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Cohen–Macaulay modules, (co)torsion pairs and virtually Gorenstein algebras

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Abstract

We use torsion pairs in stable categories and cotorsion pairs in modules categories to study, in general infinitely generated, Cohen–Macaulay modules and (a generalization of) modules of finite projective or injective dimension over an Artin algebra. We concentrate our investigation to the study of *virtually Gorenstein* algebras which provide a common generalization of Gorenstein algebras and algebras of finite representation or Cohen–Macaulay type. This class of algebras on the one hand has rich homological structure and satisfies several representation/torsion theoretic finiteness conditions, and on the other hand it is closed under various operations, for instance derived equivalences and stable equivalences of Morita type. In addition virtual Gorensteinness provides a useful tool for the study of the Gorenstein Symmetry Conjecture and modified versions of the Telescope Conjecture for module or stable categories.

Keywords: Artin algebras; Cohen–Macaulay modules; Gorenstein rings; Stable categories; Covariantly, contravariantly finite and definable subcategories; Torsion pairs and cotorsion pairs; Triangulated categories; Compact objects; Telescope Conjecture; Gorenstein Symmetry Conjecture

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Contents

1. Introduction

Since the ubiquity fundamental paper of Bass [15] commutative Noetherian Gorenstein rings and Cohen–Macaulay modules are well established as central notions in commutative algebra bearing important connections with algebraic geometry. During the last decade there is an increasing growth of interest in non-commutative algebraic geometry, and, in this connection, several definitions of Gorensteinness were proposed by various authors in various settings. In particular in the representation theory of Artin algebras, Auslander–Reiten [9,11] introduced Gorenstein algebras as the Artin algebras with finite self-injective dimension from both sides, and they showed that much of the commutative theory carries over to Artin algebras. Also Happel [33] studied Gorenstein algebras in connection with Auslander–Reiten theory in derived categories. The class of Gorenstein algebras gains its importance from the fact that on the one hand it includes algebras with finite global dimension and self-injective algebras as special cases and on the other hand the finitely generated Cohen–Macaulay modules over them have rich homological structure and behave very well with respect to many natural operations and constructions at the level of the module or the derived category. In addition Gorenstein algebras have intimate connections with tilting theory and provide positive examples for many of the homological conjectures in the representation theory of Artin algebras.

Our aim in this paper is to study, in general infinitely generated, Cohen–Macaulay or CoCohen–Macaulay modules and modules of virtually finite projective or injective dimension over an arbitrary Artin algebra Λ. Using the terminology and notation of [22], see also [4,9], we denote by $\mathrm{CM}(\mathbf{P}_\Lambda)$, respectively $\mathrm{CoCM}(\mathbf{I}_\Lambda)$, the maximal subcategory of the category Mod-Λ of all right Λ-modules which admits the full subcategory $\mathbf{P}_\Lambda$, respectively $\mathbf{I}_\Lambda$, of projective, respectively injective, modules as an Ext-injective cogenerator, respectively Ext-projective generator. We call the modules in $\mathrm{CM}(\mathbf{P}_\Lambda)$, respectively $\mathrm{CoCM}(\mathbf{I}_\Lambda)$, *Cohen–Macaulay*, respectively *CoCohen–Macaulay*, *modules*. Then the full subcategory $\mathfrak{P}_\Lambda^{\prec\infty}$, respectively $\mathfrak{I}_\Lambda^{\prec\infty}$, of *modules of virtually finite projective*, respectively *injective*, *dimension* is defined to be the right, respectively left, Ext-orthogonal subcategory of $\mathrm{CM}(\mathbf{P}_\Lambda)$, respectively $\mathrm{CoCM}(\mathbf{I}_\Lambda)$. Note that Cohen–Macaulay modules provide a generalization of finitely generated modules of G-dimension zero [5] and the modules of virtually finite projective/injective dimension provide a natural generalization of modules of finite projective/injective dimension. Our main tools for their study are the theory of approximations of modules and the effective use of cotorsion pairs in the module category and torsion pairs in the stable module category combined with recent methods from compactly generated triangulated categories.

We concentrate our investigation to the study of virtually Gorenstein algebras, introduced in [22], which provide a natural enlargement of the class of Gorenstein algebras giving at the same time a homological generalization of algebras of finite representation type and more generally of algebras of finite Cohen–Macaulay type. Recall from [22] that Λ is called virtually Gorenstein if: $\mathfrak{P}_\Lambda^{\prec\infty} = \mathfrak{I}_\Lambda^{\prec\infty}$. Note that Λ is Gorenstein iff $\mathbf{P}_\Lambda^{<\infty} = \mathbf{I}_\Lambda^{<\infty}$, where $\mathbf{P}_\Lambda^{<\infty}$, respectively $\mathbf{I}_\Lambda^{<\infty}$, is the full subcategory of all modules with finite projective, respectively injective, dimension. We stress that virtually Gorenstein algebras are defined by imposing representation theoretic finiteness conditions on Cohen–Macaulay modules or on modules of virtually finite projective or injective dimension whereas Gorenstein algebras are defined by imposing homological finiteness conditions on the ring. In this connection it is a long-standing open problem if one-sided finiteness of the self-injective dimension of an Artin algebra is sufficient for Gorensteinness. In the literature this problem is usually referred to as the *Gorenstein Symmetry Conjecture*, (GSC) for short, see [13, Conjecture (13)], [22]. As a consequence of our results we give an affirmative answer to (GSC) for the class of virtually Gorenstein algebras.

Virtually Gorenstein algebras share many properties with genuine Gorenstein algebras. For instance we show that they are stable under various operations like derived equivalences or stable equivalences of Morita type. In addition virtually Gorenstein algebras enable us to have homological control on the (stable) module category which is satisfactory from many aspects. This controllability is expressed by the existence of well-behaved (co)torsion pairs in the (stable) module category which restrict to (co)torsion pairs of finitely generated modules; in other words the (stable) category of all or finitely generated modules admits well-behaved "semi-orthogonal decompositions" in the sense of Bondal–Kapranov [24,25]. Here torsion pair in a stable category is meant in the sense of [22] and by a cotorsion pair in an abelian category we mean complete hereditary cotorsion pair in the sense of [57]. As a consequence we show that the finitely generated (Co)Cohen–Macaulay modules and the finitely generated modules of virtually finite projective or injective dimension over a virtually Gorenstein algebra form functorially finite subcategories with free Grothendieck groups and admit Auslander–Reiten sequences. We also show that virtual Gorensteinness is left–right symmetric and we give a host of characterizations of virtually Gorenstein algebras, in particular of Gorenstein algebras, in various contexts ranging from module categories to stable or derived categories, or using (co)torsion theoretic conditions. We would like to stress that although the class of virtually Gorenstein algebras is very large since it includes Gorenstein algebras and algebras of finite representation or Cohen–Macaulay type, we don't know of any example of an Artin algebra that is not virtually Gorenstein. However we show that all Artin algebras are "locally", that is, at the finitely generated level, virtually Gorenstein.

As with Gorenstein algebras, there is a nice relationship between the module category and the stable module category for a virtually Gorenstein algebra. This fruitful interplay is a consequence of the fact that the full subcategory of Cohen–Macaulay modules is definable and locally finitely presented, whereas the induced stable category is a smashing subcategory [37,42] of the stable module category which is a compactly generated triangulated category with compact generators induced by finitely generated modules. Actually these facts characterize the class of virtually Gorenstein algebras. In general it is an important open problem if a smashing subcategory of the stable module category is generated by compact objects coming from the stable category. In the context of compactly generated triangulated categories this is precisely the content of the famous Telescope Conjecture [42] which has its origin in stable homotopy theory of CW-complexes [26,52]. Our results give some information on the problem in the context of the stable module category of an Artin algebra. For instance we show that the Telescope Conjecture holds for the stable category of Cohen–Macaulay modules iff for any cotorsion pair $(\mathcal{X}, \mathcal{Y})$ is Mod-Λ, the subcategory $\mathcal{X}$ is the limit closure of the finitely generated modules it contains provided that $\mathcal{X} \cap \mathcal{Y}$ are the projectives and $\mathcal{Y}$ is closed under coproducts. And this property is invariant under derived equivalences or stable equivalences of Morita type.

We now give a short description of the organization of the paper which is divided roughly in four parts. In the first part, consisting of Sections 2–5, we study the structure and behavior under various operations and constructions of the (co)torsion pairs induced in a natural way by the Cohen–Macaulay modules. This part sets the necessary material for the rest of the paper. In the second part, consisting of Sections 6–8, we study finiteness conditions on the Cohen–Macaulay (co)torsion pairs, and in particular we investigate when they are of (co)finite type in an appropriate sense. This leads us naturally to Section 8 where we study virtually Gorenstein algebras and derive their main properties. In the third part, consisting of Sections 9 and 10, we give relative versions of the theory developed previously by giving methods for constructing (co)torsion pairs arising from Cohen–Macaulay modules. Then we present applications to the Telescope Conjecture for stable categories. In the last part of the paper which consists of Section 11 we study Artin algebras with finite right self-injective dimension in connection with virtual Gorensteinness. In particular we show that (GSC) holds for any algebra lying in the derived or stable equivalence class of a virtually Gorenstein algebra.

Convention. Although many of our results hold for Noetherian and/or left coherent and right perfect rings (even for suitable abelian categories), for simplicity and concreteness we work in the context of Artin algebras. Throughout the paper the composition of morphisms in a given category is meant in the diagrammatic order: the composition of $f : A \to B$ with $g : B \to C$ is denoted by $f \circ g : A \to C$.

2. Preliminaries: Artin algebras, torsion and cotorsion pairs

In this section we fix notation and recall some basic concepts and results concerning pretriangulated categories, torsion and cotorsion pairs which will be useful in the rest of the paper.

2.1. Pretriangulated and stable categories

Let $\mathcal{C}$ be an additive category.

If $\Omega : \mathcal{C} \to \mathcal{C}$, respectively $\Sigma : \mathcal{C} \to \mathcal{C}$, is an additive functor, then we can consider the category $\mathcal{LT}(\mathcal{C}, \Omega)$, respectively $\mathcal{RT}(\mathcal{C}, \Sigma)$, with objects the collection of all diagrams in $\mathcal{C}$ of the form $\Omega(C) \to A \to B \to C$, respectively $A \to B \to C \to \Sigma(A)$, with the obvious morphisms. A **left**, respectively **right**, **triangulation** of the pair $(\mathcal{C}, \Omega)$, respectively $(\mathcal{C}, \Sigma)$, is a strict full subcategory Δ, respectively ∇, of $\mathcal{LT}(\mathcal{C}, \Omega)$, respectively $\mathcal{RT}(\mathcal{C}, \Sigma)$, which satisfies all the axioms of a triangulated category except that Ω, respectively Σ, is not necessarily an equivalence. Then the triple $(\mathcal{C}, \Omega, \Delta)$, respectively $(\mathcal{C}, \Sigma, \nabla)$, is called a **left**, respectively **right**, **triangulated category**, Ω, respectively Σ, is the **loop**, respectively **suspension**, **functor** and the diagrams in Δ, respectively ∇, are the **left**, respectively **right**, **triangles**. An additive functor $F : \mathcal{C} \to \mathcal{D}$ between left, respectively right, triangulated categories is called **left exact**, respectively **right exact**, if F sends left, respectively right, triangles to left, respectively right, triangles and commutes with the loop, respectively suspension, functors.

Now let $\mathcal{C}$ be an additive category equipped with an adjoint pair (Σ, Ω) of additive endofunctors. If the pair $(\mathcal{C}, \Omega)$ admits a left triangulation Δ and the pair $(\mathcal{C}, \Sigma)$ admits a right triangulation ∇ such that certain compatibility conditions between Δ and ∇ are satisfied, see [22], then the quintuple $(\mathcal{C}, \Sigma, \Omega, \Delta, \nabla)$ or simply $\mathcal{C}$ is called a **pretriangulated category**. An important source of examples of pretriangulated categories emerge from functorially finite subcategories.

Let $\mathcal{V}$ be a full subcategory of an abelian category $\mathcal{A}$. A morphism $f : A \to B$ in $\mathcal{A}$ is called **$\mathcal{V}$-epic** if the map $\mathcal{A}(\mathcal{V}, f) : \mathcal{A}(\mathcal{V}, A) \to \mathcal{A}(\mathcal{V}, B)$ is surjective. The subcategory $\mathcal{V}$ is called **contravariantly finite** if there exists a $\mathcal{V}$-epic $f_A : V_A \to A$ with V_A in $\mathcal{V}$. Then f_A is called a **right $\mathcal{V}$-approximation** of A. *Covariantly finite* subcategories, *$\mathcal{V}$-monics* and *left $\mathcal{V}$-approximations* are defined dually. A subcategory is called **functorially finite** provided that it is both contravariantly and covariantly finite; we refer to [6] for details. The **stable category** $\mathcal{A}/\mathcal{V}$ of $\mathcal{A}$ modulo a subcategory $\mathcal{V}$ has as objects the objects of $\mathcal{A}$ and morphism spaces $\mathcal{A}/\mathcal{V}(A, B) := \mathcal{A}(A, B)/\mathcal{A}_{\mathcal{V}}(A, B)$ where $\mathcal{A}_{\mathcal{V}}(A, B)$ is the subgroup of $\mathcal{A}(A, B)$ consisting of all maps factorizing through an object from $\mathcal{V}$. If A, respectively f, is an object, respectively morphism, in $\mathcal{A}$, then we denote by $\underline{A}$, respectively $\underline{f}$, the object A considered as an object in $\mathcal{A}/\mathcal{V}$, respectively the equivalence class of f. Then $\pi : \mathcal{A} \to \mathcal{A}/\mathcal{V}$, where $\pi(A) = \underline{A}$ and $\pi(f) = \underline{f}$, is an additive functor with $\pi(\mathcal{V}) = 0$. A nice situation occurs when $\mathcal{V}$ is functorially finite: in this case the stable category $\mathcal{A}/\mathcal{V}$ is in a natural way a pretriangulated category. The adjoint pair of loop and suspension functors and the left/right triangles are defined via left and right $\mathcal{V}$-approximations, see [22] for details.

2.2. Torsion pairs in pretriangulated categories

Pretriangulated categories provide the proper setting for the study of torsion pairs in the sense of the following definition which generalizes the notion of usual torsion pairs in an abelian category.

Definition 2.1 [22]. Let $\mathcal{C}$ be a pretriangulated category and let $\mathcal{X}$, $\mathcal{Y}$ be full additive subcategories of $\mathcal{C}$ closed under isomorphisms and direct summands. The pair $(\mathcal{X},\mathcal{Y})$ is called a **torsion pair** in $\mathcal{C}$, and then $\mathcal{X}$ is called a **torsion class** and $\mathcal{Y}$ is called a **torsion-free class**, if:

(i) $\mathcal{C}(\mathcal{X},\mathcal{Y})=0$, i.e., $\mathcal{C}(X,Y)=0$, $\forall X\in\mathcal{X}$ and $\forall Y\in\mathcal{Y}$.
(ii) $\Sigma(\mathcal{X})\subseteq\mathcal{X}$ and $\Omega(\mathcal{Y})\subseteq\mathcal{Y}$.
(iii) For any object C in $\mathcal{C}$ there exist objects $X_C\in\mathcal{X}$ and $Y^C\in\mathcal{Y}$, and triangles:

$$\Omega\big(Y^C\big)\xrightarrow{g_C}X_C\xrightarrow{f_C}C\xrightarrow{g^C}Y^C\in\Delta\quad\text{and}\quad X_C\xrightarrow{f_C}C\xrightarrow{g^C}Y^C\xrightarrow{f^C}\Sigma(X_C)\in\nabla.$$

If $(\mathcal{X},\mathcal{Y})$ is a torsion pair in $\mathcal{C}$, then $\mathcal{X}^{\perp}:=\{C\in\mathcal{C}\mid \mathrm{Hom}_{\mathcal{C}}(\mathcal{X},C)=0\}=\mathcal{Y}$ and ${}^{\perp}\mathcal{Y}:=\{C\in\mathcal{C}\mid \mathrm{Hom}_{\mathcal{C}}(C,\mathcal{Y})=0\}=\mathcal{X}$. Moreover the assignment $C\to X_C$ gives a right adjoint $\mathbf{R}_{\mathcal{X}}:\mathcal{C}\to\mathcal{X}$ of the inclusion $\mathbf{i}_{\mathcal{X}}:\mathcal{X}\hookrightarrow\mathcal{C}$ and the assignment $C\to Y^C$ gives a left adjoint $\mathbf{L}_{\mathcal{X}}:\mathcal{C}\to\mathcal{Y}$ of the inclusion $\mathbf{i}_{\mathcal{Y}}:\mathcal{Y}\hookrightarrow\mathcal{C}$, see [22]. The torsion pair $(\mathcal{X},\mathcal{Y})$ is called **hereditary**, respectively **cohereditary**, if the idempotent functor $\mathbf{i}_{\mathcal{X}}\mathbf{R}_{\mathcal{X}}:\mathcal{C}\to\mathcal{C}$, respectively $\mathbf{i}_{\mathcal{Y}}\mathbf{L}_{\mathcal{Y}}:\mathcal{C}\to\mathcal{C}$, is left, respectively right, exact.

2.3. Cotorsion pairs

Let $\mathcal{A}$ be an abelian category. For a subcategory $\mathcal{V}$ of $\mathcal{A}$ we denote by ${}^{\perp}\mathcal{V}:=\{A\in\mathcal{A}\mid \mathrm{Ext}^n(A,V)=0,\forall n\geqslant 1,\forall V\in\mathcal{V}\}$ the **left Ext-orthogonal** subcategory of $\mathcal{V}$ and by $\mathcal{V}^{\perp}:=\{A\in\mathcal{A}\mid \mathrm{Ext}^n(V,A)=0,\forall n\geqslant 1,\forall V\in\mathcal{V}\}$ the **right Ext-orthogonal** subcategory of $\mathcal{V}$. A subcategory $\mathcal{U}\subseteq\mathcal{V}$ is called an **Ext-injective cogenerator** of $\mathcal{V}$ if for any object V in $\mathcal{V}$ there exists an exact sequence $0\to V\to U\to V'\to 0$ where V' lies in $\mathcal{V}$ and U lies in $\mathcal{U}$ and is Ext-injective in $\mathcal{V}$, i.e., $U\in\mathcal{V}^{\perp}$. **Ext-projective generators** are defined dually.

