence also $\mathbf{D}(\Lambda^{\text{op}})$ is pure semisimple. By Proposition 12.3 again, Λ is left pure semisimple, which is not true. Hence PSC is true. Note that by Proposition 12.3, for (left and right) hereditary rings, DPSC is equivalent to PSC.

EXAMPLE 12.19. The following example provides a *negative answer* to Roos's Problem 11.22, to Question 11.21, and to the derived pure semisimple conjecture for "rings with several objects" in the sense of Mitchell [53].

Consider \mathfrak{K}_t , $t \geq 0$ as a totally ordered set, let k be a field of cardinality \aleph_0 , and let $[\mathfrak{K}_t, \text{Mod}(k)]$ be the category of k -linear representations of \mathfrak{K}_t . Let $k(\mathfrak{K}_t)$ be the ringoid which yields an equivalence $[\mathfrak{K}_t, \text{Mod}(k)] \approx \text{Mod}(k(\mathfrak{K}_t)^{\text{op}})$. By a result of Brune [17], $\text{Mod}(k(\mathfrak{K}_t)^{\text{op}})$ is hereditary and pure semisimple and $\text{p.g.l.dim Mod}(k(\mathfrak{K}_t)) = t + 1$ if $t < \aleph_0$ and ∞ if $t \geq \aleph_0$. As in Proposition 12.3, it follows that the (compactly generated) derived category $\mathbf{D}(\text{Mod}(k(\mathfrak{K}_t)^{\text{op}}))$ is pure semisimple but the derived category $\mathbf{D}(\text{Mod}(k(\mathfrak{K}_t)))$ is not. Using Proposition 12.15 we infer that $\mathbf{D}(\text{Mod}(k(\mathfrak{K}_t)^{\text{op}}))$ is pure semisimple but not of finite type.

On the other hand since $\mathbf{D}(\text{Mod}(k(\mathfrak{K}_t)^{\text{op}}))$ is pure semisimple, we have that the Grothendieck category $\text{Mod}(\mathbf{D}(\text{Mod}(k(\mathfrak{K}_t)^{\text{op}}))^b)$ is Frobenius but not locally finite since otherwise, $\mathbf{D}(\text{Mod}(k(\mathfrak{K}_t)^{\text{op}}))$ would be of finite type. Note that by the Faith–Walker theorem [27], the answer to Roos's problem is affirmative for rings.

For the representation-theoretic notions and arguments used in the following result (which shows that for Artin algebras, DPSC is true), we refer to [33].

THEOREM 12.20. *For an Artin algebra Λ the following are equivalent:*

- (i) $\mathbf{D}(\Lambda)$ is of finite type.
- (ii) $\mathbf{D}(\Lambda)$ is pure semisimple.
- (iii) Λ is an iterated tilted algebra of Dynkin type.
- (iv) Λ is derived equivalent to a representation finite hereditary algebra.

In this case, the set of isoclasses of indecomposable compact objects of $\mathbf{D}(\Lambda)$ up to shift is finite.

Proof. By Proposition 12.15, (i) \Rightarrow (ii) and by [33], (iii) \Rightarrow (iv). By Corollary 12.16, (iv) \Rightarrow (i). So it remains to prove that (ii) implies (iii).

(ii) \Rightarrow (iii) By Corollary 12.16, Λ is a representation finite algebra of finite global dimension. Let $\hat{\Lambda}$ be the repetitive algebra of Λ . By [33] we have a triangle equivalence $\mathcal{H}^b(\mathcal{P}_{\hat{\Lambda}}) = \mathbf{D}^b(\text{mod}(\Lambda)) \cong \underline{\text{mod}}(\hat{\Lambda})$. We shall show that $\hat{\Lambda}$ is locally representation-finite. Let $\mathcal{P}_{\hat{\Lambda}}$ be the category of

projective objects in $\text{mod}(\hat{\Lambda})$. Since $\text{mod}(\hat{\Lambda})$ is a Frobenius category, by Proposition 9.2, $\mathcal{P}_{\hat{\Lambda}}$ is a weak abelian category. Since $\text{mod}(\hat{\Lambda})$ is a length category (any object has finite length), the module category $\text{Mod}(\mathcal{P}_{\hat{\Lambda}})$ is locally Noetherian; hence it is perfect according to Proposition 9.2. Since $\mathcal{H}^b(\mathcal{P}_{\hat{\Lambda}}) \cong \text{mod}(\hat{\Lambda})$, we have that $\text{Mod}(\overline{\text{mod}(\hat{\Lambda})})$ is also perfect. By a result of Simson [66], since $\text{Mod}(\overline{\text{mod}(\hat{\Lambda})})$ and $\overline{\text{Mod}(\mathcal{P}_{\hat{\Lambda}})}$ are perfect, the module category $\text{Mod}(\text{mod}(\hat{\Lambda}))$ is also perfect or equivalently locally Noetherian. Let $\Gamma \subseteq \hat{\Lambda}$ be a finite full convex subcategory. Then we have an inclusion $\text{mod}(\Gamma) \subseteq \text{mod}(\hat{\Lambda})$. This inclusion implies that $\text{Mod}(\text{mod}(\Gamma))$ is a localizing subcategory of $\text{Mod}(\text{mod}(\hat{\Lambda}))$. Since the latter is a locally Noetherian category, so is $\text{Mod}(\text{mod}(\Gamma))$. But since Γ is an Artin algebra, by a well-known result of Auslander [39], we have that Γ is representation-finite. Hence $\hat{\Lambda}$ is locally representation-finite. By [33], Λ is an iterated tilted algebra of Dynkin type. The last claim is trivial and is left to the reader. ■

The next corollaries follow from [7, 8] and the above results.

COROLLARY 12.21. *Let Λ be a finite dimensional hereditary k -algebra over an algebraically closed field k of cardinality \aleph_t . Then we have the following:*

- (i) *If $t = 0$, then $\text{p.gl.dim } \mathbf{D}(\Lambda) = 0$ or 1 . (0 occurs iff Λ is representation-finite.)*
- (ii) *If $t > 0$, then:*
 - (α) *If Λ is representation-finite, then $\text{p.gl.dim } \mathbf{D}(\Lambda) = 0$.*
 - (β) *If Λ is tame, then $\text{p.gl.dim } \mathbf{D}(\Lambda) = 2$.*
 - (γ) *If Λ is wild, then $\text{p.gl.dim } \mathbf{D}(\Lambda) = t + 1$.*

COROLLARY 12.22. *Let \mathcal{Q} be a quiver with or without oriented cycles and let $\Lambda = k[\mathcal{Q}]$ be its quiver-algebra over an algebraically closed field k of cardinality \aleph_t . Then:*

- (i) *If $t = 0$, then $\text{p.gl.dim } \mathbf{D}(\Lambda) = 0$ or 1 , the value 0 occurs iff \mathcal{Q} is Dynkin.*
- (ii) *If $t > 0$, then:*
 - (α) *If \mathcal{Q} is a Dynkin diagram, then $\text{p.gl.dim } \mathbf{D}(\Lambda) = 0$.*
 - (β) *If \mathcal{Q} is an oriented cycle, then $\text{p.gl.dim } \mathbf{D}(\Lambda) = 1$.*
 - (γ) *If \mathcal{Q} is extended Dynkin not an oriented cycle, then $\text{p.gl.dim } \mathbf{D}(\Lambda) = 2$.*
 - (δ) *In all the remaining cases, \mathcal{Q} is a wild quiver and $\text{p.gl.dim } \mathbf{D}(\Lambda) = t + 1$.*

COROLLARY 12.23. *Let Λ be a finite dimensional local or radical squared zero k -algebra over an algebraically closed field of uncountable cardinality \aleph_t . If $\mathbf{D}(\Lambda)$ is Brown, then Λ is representation-finite.*

12.3. Stable Module Categories

Let \mathcal{E} be a skeletally small additive category such that the module category $\text{Mod}(\mathcal{E})$ is Frobenius. It is not difficult to see that the stable category $\underline{\text{Mod}}(\mathcal{E})$ is compactly generated and $\underline{\text{Mod}}(\mathcal{E})^b = \underline{\text{mod}}(\mathcal{E})$. If $\text{Mod}(\mathcal{E}^{\text{op}})$ is also Frobenius, then the Auslander–Bridger transpose duality functor $\text{Tr}: \underline{\text{mod}}(\mathcal{E})^{\text{op}} \rightarrow \underline{\text{mod}}(\mathcal{E}^{\text{op}})$, see [3], shows that $(\underline{\text{Mod}}(\mathcal{E})^b)^{\text{op}} \approx \underline{\text{Mod}}(\mathcal{E}^{\text{op}})^b$.