Ext-projective generators and Ext-injective cogenerators emerge naturally from cotorsion pairs. First recall that for a full subcategory $\mathcal{U}$ of $\mathcal{A}$, a right $\mathcal{U}$-approximation $f:U_A\to A$, respectively left $\mathcal{U}$-approximation $g:A\to U^A$, of A is called **special** if $\mathrm{Ext}^1(\mathcal{U},\mathrm{Ker}\,f_A)=0$, respectively $\mathrm{Ext}^1(\mathrm{Coker}\,g^A,\mathcal{U})=0$. Important examples of special approximations are the minimal ones. Recall that a map $f:A\to B$ in $\mathcal{A}$ is called **right**, respectively **left**, **minimal**, if any endomorphism $\alpha:A\to A$, respectively $\beta:B\to B$, is invertible provided that $\alpha\circ f=f$, respectively $f\circ\beta=f$. A minimal right, respectively left, approximation is a right, respectively left, approximation which is right, respectively left, minimal. Note that minimal approximations are unique up to isomorphism and, by Wakamatsu's Lemma [13], for any extension closed full subcategory $\mathcal{U}$ of $\mathcal{A}$, any minimal right, respectively left, $\mathcal{U}$-approximation is special.

Definition 2.2. A pair $(\mathcal{X},\mathcal{Y})$ of full subcategories of $\mathcal{A}$ is called a **cotorsion pair**, and then $\mathcal{X}$ is called a **cotorsion class** and $\mathcal{Y}$ a **cotorsion-free class**, if:

(i) $\mathcal{X}^{\perp}=\mathcal{Y}$ and ${}^{\perp}\mathcal{Y}=\mathcal{X}$.
(ii) Any object of $\mathcal{A}$ admits a special right $\mathcal{X}$-, respectively left $\mathcal{Y}$-, approximation.

If $(\mathcal{X}, \mathcal{Y})$ is a cotorsion pair in the abelian category $\mathcal{A}$, then, by [22], $\mathcal{X} \cap \mathcal{Y}$ is an Ext-injective cogenerator of $\mathcal{X}$ and an Ext-projective generator of $\mathcal{Y}$. If $\mathcal{X} \cap \mathcal{Y}$ is functorially finite, so the stable category $\mathcal{A}/\mathcal{X} \cap \mathcal{Y}$ is pretriangulated, then the pair $(\mathcal{X}/\mathcal{X} \cap \mathcal{Y}, \mathcal{Y}/\mathcal{X} \cap \mathcal{Y})$ is a torsion pair in $\mathcal{A}/\mathcal{X} \cap \mathcal{Y}$.

The class of cotorsion, respectively torsion, pairs in an abelian, respectively pretriangulated, category $\mathcal{A}$, respectively $\mathcal{C}$, is partially ordered: $(\mathcal{X}_1, \mathcal{Y}_1) \preccurlyeq (\mathcal{X}_2, \mathcal{Y}_2)$ iff $\mathcal{X}_1 \subseteq \mathcal{X}_2$. With respect to $\preccurlyeq$, the least, respectively greatest, cotorsion pair is $(\mathcal{P}, \mathcal{A})$, respectively $(\mathcal{A}, \mathcal{I})$, where $\mathcal{P}$, respectively $\mathcal{I}$, are the projectives, respectively injectives, of $\mathcal{A}$, and the least, respectively greatest, torsion pair is $(0, \mathcal{C})$, respectively $(\mathcal{C}, 0)$.

Cotorsion(-free) classes are examples of (co)resolving subcategories. Recall that a full subcategory of $\mathcal{A}$ is called **resolving** provided that it is closed under extensions, kernels of epics and contains the projectives. **Coresolving** subcategories are defined dually. Having a (co)resolving subcategory we can define the notion of (co)resolution dimension of $\mathcal{A}$. If $\mathcal{X}$ is a resolving subcategory of $\mathcal{A}$, then the **$\boldsymbol{\mathcal{X}}$-resolution dimension** of an object A in $\mathcal{A}$, written $\operatorname{res.dim}_{\mathcal{X}} A$, is defined inductively as follows. If A is in $\mathcal{X}$, then $\operatorname{res.dim}_{\mathcal{X}} A = 0$. If $t \geqslant 1$, then $\operatorname{res.dim}_{\mathcal{X}} A \leqslant t$ if there exists an exact sequence $0 \to X_t \to \cdots \to X_0 \to A \to 0$ where $\operatorname{res.dim}_{\mathcal{X}} X_i = 0$, for $0 \leqslant i \leqslant t$. Then $\operatorname{res.dim}_{\mathcal{X}} A = t$ if $\operatorname{res.dim}_{\mathcal{X}} A \leqslant t$ and $\operatorname{res.dim}_{\mathcal{X}} A \nleqslant t-1$. Finally if $\operatorname{res.dim}_{\mathcal{X}} A \neq t$ for any $t \geqslant 0$, then define $\operatorname{res.dim}_{\mathcal{X}} A = \infty$. The **$\boldsymbol{\mathcal{X}}$-resolution dimension** of $\mathcal{A}$ is defined by $\operatorname{res.dim}_{\mathcal{X}} \mathcal{A} := \sup\{\operatorname{res.dim}_{\mathcal{X}} A \mid A \in \mathcal{A}\}$. The **$\boldsymbol{\mathcal{Y}}$-coresolution dimension** of $\mathcal{A}$, which is denoted by $\operatorname{cores.dim}_{\mathcal{Y}} \mathcal{A}$, for a coresolving subcategory $\mathcal{Y}$ of $\mathcal{A}$, is defined dually. We denote by $\widehat{\mathcal{X}} := \{A \in \mathcal{A} \mid \operatorname{res.dim}_{\mathcal{X}} A < \infty\}$, respectively $\widetilde{\mathcal{Y}} := \{A \in \mathcal{A} \mid \operatorname{cores.dim}_{\mathcal{Y}} A < \infty\}$, the full subcategory of $\mathcal{A}$ consisting of all objects which admit finite exact resolutions by objects from $\mathcal{X}$, respectively finite exact coresolutions by objects from $\mathcal{Y}$.

2.4. *Artin algebras*

The basic examples of pretriangulated categories in this paper emerge from Artin algebras. From now on we fix an Artin R-algebra Λ over a commutative Artin ring R with radical $\operatorname{Rad} R$. The radical of Λ is denoted by $\mathbf{r}$. Let $\text{Mod-}\Lambda$ be the category of all right Λ-modules and let $\text{mod-}\Lambda$ be the category of finitely generated right Λ-modules. We view left Λ-modules as right Λ^{op}-modules and we denote by $\mathrm{D} : \text{Mod-}\Lambda \to \text{Mod-}\Lambda^{\text{op}}$ the usual duality which is given by $\mathrm{D} = \operatorname{Hom}_R(-, \mathbb{E})$ where $\mathbb{E} := R/\operatorname{Rad} R$ is the injective envelope of $R/\operatorname{Rad} R$.

Since Λ is left coherent and right perfect, the category $\mathbf{P}_\Lambda$ of projective Λ-modules is functorially finite and any module admits a minimal left and a minimal right projective approximation [22]. Hence the stable category $\underline{\text{Mod}}\text{-}\Lambda$ modulo projectives is pretriangulated with induced adjoint pair of endofunctors $(\Sigma_{\mathbf{P}}, \Omega)$, where Ω is the usual loop functor and $\Sigma_{\mathbf{P}}(\underline{A})$ is the cokernel of a left projective approximation of A. Since the category $\mathcal{P}_\Lambda$ of finitely generated projective modules is functorially finite, the stable category $\underline{\text{mod}}\text{-}\Lambda$ modulo projectives is pretriangulated with induced adjoint pair of endofunctors $(\Sigma_{\mathcal{P}}, \Omega)$ which is the restriction of the first one on the full subcategory $\underline{\text{mod}}\text{-}\Lambda$.

Dually since Λ is right Noetherian, the category $\mathbf{I}_\Lambda$ of injective Λ-modules is functorially finite and any module admits a minimal left and a minimal right injective approximation. Hence the stable category $\overline{\text{Mod}}\text{-}\Lambda$ modulo injectives is pretriangulated with induced adjoint pair of endofunctors $(\Sigma, \Omega_{\mathbf{I}})$ where Σ is the usual suspension functor and $\Omega_{\mathbf{I}}(\overline{A})$ is the kernel of a right injective approximation of A. Since the category $\mathcal{I}_\Lambda$ of finitely generated injective modules is functorially finite, the stable category $\overline{\text{mod}}\text{-}\Lambda$ modulo injectives is pretriangulated with induced adjoint pair of endofunctors $(\Sigma, \Omega_{\mathcal{I}})$ which is the restriction of the first one on the full subcategory $\overline{\text{mod}}\text{-}\Lambda$.

In the sequel we shall need the following observations.

Remark 2.3. (i) By construction, $\forall A \in \text{Mod-}\Lambda$, there exists an exact sequence $A \to P^A \to \Sigma_{\mathbf{P}}(A) \to 0$, where $g^A : A \to P^A$ is a left $\mathbf{P}_\Lambda$-approximation of A. If $A \xrightarrow{\varepsilon} \Omega\Sigma_{\mathbf{P}}(A) \xrightarrow{\mu} P^A$ is the canonical factorization of g^A, then the map $\underline{\varepsilon} : \underline{A} \to \Omega\Sigma_{\mathbf{P}}(\underline{A})$ is the reflection of $\underline{A}$ in the full subcategory $\operatorname{Im}\Omega$ of $\underline{\text{Mod}}\text{-}\Lambda$ consisting of the syzygy modules. Similar remarks hold for the functor $\Omega_{\mathbf{I}}$.

(ii) By [18], we have $\Sigma_{\mathcal{P}} = \Sigma_{\mathbf{P}}|_{\underline{\text{mod}}\text{-}\Lambda} = \operatorname{Tr}\Omega\operatorname{Tr}$ and $\Omega_{\mathcal{I}} = \Omega_{\mathbf{I}}|_{\overline{\text{mod}}\text{-}\Lambda} = \operatorname{DTr}\Omega\operatorname{TrD}$, where $\operatorname{Tr} : (\underline{\text{mod}}\text{-}\Lambda)^{\text{op}} \xrightarrow{\approx} \underline{\text{mod}}\text{-}\Lambda^{\text{op}}$ is the Auslander–Bridger transpose duality functor [5]. Recall that for any T in mod-Λ and any A in Mod-Λ we have the Auslander–Reiten formulas [8]:

$$\operatorname{Ext}^1_\Lambda\big(A, \operatorname{DTr}(T)\big) \cong \operatorname{D}\underline{\operatorname{Hom}}_\Lambda(T, A) \quad \text{and} \quad \operatorname{DExt}^1_\Lambda(T, A) \cong \overline{\operatorname{Hom}}_\Lambda\big(A, \operatorname{DTr}(T)\big).$$

We denote by $\mathsf{N}^+ : \text{Mod-}\Lambda \to \text{Mod-}\Lambda$ the Nakayama functor defined by $\mathsf{N}^+(A) = A \otimes_\Lambda \operatorname{D}(\Lambda)$ and by $\mathsf{N}^- : \text{Mod-}\Lambda \to \text{Mod-}\Lambda$ its right adjoint defined by $\mathsf{N}^-(A) = \operatorname{Hom}_\Lambda(\operatorname{D}(\Lambda), A)$. It is not difficult to see that the adjoint pair $(\mathsf{N}^+, \mathsf{N}^-)$ induces an equivalence $\mathsf{N}^+ : \mathbf{P}_\Lambda \xrightarrow{\approx} \mathbf{I}_\Lambda$ with quasi-inverse $\mathsf{N}^- : \mathbf{I}_\Lambda \xrightarrow{\approx} \mathbf{P}_\Lambda$. Krause observed that this fact can be used to show that the Auslander–Reiten operators DTr and TrD can be extended to the big module category. Let A and B be arbitrary modules. Let $0 \to \Omega^2(A) \to P_1 \to P_0 \to A \to 0$ be the start of a (minimal) projective resolution of A, and let $0 \to B \to I^0 \to I^1 \to \Sigma^2(B) \to 0$ be the start of a (minimal) injective coresolution of B. Following [41] we define the modules $\tau^+(A)$ and $\tau^-(B)$ by the exact sequences:

$$0 \to \tau^+(A) \to \mathsf{N}^+(P_1) \to \mathsf{N}^+(P_0) \to \mathsf{N}^+(A) \to 0,$$
$$0 \to \mathsf{N}^-(B) \to \mathsf{N}^-\big(I^0\big) \to \mathsf{N}^-\big(I^1\big) \to \tau^-(B) \to 0.$$

We call τ^+ and τ^- the **Auslander–Reiten operators** of Mod-Λ. By [41] the operators τ^+ and τ^- induce an adjoint pair of stable equivalences $(\tau^+, \tau^-) : \underline{\text{Mod}}\text{-}\Lambda \xrightarrow{\approx} \overline{\text{Mod}}\text{-}\Lambda$, and coincide with DTr and TrD respectively in case we work with finitely generated modules.

Remark 2.4. (i) If Λ is a finite-dimensional k-algebra over a perfect field k, then the Auslander–Reiten operators can be made functorial. That is, there exists an adjoint pair (F, G) of endofunctors of Mod-Λ such that F sends injectives to projectives and $F \cong \tau^- : \overline{\text{Mod}}\text{-}\Lambda \to \underline{\text{Mod}}\text{-}\Lambda$, and G sends projectives to injectives and $G \cong \tau^+ : \underline{\text{Mod}}\text{-}\Lambda \to \overline{\text{Mod}}\text{-}\Lambda$, see [10,17].

(ii) Since the Nakayama functors $\mathsf{N}^\pm$ preserve filtered colimits, products and finitely presented modules, it follows that they preserve pure short exact sequences and pure-injective modules.

For a Λ-module A, the projective, respectively injective, dimension of A is denoted by $\operatorname{pd} A$, respectively $\operatorname{id} A$. We denote by $\widehat{\mathbf{P}_\Lambda} := \mathbf{P}_\Lambda^{<\infty}$, respectively $\widehat{\mathcal{P}_\Lambda} := \mathcal{P}_\Lambda^{<\infty}$, the full subcategory of all, respectively finitely generated, modules with finite projective dimension. Dually $\widetilde{\mathbf{I}_\Lambda} := \mathbf{I}_\Lambda^{<\infty}$, respectively $\widetilde{\mathcal{I}_\Lambda} := \mathcal{I}_\Lambda^{<\infty}$, denotes the full subcategory of all, respectively finitely generated, modules with finite injective dimension. The **big projective finitistic dimension** $\operatorname{FPD}(\Lambda)$ of Λ is defined by $\operatorname{FPD}(\Lambda) = \sup\{\operatorname{pd} C \mid C \in \mathbf{P}_\Lambda^{<\infty}\}$ and the **little projective finitistic dimension** $\operatorname{fpd}(\Lambda)$ of Λ is defined by $\operatorname{fpd}(\Lambda) = \sup\{\operatorname{pd} C \mid C \in \mathcal{P}_\Lambda^{<\infty}\}$. The big, respectively little, injective finitistic dimension $\operatorname{FID}(\Lambda)$, respectively $\operatorname{fid}(\Lambda)$, are defined dually. The global dimension of Λ is denoted by $\operatorname{gl.dim} \Lambda$. Finally if $\mathcal{V}$ is a full subcategory of $\operatorname{Mod-}\Lambda$, then $\operatorname{Prod}(\mathcal{V})$, respectively $\operatorname{Add}(\mathcal{V})$, respectively $\operatorname{add}(\mathcal{V})$, denotes the full subcategory consisting of all direct summands of products, respectively coproducts, respectively finite coproducts, of modules from $\mathcal{V}$.

An object T in an additive category $\mathcal{C}$ which admits all small coproducts is called **compact** if the representable functor $\mathcal{C}(T,-):\mathcal{C} \to \mathcal{A}b$ preserves all small coproducts. The full subcategory of compact objects of $\mathcal{C}$ is denoted by $\mathcal{C}^{\mathrm{b}}$. If $\mathcal{U} \subseteq \operatorname{Mod-}\Lambda$, we denote by $\mathcal{U}^{\mathrm{fin}}$ the full subcategory:

$$\mathcal{U}^{\mathrm{fin}} := \mathcal{U} \cap \operatorname{mod-}\Lambda.$$

Note that if $\mathcal{U}$ is closed under coproducts in $\operatorname{Mod-}\Lambda$, then we have $\mathcal{U}^{\mathrm{fin}} \subseteq \mathcal{U}^{\mathrm{b}}$. And clearly we have equalities $(\operatorname{Mod-}\Lambda)^{\mathrm{b}} = \operatorname{mod-}\Lambda = (\operatorname{Mod-}\Lambda)^{\mathrm{fin}}$ since Λ is right Noetherian.

3. Cohen–Macaulay modules and (co)torsion pairs

In this section we fix notation and recall some basic concepts concerning (Co)Cohen–Macaulay modules. In addition we prove several useful results, mainly of homological nature, related to (Co)Cohen–Macaulay modules which will be essential in the rest of the paper and we give the connections with torsion and cotorsion pairs.

From now on and throughout the rest of the paper we fix an Artin algebra Λ.

Remark 3.1. It is easy to see that we have the following identifications.

$$ {}^{\perp}\Lambda = {}^{\perp}\mathbf{P}_\Lambda = \{A \in \operatorname{Mod-}\Lambda \mid \mathrm{L}_n\mathsf{N}^+(A) = \operatorname{Tor}_n^\Lambda(A, \mathrm{D}(\Lambda)) = 0,\ \forall n \geqslant 1\},$$
$$\mathrm{D}(\Lambda)^{\perp} = \mathbf{I}_\Lambda^{\perp} = \{A \in \operatorname{Mod-}\Lambda \mid \mathrm{R}^n\mathsf{N}^-(A) = \operatorname{Ext}_\Lambda^n(\mathrm{D}(\Lambda), A) = 0,\ \forall n \geqslant 1\}.$$

Let $\operatorname{CM}(\mathbf{P}_\Lambda)$, respectively $\operatorname{CoCM}(\mathbf{I}_\Lambda)$, be the maximal subcategory of $\operatorname{Mod-}\Lambda$ which admits the full subcategory of projective, respectively injective, modules as an Ext-injective cogenerator, respectively Ext-projective generator. Clearly we have:

$$\begin{aligned}\operatorname{CM}(\mathbf{P}_\Lambda) = \{X \in \operatorname{Mod-}\Lambda \mid\ & \text{there exists an exact sequence} \\ & 0 \to X \to P^0 \xrightarrow{f^0} P^1 \xrightarrow{f^1} P^2 \to \cdots,\ \text{where } P^s \in \mathbf{P}_\Lambda, \\ & \forall s \geqslant 0,\ \text{and } \operatorname{Ker}(f^n) \in {}^{\perp}\Lambda,\ \forall n \geqslant 0\},\end{aligned}$$

$$\mathrm{CoCM}(\mathbf{I}_\Lambda) = \Big\{ Z \in \text{Mod-}\Lambda \;\Big|\; \text{there exists an exact sequence} \\ \cdots \to I^{-2} \xrightarrow{g^{-1}} I^{-1} \xrightarrow{g^{0}} I^{0} \to Z \to 0, \text{ where } I^{-s} \in \mathbf{I}_\Lambda, \\ \forall s \geqslant 0, \text{ and } \mathrm{Coker}\big(g^{-n}\big) \in \mathrm{D}(\Lambda)^{\perp}, \ \forall n \geqslant 0 \Big\}.$$

The stable category of Cohen–Macaulay modules modulo projectives is denoted by $\underline{\mathrm{CM}}(\mathbf{P}_\Lambda)$ and the stable category of CoCohen–Macaulay modules modulo injectives is denoted by $\overline{\mathrm{CoCM}}(\mathbf{I}_\Lambda)$.

Note. Obviously we have: $\mathrm{CM}(\mathbf{P}_\Lambda) \cap \mathbf{I}_\Lambda = \mathbf{P}_\Lambda \cap \mathbf{I}_\Lambda = \mathbf{P}_\Lambda \cap \mathrm{CoCM}(\mathbf{I}_\Lambda)$.

Inspired by the work of Auslander–Bridger [5] and following [22] it is now natural to make the following definition.

Definition 3.2. The modules in $\mathrm{CM}(\mathbf{P}_\Lambda)$ are called **Cohen–Macaulay modules** and the modules in $\mathrm{CoCM}(\mathbf{I}_\Lambda)$ are called **CoCohen–Macaulay modules**.

We also consider the finitely generated versions of the above definitions. We set

$$\mathrm{CM}(\Lambda) = \mathrm{CM}(\mathbf{P}_\Lambda)^{\mathrm{fin}} = \text{mod-}\Lambda \cap \mathrm{CM}(\mathbf{P}_\Lambda),$$
$$\mathrm{CoCM}\big(\mathrm{D}(\Lambda)\big) = \mathrm{CoCM}(\mathbf{I}_\Lambda)^{\mathrm{fin}} = \text{mod-}\Lambda \cap \mathrm{CoCM}(\mathbf{I}_\Lambda),$$
$$\underline{\mathrm{CM}}(\Lambda) = \underline{\mathrm{mod}}\text{-}\Lambda \cap \underline{\mathrm{CM}}(\mathbf{P}_\Lambda), \qquad \overline{\mathrm{CoCM}}\big(\mathrm{D}(\Lambda)\big) = \overline{\mathrm{mod}}\text{-}\Lambda \cap \overline{\mathrm{CoCM}}(\mathbf{I}_\Lambda).$$

By [22], $\mathrm{CM}(\mathbf{P}_\Lambda)$, respectively $\mathrm{CM}(\Lambda)$, is a resolving subcategory of Mod-Λ, respectively mod-Λ, with $\underline{\mathrm{CM}}(\mathbf{P}_\Lambda)$, respectively $\underline{\mathrm{CM}}(\Lambda)$, triangulated. And $\mathrm{CoCM}(\mathbf{I}_\Lambda)$, respectively $\mathrm{CoCM}(\mathrm{D}(\Lambda))$, is a coresolving subcategory of Mod-Λ, respectively mod-Λ, with $\overline{\mathrm{CoCM}}(\mathbf{I}_\Lambda)$, respectively $\overline{\mathrm{CoCM}}(\mathrm{D}(\Lambda))$ triangulated. The following result gives a convenient description of the (Co)Cohen–Macaulay modules which will be useful later.

Lemma 3.3.

(1) *A module A is Cohen–Macaulay iff*
 - (α) $\mathrm{L}_n\mathsf{N}^+(A) = 0$, $\forall n \geqslant 1$,
 - (β) $\mathrm{R}^n\mathsf{N}^-(\mathsf{N}^+(A)) = 0$, $\forall n \geqslant 1$, *and*
 - (γ) *the natural map $A \to \mathsf{N}^-\mathsf{N}^+(A)$ is invertible.*

(2) *A module A is CoCohen–Macaulay iff*
 - (α) $\mathrm{R}^n\mathsf{N}^-(A) = 0$, $\forall n \geqslant 1$,
 - (β) $\mathrm{L}_n\mathsf{N}^+(\mathsf{N}^-(A)) = 0$, $\forall n \geqslant 1$, *and*
 - (γ) *the natural map $\mathsf{N}^+\mathsf{N}^-(A) \to A$ is invertible.*

Proof. We prove only (1) since part (2) is dual. If A is Cohen–Macaulay then the conditions hold by [22]. Conversely, by Remark 3.1, condition (α) implies that $A \in {}^{\perp}\Lambda$. Let $0 \to \mathsf{N}^+(A) \to I^0 \to I^1 \to \cdots$ be an injective coresolution of $\mathsf{N}^+(A)$. Applying N^- and using

(β) and (γ) we obtain an exact coresolution $0 \to A \to \mathsf{N}^-(I^0) \to \mathsf{N}^-(I^1) \to \cdots$ of A by projectives. Denoting by A^n the kernel of $\mathsf{N}^-(I^n) \to \mathsf{N}^-(I^{n+1})$ for $n \geqslant 0$ and applying to the exact sequence $0 \to A \to \mathsf{N}^-(I^0) \to A^1 \to 0$ the functor N^+, we get $\mathrm{L}_n\mathsf{N}^+(A^1) = 0$, $\forall n \geqslant 2$ and an exact sequence $0 \to \mathrm{L}_1\mathsf{N}^+(A^1) \to \mathsf{N}^+(A) \to I^0$ which shows that $\mathrm{L}_1\mathsf{N}^+(A^1) = 0$ since $\mathsf{N}^+(A) \to I^0$ is a monomorphism. Hence $\mathrm{L}_n\mathsf{N}^+(A^1) = 0$, $\forall n \geqslant 1$. Inductively we see that $\mathrm{L}_n\mathsf{N}^+(A^t) = 0$, $\forall n \geqslant 1$, $\forall t \geqslant 2$. Hence A is Cohen–Macaulay. □

The following result from [22] indicates an interesting interplay between (Co)Cohen–Macaulay modules, Nakayama functors and the Auslander–Reiten operators which will be useful later.

Proposition 3.4. *The adjoint pair* $(\mathsf{N}^+, \mathsf{N}^-)$ *induces quasi-inverse equivalences*

$$\mathsf{N}^+ : \mathrm{CM}(\mathbf{P}_\Lambda) \leftrightarrows \mathrm{CoCM}(\mathbf{I}_\Lambda) : \mathsf{N}^- \quad \textit{and} \quad \mathsf{N}^+ : \mathrm{CM}(\Lambda) \leftrightarrows \mathrm{CoCM}\big(\mathrm{D}(\Lambda)\big) : \mathsf{N}^-$$

which in turn induce exact quasi-inverse equivalences of triangulated categories

$$\mathsf{N}^+ : \underline{\mathrm{CM}}(\mathbf{P}_\Lambda) \leftrightarrows \overline{\mathrm{CoCM}}(\mathbf{I}_\Lambda) : \mathsf{N}^- \quad \textit{and} \quad \mathsf{N}^+ : \underline{\mathrm{CM}}(\Lambda) \leftrightarrows \overline{\mathrm{CoCM}}\big(\mathrm{D}(\Lambda)\big) : \mathsf{N}^-.$$

Moreover the Auslander–Reiten operators τ^+ *and* τ^- *restrict to quasi-inverse triangle equivalences*

$$\tau^+ : \underline{\mathrm{CM}}(\mathbf{P}_\Lambda) \leftrightarrows \overline{\mathrm{CoCM}}(\mathbf{I}_\Lambda) : \tau^- \quad \textit{and} \quad \tau^- : \underline{\mathrm{CM}}(\Lambda) \leftrightarrows \overline{\mathrm{CoCM}}\big(\mathrm{D}(\Lambda)\big) : \tau^-$$

such that $\Sigma^{-2}\mathsf{N}^+ \cong \tau^+\mathbf{i}_{\mathrm{CM}}$ *and* $\Omega^{-2}\mathsf{N}^- \cong \tau^-\mathbf{i}_{\mathrm{CoCM}}$, *where* $\mathbf{i}_{\mathrm{CM}}$ *and* $\mathbf{i}_{\mathrm{CoCM}}$ *are the inclusion functors.*

Our analysis of (Co)Cohen–Macaulay modules is based on the following result from [22].

Theorem 3.5 [22]. *If* Λ *is an Artin algebra, then we have the following.*

(i) *There is a cotorsion pair* $(\mathrm{CM}(\mathbf{P}_\Lambda), \mathfrak{P}_\Lambda^{\prec\propto})$ *in* Mod-Λ. *The full subcategory* $\mathrm{CM}(\mathbf{P}_\Lambda)$ *of Cohen–Macaulay modules is functorially finite resolving and we have*:

$$\mathrm{CM}(\mathbf{P}_\Lambda) \cap \mathfrak{P}_\Lambda^{\prec\propto} = \mathbf{P}_\Lambda \quad \textit{and} \quad \mathbf{P}_\Lambda^{<\infty} = \mathfrak{P}_\Lambda^{\prec\propto} \cap \widehat{\mathrm{CM}(\mathbf{P}_\Lambda)}.$$

(ii) *There is a cotorsion pair* $(\mathfrak{I}_\Lambda^{\prec\propto}, \mathrm{CoCM}(\mathbf{I}_\Lambda))$ *in* Mod-Λ. *The full subcategory* $\mathrm{CoCM}(\mathbf{I}_\Lambda)$ *of CoCohen–Macaulay modules is functorially finite coresolving and we have*:

$$\mathrm{CoCM}(\mathbf{I}_\Lambda) \cap \mathfrak{I}_\Lambda^{\prec\propto} = \mathbf{I}_\Lambda \quad \textit{and} \quad \mathbf{I}_\Lambda^{<\infty} = \mathfrak{I}_\Lambda^{\prec\propto} \cap \widetilde{\mathrm{CoCM}(\mathbf{I}_\Lambda)}.$$

(iii) *The subcategories* $\mathfrak{P}_\Lambda^{\prec\alpha}$ *and* $\mathfrak{I}_\Lambda^{\prec\alpha}$ *are resolving and coresolving. In particular we have*:

$$\mathbf{P}_\Lambda^{<\infty} \subseteq \mathfrak{P}_\Lambda^{\prec\alpha} \supseteq \mathbf{I}_\Lambda^{<\infty} \quad \textit{and} \quad \mathbf{P}_\Lambda^{<\infty} \subseteq \mathfrak{I}_\Lambda^{\prec\alpha} \supseteq \mathbf{I}_\Lambda^{<\infty}.$$

(iv) *The pair* $(\underline{\mathrm{CM}}(\mathbf{P}_\Lambda), \underline{\mathfrak{P}}_\Lambda^{\prec\alpha})$ *is a hereditary torsion pair in* $\underline{\mathrm{Mod}}$-Λ *and the pair* $(\overline{\mathfrak{I}}_\Lambda^{\prec\alpha}, \overline{\mathrm{CoCM}}(\mathbf{I}_\Lambda))$ *is a cohereditary torsion pair in* $\overline{\mathrm{Mod}}$-Λ.

It is now natural to call the modules in $\mathfrak{P}_\Lambda^{\prec\alpha}$ the modules **of virtually finite projective dimension** and the modules in $\mathfrak{I}_\Lambda^{\prec\alpha}$ the modules **of virtually finite injective dimension**.

Remark 3.6. By [22] for any module A, there exist exact commutative diagrams and sequences:

$$\begin{tikzcd} & & 0 \arrow[d] & 0 \arrow[d] & \\ 0 \arrow[r] & Y_A \arrow[r, "g_A"] \arrow[d, equal] & X_A \arrow[r, "f_A"] \arrow[d, "\kappa"'] & A \arrow[r] \arrow[d, "g^A"'] & 0 \\ 0 \arrow[r] & Y_A \arrow[r, "\lambda"] & P_A \arrow[r, "\mu"] \arrow[d, "\nu"'] & Y^A \arrow[r] \arrow[d, "f^A"'] & 0 \\ & & X^A \arrow[r, equal] \arrow[d] & X^A \arrow[d] & \\ & & 0 & 0 & \end{tikzcd}$$

$$0 \longrightarrow X_A \xrightarrow{(f_A,-\kappa)} A \oplus P_A \xrightarrow{\binom{g^A}{\mu}} Y^A \longrightarrow 0$$

$$0 \longrightarrow W_A \xrightarrow{(f_A^*,-\rho)} A \oplus I^A \xrightarrow{\binom{g_*^A}{\sigma}} Z^A \longrightarrow 0$$

$$\begin{tikzcd} & 0 \arrow[d] & 0 \arrow[d] & & \\ & Z_A \arrow[r, equal] \arrow[d, "g_A^*"'] & Z_A \arrow[d, "\xi"'] & & \\ 0 \arrow[r] & W_A \arrow[r, "\rho"] \arrow[d, "f_A^*"'] & I^A \arrow[r, "\tau"] \arrow[d, "\sigma"'] & W^A \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & A \arrow[r, "g_*^A"] \arrow[d] & Z^A \arrow[r, "f_*^A"] \arrow[d] & W^A \arrow[r] & 0 \\ & 0 & 0 & & \end{tikzcd}$$

where $f_A : X_A \to A$, respectively $f_A^* : W_A \to A$, is a special right $\mathrm{CM}(\mathbf{P}_\Lambda)$-, respectively $\mathfrak{I}_\Lambda^{\prec\alpha}$-, approximation of A and $g^A : A \to Y^A$, respectively $g_*^A : A \to Z^A$, is a special left $\mathfrak{P}_\Lambda^{\prec\alpha}$-, respectively $\mathrm{CoCM}(\mathbf{I}_\Lambda)$-, approximation of A. Also $\kappa : X_A \to P_A$, respectively $\sigma : I^A \to Z^A$, is a left $\mathbf{P}_\Lambda$-, respectively right $\mathbf{I}_\Lambda$-, approximation of X_A, respectively Z^A.

Remark–Notation 3.7. Clearly $\mathrm{CM}(\mathbf{P}_\Lambda) = \mathbf{P}_\Lambda$ iff $\mathrm{CoCM}(\mathbf{I}_\Lambda) = \mathbf{I}_\Lambda$ iff $\widehat{\mathrm{CM}(\mathbf{P}_\Lambda)} = \mathbf{P}_\Lambda^{<\infty}$ iff $\widetilde{\mathrm{CoCM}(\mathbf{I}_\Lambda)} = \mathbf{I}_\Lambda^{<\infty}$ iff $\mathfrak{P}_\Lambda^{\prec\alpha} = \mathrm{Mod}\text{-}\Lambda$ iff $\mathrm{Mod}\text{-}\Lambda = \mathfrak{I}_\Lambda^{\prec\alpha}$. This happens if $\mathrm{gl.dim}\,\Lambda < \infty$. Also Λ is self-injective iff $\mathrm{D}(\Lambda) \in \mathrm{CM}(\mathbf{P}_\Lambda)$ iff $\Lambda \in \mathrm{CoCM}(\mathbf{I}_\Lambda)$; in this case we have: $\mathrm{CM}(\mathbf{P}_\Lambda) = \mathrm{Mod}\text{-}\Lambda = \mathrm{CoCM}(\mathbf{I}_\Lambda)$ and $\mathfrak{P}_\Lambda^{\prec\alpha} = \mathbf{P}_\Lambda = \mathbf{I}_\Lambda = \mathfrak{I}_\Lambda^{\prec\alpha}$. To avoid trivialities we usually assume throughout the paper that Λ is a non-self-injective algebra of infinite global dimension.

In what follows we denote by:

(i) $\mathbf{R}_{\mathrm{CM}} : \underline{\mathrm{Mod}}\text{-}\Lambda \to \underline{\mathrm{CM}}(\mathbf{P}_\Lambda)$ the right adjoint of the inclusion $\mathbf{i}_{\mathrm{CM}} : \underline{\mathrm{CM}}(\mathbf{P}_\Lambda) \hookrightarrow \underline{\mathrm{Mod}}\text{-}\Lambda$.

(ii) $\mathbf{L}_{\mathrm{CoCM}}: \overline{\mathrm{Mod}}\text{-}\Lambda \to \overline{\mathrm{CoCM}}(\mathbf{I}_\Lambda)$ the left adjoint of the inclusion $\mathbf{i}_{\mathrm{CoCM}}: \overline{\mathrm{CoCM}}(\mathbf{I}_\Lambda) \hookrightarrow \overline{\mathrm{Mod}}\text{-}\Lambda$.
(iii) $\mathbf{L}_{\mathfrak{P}}: \underline{\mathrm{Mod}}\text{-}\Lambda \to \underline{\mathfrak{P}}_\Lambda^{\prec\infty}$ the left adjoint of the inclusion $\mathbf{i}_{\mathfrak{P}}: \underline{\mathfrak{P}}_\Lambda^{\prec\infty} \hookrightarrow \underline{\mathrm{Mod}}\text{-}\Lambda$.
(iv) $\mathbf{R}_{\mathfrak{I}}: \overline{\mathrm{Mod}}\text{-}\Lambda \to \overline{\mathfrak{I}}_\Lambda^{\prec\infty}$ the right adjoint of the inclusion $\mathbf{i}_{\mathfrak{I}}: \overline{\mathfrak{I}}_\Lambda^{\prec\infty} \hookrightarrow \overline{\mathrm{Mod}}\text{-}\Lambda$.

The existence of these functors follow from [22] in connection with Theorem 3.5 and the discussion in Section 2.2. Note that, by [22], the functors $\mathbf{R}_{\mathrm{CM}}$, $\mathbf{R}_{\mathfrak{I}}$ and $\mathbf{i}_{\mathfrak{P}}$ are left exact, the functors $\mathbf{L}_{\mathrm{CoCM}}$, $\mathbf{L}_{\mathfrak{P}}$ and $\mathbf{i}_{\mathfrak{I}}$ are right exact and the functors $\mathbf{i}_{\mathrm{CM}}$ and $\mathbf{i}_{\mathrm{CoCM}}$ are exact.

The following result collects some basic properties of (Co)Cohen–Macaulay modules and modules of virtually finite projective or injective dimension which will be useful in the sequel. First we recall that a full subcategory $\mathcal{D}$ of Mod-Λ is called **definable** if $\mathcal{D}$ is closed under filtered colimits, products and pure submodules. We refer to [43] for detailed information concerning definable subcategories.

Proposition 3.8.