PROPOSITION 12.24. *$\text{p.gl.dim Mod}(\mathcal{E}) = \text{p.gl.dim } \underline{\text{Mod}}(\mathcal{E})$. In particular if $w(\mathcal{E}) \leq \aleph_t$, for some $t \geq 0$, then $\text{p.gl.dim } \underline{\text{Mod}}(\mathcal{E}) \leq t + 1$.*

Proof. Since any projective right \mathcal{E} -module is pure projective, it follows trivially that $\underline{M} \in \underline{\text{Mod}}(\mathcal{E})$ is pure projective iff M is pure projective. Similarly it is not difficult to see that a short exact sequence $0 \rightarrow K \rightarrow L \rightarrow M \rightarrow 0$ in $\text{Mod}(\mathcal{E})$ is pure iff the induced triangle $\underline{K} \rightarrow \underline{L} \rightarrow \underline{M} \rightarrow \Sigma(\underline{K})$ is pure in $\underline{\text{Mod}}(\mathcal{E})$; further any pure triangle in $\underline{\text{Mod}}(\mathcal{E})$ arises in this way. It follows that a pure projective resolution of \underline{M} induces a pure projective resolution of M and any pure projective resolution of M in the stable category arises in this way. This implies that $\text{p.p.d } M = \text{p.p.d } \underline{M}$, $\forall M \in \text{Mod}(\mathcal{E})$. Hence we have $\text{p.gl.dim Mod}(\mathcal{E}) = \text{p.gl.dim } \underline{\text{Mod}}(\mathcal{E})$. ■

THEOREM 12.25. *If $\text{Mod}(\mathcal{E})$ is Frobenius then the following are equivalent:*

- (i) $\underline{\text{Mod}}(\mathcal{E})$ is pure semisimple (of finite type).
- (ii) $\text{Mod}(\mathcal{E})$ is pure semisimple (and locally Artinian).
- (iii) $\text{Mod}(\underline{\text{mod}}(\mathcal{E}))$ is locally Noetherian or perfect (and locally Artinian).

If $\underline{\text{Mod}}(\mathcal{E})$ is of finite type, then $\text{Mod}(\mathcal{E}^{\text{op}})$ is Frobenius and $\underline{\text{Mod}}(\mathcal{E}^{\text{op}})$ is of finite type. Finally if one of the above equivalent conditions is true, then:

$$\begin{aligned} \underline{\text{Mod}}(\mathcal{E}) &\cong \text{Proj}(\text{Mod}(\underline{\text{mod}}(\mathcal{E}))) = \text{Flat}(\text{Mod}(\underline{\text{mod}}(\mathcal{E}))) \\ &= \text{Inj}(\text{Mod}(\underline{\text{mod}}(\mathcal{E}))). \end{aligned}$$

Proof. By Theorem 9.3, (i) \Leftrightarrow (iii) and by Proposition 12.24, (i) \Leftrightarrow (ii). The easy proof of the remaining assertions and the parenthetical case is left to the reader. ■

If Λ is a QF-ring, then $\text{Mod}(\Lambda)$ is Frobenius. Hence $\underline{\text{Mod}}(\Lambda)$ is a compactly generated triangulated category and $\underline{\text{Mod}}(\Lambda)^b = \underline{\text{mod}}(\Lambda)$. The following consequence of Theorem 12.25 generalizes a result of [13, 48].

COROLLARY 12.26. *Let Λ be a QF-ring. Then the following are equivalent:*

- (i) $\underline{\text{Mod}}(\Lambda)$, equivalently $\underline{\text{Mod}}(\Lambda^{\text{op}})$, is pure semisimple (of finite type).
- (ii) The set of isoclasses of indecomposable compact objects of $\underline{\text{Mod}}(\Lambda)$ is finite.
- (iii) Λ is representation finite.

If Λ is an Artin algebra, we denote by $T(\Lambda) = \Lambda \rtimes D(\Lambda)$ the trivial extension of Λ by the minimal injective cogenerator [33]. The following consequence of Theorems 12.20 and 12.25 shows that the finite type property coincides for the three triangulated categories $\mathbf{D}(\Lambda)$, $\underline{\text{Mod}}(T(\Lambda))$, $\underline{\text{Mod}}(\hat{\Lambda})$, associated to Λ .