(i) *The full subcategories* $\mathrm{CM}(\mathbf{P}_\Lambda)$ *and* $\mathrm{CoCM}(\mathbf{I}_\Lambda)$ *are exact Frobenius and definable subcategories of* Mod-Λ.
(ii) $\mathfrak{P}_\Lambda^{\prec\infty}$ *and* $\mathfrak{I}_\Lambda^{\prec\infty}$ *are exact subcategories of* Mod-Λ *with enough projectives and injectives.*
(iii) $\mathrm{CM}(\mathbf{P}_\Lambda)$ *is closed under cokernels of pure monomorphisms and* $\mathrm{CoCM}(\mathbf{I}_\Lambda)$ *is closed under kernels of pure epimorphisms.*
(iv) (a) *Any module admits a minimal right* $\mathrm{CM}(\mathbf{P}_\Lambda)$-, *respectively* $\mathrm{CoCM}(\mathbf{I}_\Lambda)$-, *approximation.*
(b) *Any module admits a minimal left* $\mathfrak{P}_\Lambda^{\prec\infty}$-, *respectively* $\mathrm{CoCM}(\mathbf{I}_\Lambda)^{\perp}$-, *approximation.*
(v) $\forall A \in \mathrm{Mod}\text{-}\Lambda$, $\forall X \in \mathrm{CM}(\mathbf{P}_\Lambda)$, $\forall Z \in \mathrm{CoCM}(\mathbf{I}_\Lambda)$, *there are isomorphisms* $\forall n \geqslant 1$, $\forall m \geqslant 0$:

$$\mathrm{Ext}^n_\Lambda(X, A) \cong \underline{\mathrm{Hom}}_\Lambda\big(\Omega^n(\underline{X}), \underline{A}\big) \quad \textit{and} \quad \mathrm{Ext}^n_\Lambda(A, Z) \cong \overline{\mathrm{Hom}}_\Lambda\big(\overline{A}, \Sigma^n\big(\overline{Z}\big)\big),$$
$$\mathrm{Ext}^1_\Lambda\big(\Sigma^{m+1}_{\mathbf{P}}(A), X\big) \cong \underline{\mathrm{Hom}}_\Lambda\big(\underline{A}, \Omega^m(\underline{X})\big) \quad \textit{and}$$
$$\mathrm{Ext}^1_\Lambda\big(Z, \Omega^{m+1}_{\mathbf{I}}(A)\big) \cong \overline{\mathrm{Hom}}_\Lambda\big(\Sigma^m\big(\overline{Z}\big), \overline{A}\big).$$

(vi) $\forall T \in \mathrm{mod}\text{-}\Lambda$, $\forall X \in \mathrm{CM}(\mathbf{P}_\Lambda)$, $\forall Z \in \mathrm{CoCM}(\mathbf{I}_\Lambda)$, *there are isomorphisms,* $\forall m \geqslant 0$:

$$\mathrm{D}\underline{\mathrm{Hom}}_\Lambda\big(T, \Omega^m(X)\big) \cong \mathrm{Ext}^{m+1}_\Lambda\big(X, \mathrm{DTr}(T)\big) \cong \mathrm{DExt}^1_\Lambda\big(\Sigma^{m+1}_{\mathbf{P}}(T), X\big),$$
$$\mathrm{DExt}^{m+1}_\Lambda\big(\mathrm{TrD}(T), Z\big) \cong \mathrm{D}\overline{\mathrm{Hom}}_\Lambda\big(\Sigma^m(Z), T\big) \cong \mathrm{Ext}^1_\Lambda\big(Z, \Omega^{m+1}_{\mathbf{I}}(T)\big).$$

Proof. (i) and (ii). Since $\mathrm{CM}(\mathbf{P}_\Lambda)$ is closed under extensions and admits the projectives as an Ext-injective cogenerator, it follows directly that $\mathrm{CM}(\mathbf{P}_\Lambda)$ is an exact Frobenius subcategory of Mod-Λ having $\mathbf{P}_\Lambda$ as the full subcategory of projective–injective objects. Dually

$\mathrm{CoCM}(\mathbf{I}_\Lambda)$ is an exact Frobenius subcategory of Mod-Λ having $\mathbf{I}_\Lambda$ as the full subcategory of projective–injective objects. By Remark 2.4(ii) and Lemma 3.3 it follows directly that $\mathrm{CM}(\mathbf{P}_\Lambda)$ is closed under filtered colimits and products. We show that $\mathrm{CM}(\mathbf{P}_\Lambda)$ is closed under pure-submodules. Since $\mathrm{CM}(\mathbf{P}_\Lambda)$ is resolving, by [46] it suffices to show that $\mathrm{CM}(\mathbf{P}_\Lambda)$ is closed under the double dual functor D^2. If X lies in $\mathrm{CM}(\mathbf{P}_\Lambda)$, then there exists an exact sequence $0 \to X \to P^0 \to P^1 \to \cdots$ where the P^i are projective and $\mathrm{Ker}(P^n \to P^{n+1}) \in {}^\perp\Lambda$, $\forall n \geqslant 0$. Then $0 \to \mathrm{D}^2(X) \to \mathrm{D}^2(P^0) \to \mathrm{D}^2(P^1) \to \cdots$ is exact and the $\mathrm{D}^2(P^i)$ are projective since D^2 is exact and preserves projectives. Since Λ is pure-injective, it is easy to see that ${}^\perp\Lambda$ is definable and therefore $\mathrm{Ker}(\mathrm{D}^2(P^n) \to \mathrm{D}^2(P^{n+1})) \in {}^\perp\Lambda$, $\forall n \geqslant 0$, see [46]. We infer that $\mathrm{D}^2(X)$ lies in $\mathrm{CM}(\mathbf{P}_\Lambda)$ and therefore $\mathrm{CM}(\mathbf{P}_\Lambda)$ is definable. Since the Nakayama functors $\mathsf{N}^\pm$ induce quasi-inverse equivalences between $\mathrm{CM}(\mathbf{P}_\Lambda)$ and $\mathrm{CoCM}(\mathbf{I}_\Lambda)$, it follows easily that $\mathrm{CoCM}(\mathbf{I}_\Lambda)$ is definable. Part (ii) follows directly from the fact that $\mathfrak{P}_\Lambda^{\prec\infty}$ and $\mathfrak{I}_\Lambda^{\prec\infty}$ are resolving and coresolving.

(iii) and (iv). Let $(E): 0 \to X_1 \xrightarrow{g} X_2 \to A \to 0$ be a pure short exact sequence in Mod-Λ where the X_i are Cohen–Macaulay. Let

$$\underline{X}_1 \xrightarrow{g} \underline{X}_2 \to \underline{X} \to \Sigma_{\mathbf{P}}(\underline{X}_1)$$

be a triangle in $\underline{\mathrm{CM}}(\mathbf{P}_\Lambda)$. Since g is a pure monomorphism and any projective is pure-injective, it follows that g is $\mathbf{P}_\Lambda$-monic and therefore (E) induces a right triangle $\underline{X}_1 \to \underline{X}_2 \to \underline{A} \to \Sigma_{\mathbf{P}}(\underline{X}_1)$ in $\underline{\mathrm{Mod}}$-Λ. Since $\underline{\mathrm{CM}}(\mathbf{P}_\Lambda)$ is closed under extensions of right triangles, we infer that $\underline{A} \cong \underline{X}$ lies in $\underline{\mathrm{CM}}(\mathbf{P}_\Lambda)$, i.e., A is Cohen–Macaulay. The proof for CoCohen–Macaulay modules is similar. Part (iv) follows from [59] since by (i) the subcategories $\mathrm{CM}(\mathbf{P}_\Lambda)$ and $\mathrm{CoCM}(\mathbf{I}_\Lambda)$ are contravariantly finite and closed under filtered colimits and extensions.

(v) and (vi). We prove only the assertions for Cohen–Macaulay modules. Let $0 \to \Omega(X) \xrightarrow{g} P_0 \to X \to 0$ be exact with P_0 projective. Then we have an exact sequence $\mathrm{Hom}_\Lambda(P_0, A) \xrightarrow{g_*} \mathrm{Hom}_\Lambda(\Omega(X), A) \xrightarrow{\vartheta} \mathrm{Ext}^1_\Lambda(X, A) \to 0$ and the canonical epic $\varpi : \mathrm{Hom}_\Lambda(\Omega(X), A) \to \underline{\mathrm{Hom}}_\Lambda(\Omega(X), A)$ admits a factorization $\mathrm{Hom}_\Lambda(\Omega(X), A) \xrightarrow{\vartheta} \mathrm{Ext}^1_\Lambda(X, A) \xrightarrow{\omega} \underline{\mathrm{Hom}}_\Lambda(\Omega(X), A)$. Since $\mathrm{Ext}^1_\Lambda(X, P) = 0$, for any projective module P, it follows easily that $\mathrm{Ker}\,\varpi = \mathrm{Im}\,g_*$. This implies by diagram chasing that ω is monic. Then ω is invertible since it is always epic, and the first isomorphism follows by dimension shift. Now let $A \to P^A \to \Sigma_{\mathbf{P}}(A) \to 0$ be exact and let $A \twoheadrightarrow \Omega\Sigma_{\mathbf{P}}(A) \rightarrowtail P^A$ be the canonical factorization of the minimal left projective approximation of $A \to P^A$ of A. From the exact sequence $0 \to \mathrm{Hom}_\Lambda(\Sigma_{\mathbf{P}}(A), X) \to \mathrm{Hom}_\Lambda(P^A, X) \to \mathrm{Hom}_\Lambda(\Omega\Sigma_{\mathbf{P}}(A), X) \to \mathrm{Ext}^1_\Lambda(\Sigma_{\mathbf{P}}(A), X) \to 0$ it follows that the canonically induced map $\mathrm{Ext}^1_\Lambda(\Sigma_{\mathbf{P}}(A), X) \to \underline{\mathrm{Hom}}_\Lambda(\Omega\Sigma_{\mathbf{P}}(\underline{A}), \underline{X})$ is invertible. Since any Cohen–Macaulay module is a syzygy module, by Remark 2.3 we have $\mathrm{Ext}^1_\Lambda(\Sigma_{\mathbf{P}}(A), X) \cong \underline{\mathrm{Hom}}_\Lambda(\Omega\Sigma_{\mathbf{P}}(\underline{A}), \underline{X}) \cong \underline{\mathrm{Hom}}_\Lambda(\underline{A}, \underline{X})$. Finally $\forall m \geqslant 1$ we have:

$$\underline{\mathrm{Hom}}_\Lambda\big(\underline{A}, \Omega^m(\underline{X})\big) \cong \underline{\mathrm{Hom}}_\Lambda\big(\Sigma^m_{\mathbf{P}}(\underline{A}), \underline{X}\big) \cong \mathrm{Ext}^1_\Lambda\big(\Sigma^{m+1}_{\mathbf{P}}(A), X\big).$$

Part (vi) follows from (v) and Auslander–Reiten formulas, cf. Remark 2.3. $\square$

If $\mathcal{U}$ is a resolving, respectively coresolving, subcategory of an abelian category $\mathcal{A}$, we let $\mathcal{U}^{\leqslant n}$ be the full subcategory of $\mathcal{A}$ consisting of all objects A with $\text{res.dim}_{\mathcal{U}} A \leqslant n$, respectively $\text{cores.dim}_{\mathcal{U}} A \leqslant n$. In particular $\mathbf{P}_\Lambda^{\leqslant n}$, respectively $\mathbf{I}_\Lambda^{\leqslant n}$, is the subcategory of all modules with projective, respectively injective, dimension $\leqslant n$. By [5, Lemma 3.12], $\text{res.dim}_{\mathcal{U}} A \leqslant n$ iff $\Omega^n(A) \in \mathcal{U}$ and $\text{cores.dim}_{\mathcal{U}} A \leqslant n$ iff $\Sigma^n(A) \in \mathcal{U}$.

The following result gives an alternative way to compute self-injective and finitistic dimensions.

Proposition 3.9.

(i) $\mathrm{CM}(\mathbf{P}_\Lambda)^{\leqslant n} \cap \mathfrak{P}_\Lambda^{\prec\propto} = \mathbf{P}_\Lambda^{\leqslant n}$ *and* $\mathfrak{I}_\Lambda^{\prec\propto} \cap \mathrm{CoCM}(\mathbf{I}_\Lambda)^{\leqslant n} = \mathbf{I}_\Lambda^{\leqslant n}$.

(ii) ${}^\perp\Lambda \cap \widehat{\mathrm{CM}(\mathbf{P}_\Lambda)} = \mathrm{CM}(\mathbf{P}_\Lambda)$ *and* $\mathrm{D}(\Lambda)^\perp \cap \mathrm{Co}\widetilde{\mathrm{CM}(\mathbf{I}_\Lambda)} = \mathrm{CoCM}(\mathbf{I}_\Lambda)$.

(iii) *For any non-zero module A in $\widehat{\mathrm{CM}(\mathbf{P}_\Lambda)}$ and any non-zero module B in $\mathrm{Co}\widetilde{\mathrm{CM}(\mathbf{I}_\Lambda)}$ we have*:

$$\text{res.dim}_{\mathrm{CM}} A = \sup\bigl\{n \geqslant 0 \mid \mathrm{Ext}_\Lambda^n(A, \Lambda) \neq 0\bigr\},$$
$$\text{cores.dim}_{\mathrm{CoCM}} B = \sup\bigl\{n \geqslant 0 \mid \mathrm{Ext}_\Lambda^n\bigl(\mathrm{D}(\Lambda), B\bigr) \neq 0\bigr\}.$$

(iv) $\forall W \in \mathfrak{I}_\Lambda^{\prec\propto}$: $\mathrm{id}\, W = \text{cores.dim}_{\mathrm{CoCM}} W$ *and* $\forall Y \in \mathfrak{P}_\Lambda^{\prec\propto}$: $\mathrm{pd}\, Y = \text{res.dim}_{\mathrm{CM}} Y$.

(v) $\mathrm{id}\, \Lambda_\Lambda = \text{cores.dim}_{\mathrm{CoCM}} \Lambda$ *and* $\mathrm{pd}\, \mathrm{D}(\Lambda)_\Lambda = \text{res.dim}_{\mathrm{CM}} \mathrm{D}(\Lambda)$.

(vi) $\mathrm{FPD}(\Lambda) = \sup\{\text{res.dim}_{\mathrm{CM}} C \mid C \in \widehat{\mathrm{CM}(\mathbf{P}_\Lambda)}\}$.

(vii) $\mathrm{FID}(\Lambda) = \sup\{\text{cores.dim}_{\mathrm{CoCM}} C \mid C \in \mathrm{Co}\widetilde{\mathrm{CM}(\mathbf{I}_\Lambda)}\}$.

Proof. (i) Clearly $\mathbf{P}_\Lambda^{\leqslant n} \subseteq \mathrm{CM}(\mathbf{P}_\Lambda)^{\leqslant n} \cap \mathfrak{P}_\Lambda^{\prec\propto}$. If A lies in $\mathrm{CM}(\mathbf{P}_\Lambda)^{\leqslant n} \cap \mathfrak{P}_\Lambda^{\prec\propto}$, then, since $\mathfrak{P}_\Lambda^{\prec\propto}$ is resolving, $\Omega^n(A)$ lies in $\mathrm{CM}(\mathbf{P}_\Lambda) \cap \mathfrak{P}_\Lambda^{\prec\propto}$. Hence, by Theorem 3.5, $\Omega^n(A)$ is projective and therefore A lies in $\mathbf{P}_\Lambda^{\leqslant n}$. It follows that $\mathrm{CM}(\mathbf{P}_\Lambda)^{\leqslant n} \cap \mathfrak{P}_\Lambda^{\prec\propto} = \mathbf{P}_\Lambda^{\leqslant n}$ and dually $\mathfrak{I}_\Lambda^{\prec\propto} \cap \mathrm{CoCM}(\mathbf{I}_\Lambda)^{\leqslant n} = \mathbf{I}_\Lambda^{\leqslant n}$.

(ii) Clearly $\mathrm{CM}(\mathbf{P}_\Lambda) \subseteq {}^\perp\Lambda \cap \widehat{\mathrm{CM}(\mathbf{P}_\Lambda)}$. By [22, Proposition VI.2.3], for any module A we have $A \in \widehat{\mathrm{CM}(\mathbf{P}_\Lambda)}$ iff $Y_A \in \mathbf{P}_\Lambda^{<\infty}$. Hence if A lies in ${}^\perp\Lambda \cap \widehat{\mathrm{CM}(\mathbf{P}_\Lambda)}$, then $\mathrm{pd}\, Y_A < \infty$. This implies, using dimension shifting on a projective resolution of Y_A, that $\mathrm{Ext}_\Lambda^1(A, Y_A) = 0$ and therefore the special right $\mathrm{CM}(\mathbf{P}_\Lambda)$-approximation sequence $0 \to Y_A \to X_A \to A \to 0$ of A splits. Hence A is Cohen–Macaulay and consequently $\mathrm{CM}(\mathbf{P}_\Lambda) = {}^\perp\Lambda \cap \widehat{\mathrm{CM}(\mathbf{P}_\Lambda)}$. The second equality is proved similarly.

(iii) Let $\delta = \sup_{n \geqslant 0}\{\mathrm{Ext}_\Lambda^n(A, \Lambda) \neq 0\}$ and $d = \text{res.dim}_{\mathrm{CM}} A < \infty$. Since $\Omega^d(A)$ lies in $\mathrm{CM}(\mathbf{P}_\Lambda)$ we have $\mathrm{Ext}_\Lambda^n(A, \Lambda) = 0$, $\forall n \geqslant d+1$. If $\mathrm{Ext}_\Lambda^d(A, \Lambda) = 0$, then pulling-back the exact sequence $0 \to \Omega^d(A) \to P^{d-1} \to \Omega^{d-1}(A) \to 0$, where P^{d-1} is projective, along the left projective approximation $\Omega^d(A) \to P^{\Omega^d(A)}$ of $\Omega^d(A)$, we get an exact sequence $0 \to P^{\Omega^d(A)} \to \Sigma_{\mathbf{P}}(\Omega^d(A)) \oplus P^{d-1} \to \Omega^{d-1}(A) \to 0$ which splits by our assumption. Since $\Sigma_{\mathbf{P}}(\Omega^d(A))$ is Cohen–Macaulay, we infer that so is $\Omega^{d-1}(A)$ and therefore $\text{res.dim}_{\mathrm{CM}}(A) \leqslant d-1$, a contradiction. Hence $\mathrm{Ext}_\Lambda^d(A, \Lambda) \neq 0$ and therefore $\delta \leqslant d$. If $\delta < d$, then clearly $\Omega^\delta(A)$ lies in ${}^\perp\Lambda \cap \widehat{\mathrm{CM}(\mathbf{P}_\Lambda)}$ which is equal to $\mathrm{CM}(\mathbf{P}_\Lambda)$ by (ii). This implies that $\operatorname{res.dim}_{\mathrm{CM}} A \leqslant \delta$, a contradiction. Hence $\delta = d$. The second equality is proved similarly.

(iv) and (v). Clearly $\operatorname{cores.dim}_{\mathrm{CoCM}} W \leqslant \operatorname{id} W$ and it suffices to show that $\operatorname{id} W \leqslant \operatorname{cores.dim}_{\mathrm{CoCM}} W$ if the latter, say n, is finite. If $n = 0$, then $W \in \mathfrak{I}_\Lambda^{\prec\propto} \cap \mathrm{CoCM}(\mathbf{I}_\Lambda) = \mathbf{I}_\Lambda$ and therefore $\operatorname{id} W = 0$. If $n > 1$, then since $\Sigma^n(W) \in \mathrm{CoCM}(\mathbf{I}_\Lambda)$ and $\mathfrak{I}_\Lambda^{\prec\propto}$ is coresolving, we infer that $\Sigma^n(W) \in \mathfrak{I}_\Lambda^{\prec\propto} \cap \mathrm{CoCM}(\mathbf{I}_\Lambda) = \mathbf{I}_\Lambda$ and therefore $\operatorname{id} W \leqslant n$. Hence $\operatorname{id} W = \operatorname{cores.dim}_{\mathrm{CoCM}} W$. The second equality is proved similarly. Then part (v) follows directly from (iv) since $\Lambda \in \mathfrak{I}_\Lambda^{\prec\propto}$ and $\mathrm{D}(\Lambda) \in \mathfrak{P}_\Lambda^{\prec\propto}$.

(vi) and (vii). Let $\sup\{\operatorname{res.dim}_{\mathrm{CM}} C \mid C \in \widehat{\mathrm{CM}(\mathbf{P}_\Lambda)}\} = d$ and $\mathrm{FPD}(\Lambda) = \delta$. If $d < \infty$, then $\widehat{\mathrm{CM}(\mathbf{P}_\Lambda)} = \mathrm{CM}(\mathbf{P}_\Lambda)^{\leqslant d}$ and therefore, using (i),

$$\mathbf{P}_\Lambda^{<\infty} = \widehat{\mathrm{CM}(\mathbf{P}_\Lambda)} \cap \mathfrak{P}_\Lambda^{\prec\propto} = \mathrm{CM}(\mathbf{P}_\Lambda)^{\leqslant d} \cap \mathfrak{P}_\Lambda^{\prec\propto} = \mathbf{P}_\Lambda^{\leqslant d}.$$

Hence $\delta \leqslant d$. On the other hand if $\delta < \infty$, so $\mathbf{P}_\Lambda^{\leqslant\delta} = \mathbf{P}_\Lambda^{<\infty}$, then let $A \in \widehat{\mathrm{CM}(\mathbf{P}_\Lambda)}$ and let $0 \to A \to Y^A \to X^A \to 0$ be a special left $\mathfrak{P}_\Lambda^{\prec\propto}$-approximation sequence of A. Clearly Y^A lies in $\widehat{\mathrm{CM}(\mathbf{P}_\Lambda)} \cap \mathfrak{P}_\Lambda^{\prec\propto} = \mathbf{P}_\Lambda^{<\infty} = \mathbf{P}_\Lambda^{\leqslant\delta}$. Since $\operatorname{pd} Y^A \leqslant \delta$, by the first diagram in Remark 3.6 it follows that $Y_A = \Omega(Y^A)$ lies in $\mathbf{P}_\Lambda^{\leqslant\delta-1}$ and this implies that $\operatorname{res.dim}_{\mathrm{CM}} A \leqslant \delta$. Hence $d \leqslant \delta$. Part (vi) is proved similarly. $\square$

Note. Since $\mathrm{CM}(\mathbf{P}_\Lambda)$ is contravariantly finite and $\mathrm{CoCM}(\mathbf{I}_\Lambda)$ is covariantly finite, it follows by [22, Lemma IX.3.1] that $\forall A \in \text{Mod-}\Lambda$: $\operatorname{res.dim}_{\mathrm{CM}} A = \text{CM-pd}\, A$ and $\operatorname{cores.dim}_{\mathrm{CoCM}} A = \text{CoCM-id}\, A$, where CM-pd, respectively CoCM-id, denotes relative projective, respectively injective, dimension with respect to the relative homological algebra in Mod-Λ induced by the (Co)Cohen–Macaulay modules.

We close this section with some remarks on Gorenstein algebras. Recall that an Artin algebra is called **Gorenstein** if $\operatorname{id}_\Lambda \Lambda < \infty$ and $\operatorname{id} \Lambda_\Lambda < \infty$. Equivalently $\mathcal{P}_\Lambda^{<\infty} = \mathcal{I}_\Lambda^{<\infty}$, or $\mathbf{P}_\Lambda^{<\infty} = \mathbf{I}_\Lambda^{<\infty}$. It is well known that for a Gorenstein algebra Λ we have $\operatorname{id}_\Lambda \Lambda = \operatorname{id} \Lambda_\Lambda$. Recall that $T \in \text{Mod-}\Lambda$ is called a **tilting** module if:

(α) T has finite projective dimension,
(β) $\operatorname{Ext}_\Lambda^n(T, T^{(I)}) = 0$, for any $n \geqslant 1$ and index set I, and
(γ) $\Lambda \in \widetilde{\operatorname{Add}(T)}$.

Dually $T \in \text{Mod-}\Lambda$ is called a **cotilting** module if:

(α) T has finite injective dimension,
(β) $\operatorname{Ext}_\Lambda^n(T^I, T) = 0$, for any $n \geqslant 1$ and index set I, and
(γ) $\mathrm{D}(\Lambda) \in \widehat{\operatorname{Prod}(T)}$.

More generally a subcategory $\mathcal{T}$, respectively $\mathcal{C}$, of Mod-Λ is called **tilting**, respectively **cotilting**, **subcategory** if there exists a tilting, respectively cotilting, module T, respectively C, such that $\mathcal{T} = T^\perp$, respectively $\mathcal{C} = {}^\perp C$, see [3].

In the sequel we shall need the following characterizations of Gorensteinness, many of them well known for finitely generated modules, which follow from Theorem 3.5, Proposition 3.8 and [4,18,22]. In essence for Gorenstein algebras “all dimensions are finite and equal”.

Proposition 3.10. *The following are equivalent*:

(i) Λ *is Gorenstein.*
(ii) $\mathrm{cores.dim}_{\mathrm{CoCM}}\,\Lambda < \infty$ *and* $\mathrm{res.dim}_{\mathrm{CM}}\,\mathrm{D}(\Lambda) < \infty$.
(iii) $\mathrm{fpd}(\Lambda^{\mathrm{op}}) < \infty$ *and* $\mathrm{id}\,\Lambda_\Lambda < \infty$, *equivalently* $\mathrm{fpd}(\Lambda) < \infty$ *and* $\mathrm{id}\,{}_\Lambda\Lambda < \infty$.
(iv) $\widehat{\mathrm{CM}(\mathbf{P}_\Lambda)} \cap \mathfrak{P}_\Lambda^{\prec\infty} = \mathfrak{I}_\Lambda^{\prec\infty} \cap \widetilde{\mathrm{CoCM}(\mathbf{I}_\Lambda)}$.
(v) $\mathrm{res.dim}_{\mathrm{CM}}\,\mathrm{Mod}\text{-}\Lambda < \infty$ *or equivalently* $\mathrm{cores.dim}_{\mathrm{CoCM}}\,\mathrm{Mod}\text{-}\Lambda < \infty$.
(vi) $\mathfrak{P}_\Lambda^{\prec\infty} = \mathbf{P}_\Lambda^{<\infty}$ *or equivalently* $\widehat{\mathrm{CM}(\mathbf{P}_\Lambda)} = \mathrm{Mod}\text{-}\Lambda$.
(vii) $\mathfrak{I}_\Lambda^{\prec\infty} = \mathbf{I}_\Lambda^{<\infty}$ *or equivalently* $\widetilde{\mathrm{CoCM}(\mathbf{I}_\Lambda)} = \mathrm{Mod}\text{-}\Lambda$.
(viii) $(\mathrm{CM}(\mathbf{P}_\Lambda), \mathbf{P}_\Lambda^{<\infty})$ *is a cotorsion pair in* $\mathrm{Mod}\text{-}\Lambda$.
(ix) $(\mathbf{I}_\Lambda^{<\infty}, \mathrm{CoCM}(\mathbf{I}_\Lambda))$ *is a cotorsion pair in* $\mathrm{Mod}\text{-}\Lambda$.
(x) *The full subcategory* $\mathrm{CM}(\mathbf{P}_\Lambda)$, *respectively* $\mathrm{CoCM}(\mathbf{I}_\Lambda)$, *is cotilting, respectively tilting.*
(xi) Λ *is a cotilting module, or equivalently* $\mathrm{D}(\Lambda)$ *is a tilting module.*

If Λ *is Gorenstein, then* $\mathrm{CM}(\mathbf{P}_\Lambda) = {}^\perp\Lambda$ *and* $\mathrm{CoCM}(\mathbf{I}_\Lambda) = \mathrm{D}(\Lambda)^\perp$, *and moreover*:

$$\begin{aligned}\mathrm{id}\,{}_\Lambda\Lambda &= \mathrm{id}\,\Lambda_\Lambda = \mathrm{fpd}(\Lambda) = \mathrm{fid}(\Lambda) = \mathrm{FPD}(\Lambda) = \mathrm{FID}(\Lambda)\\ &= \mathrm{cores.dim}_{\mathrm{CoCM}}\,\Lambda = \mathrm{res.dim}_{\mathrm{CM}}\,\mathrm{D}(\Lambda) = \mathrm{res.dim}_{\mathrm{CM}}\,\mathrm{Mod}\text{-}\Lambda\\ &= \mathrm{cores.dim}_{\mathrm{CoCM}}\,\mathrm{Mod}\text{-}\Lambda < \infty.\end{aligned}$$

In view of Propositions 3.9 and 3.10, it is now natural to define the **virtual finitistic projective**, respectively **injective**, **dimension** $\mathrm{vFPD}(\Lambda)$, respectively $\mathrm{vFID}(\Lambda)$ of Λ as follows:

$$\mathrm{vFPD}(\Lambda) := \sup\{\mathrm{pd}\,Y \mid Y \in \mathfrak{P}_\Lambda^{\prec\infty}\} = \mathrm{res.dim}_{\mathrm{CM}}\,\mathfrak{P}_\Lambda^{\prec\infty},$$
$$\mathrm{vFID}(\Lambda) := \sup\{\mathrm{id}\,Y \mid Y \in \mathfrak{I}_\Lambda^{\prec\infty}\} = \mathrm{cores.dim}_{\mathrm{CoCM}}\,\mathfrak{I}_\Lambda^{\prec\infty}.$$

Clearly $\mathrm{FPD}(\Lambda) \leqslant \mathrm{vFPD}(\Lambda)$ and $\mathrm{FID}(\Lambda) \leqslant \mathrm{vFID}(\Lambda)$. Since $\mathrm{vFPD}(\Lambda) < \infty$, respectively $\mathrm{vFID}(\Lambda) < \infty$, is equivalent to $\mathfrak{P}_\Lambda^{\prec\infty} = \mathbf{P}_\Lambda^{<\infty}$, respectively $\mathfrak{I}_\Lambda^{\prec\infty} = \mathbf{I}_\Lambda^{<\infty}$, we have the following consequence which shows that Gorenstein algebras are characterized by the finiteness of the virtual finitistic dimensions.

Corollary 3.11. Λ *is Gorenstein iff* $\mathrm{vFPD}(\Lambda) < \infty$ *iff* $\mathrm{vFPD}(\Lambda) < \infty$.

The following result shows that for modules with finite (Co)Cohen–Macaulay (co)resolution dimension we have an explicit description of their (Co)Cohen–Macaulay approximations.

Proposition 3.12. *For a module A in $\widehat{\mathrm{CM}(\mathbf{P}_\Lambda)}$ and a module B in $\widetilde{\mathrm{CoCM}(\mathbf{I}_\Lambda)}$, we have the following.*

(i) *The counit $\Sigma^d_{\mathbf{P}}\Omega^d(\underline{A}) \to \underline{A}$ is the coreflection of $\underline{A}$ in $\underline{\mathrm{CM}}(\mathbf{P}_\Lambda)$, where $d =$* res.dim$_{\mathrm{CM}}$ A.
(ii) *The unit $\overline{B} \to \Omega^\delta_{\mathbf{I}}\Sigma^\delta(\overline{B})$ is the reflection of $\overline{B}$ in $\overline{\mathrm{CoCM}}(\mathbf{I}_\Lambda)$, where $\delta =$* cores.dim$_{\mathrm{CoCM}}$ B.

Proof. Since $\Omega^d(A) \in \mathrm{CM}(\mathbf{P}_\Lambda)$ and, by Proposition 3.8(iii), $\mathrm{CM}(\mathbf{P}_\Lambda)$ is closed under left projective approximations we infer that $\Sigma^d_{\mathbf{P}}\Omega^d(A)$ is Cohen–Macaulay. Let $\varepsilon^d_{\underline{A}} : \Sigma^d_{\mathbf{P}}\Omega^d(\underline{A}) \to \underline{A}$ be the counit of the adjoint pair $(\Sigma^d_{\mathbf{P}}, \Omega^d)$ in $\underline{\mathrm{Mod}}$-Λ and let $\underline{\alpha} : \underline{X} \to \underline{A}$ be a morphism where X lies in $\mathrm{CM}(\mathbf{P}_\Lambda)$. Since the natural map $\varepsilon^d_{\underline{X}} : \Sigma^d_{\mathbf{P}}\Omega^d(\underline{X}) \to \underline{X}$ is invertible, we deduce directly that $\underline{\alpha} = (\varepsilon^d_{\underline{X}})^{-1} \circ \Sigma^d_{\mathbf{P}}\Omega^d(\underline{\alpha}) \circ \varepsilon^d_{\underline{A}}$. Hence $\underline{\alpha}$ factors through $\varepsilon^d_{\underline{A}}$ and clearly this factorization is unique since $\Sigma_{\mathbf{P}}|_{\underline{\mathrm{CM}}(\mathbf{P}_\Lambda)}$ is an equivalence. Hence $\Sigma^d_{\mathbf{P}}\Omega^d(\underline{A})$ is evaluation of the coreflection of $\underline{A}$ in $\underline{\mathrm{CM}}(\mathbf{P}_\Lambda)$. Part (ii) is similar. $\square$

The above result in connection with Proposition 3.10 admits the following consequence, see also [18], which shows in particular that for Gorenstein algebras the functors $\mathbf{R}_{\mathrm{CM}} : \underline{\mathrm{Mod}}\text{-}\Lambda \to \underline{\mathrm{CM}}(\mathbf{P}_\Lambda)$ and $\mathbf{L}_{\mathrm{CoCM}} : \overline{\mathrm{Mod}}\text{-}\Lambda \to \overline{\mathrm{CoCM}}(\mathbf{I}_\Lambda)$ preserves products and coproducts. These conditions will play an important role later in the paper in connection with virtually Gorenstein algebras.

Corollary 3.13. *Let Λ be a Gorenstein algebra with* $\mathrm{id}\,\Lambda = d$. *Then there exist isomorphisms of functors*

$$\mathbf{R}_{\mathrm{CM}} \cong \Sigma^d_{\mathbf{P}}\Omega^d : \underline{\mathrm{Mod}}\text{-}\Lambda \to \underline{\mathrm{CM}}(\mathbf{P}_\Lambda) \quad \textit{and} \quad \mathbf{L}_{\mathrm{CoCM}} \cong \Omega^d_{\mathbf{I}}\Sigma^d : \overline{\mathrm{Mod}}\text{-}\Lambda \to \overline{\mathrm{CoCM}}(\mathbf{I}_\Lambda).$$

In particular $\mathbf{R}_{\mathrm{CM}}$, respectively $\mathbf{L}_{\mathrm{CoCM}}$, preserves coproducts, respectively products.

4. Cohen–Macaulay modules and (co)stabilizations

In this section we discuss structural properties of the (Co)Cohen–Macaulay modules and (co)torsion pairs in connection with the behavior of the (co)universal triangulated categories associated to the stable module categories modulo projectives/injectives. These properties will be useful later in connection with virtually Gorenstein algebras and algebras with finite left or right self-injective dimension.

4.1. Stabilizations

Recall from [18] that to a given left triangulated category $\mathcal{C}$, there is associated in a universal way a triangulated category $\mathcal{T}(\mathcal{C})$ which reflects many important homological properties of $\mathcal{C}$. More precisely there exists a left exact functor $\mathsf{T} : \mathcal{C} \to \mathcal{T}(\mathcal{C})$ such that for any left exact functor $F : \mathcal{C} \to \mathcal{T}$ to a triangulated category $\mathcal{T}$, there exists a unique up to isomorphism exact functor $F^* : \mathcal{T}(\mathcal{C}) \to \mathcal{T}$ such that $F^*\mathsf{T} \cong F$. The category $\mathcal{T}(\mathcal{C})$ is called the **stabilization** of $\mathcal{C}$ and the functor T is called the **stabilization functor**. We refer to [18,40] for details.

Here we need only the following facts. The objects of $\mathcal{T}(\mathcal{C})$ are pairs (C, n), where C is an object in $\mathcal{C}$ and $n \in \mathbb{Z}$. The space of morphisms $\mathcal{T}(\mathcal{C})[(A, n), (B, m)]$ is identified with the direct limit:

$$\mathcal{T}(\mathcal{C})\big[(A, n), (B, m)\big] = \varinjlim_{k \geqslant n, k \geqslant m} \mathcal{C}\big[\Omega^{k-n}(A), \Omega^{k-m}(B)\big].$$

The loop functor $\Omega : \mathcal{T}(\mathcal{C}) \to \mathcal{T}(\mathcal{C})$ is defined by $\Omega(C, n) = (C, n-1)$ and the stabilization functor T is defined by $\mathsf{T}(C) = (C, 0)$. Finally the extension of the left exact functor F above is defined by $F^*(C, n) = \Omega^n F(C)$. Dually any right triangulated category admits its stabilization which has a dual description and dual properties. In particular if $\mathcal{C}$ is a pretriangulated category, then $\mathcal{C}$ admits a stabilization $\mathsf{T}_l : \mathcal{C} \to \mathcal{T}_l(\mathcal{C})$ when considered as a left triangulated category, and a stabilization $\mathsf{T}_r : \mathcal{C} \to \mathcal{T}_r(\mathcal{C})$ when considered as a right triangulated category.