COROLLARY 12.27. *For an Artin algebra Λ the following are equivalent.*

- (i) The derived category $\mathbf{D}(\Lambda)$ of Λ is of finite type.
- (ii) The stable module category $\underline{\text{Mod}}(T(\Lambda))$ is of finite type.
- (iii) The stable module category $\underline{\text{Mod}}(\hat{\Lambda})$ of its repetitive algebra $\hat{\Lambda}$ is of finite type.
- (iv) Λ is an iterated tilted algebra of Dynkin type.

COROLLARY 12.28. *Let Λ be a finite dimensional self-injective local k -algebra, over an algebraically closed field k . Then the following statements are equivalent:*

- (i) $\underline{\text{Mod}}(\Lambda)$ is a Brown category, i.e., $\text{p.gl.dim } \underline{\text{Mod}}(\Lambda) \leq 1$.
- (ii) Either Λ is representation finite or else k is a countable field.

Proof. That (ii) implies (i) follows from Proposition 12.24. Suppose that $\underline{\text{Mod}}(\Lambda)$ is Brown. If $\text{p.gl.dim } \underline{\text{Mod}}(\Lambda) = 0$, then Λ is representation-finite. If $\text{p.gl.dim } \underline{\text{Mod}}(\Lambda) = 1$, then by Proposition 12.24, $\text{r.p.gl.dim } \Lambda = 1$. Then by [8], k is countable. ■

Using induction on the p -Sylow subgroups, we have the following consequence first observed by Benson and Gnacadja.

COROLLARY 12.29 [15]. *Let $\Lambda = kG$ be the group algebra of a finite group G over an algebraically closed field k with $\text{char}(k) = p$. Then the following are equivalent:*

- (i) $\underline{\text{Mod}}(\Lambda)$ is a Brown category, i.e., $\text{p.gl.dim } \underline{\text{Mod}}(\Lambda) \leq 1$.
- (ii) Either G has cyclic p -Sylow subgroups or else k is a countable field.

If Λ and Γ are derived equivalent rings, then $\text{p.gl.dim } \mathbf{D}(\Lambda) = \text{p.gl.dim } \mathbf{D}(\Gamma)$. If in addition Λ, Γ are self-injective Artin algebras, then by [63], there exists a triangle equivalence $\underline{\text{Mod}}(\Lambda) \approx \underline{\text{Mod}}(\Gamma)$. Hence $\text{p.gl.dim } \underline{\text{Mod}}(\Lambda) = \text{p.gl.dim } \underline{\text{Mod}}(\Gamma)$. Finally if Λ is an Artin algebra of finite global dimension, then by Happel's theorem [33], there exists a triangle equivalence $\mathbf{D}(\Lambda) \approx \underline{\text{Mod}}(\hat{\Lambda})$, where $\hat{\Lambda}$ is the repetitive algebra. Since $\underline{\text{Mod}}(\hat{\Lambda})$ is Frobenius, in case $\text{gl.dim } \Lambda < \infty$, we have $\text{p.gl.dim } \mathbf{D}(\Lambda) = \text{p.gl.dim } \underline{\text{Mod}}(\hat{\Lambda})$. It is interesting to have characterizations in representation-theoretic terms of Artin algebras, resp. self-injective Artin algebras, with Brown derived categories, resp. stable module categories.

If the derived (resp. stable module) category of a ring (resp. QF-ring) is a Brown category, then the results of Sections 10 and 11 are true for the triangulated category $\mathbf{D}(\Lambda)$ (resp. $\underline{\text{Mod}}(\Lambda)$). We leave the formulations of these results to the reader.

Remark 12.30. The results of this section indicate that finite type is the appropriate notion of “representation-finiteness” for derived and stable categories. It is unclear what are the appropriate notions for tameness and wildness in these categories in analogy with the tame–wild dichotomy of finite dimensional algebras.

12.4. Localization and the Derived Category

Let Λ be an associative ring. We denote by $\mathcal{H}(\Lambda)$ be the unbounded homotopy category of all right Λ -modules and by $\mathcal{H}(\text{Proj}(\Lambda))$ the homotopy category of all projective right Λ -modules.

DEFINITION 12.31. A complex P^\bullet is called a *Cartan–Eilenberg projective complex* (CE-projective complex, for short), if P^\bullet is homotopy equivalent to a complex having projective components and zero differential.