Now the stable module categories $\underline{\mathrm{Mod}}$-Λ and $\overline{\mathrm{Mod}}$-Λ are both pretriangulated. To avoid confusion, we shall use the following notations:

- $\mathsf{P}_l : \underline{\mathrm{Mod}}\text{-}\Lambda \to \mathcal{T}_l(\underline{\mathrm{Mod}}\text{-}\Lambda)$ is the **left projective stabilization functor**, i.e., the stabilization functor of the left triangulated category $\underline{\mathrm{Mod}}$-Λ.
- $\mathsf{P}_r : \underline{\mathrm{Mod}}\text{-}\Lambda \to \mathcal{T}_r(\underline{\mathrm{Mod}}\text{-}\Lambda)$ is the **right projective stabilization functor**, i.e., the stabilization functor of the right triangulated category $\underline{\mathrm{Mod}}$-Λ.
- $\mathsf{Q}_r : \overline{\mathrm{Mod}}\text{-}\Lambda \to \mathcal{T}_r(\overline{\mathrm{Mod}}\text{-}\Lambda)$ is the **right injective stabilization functor**, i.e., the stabilization functor of the right triangulated category $\overline{\mathrm{Mod}}$-Λ.
- $\mathsf{Q}_l : \overline{\mathrm{Mod}}\text{-}\Lambda \to \mathcal{T}_l(\overline{\mathrm{Mod}}\text{-}\Lambda)$ is the **left injective stabilization functor**, i.e., the stabilization functor of the left triangulated category $\overline{\mathrm{Mod}}$-Λ.

Note that $\underline{\mathrm{mod}}$-Λ and $\overline{\mathrm{mod}}$-Λ are pretriangulated subcategories of $\underline{\mathrm{Mod}}$-Λ and $\overline{\mathrm{Mod}}$-Λ with stabilization functors the restrictions of the stabilization functors of $\underline{\mathrm{Mod}}$-Λ and $\overline{\mathrm{Mod}}$-Λ to $\underline{\mathrm{mod}}$-Λ and $\overline{\mathrm{mod}}$-Λ respectively. The inclusions $\underline{\mathrm{mod}}\text{-}\Lambda \hookrightarrow \underline{\mathrm{Mod}}\text{-}\Lambda$ and $\overline{\mathrm{mod}}\text{-}\Lambda \hookrightarrow \overline{\mathrm{Mod}}\text{-}\Lambda$ extend to inclusions $\mathcal{T}_*(\underline{\mathrm{mod}}\text{-}\Lambda) \hookrightarrow \mathcal{T}_*(\underline{\mathrm{Mod}}\text{-}\Lambda)$ and $\mathcal{T}_*(\overline{\mathrm{mod}}\text{-}\Lambda) \hookrightarrow \mathcal{T}_*(\overline{\mathrm{Mod}}\text{-}\Lambda)$ respectively, where $* = r, l$.

In the sequel we shall need the following useful results.

Lemma 4.1 [18]. *Let $\mathcal{C}$ be a right, respectively left, triangulated category with suspension, respectively loop, functor Σ, respectively Ω. Then the right, respectively left, stabilization $\mathcal{T}_r(\mathcal{C})$, respectively $\mathcal{T}_l(\mathcal{C})$, of $\mathcal{C}$ is trivial if and only if for any object C in $\mathcal{C}$ there exists $n = n_C \geqslant 0$ such that $\Sigma^n(C) = 0$, respectively $\Omega^n(C) = 0$.*

In particular $\mathcal{T}_l(\underline{\mathrm{Mod}}\text{-}\Lambda) = 0$ iff $\operatorname{gl.dim} \Lambda < \infty$ *iff $\mathcal{T}_r(\overline{\mathrm{Mod}}\text{-}\Lambda) = 0$.*

As a direct consequence of the above lemma we have:

- $\operatorname{Ker} \mathsf{P}_l = \mathbf{P}_\Lambda^{<\infty}/\mathbf{P}_\Lambda$ and $\operatorname{Ker} \mathsf{P}_r = \{\underline{A} \in \underline{\mathrm{Mod}}\text{-}\Lambda \mid \operatorname{id} \tau^+ A < \infty\}$.
- $\operatorname{Ker} \mathsf{Q}_l = \{\underline{A} \in \underline{\mathrm{Mod}}\text{-}\Lambda \mid \operatorname{pd} \tau^- A < \infty\}$ and $\operatorname{Ker} \mathsf{Q}_r = \mathbf{I}_\Lambda^{<\infty}/\mathbf{I}_\Lambda$.

Lemma 4.2 [18].

(1) *The map* $\underline{\mathrm{Hom}}_\Lambda(\underline{X}, \underline{A}) \to \mathrm{Hom}[\mathsf{P}_l(\underline{X}), \mathsf{P}_l(\underline{A})]$ *is invertible,* $\forall X \in {}^\perp\Lambda$ *and* $\forall A \in \text{Mod-}\Lambda$. *In particular the exact functor* $\mathsf{P}_l\mathbf{i}_{\mathrm{CM}} : \underline{\mathrm{CM}}(\mathbf{P}_\Lambda) \to \mathcal{T}_l(\underline{\mathrm{Mod}}\text{-}\Lambda)$ *is fully faithful.*

(2) *The map* $\overline{\mathrm{Hom}}_\Lambda(\overline{A}, \overline{Z}) \to \mathrm{Hom}[\mathsf{Q}_r(\overline{A}), \mathsf{Q}_r(\overline{Z})]$ *is invertible for any module* $Z \in \mathrm{D}(\Lambda)^\perp$ *and any module* A. *In particular the exact functor* $\mathsf{Q}_r\mathbf{i}_{\mathrm{CoCM}} : \overline{\mathrm{CoCM}}(\mathbf{I}_\Lambda) \to \mathcal{T}_r(\overline{\mathrm{Mod}}\text{-}\Lambda)$ *is fully faithful.*

The next remark gives more concrete realizations of the stabilization categories.

Remark 4.3. Let $\mathcal{H}^{\mathrm{b}}(\mathbf{P}_\Lambda)$, respectively $\mathcal{H}^{\mathrm{b}}(\mathbf{I}_\Lambda)$, be the bounded homotopy category of complexes of projective, respectively injective, modules. Also let $\mathbf{D}^{\mathrm{b}}(\text{Mod-}\Lambda)$ be the bounded derived category of Mod-Λ. Then $\mathcal{H}^{\mathrm{b}}(\mathbf{P}_\Lambda)$ and $\mathcal{H}^{\mathrm{b}}(\mathbf{I}_\Lambda)$ are thick subcategories of $\mathbf{D}^{\mathrm{b}}(\text{Mod-}\Lambda)$ and by a result of Keller–Vossieck [40], see also [18], the canonical functors $\mathbf{D}^{\mathrm{b}}(\text{Mod-}\Lambda)/\mathcal{H}^{\mathrm{b}}(\mathbf{I}_\Lambda) \leftarrow \text{Mod-}\Lambda \to \mathbf{D}^{\mathrm{b}}(\text{Mod-}\Lambda)/\mathcal{H}^{\mathrm{b}}(\mathbf{P}_\Lambda)$ from Mod-Λ to the corresponding Verdier quotients induce triangle equivalences:

$$\mathcal{T}_l(\underline{\mathrm{Mod}}\text{-}\Lambda) \xrightarrow{\approx} \mathbf{D}^{\mathrm{b}}(\text{Mod-}\Lambda)/\mathcal{H}^{\mathrm{b}}(\mathbf{P}_\Lambda) \quad \text{and} \quad \mathcal{T}_r\big(\overline{\mathrm{Mod}}\text{-}\Lambda\big) \xrightarrow{\approx} \mathbf{D}^{\mathrm{b}}(\text{Mod-}\Lambda)/\mathcal{H}^{\mathrm{b}}(\mathbf{I}_\Lambda).$$

If $\mathcal{H}^{\mathrm{b}}(\mathcal{P}_\Lambda)$, respectively $\mathcal{H}^{\mathrm{b}}(\mathcal{I}_\Lambda)$, is the bounded homotopy category of complexes of finitely generated projective, respectively injective, modules and $\mathbf{D}^{\mathrm{b}}(\text{mod-}\Lambda)$ is the bounded derived category of mod-Λ, then the above triangle equivalences restrict to triangle equivalences

$$\mathcal{T}_l(\underline{\mathrm{mod}}\text{-}\Lambda) \xrightarrow{\approx} \mathbf{D}^{\mathrm{b}}(\text{mod-}\Lambda)/\mathcal{H}^{\mathrm{b}}(\mathcal{P}_\Lambda) \quad \text{and} \quad \mathcal{T}_r\big(\overline{\mathrm{mod}}\text{-}\Lambda\big) \xrightarrow{\approx} \mathbf{D}^{\mathrm{b}}(\text{mod-}\Lambda)/\mathcal{H}^{\mathrm{b}}(\mathcal{I}_\Lambda).$$

We denote by $\Omega^\infty(\text{Mod-}\Lambda)$, respectively $\Omega^n(\text{Mod-}\Lambda)$, the full subcategory of Mod-Λ consisting of the projectives and the arbitrary syzygy, respectively nth-syzygy, modules. The induced stable categories modulo projectives are denoted by $\Omega^\infty(\underline{\mathrm{Mod}}\text{-}\Lambda)$ and $\Omega^n(\underline{\mathrm{Mod}}\text{-}\Lambda)$ respectively. Dually we denote by $\Sigma^\infty(\text{Mod-}\Lambda)$, respectively $\Sigma^n(\text{Mod-}\Lambda)$, the full subcategory of Mod-Λ consisting of the injectives and the arbitrary cosyzygy, respectively nth-cosyzygy, modules. The induced stable categories modulo injectives are denoted by $\Sigma^\infty(\overline{\mathrm{Mod}}\text{-}\Lambda)$, respectively $\Sigma^n(\overline{\mathrm{Mod}}\text{-}\Lambda)$ respectively. Part (i) of the following result generalizes a result of Happel [33] and part (iv) generalizes a result of Auslander–Buchweitz [4].

Proposition 4.4. *If* $\mathrm{id}\,\Lambda_\Lambda = d < \infty$, *then we have the following.*

(i) *The stabilization functors*

$$\mathsf{Q}_r : \overline{\mathrm{Mod}}\text{-}\Lambda \to \mathcal{T}_r\big(\overline{\mathrm{Mod}}\text{-}\Lambda\big) \quad \textit{and} \quad \mathsf{P}_l : \underline{\mathrm{Mod}}\text{-}\Lambda^{\mathrm{op}} \to \mathcal{T}_l\big(\underline{\mathrm{Mod}}\text{-}\Lambda^{\mathrm{op}}\big)$$

are surjective on objects: $\forall(\overline{A}, n) \in \mathcal{T}_r(\overline{\text{Mod}}\text{-}\Lambda)$, $\forall(\underline{A}, n) \in \mathcal{T}_l(\underline{\text{Mod}}\text{-}\Lambda^{\text{op}})$ *we have*:

$$\big(\overline{A}, n\big) \cong \begin{cases} \mathsf{Q}_r(\Sigma^n(\overline{A})), & \textit{if } n \geqslant 0, \\ \mathsf{Q}_r(\overline{\Omega^{-n}(A)}), & \textit{if } n < 0. \end{cases} \quad \textit{and} \quad (\underline{A}, n) \cong \begin{cases} \mathsf{P}_l(\Omega^{-n}(\underline{A}), & \textit{if } n \leqslant 0, \\ \mathsf{P}_l(\underline{\Sigma^n(A)}), & \textit{if } n > 0. \end{cases}$$

(ii) $\Omega^\infty(\text{Mod-}\Lambda) = \text{CM}(\mathbf{P}_\Lambda) \subseteq \Omega^d(\text{Mod-}\Lambda) \subseteq {}^\perp\Lambda$.
(iii) $\Sigma^\infty(\text{Mod-}\Lambda^{\text{op}}) = \text{CoCM}(\mathbf{I}_{\Lambda^{\text{op}}}) \subseteq \Sigma^d(\text{Mod-}\Lambda^{\text{op}}) \subseteq \text{D}(\Lambda)^\perp$.
(iv) $\text{CM}(\mathbf{P}_{\Lambda^{\text{op}}}) = {}^\perp\Lambda^{\text{op}}$ *and* $\text{CoCM}(\mathbf{I}_\Lambda) = \text{D}(\Lambda)^\perp$.
(v) $\widehat{{}^\perp\Lambda} = \text{Mod-}\Lambda$ *and* $\widetilde{\text{D}(\Lambda)^\perp} = \text{Mod-}\Lambda^{\text{op}}$.

Proof. (i) Let $(\overline{A}, n)$ be in $\mathcal{T}_r(\overline{\text{Mod}}\text{-}\Lambda)$ and let $\cdots \to P_1 \to P_0 \to A \to 0$ be a projective resolution of A. Applying the stabilization functor Q_r to the right triangle $\overline{\Omega(A)} \to \overline{P}_0 \to \overline{A} \to \Sigma\overline{\Omega(A)}$ in $\overline{\text{Mod}}\text{-}\Lambda$ and using that:

(α) any projective module has finite injective dimension by hypothesis, and
(β) Q_r kills the modules of finite injective dimension, we have an isomorphism $\mathsf{Q}_r(\overline{A}) \cong \mathsf{Q}_r(\Sigma\overline{\Omega(A)})$ in $\mathcal{T}_r(\overline{\text{Mod}}\text{-}\Lambda)$.

Similarly considering the right triangle $\overline{\Omega^2(A)} \to \overline{P}_1 \to \overline{\Omega(A)} \to \Sigma\overline{\Omega^2(A)}$ in $\overline{\text{Mod}}\text{-}\Lambda$ we have an isomorphism $\mathsf{Q}_r(\overline{\Omega(A)}) \cong \mathsf{Q}_r(\Sigma\overline{\Omega^2(A)})$ in $\mathcal{T}_r(\overline{\text{Mod}}\text{-}\Lambda)$. Then we have isomorphisms

$$\mathsf{Q}_r\big(\overline{A}\big) \cong \mathsf{Q}_r\big(\Sigma\overline{\Omega(A)}\big) \cong \Sigma\mathsf{Q}_r\big(\overline{\Omega(A)}\big) \cong \mathsf{Q}_r\big(\Sigma^2\overline{\Omega^2(A)}\big).$$

Inductively we obtain isomorphisms $\mathsf{Q}_r(\overline{A}) \cong \mathsf{Q}_r(\Sigma^t\overline{\Omega^t(A)})$, $\forall t \geqslant 0$. If $n \geqslant 0$, then

$$\big(\overline{A}, n\big) = \Sigma^n\big(\overline{A}, 0\big) = \Sigma^n\mathsf{Q}_r\big(\overline{A}\big) = \mathsf{Q}_r\big(\Sigma^n\big(\overline{A}\big)\big).$$

If $n < 0$, then

$$\begin{aligned}\big(\overline{A}, n\big) &= \Sigma^n\big(\overline{A}, 0\big) = \Sigma^n\mathsf{Q}_r\big(\overline{A}\big) \cong \Sigma^n\mathsf{Q}_r\big(\Sigma^{-n}\overline{\Omega^{-n}(A)}\big) = \Sigma^n\Sigma^{-n}\mathsf{Q}_r\big(\overline{\Omega^{-n}(A)}\big) \\ &= \mathsf{Q}_r\big(\overline{\Omega^{-n}(A)}\big).\end{aligned}$$

Hence Q_r is surjective on objects. The other assertion follows by duality.

(ii), (iii) and (v). For any module C we have $\text{Ext}^n(\Omega^d(C), \Lambda) = \text{Ext}^{n+d}(C, \Lambda) = 0$, $\forall n \geqslant 1$. Hence $\Omega^d(C) \in {}^\perp\Lambda$. It follows that $\Omega^\infty(\text{Mod-}\Lambda) \subseteq \Omega^d(\text{Mod-}\Lambda) \subseteq {}^\perp\Lambda$. This implies that $\widehat{{}^\perp\Lambda} = \text{Mod-}\Lambda$ and $\Omega^\infty(\text{Mod-}\Lambda) \subseteq \text{CM}(\mathbf{P}_\Lambda) \subseteq \Omega^\infty(\text{Mod-}\Lambda)$; hence

$$\Omega^\infty(\text{Mod-}\Lambda) = \text{CM}(\mathbf{P}_\Lambda).$$

Finally the equalities

$$\Sigma^\infty\big(\text{Mod-}\Lambda^{\text{op}}\big) = \text{CoCM}(\mathbf{I}_{\Lambda^{\text{op}}}) \quad \text{and} \quad \widetilde{\text{D}(\Lambda)^\perp} = \text{Mod-}\Lambda^{\text{op}}$$

are proved similarly.

(iv) Clearly $\mathrm{CoCM}(\mathbf{I}_\Lambda) \subseteq \mathrm{D}(\Lambda)^\perp$. If $A \in \mathrm{D}(\Lambda)^\perp$ and $(\dagger): 0 \to A \to I_0 \to I_1 \to \cdots$ is an injective coresolution of A, then we have an acyclic coresolution $(\dagger\dagger): 0 \to \mathsf{N}^-(A) \to \mathsf{N}^-(I_0) \to \mathsf{N}^-(I_1) \to \cdots$ of $\mathsf{N}^-(A)$ by projectives. It follows that $\mathsf{N}^-(A)$ is an arbitrary syzygy module, i.e., $\mathsf{N}^-(A) \in \Omega^\infty(\text{Mod-}\Lambda)$ and therefore $\mathsf{N}^-(A)$ is Cohen–Macaulay by (ii). Then $\mathsf{N}^+\mathsf{N}^-(A)$ is a CoCohen–Macaulay module. If $\Sigma^n(A)$ are the cosyzygies of A, then since $\mathrm{D}(\Lambda)^\perp$ is coresolving, we have $\Sigma^n(A) \in \mathrm{D}(\Lambda)^\perp$, hence $\mathsf{N}^-(\Sigma^n(A)) = \operatorname{Im} \mathsf{N}^-(I_n \to I_{n+1}) := K_n$. Observe that the modules K_n are also arbitrary syzygy modules, hence they are Cohen–Macaulay. We now show that the natural map $\mathsf{N}^+\mathsf{N}^-(A) \to A$ is invertible. Applying N^+ to the exact sequence $(\dagger\dagger)$ and using that K_n lies in $\mathrm{CM}(\mathbf{P}_\Lambda)$, we infer that $\mathrm{L}_t\mathsf{N}^+(K_n) = 0$, $\forall t \geqslant 1$, $\forall n \geqslant 0$. Hence the sequence $0 \to \mathsf{N}^+\mathsf{N}^-(A) \to \mathsf{N}^+\mathsf{N}^-(I_0) \to \mathsf{N}^+\mathsf{N}^-(I_1) \to \cdots$ is exact. Since the natural maps $\mathsf{N}^+\mathsf{N}^-(I_n) \to I_n$ are invertible $\forall n \geqslant 0$, so is the natural map $\mathsf{N}^+\mathsf{N}^-(A) \to A$. Since $\mathsf{N}^+\mathsf{N}^-(A)$ lies in $\mathrm{CoCM}(\mathbf{I}_\Lambda)$, we infer that so does A. We conclude that $\mathrm{D}(\Lambda)^\perp \subseteq \mathrm{CoCM}(\mathbf{I}_\Lambda)$. The equality $\mathrm{CM}(\mathbf{P}_{\Lambda^{\mathrm{op}}}) = {}^\perp\Lambda^{\mathrm{op}}$ is proved similarly. $\square$

4.2. Costabilizations

Dually to any left, respectively right, triangulated category $\mathcal{C}$ we can associate in a universal way a triangulated category $\mathcal{R}(\mathcal{C})$. More precisely there exists a left, respectively right, exact functor $\mathsf{Z}: \mathcal{R}(\mathcal{C}) \to \mathcal{C}$ such that for any left, respectively right, exact functor $F: \mathcal{R} \to \mathcal{C}$ from a triangulated category $\mathcal{R}$, there exists an exact functor $F^*: \mathcal{R} \to \mathcal{R}(\mathcal{C})$, unique up to isomorphism, such that $\mathsf{Z}F^* \cong F$. The category $\mathcal{R}(\mathcal{C})$ is called the **costabilization** of $\mathcal{C}$ and the functor Z is called the **costabilization functor**. By [18] any left or right triangulated category admits its costabilization which can be realized as an appropriate category of spectra of objects.