The full subcategory of $\mathcal{H}(\Lambda)$ of all CE-projective complexes is denoted by \mathcal{P}_{CE} . Setting $\mathcal{E} := \mathcal{E}(\mathcal{P}_{\text{CE}})$, we obtain a proper class of triangles in $\mathcal{H}(\Lambda)$, and $\mathcal{H}(\Lambda)$ has enough projectives and $\mathcal{P}(\mathcal{E}) = \mathcal{P}_{\text{CE}}$ (see for instance [23, 26]). Choosing $\mathcal{E} = \mathcal{H}(\Lambda)$ and $\mathcal{S} = \{\Sigma^n(\Lambda) \mid n \in \mathbb{Z}\}$ in Theorem 6.5 and keeping the terminology and notation of Section 6, we have the following result proved in [16, 42, 67, 71], by using essentially the natural closed model structure of the category of complexes.

THEOREM 12.32. (1) $\widehat{\mathcal{P}(\mathcal{E})}^{*\infty}$ is the full subcategory $\mathcal{H}_{\mathcal{P}}(\Lambda)$ of $\mathcal{H}(\Lambda)$ consisting of all complexes A^\bullet which are homotopy equivalent to complexes P^\bullet with the following property. There exists a tower of complexes

$$0 \subseteq P_0^\bullet \subseteq P_1^\bullet \subseteq \cdots \subseteq P_n^\bullet \subseteq P_{n+1}^\bullet \subseteq \cdots \subseteq P^\bullet$$

such that:

(†) $P^\bullet = \bigcup_{n \geq 0} P_n^\bullet$, each inclusion $P_n^\bullet \subseteq P_{n+1}^\bullet$ splits in each component, and each quotient complex $P_{n+1}^\bullet/P_n^\bullet$ is a CE-projective complex.

(2) $\mathcal{D}_{\mathcal{E}}(\mathcal{H}(\Lambda)) = \mathbf{D}(\Lambda)$, the quotient functor $\mathcal{H}(\Lambda) \rightarrow \mathbf{D}(\Lambda)$ admits a fully faithful left adjoint, and $\mathcal{A}_{\mathcal{E}}(\mathcal{H}(\Lambda))$ is the subcategory $\mathcal{H}_{\mathcal{A}c}(\Lambda)$ of acyclic complexes.

(3) The canonical functor $\mathcal{H}_{\mathcal{P}}(\Lambda) \rightarrow \mathbf{D}(\Lambda)$ is a triangle equivalence.

(4) The inclusion $\mathcal{H}_{\mathcal{A}c}(\Lambda) \hookrightarrow \mathcal{H}(\Lambda)$ has a left adjoint \mathbf{a} , the inclusion $\mathcal{H}_{\mathcal{P}}(\Lambda) \hookrightarrow \mathcal{H}(\Lambda)$ has a right adjoint \mathbf{p} , and any complex A^\bullet is contained in a triangle

$$\mathbf{p}(A^\bullet) \rightarrow A^\bullet \rightarrow \mathbf{a}(A^\bullet) \rightarrow \Sigma(\mathbf{p}(A^\bullet)),$$

where

$$\mathbf{p}(A^\bullet) \in \mathcal{H}_{\mathcal{P}}(\Lambda) \quad \text{and} \quad \mathbf{a}(A^\bullet) \in \mathcal{H}_{\mathcal{A}c}(\Lambda).$$

(5) Any additive functor $F: \text{Mod}(\Lambda) \rightarrow \text{Mod}(\Gamma)$ has a total left derived functor

$$\underline{\underline{L}}F: \mathbf{D}(\Lambda) \rightarrow \mathbf{D}(\Gamma).$$

It is clear from Section 6 that the above theorem holds in much more general situations, for instance, in the stable homotopy category, and in the derived category of a Grothendieck category with projectives, where as noted in [16, 42, 67, 71], the theory implies the existence of total left derived functors defined on unbounded complexes. The dual result concerning injectives and total right derived functors also holds, and can be applied for instance to the derived category of any Grothendieck category, e.g., to categories of sheaves, since the theory of the previous sections can be dualized (compactness can be avoided using the cohomology functor as in Remark 6.6). The main theorem of Section 6 also covers the construction of the various derived categories associated with projective classes in [20].

12.5. The Stable Derived Category

For simplicity we assume that Λ is an Artin algebra. We denote by $\mathbf{D}^b(\Lambda)$ the bounded derived category $\mathbf{D}^b(\text{mod}(\Lambda))$. It is well known that $\mathbf{D}^b(\Lambda) = \mathcal{H}^{-,b}(\mathcal{P}_\Lambda) = \mathcal{H}^{+,b}(\mathcal{I}_\Lambda)$, where \mathcal{I}_Λ is the full subcategory of $\text{mod}(\Lambda)$ consisting of injective modules and $\mathcal{H}^{-,b}(\mathcal{P}_\Lambda)$ ($\mathcal{H}^{+,b}(\mathcal{I}_\Lambda)$) is the full subcategory of $\mathcal{H}^-(\mathcal{P}_\Lambda)$ ($\mathcal{H}^+(\mathcal{I}_\Lambda)$), consisting of complexes with bounded cohomology. As in Section 12.4, the Cartan–Eilenberg projectives \mathcal{P}_{CE} in $\mathbf{D}^b(\Lambda)$ form the full subcategory of \mathcal{E} -projectives of a proper class

of triangles \mathcal{E} , and moreover using the above identifications, $\mathbf{D}^b(\Lambda)$ has enough \mathcal{E} -injectives, namely the full subcategory \mathcal{F}_{CE} of the Cartan–Eilenberg injective complexes in $\mathbf{D}^b(\Lambda)$.

The projectively stable (bounded) derived category $\underline{\mathbf{D}}^b(\Lambda)$ of Λ is defined to be the stable category $\mathbf{D}^b(\Lambda)/\mathcal{P}_{\text{CE}}$, and similarly the injectively stable (bounded) derived category $\overline{\mathbf{D}}^b(\Lambda)$ of Λ is defined to be the stable category $\mathbf{D}^b(\Lambda)/\mathcal{F}_{\text{CE}}$. By Section 7, we have that $\underline{\mathbf{D}}^b(\Lambda)$ is left triangulated and $\overline{\mathbf{D}}^b(\Lambda)$ is right triangulated.

PROPOSITION 12.33. *Λ is self-injective iff $\underline{\mathbf{D}}^b(\Lambda)$ is triangulated iff $\overline{\mathbf{D}}^b(\Lambda)$ is triangulated. If Λ is self-injective, then $\underline{\mathbf{D}}^b(\Lambda) = \overline{\mathbf{D}}^b(\Lambda)$.*

Proof. Let $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ be an exact sequence in $\text{mod}(\Lambda)$. It induces a triangle $A[0] \rightarrow B[0] \rightarrow C[0] \rightarrow A[1] \in \mathcal{E}$ in $\mathbf{D}^b(\Lambda)$. If $\underline{\mathbf{D}}^b(\Lambda)$ is triangulated, then $\Lambda[0] \in \mathcal{P}_{\text{CE}} \subseteq \mathcal{F}_{\text{CE}}$. Hence the sequence $0 \rightarrow (C[0], \Lambda[0]) \rightarrow (B[0], \Lambda[0]) \rightarrow (A[0], \Lambda[0]) \rightarrow 0$, equivalently the sequence $0 \rightarrow (C, \Lambda) \rightarrow (B, \Lambda) \rightarrow (A, \Lambda) \rightarrow 0$, is exact and Λ is self-injective. Conversely if Λ is self-injective, then $\mathcal{P}_{\Lambda} = \mathcal{F}_{\Lambda}$. Then obviously $\mathcal{P}_{\text{CE}} = \mathcal{F}_{\text{CE}}$ and by Theorem 7.2, $\underline{\mathbf{D}}^b(\Lambda)$ is triangulated. ■

Our aim is to show that the stable derived category $\underline{\mathbf{D}}^b(\Lambda)$ is connected with the stable module category $\underline{\text{mod}}(\Lambda)$, which is also left triangulated, in the same way as $\mathbf{D}^b(\Lambda)$ is connected with $\text{mod}(\Lambda)$. Consider the projective resolution functor $\varrho: \text{mod}(\Lambda) \rightarrow \mathbf{D}^b(\Lambda)$, which assigns a deleted projective resolution to a Λ -module. Trivially ϱ sends projective modules to CE-projective complexes; hence it induces a functor $\mathfrak{R}: \underline{\text{mod}}(\Lambda) \rightarrow \underline{\mathbf{D}}^b(\Lambda)$, making the following diagram commutative

$$\begin{array}{ccccc} \mathcal{P}_{\Lambda} & \longrightarrow & \text{mod}(\Lambda) & \longrightarrow & \underline{\text{mod}}(\Lambda) \\ \downarrow & & \varrho \downarrow & & \mathfrak{R} \downarrow \\ \mathcal{P}_{\text{CE}} & \longrightarrow & \mathbf{D}^b(\Lambda) & \longrightarrow & \underline{\mathbf{D}}^b(\Lambda) \end{array}$$