By [18] the costabilization of the stable category $\underline{\mathrm{Mod}}\text{-}\Lambda$ considered as a left triangulated category, which we call the **(projective) costabilization category** of Λ, is the homotopy category $\mathcal{H}_{\mathsf{Ac}}(\mathbf{P}_\Lambda)$ of unbounded acyclic complexes of projective modules. The costabilization functor

$$\underline{\mathsf{Z}}: \mathcal{H}_{\mathsf{Ac}}(\mathbf{P}_\Lambda) \to \underline{\mathrm{Mod}}\text{-}\Lambda$$

which we call **(projective) costabilization functor**, is given as follows. If $P^\bullet \to \cdots \to P^{-1} \to P^0 \to P^1 \to \cdots$ is an acyclic complex of projectives, then

$$\underline{\mathsf{Z}}(P^\bullet) = \underline{\operatorname{Im}(P^{-1} \to P^0)}.$$

Then $\underline{\mathsf{Z}}$ is left exact in the sense that $\underline{\mathsf{Z}}$ sends triangles in $\mathcal{H}_{\mathsf{Ac}}(\mathbf{P}_\Lambda)$ to left triangles in $\underline{\mathrm{Mod}}\text{-}\Lambda$ and satisfies $\underline{\mathsf{Z}}(P^\bullet[-1]) \xrightarrow{\cong} \Omega\underline{\mathsf{Z}}(P^\bullet)$, $\forall P^\bullet \in \mathcal{H}_{\mathsf{Ac}}(\mathbf{P}_\Lambda)$. In particular $\operatorname{Im}\underline{\mathsf{Z}} \subseteq \Omega^\infty(\underline{\mathrm{Mod}}\text{-}\Lambda)$.

By a basic result of Jørgensen [36], the costabilization functor admits a left adjoint

$$\underline{\mathsf{Sp}}: \underline{\mathrm{Mod}}\text{-}\Lambda \to \mathcal{H}_{\mathsf{Ac}}(\mathbf{P}_\Lambda)$$

called the **(projective) spectrification functor**, which is right exact. That is, $\underline{\mathsf{Sp}}$ sends right triangles in $\underline{\mathrm{Mod}}\text{-}\Lambda$ to triangles in $\mathcal{H}_{\mathsf{Ac}}(\mathbf{P}_\Lambda)$ and satisfies $\underline{\mathsf{Sp}}(\Sigma_{\mathbf{P}}(\underline{A})) \cong \underline{\mathsf{Sp}}(\underline{A})[1]$, $\forall \underline{A} \in \underline{\mathrm{Mod}}\text{-}\Lambda$.

By the universal property of stabilizations there exists a unique up to isomorphism triangulated functor $\underline{\mathsf{Sp}}^* : \mathcal{T}_r(\underline{\mathrm{Mod}}\text{-}\Lambda) \to \mathcal{H}_{\mathsf{Ac}}(\mathbf{P}_\Lambda)$ such that $\underline{\mathsf{Sp}}^*\mathsf{P}_r = \underline{\mathsf{Sp}}$. Setting $\mathcal{H}_{\mathsf{Ac}}(\mathbf{P}_\Lambda)^{\mathrm{b}} := \mathcal{H}^{\mathrm{b}}_{\mathsf{Ac}}(\mathbf{P}_\Lambda)$, we have the following result which gives a description of the compact objects of the projective costabilization.

Proposition 4.5 [36]. *The spectrification functor* $\underline{\mathsf{Sp}}$ *induces a triangle equivalence*:

$$\underline{\mathsf{Sp}}^{\mathrm{b}} := \underline{\mathsf{Sp}}^*|_{\mathcal{T}_r(\underline{\mathrm{mod}}\text{-}\Lambda)} : \mathcal{T}_r(\underline{\mathrm{mod}}\text{-}\Lambda) \xrightarrow{\approx} \mathcal{H}^{\mathrm{b}}_{\mathsf{Ac}}(\mathbf{P}_\Lambda).$$

We are interested in the relations between the projective costabilization and Cohen–Macaulay modules. In this connection the following result will be useful.

Proposition 4.6. *The triangulated functor* $\underline{\mathsf{Sp}}\mathbf{i}_{\mathrm{CM}} : \underline{\mathrm{CM}}(\mathbf{P}_\Lambda) \to \mathcal{H}_{\mathsf{Ac}}(\mathbf{P}_\Lambda)$ *is fully faithful and admits the functor* $\mathbf{R}_{\mathrm{CM}}\underline{\mathsf{Z}}$ *as a right adjoint. Moreover there exists a torsion pair* $(\underline{\mathrm{CM}}(\mathbf{P}_\Lambda), \underline{\mathsf{Z}}^{-1}(\underline{\mathfrak{P}}_\Lambda^{\prec\infty}))$ *in* $\mathcal{H}_{\mathsf{Ac}}(\mathbf{P}_\Lambda)$ *identifying* $\underline{\mathrm{CM}}(\mathbf{P}_\Lambda)$ *with* $\operatorname{Im}\underline{\mathsf{Sp}}\mathbf{i}_{\mathrm{CM}}$ *and* $\underline{\mathsf{Z}}^{-1}(\underline{\mathfrak{P}}_\Lambda^{\prec\infty}) = \{P^\bullet \in \mathcal{H}_{\mathsf{Ac}}(\mathbf{P}_\Lambda) \mid \underline{\mathsf{Z}}(P^\bullet) \in \underline{\mathfrak{P}}_\Lambda^{\prec\infty}\}$.

Proof. For any module $X \in \mathrm{CM}(\mathbf{P}_\Lambda)$ and any complex $P^\bullet \in \mathcal{H}_{\mathsf{Ac}}(\mathbf{P}_\Lambda)$ we have isomorphisms:

$$\mathrm{Hom}\big(\underline{\mathsf{Sp}}\mathbf{i}_{\mathrm{CM}}(\underline{X}), P^\bullet\big) \cong \underline{\mathrm{Hom}}_\Lambda\big(\mathbf{i}_{\mathrm{CM}}(\underline{X}), \underline{\mathsf{Z}}\big(P^\bullet\big)\big) \cong \underline{\mathrm{Hom}}_\Lambda\big(\underline{X}, \mathbf{R}_{\mathrm{CM}}\underline{\mathsf{Z}}\big(P^\bullet\big)\big).$$

Hence $\underline{\mathsf{Sp}}\mathbf{i}_{\mathrm{CM}}$ is left adjoint to $\mathbf{R}_{\mathrm{CM}}\underline{\mathsf{Z}}$. Clearly $\underline{\mathsf{Sp}}\mathbf{i}_{\mathrm{CM}}$, hence $\mathbf{R}_{\mathrm{CM}}\underline{\mathsf{Z}}$, is triangulated. By the construction of $\underline{\mathsf{Sp}}$ in [36], it follows that for X in $\mathrm{CM}(\mathbf{P}_\Lambda)$, $\underline{\mathsf{Sp}}\mathbf{i}_{\mathrm{CM}}(\underline{X})$ is the complex obtained by splicing a projective resolution $\cdots \to P^{-2} \to P^{-1} \to X \to 0$ of X and an exact coresolution $0 \to X \to P^0 \to P_1 \to \cdots$ of X where $X \to P^0$ and the maps $\mathrm{Im}(P^n \to P^{n+1}) \rightarrowtail P^{n+1}$ are left projective approximations, $\forall n \geqslant 0$. This clearly implies that $\underline{\mathsf{Sp}}\mathbf{i}_{\mathrm{CM}}$ is fully faithful. Consequently we have a torsion pair $(\operatorname{Im}\underline{\mathsf{Sp}}\mathbf{i}_{\mathrm{CM}}, \operatorname{Ker}\mathbf{R}_{\mathrm{CM}}\underline{\mathsf{Z}})$ in $\mathcal{H}_{\mathsf{Ac}}(\mathbf{P}_\Lambda)$, where $\operatorname{Im}\underline{\mathsf{Sp}}\mathbf{i}_{\mathrm{CM}}$ is triangle equivalent to $\underline{\mathrm{CM}}(\mathbf{P}_\Lambda)$. Finally $P^\bullet$ lies in $\operatorname{Ker}\mathbf{R}_{\mathrm{CM}}\underline{\mathsf{Z}}$ iff $\mathbf{R}_{\mathrm{CM}}\underline{\mathsf{Z}}(P^\bullet) = 0$ iff $\underline{\mathsf{Z}}(P^\bullet)$ lies in $\underline{\mathfrak{P}}_\Lambda^{\prec\infty}$. $\square$

Corollary 4.7. *The following are equivalent.*

(i) *The functor* $\underline{\mathsf{Sp}}\mathbf{i}_{\mathrm{CM}} : \underline{\mathrm{CM}}(\mathbf{P}_\Lambda) \to \mathcal{H}_{\mathsf{Ac}}(\mathbf{P}_\Lambda)$ *is a triangle equivalence.*
(ii) $\Omega^\infty(\mathrm{Mod}\text{-}\Lambda) \subseteq {}^\perp\Lambda$. *That is* Λ *is* right CoGorenstein *in the sense of* [18].
(iii) $\Omega^\infty(\mathrm{Mod}\text{-}\Lambda) \cap \mathfrak{P}_\Lambda^{\prec\infty} = \mathbf{P}_\Lambda$.

If (i) *holds, then* Λ *satisfies the Nunke condition*: *if* $A \in \mathrm{Mod}\text{-}\Lambda$ *is such that*

$$\mathrm{Ext}^n_\Lambda\big(\mathsf{D}(\Lambda), A\big) = 0, \quad \forall n \geqslant 0,$$

then $A = 0$. *In particular* Λ *satisfies the* (*generalized*) *Nakayama conjecture* [13].

Proof. The equivalence (i) $\Leftrightarrow$ (iii) follows from the torsion pair $(\operatorname{Im}\underline{\mathsf{Spi}}_{\mathrm{CM}}, \operatorname{Ker}\mathbf{R}_{\mathrm{CM}}\underline{\mathsf{Z}})$ in $\mathcal{H}_{\mathsf{Ac}}(\mathbf{P}_\Lambda)$. Clearly (ii) implies that

$$\Omega^\infty(\mathrm{Mod}\text{-}\Lambda) = \mathrm{CM}(\mathbf{P}_\Lambda).$$

Then the counit $\underline{\mathsf{Spi}}_{\mathrm{CM}}\mathbf{R}_{\mathrm{CM}}\underline{\mathsf{Z}} \to \mathrm{Id}_{\mathcal{H}_{\mathsf{Ac}}(\mathbf{P}_\Lambda)}$ is invertible. Hence $\underline{\mathsf{Spi}}_{\mathrm{CM}}$ is a triangle equivalence. The implication (i) $\Rightarrow$ (ii) follows from [18]. If (i) holds, then let $0 \to A \to I^0 \to I^1 \to \cdots$ be an injective coresolution of A. Applying N^- we obtain an acyclic complex of projectives $0 \to \mathsf{N}^-(I^0) \to \mathsf{N}^-(I^1) \to \cdots$. Applying N^+ we obtain an exact sequence $0 \to \mathrm{L}_1\mathsf{N}^+(K) \to \mathsf{N}^+\mathsf{N}^-(I^0) \to \mathsf{N}^+\mathsf{N}^-(I^1) \to \mathsf{N}^+(K) \to 0$, where $K := \operatorname{Coker}(\mathsf{N}^-(I^0) \to \mathsf{N}^-(I^1))$, which implies that $\mathrm{L}_1\mathsf{N}^+(K) \cong A$. Then $A = 0$ since K lies in $\Omega^\infty(\mathrm{Mod}\text{-}\Lambda)$. $\square$

Combining Proposition 4.4(ii), Proposition 4.5 and Corollary 4.7, we have the following consequence which will be useful in the last section.

Corollary 4.8. *If* $\mathrm{id}\,\Lambda_\Lambda < \infty$, *then we have the following.*

(i) *The costabilization functor* $\underline{\mathsf{Z}} : \mathcal{H}_{\mathsf{Ac}}(\mathbf{P}_\Lambda) \to \underline{\mathrm{Mod}}\text{-}\Lambda$ *admits a factorization*

$$\underline{\mathsf{Z}} = \mathbf{i}_{\mathrm{CM}}\underline{\mathsf{Z}}_{\mathrm{CM}} : \mathcal{H}_{\mathsf{Ac}}(\mathbf{P}_\Lambda) \xrightarrow{\underline{\mathsf{Z}}_{\mathrm{CM}}} \underline{\mathrm{CM}}(\mathbf{P}_\Lambda) \xrightarrow{\mathbf{i}_{\mathrm{CM}}} \underline{\mathrm{Mod}}\text{-}\Lambda$$

where the exact functor $\underline{\mathsf{Z}}_{\mathrm{CM}} : \mathcal{H}_{\mathsf{Ac}}(\mathbf{P}_\Lambda) \xrightarrow{\approx} \underline{\mathrm{CM}}(\mathbf{P}_\Lambda)$ *is a triangle equivalence with quasi-inverse* $\underline{\mathsf{Spi}}_{\mathrm{CM}} : \underline{\mathrm{CM}}(\mathbf{P}_\Lambda) \xrightarrow{\approx} \mathcal{H}_{\mathsf{Ac}}(\mathbf{P}_\Lambda)$. *In particular* $\underline{\mathsf{Z}}$ *is fully faithful.*

(ii) $\underline{\mathsf{Z}}$ *induces a triangle equivalence* $\underline{\mathsf{Z}}^{\mathrm{b}}_{\mathrm{CM}} : \mathcal{H}^{\mathrm{b}}_{\mathsf{Ac}}(\mathbf{P}_\Lambda) \xrightarrow{\approx} \underline{\mathrm{CM}}(\mathbf{P}_\Lambda)^{\mathrm{b}}$.

(iii) *The spectrification functor* $\underline{\mathsf{Sp}} : \underline{\mathrm{Mod}}\text{-}\Lambda \to \mathcal{H}_{\mathsf{Ac}}(\mathbf{P}_\Lambda)$ *induces a triangle equivalence*:

$$\underline{\mathsf{Z}}^{\mathrm{b}}_{\mathrm{CM}}\underline{\mathsf{Sp}}^{\mathrm{b}} : \mathcal{T}_r(\underline{\mathrm{mod}}\text{-}\Lambda) \xrightarrow{\approx} \underline{\mathrm{CM}}(\mathbf{P}_\Lambda)^{\mathrm{b}}.$$

Remark 4.9. The above results admit dual versions concerning the injective spectrification functor and the costabilization of the stable category modulo injectives which is equivalent to the unbounded homotopy category $\mathcal{H}_{\mathsf{Ac}}(\mathbf{I}_\Lambda)$ of acyclic complexes of injectives. We state only the following equivalent conditions ensuring that the projective costabilization is triangle equivalent to the injective costabilization. Details are left to the reader (note that these conditions hold for Gorenstein algebras).

(i) The Nakayama functors $\mathsf{N}^\pm$ induce triangle equivalences $(\mathsf{N}^+, \mathsf{N}^-) : \mathcal{H}_{\mathsf{Ac}}(\mathbf{P}_\Lambda) \xrightarrow{\approx} \mathcal{H}_{\mathsf{Ac}}(\mathbf{I}_\Lambda)$.

(ii) $\Omega^\infty(\mathrm{Mod}\text{-}\Lambda) \subseteq {}^\perp\Lambda$ and $\Sigma^\infty(\mathrm{Mod}\text{-}\Lambda) \subseteq \mathrm{D}(\Lambda)^\perp$.

(iii) $\Omega^\infty(\mathrm{Mod}\text{-}\Lambda) \cap \mathfrak{P}^{\prec\infty}_\Lambda = \mathbf{P}_\Lambda$ and $\Sigma^\infty(\mathrm{Mod}\text{-}\Lambda) \cap \mathfrak{I}^{\prec\infty}_\Lambda = \mathbf{I}_\Lambda$.

5. Auslander–Reiten operators and exact sequences of stabilization categories

In this section we study the behavior of the (Co)Cohen–Macaulay (co)torsion pairs and the Auslander–Reiten operators under stabilizations. More precisely we show that the torsion pairs $(\underline{\mathrm{CM}}(\mathbf{P}_\Lambda), \underline{\mathfrak{P}}_\Lambda^{\prec\infty})$ and $(\bar{\mathfrak{I}}_\Lambda^{\prec\infty}, \overline{\mathrm{CoCM}}(\mathbf{I}_\Lambda))$ in the respective stable module categories can be lifted to torsion pairs in the triangulated stabilizations. In addition the stable equivalences $\tau^{\pm}$ allow us to obtain several identifications which will be useful later.

5.1. Nakayama functors and stabilization categories

Since the adjoint pair $(\mathsf{N}^+, \mathsf{N}^-)$ of Nakayama functors induces an equivalence between $\mathbf{P}_\Lambda$ and $\mathbf{I}_\Lambda$, it induces an adjoint pair $(\mathsf{N}^+, \mathsf{N}^-)$ of functors $\mathsf{N}^+ : \underline{\mathrm{Mod}}\text{-}\Lambda \leftrightarrows \overline{\mathrm{Mod}}\text{-}\Lambda : \mathsf{N}^-$.