The cohomology functors $H^n: \mathbf{D}^b(\Lambda) \rightarrow \text{mod}(\Lambda)$, $n \in \mathbb{Z}$, send the class of triangles \mathcal{E} in $\mathbf{D}^b(\Lambda)$ to short exact sequences in $\text{mod}(\Lambda)$ and they send \mathcal{P}_{CE} to \mathcal{P}_{Λ} ; hence there are induced functors $\underline{H}^n: \underline{\mathbf{D}}^b(\Lambda) \rightarrow \underline{\text{mod}}(\Lambda)$. We denote by $H := \bigoplus_{n \in \mathbb{Z}} H^n$ and $\underline{H} := \bigoplus_{n \in \mathbb{Z}} \underline{H}^n$. Using the definition of the functor \mathfrak{R} and the construction of the triangulation of $\underline{\mathbf{D}}^b(\Lambda)$ as in Section 7, it is easy to deduce the following.

PROPOSITION 12.34. *The functors $\mathfrak{R}, \underline{H}$ are exact, \mathfrak{R} is fully faithful, and $\underline{H}\mathfrak{R} = \text{Id}_{\underline{\text{mod}}(\Lambda)}$. Further $\mathcal{E}\text{-gl.dim } \mathbf{D}^b(\Lambda) = \text{gl.dim } \Lambda$ and the functor \mathfrak{R} induces an isomorphism $K_0(\underline{\text{mod}}(\Lambda)) \cong K_0(\underline{\mathbf{D}}^b(\Lambda))$ between the Grothendieck groups of the left triangulated categories $\underline{\text{mod}}(\Lambda)$ and $\underline{\mathbf{D}}^b(\Lambda)$.*

LEMMA 12.35. *A morphism $f^\bullet: A^\bullet \rightarrow B^\bullet$ in $\mathbf{D}^b(\Lambda)$ is an isomorphism in $\underline{\mathbf{D}}^b(\Lambda)$ iff $\forall n \in \mathbb{Z}, \underline{H}^n(f^\bullet): \underline{H}^n(A^\bullet) \rightarrow \underline{H}^n(B^\bullet)$ are isomorphisms in $\underline{\text{mod}}(\Lambda)$.*

Proof. If f^\bullet is an isomorphism in $\underline{\mathbf{D}}^b(\Lambda)$, then there exists $g^\bullet: B^\bullet \rightarrow A^\bullet$ such that $1_{A^\bullet} - f^\bullet \circ g^\bullet, 1_{B^\bullet} - g^\bullet \circ f^\bullet$ factor through some CE-projective complex. Then $1_{H^n(A^\bullet)} - H^n(f^\bullet) \circ H^n(g^\bullet)$ and $1_{H^n(B^\bullet)} - H^n(g^\bullet) \circ H^n(f^\bullet)$ factor through some projective module. Hence $\underline{H}^n(f^\bullet)$ are isomorphisms in $\underline{\text{mod}}(\Lambda), \forall n \in \mathbb{Z}$. Conversely since $\mathbf{D}^b(\Lambda)$ is a Krull–Schmidt category, we can assume that A^\bullet, B^\bullet do not have summands with projective homology. If $\underline{H}^n(f^\bullet)$ are isomorphisms in $\underline{\text{mod}}(\Lambda)$, then so are $H^n(f^\bullet)$. Hence f^\bullet is invertible in $\underline{\mathbf{D}}^b(\Lambda)$, hence in $\underline{\mathbf{D}}^b(\Lambda)$. ■

In the following result we denote by $\text{mod}(\Lambda)^b$ the full subcategory of the \mathbb{Z} -graded category $\text{mod}(\Lambda)^\mathbb{Z}$ consisting of all the bounded objects.

COROLLARY 12.36. (1) *The homology functor $H: \mathbf{D}^b(\Lambda) \rightarrow \text{mod}(\Lambda)^b$ is a representation equivalence iff the functor \mathfrak{R} is an equivalence iff $\text{gl.dim } \Lambda \leq 1$.*

(2) *If Λ is indecomposable, then $\underline{\mathbf{D}}^b(\Lambda)$ is abelian iff Λ is hereditary 1-Gorenstein.*

Proof. Part (1) follows as in Theorem 10.2, using Proposition 12.34 and the fact that Λ is hereditary iff any indecomposable complex in $\mathbf{D}^b(\Lambda)$ is a stalk complex in some degree [33]. If Λ is indecomposable, then by [4], Λ is hereditary 1-Gorenstein iff $\underline{\text{mod}}(\Lambda)$ is abelian. Part (2) follows from this, using (1) and Proposition 12.34. ■

The following generalizes results of Wheeler [72], in which Lemma 12.35 and the next result are proved in the case of a self-injective algebra. We note that by Proposition 12.33, in case Λ is self-injective, the stable derived category is triangulated. This is also a result of Wheeler [73], proved by different methods.

THEOREM 12.37. *Let Λ, Γ be Artin algebras, and let (F, G) be an adjoint pair of exact functors, $F: \text{mod}(\Lambda) \rightleftarrows \text{mod}(\Gamma): G$. Then the following are equivalent:*

(i) *The functors F, G preserve projectives and the induced functors $\underline{F}: \underline{\text{mod}}(\Lambda) \rightarrow \underline{\text{mod}}(\Gamma)$ and $\underline{G}: \underline{\text{mod}}(\Gamma) \rightarrow \underline{\text{mod}}(\Lambda)$ are inverse triangle equivalences.*

(ii) *The induced functors $F^\bullet: \mathbf{D}^b(\Lambda) \rightarrow \mathbf{D}^b(\Gamma)$ and $G^\bullet: \mathbf{D}^b(\Gamma) \rightarrow \mathbf{D}^b(\Lambda)$ preserve CE-projectives and the induced functors $\underline{F}^\bullet: \underline{\mathbf{D}}^b(\Lambda) \rightarrow \underline{\mathbf{D}}^b(\Gamma)$ and $\underline{G}^\bullet: \underline{\mathbf{D}}^b(\Gamma) \rightarrow \underline{\mathbf{D}}^b(\Lambda)$ are inverse triangle equivalences.*

Proof. Fix counit $\varepsilon: FG \rightarrow \text{Id}_{\text{mod}(\Gamma)}$ and unit $\delta: \text{Id}_{\text{mod}(\Lambda)} \rightarrow GF$. Let F^\bullet, G^\bullet be the induced functors on the derived categories. Then ε, δ induce morphisms $\varepsilon^\bullet, \delta^\bullet$ which are counit and unit, respectively, of an adjoint pair (F^\bullet, G^\bullet) . If condition (i) holds, then since F, G preserve projectives, the functors F^\bullet, G^\bullet preserve CE-projectives, so we have induced functors $\underline{F}^\bullet, \underline{G}^\bullet$ on the stable derived categories. It is easy to see that $(\underline{F}^\bullet, \underline{G}^\bullet)$ is also an adjoint pair, with counit $\underline{\varepsilon}^\bullet$ and unit $\underline{\delta}^\bullet$. By Lemma 12.35, it suffices to show that $H^n(\varepsilon^\bullet), H^n(\delta^\bullet)$ are isomorphisms. This is immediate, from the fact that the induced morphisms $\underline{\varepsilon}, \underline{\delta}$ are isomorphisms in the stable module categories by hypothesis. The converse follows similarly. ■

Since by [63], derived equivalent self-injective algebras are stably equivalent by an equivalence satisfying condition (i) of the above theorem, we have the following.

COROLLARY 12.38. *Let Λ, Γ be derived equivalent self-injective Artin algebras. Then the triangulated categories $\underline{\mathbf{D}}^b(\Lambda)$ and $\underline{\mathbf{D}}^b(\Gamma)$ are triangle equivalent.*

Remark 12.39. The results of this section generalize to the homotopy category of complexes of injective comodules, resp. stable category of comodules modulo injectives, over commutative, resp. finite dimensional, Hopf algebras over a field. Also most of the results concerning the stable derived category generalize to the unbounded case and are valid for any ring. Details are left to the reader.

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