Lemma 5.1. *For an exact sequence $A \to B \to C \to 0$ in* Mod-Λ*, the following are equivalent*:

(i) *$A \to B$ is $\mathbf{P}_\Lambda$-monic, i.e., any map $A \to P$ with P projective, factors through $A \to B$.*
(ii) *The sequence $0 \to \mathsf{N}^+(A) \to \mathsf{N}^+(B) \to \mathsf{N}^+(C) \to 0$ is exact.*

In particular the Nakayama functor $\mathsf{N}^+ : \underline{\mathrm{Mod}}\text{-}\Lambda \to \overline{\mathrm{Mod}}\text{-}\Lambda$ is right exact. Moreover $\forall A \in$ Mod-Λ*, $\forall X \in \mathrm{CM}(\mathbf{P}_\Lambda)$, there are isomorphisms, $\forall n \geqslant 1$*:

$$\mathrm{Ext}^n_\Lambda\big(A, \mathsf{N}^+(X)\big) \cong \mathrm{Ext}^n_\Lambda\big(\mathsf{N}^-\big(Z^A\big), X\big) \quad \textit{and} \quad \overline{\mathrm{Hom}}_\Lambda\big(A, \mathsf{N}^+(X)\big) \cong \underline{\mathrm{Hom}}_\Lambda\big(\mathsf{N}^-\big(Z^A\big), X\big).$$

Proof. The equivalence (i) $\Leftrightarrow$ (ii) is straightforward. Now let $(T) : \underline{A} \to \underline{B} \to \underline{C} \to \Sigma_{\mathbf{P}}(\underline{A})$ be a right triangle in $\underline{\mathrm{Mod}}\text{-}\Lambda$. Then (T) is induced by a right exact sequence $A \xrightarrow{g} B \to C \to 0$ in Mod-Λ where g is $\mathbf{P}_\Lambda$-monic. Hence we have a short exact sequence $0 \to \mathsf{N}^+(A) \to \mathsf{N}^+(B) \to \mathsf{N}^+(C) \to 0$ in Mod-Λ, which induces a right triangle $\mathsf{N}^+(\underline{A}) \to \mathsf{N}^+(\underline{B}) \to \mathsf{N}^+(\underline{C}) \to \Sigma\mathsf{N}^+(\underline{A})$ in $\overline{\mathrm{Mod}}\text{-}\Lambda$. If B is projective, then $\underline{C}$ is isomorphic to $\Sigma_{\mathbf{P}}(\underline{A})$ and therefore $\mathsf{N}^+(\Sigma_{\mathbf{P}}(\underline{A})) \cong \Sigma\mathsf{N}^+(\underline{A}))$. Hence the induced functor $\mathsf{N}^+ : \underline{\mathrm{Mod}}\text{-}\Lambda \to \overline{\mathrm{Mod}}\text{-}\Lambda$ is right exact. Now by Proposition 3.8 we have isomorphisms, $\forall n \geqslant 1$:

$$\begin{aligned}\mathrm{Ext}^n_\Lambda\big(A, \mathsf{N}^+(X)\big) &\cong \overline{\mathrm{Hom}}_\Lambda\big(A, \Sigma^n\big(\mathsf{N}^+(\underline{X})\big)\big) \cong \overline{\mathrm{Hom}}_\Lambda\big(\mathbf{L}_{\mathrm{CoCM}}\big(\overline{A}\big), \mathsf{N}^+\big(\Sigma^n_{\mathbf{P}}(\underline{X})\big)\big)\\ &\cong \underline{\mathrm{Hom}}_\Lambda\big(\mathsf{N}^-\mathbf{L}_{\mathrm{CoCM}}\big(\overline{A}\big), \Sigma^n_{\mathbf{P}}(\underline{X})\big) \cong \underline{\mathrm{Hom}}_\Lambda\big(\Omega^n\mathsf{N}^-\mathbf{L}_{\mathrm{CoCM}}\big(\overline{A}\big), \underline{X}\big)\\ &\cong \mathrm{Ext}^n_\Lambda\big(\mathsf{N}^-\big(Z^A\big), X\big),\end{aligned}$$

for any $X \in \mathrm{CM}(\mathbf{P}_\Lambda)$ and $A \in$ Mod-Λ. The second isomorphism follows similarly. $\square$

Let $\mathsf{P}_r : \underline{\mathrm{Mod}}\text{-}\Lambda \to \mathcal{T}_r(\underline{\mathrm{Mod}}\text{-}\Lambda)$, respectively $\mathsf{Q}_r : \overline{\mathrm{Mod}}\text{-}\Lambda \to \mathcal{T}_r(\overline{\mathrm{Mod}}\text{-}\Lambda)$, be the right stabilization functor of the right triangulated category $\underline{\mathrm{Mod}}\text{-}\Lambda$, respectively $\overline{\mathrm{Mod}}\text{-}\Lambda$. The following result shows that the right stabilization categories $\mathcal{T}_r(\underline{\mathrm{Mod}}\text{-}\Lambda)$ and $\mathcal{T}_r(\overline{\mathrm{Mod}}\text{-}\Lambda)$ are triangle equivalent.

Theorem 5.2. *The stable equivalence* $\tau^+ : \underline{\mathrm{Mod}}\text{-}\Lambda \xrightarrow{\approx} \overline{\mathrm{Mod}}\text{-}\Lambda$ *and the stable Nakayama functor* $\mathsf{N}^+ : \underline{\mathrm{Mod}}\text{-}\Lambda \to \overline{\mathrm{Mod}}\text{-}\Lambda$ *are right exact and induce triangle equivalences* $\widetilde{\tau}^+, \widetilde{\mathsf{N}}^+ \cong \Sigma^2\widetilde{\tau}^+ : \mathcal{T}_r(\underline{\mathrm{Mod}}\text{-}\Lambda) \xrightarrow{\approx} \mathcal{T}_r(\overline{\mathrm{Mod}}\text{-}\Lambda)$, *making the following diagram commutative*:

$$\begin{tikzcd}
\underline{\mathrm{Mod}}\text{-}\Lambda \arrow[r, "\tau^+{,}\mathsf{N}^+"] \arrow[d, "\mathsf{P}_r"'] & \overline{\mathrm{Mod}}\text{-}\Lambda \arrow[d, "\mathsf{Q}_r"] \\
\mathcal{T}_r(\underline{\mathrm{Mod}}\text{-}\Lambda) \arrow[r, "\widetilde{\tau}^+{,}\widetilde{\mathsf{N}}^+"] & \mathcal{T}_r(\overline{\mathrm{Mod}}\text{-}\Lambda).
\end{tikzcd}$$

Proof. Let A be in Mod-Λ and let $A \to P_0^A \to \Sigma_{\mathbf{P}}(A) \to 0$ be a right exact sequence, where $g^A : A \to P_0^A$ is a right projective approximation of A. Let $g^A := \varepsilon \circ \mu : A \xrightarrow{\varepsilon} \Omega\Sigma_{\mathbf{P}}(A) \xrightarrow{\mu} P_0^A$ be the canonical factorization of g^A. By Lemma 5.1, the sequence $0 \to \mathsf{N}^+(A) \to \mathsf{N}^+(P_0^A) \to \mathsf{N}^+(\Sigma_{\mathbf{P}}(A)) \to 0$ is exact. It follows that the map $\mathsf{N}^+(\varepsilon) : \mathsf{N}^+(A) \to \mathsf{N}^+(\Omega\Sigma_{\mathbf{P}}(A))$ is invertible. Now let $P_2 \xrightarrow{\lambda} P_1 \xrightarrow{\kappa} A \to 0$ be a projective presentation of A. Then

$$P_1 \xrightarrow{\kappa \circ g^A} P_0^A \to \Sigma_{\mathbf{P}}(A) \to 0$$

is a projective presentation of $\Sigma_{\mathbf{P}}(A)$ and by construction of τ^+, we have $\tau^+(A) \cong \mathrm{Ker}(\mathsf{N}^+(\lambda))$ and $\tau^+(\Sigma_{\mathbf{P}}(A)) \cong \mathrm{Ker}(\mathsf{N}^+(\kappa \circ \varepsilon))$. Since $\mathsf{N}^+(\varepsilon)$ is invertible, we have:

$$\tau^+\big(\Sigma_{\mathbf{P}}(A)\big) \cong \mathrm{Ker}\,\mathsf{N}^+(\kappa) \cong \mathrm{Im}\,\mathsf{N}^+(\lambda)$$

and a short exact sequence $0 \to \tau^+(A) \to \mathsf{N}^+(P_2) \to \tau^+(\Sigma_{\mathbf{P}}(A)) \to 0$. Since $\mathsf{N}^+(P_2)$ is injective, it follows that $\Sigma\tau^+(\underline{A}) \xrightarrow{\cong} \tau^+(\Sigma_{\mathbf{P}}(\underline{A}))$. Therefore we have a natural isomorphism $\tau^+\Sigma_{\mathbf{P}} \xrightarrow{\cong} \Sigma\tau^+ : \underline{\mathrm{Mod}}\text{-}\Lambda \to \overline{\mathrm{Mod}}\text{-}\Lambda$. Since τ^+ commutes with the suspension functors, by the universal property of the stabilizations (without considering the involved triangulated structures), it follows that there exists a unique functor $\widetilde{\tau}^+ : \mathcal{T}_r(\underline{\mathrm{Mod}}\text{-}\Lambda) \to \mathcal{T}_r(\overline{\mathrm{Mod}}\text{-}\Lambda)$ which commutes with the suspension functors and is such that: $\widetilde{\tau}^+\mathsf{P}_r = \mathsf{Q}_r\tau^+$. Since τ^+ is an equivalence, it follows that $\widetilde{\tau}^+$ is an equivalence. We show that $\widetilde{\tau}^+$ is triangulated. Since the functor $\mathsf{N}^+ : \underline{\mathrm{Mod}}\text{-}\Lambda \to \overline{\mathrm{Mod}}\text{-}\Lambda$ is right exact, by the universal property of the stabilizations, there exists a unique exact functor $\widetilde{\mathsf{N}}^+ : \mathcal{T}_r(\underline{\mathrm{Mod}}\text{-}\Lambda) \to \mathcal{T}_r(\overline{\mathrm{Mod}}\text{-}\Lambda)$ making the square above commutative. By construction $\widetilde{\mathsf{N}}^+$ is given by $\widetilde{\mathsf{N}}^+(\underline{A}, n) = (\mathsf{N}^+(\underline{A}), n)$, and $\widetilde{\tau}^+$ is given by $\widetilde{\tau}^+(\underline{A}, n) = (\tau^+(\underline{A}), n)$. From the exact sequence $0 \to \tau^+(A) \to \mathsf{N}^+(P_1) \to \mathsf{N}^+(P_0) \to \mathsf{N}^+(A) \to 0$ defining the object $\tau^+(A)$ up to injective summands, it follows that $\Sigma^2\tau^+(\underline{A}) \cong \mathsf{N}^+(\underline{A})$ in $\overline{\mathrm{Mod}}$-Λ. Then for any object $(\underline{A}, n)$ in $\mathcal{T}_r(\underline{\mathrm{Mod}}\text{-}\Lambda)$ we have isomorphisms:

$$\Sigma^2\widetilde{\tau}^+(\underline{A}, n) \cong \Sigma^2\big(\tau^+(\underline{A}), n\big) \cong \big(\Sigma^2\tau^+(\underline{A}), n\big) \cong \big(\mathsf{N}^+(\underline{A}), n\big) \cong \widetilde{\mathsf{N}}^+(\underline{A}, n).$$

Hence we have a natural isomorphism of functors $\Sigma^2\widetilde{\tau}^+ \xrightarrow{\cong} \widetilde{\mathsf{N}}^+$. Since Σ^2 and $\widetilde{\tau}^+$ are equivalences, so is $\widetilde{\mathsf{N}}^+$. Since the latter is exact, we infer that $\widetilde{\mathsf{N}}^+$ is a triangle equivalence and then so is $\widetilde{\tau}^+$. □

For later use we state below without proof the dual versions of the above results.

Proposition 5.3.

(1) *If* $0 \to A \to B \to C$ *is exact in* Mod-Λ, *then the following are equivalent*:
 (i) $B \to C$ *is* $\mathbf{I}_\Lambda$*-epic, i.e., any map* $I \to C$ *with* I *injective, factors through* $B \to C$.
 (ii) *The sequence* $0 \to \mathsf{N}^-(A) \to \mathsf{N}^-(B) \to \mathsf{N}^-(C) \to 0$ *is exact.*
 In particular the Nakayama functor $\mathsf{N}^- : \overline{\text{Mod}}\text{-}\Lambda \to \underline{\text{Mod}}\text{-}\Lambda$ *is left exact. Moreover* $\forall B \in \text{Mod-}\Lambda$, $\forall Z \in \text{CoCM}(\mathbf{I}_\Lambda)$, *we have isomorphisms*, $\forall n \geqslant 1$:

$$\operatorname{Ext}^n_\Lambda\big(\mathsf{N}^-(Z), B\big) \cong \operatorname{Ext}^n_\Lambda\big(Z, \mathsf{N}^+(X_B)\big) \quad \textit{and}$$
$$\underline{\operatorname{Hom}}_\Lambda\big(\mathsf{N}^-(Z), B\big) \cong \overline{\operatorname{Hom}}_\Lambda\big(Z, \mathsf{N}^+(X_B)\big).$$

(2) *The stable equivalence* $\tau^- : \overline{\text{Mod}}\text{-}\Lambda \xrightarrow{\approx} \underline{\text{Mod}}\text{-}\Lambda$ *and the left exact Nakayama functor* $\mathsf{N}^- : \overline{\text{Mod}}\text{-}\Lambda \xrightarrow{\approx} \underline{\text{Mod}}\text{-}\Lambda$ *induce triangle equivalences*

$$\widetilde{\tau}^-, \widetilde{\mathsf{N}}^- : \mathcal{T}_l\big(\overline{\text{Mod}}\text{-}\Lambda\big) \xrightarrow{\approx} \mathcal{T}_l(\underline{\text{Mod}}\text{-}\Lambda), \qquad \Omega^2\widetilde{\tau}^- \cong \widetilde{\mathsf{N}}^-$$

which commute with the stabilization functors Q_l *and* P_l.

5.2. Exact sequences of stabilization categories

We have seen that the inclusion $\mathbf{i}_{\text{CM}} : \underline{\text{CM}}(\mathbf{P}_\Lambda) \hookrightarrow \underline{\text{Mod}}\text{-}\Lambda$ admits a right adjoint $\mathbf{R}_{\text{CM}}$, with kernel $\operatorname{Ker}\mathbf{R}_{\text{CM}} = \underline{\mathfrak{P}}^{<\infty}_\Lambda$. The following result shows that $\mathbf{i}_{\text{CM}}$ admits a left adjoint $\mathbf{L}_{\text{CM}}$ with kernel ${}^\perp\underline{\text{CM}}(\mathbf{P}_\Lambda)$.

Lemma 5.4. *The inclusion* $\mathbf{i}_{\text{CM}} : \underline{\text{CM}}(\mathbf{P}_\Lambda) \hookrightarrow \underline{\text{Mod}}\text{-}\Lambda$ *admits a right exact left adjoint* $\mathbf{L}_{\text{CM}} : \underline{\text{Mod}}\text{-}\Lambda \to \underline{\text{CM}}(\mathbf{P}_\Lambda)$. *Moreover* $\mathbf{L}_{\text{CM}} = \mathsf{N}^-\mathbf{L}_{\text{CoCM}}\mathsf{N}^+$ *and* $\operatorname{Ker}\mathbf{L}_{\text{CM}} = {}^\perp\underline{\text{CM}}(\mathbf{P}_\Lambda)$ *is a right triangulated subcategory of* $\underline{\text{Mod}}\text{-}\Lambda$ *closed under coproducts.*

Proof. If $A \in \text{Mod-}\Lambda$, let $0 \to \mathsf{N}^+(A) \to Z^{\mathsf{N}^+(A)} \to W^{\mathsf{N}^+(A)} \to 0$ be a special left $\text{CoCM}(\mathbf{I}_\Lambda)$-approximation of $\mathsf{N}^+(A)$. Then $\mathsf{N}^-(Z^{\mathsf{N}^+(A)})$ is Cohen–Macaulay; if A is projective, then $Z^{\mathsf{N}^+(A)} = \mathsf{N}^+(A) \oplus W^{\mathsf{N}^+(A)}$ which implies that $Z^{\mathsf{N}^+(A)}$ is injective, hence $\mathsf{N}^-(Z^{\mathsf{N}^+(A)})$ is projective. Setting $\mathbf{L}_{\text{CM}}(\underline{A}) = \mathsf{N}^-(\overline{Z}^{\mathsf{N}^+(A)})$, it is easy to see that in this way we obtain a well-defined functor $\mathbf{L}_{\text{CM}} : \underline{\text{Mod}}\text{-}\Lambda \to \underline{\text{CM}}(\mathbf{P}_\Lambda)$ and by construction we have: $\mathbf{L}_{\text{CM}} = \mathsf{N}^-\mathbf{L}_{\text{CoCM}}\mathsf{N}^+$. Using Proposition 3.4 we have, for any module A and any Cohen–Macaulay module X, the following natural isomorphisms

$$\begin{aligned}
\underline{\operatorname{Hom}}_\Lambda\big[\mathsf{N}^-\mathbf{L}_{\text{CoCM}}\mathsf{N}^+(\underline{A}), \underline{X}\big] &\cong \overline{\operatorname{Hom}}_\Lambda\big[\mathsf{N}^+\mathsf{N}^-\mathbf{L}_{\text{CoCM}}\mathsf{N}^+(\underline{A}), \mathsf{N}^+(\underline{X})\big] \\
&\cong \overline{\operatorname{Hom}}_\Lambda\big[\mathbf{L}_{\text{CoCM}}\mathsf{N}^+(\underline{A}), \mathsf{N}^+(\underline{X})\big] \\
&\cong \overline{\operatorname{Hom}}_\Lambda\big[\mathsf{N}^+(\underline{A}), \mathsf{N}^+(\underline{X})\big] \\
&\cong \underline{\operatorname{Hom}}_\Lambda\big[\underline{A}, \mathsf{N}^-\mathsf{N}^+(\underline{X})\big] \cong \underline{\operatorname{Hom}}_\Lambda(\underline{A}, \underline{X})
\end{aligned}$$

which show that $\mathbf{L}_{\mathrm{CM}}$ is left adjoint to the inclusion $\mathbf{i}_{\mathrm{CM}}$. Since the stable functors N^+ and $\mathbf{L}_{\mathrm{CoCM}}$ are right exact and the functor $\mathsf{N}^-|_{\overline{\mathrm{CoCM}}(\mathbf{I}_\Lambda)}$ is exact it follows that $\mathbf{L}_{\mathrm{CM}}$ is right exact. Finally that ${}^{\perp}\underline{\mathrm{CM}}(\mathbf{P}_\Lambda)$ is a right triangulated category with coproducts is a consequence of [22]. □

We have seen that the Auslander–Reiten operators $\tau^{\pm}$ induce quasi-inverse equivalences between $\underline{\mathrm{CM}}(\mathbf{P}_\Lambda)$ and $\overline{\mathrm{CoCM}}(\mathbf{I}_\Lambda)$. The following result, which generalizes a result of Auslander–Reiten [11, Proposition 5.5] from finitely generated modules over a Gorenstein algebra to arbitrary modules over any Artin algebra, shows that $\tau^{\pm}$ induce quasi-inverse equivalences between ${}^{\perp}\underline{\mathrm{CM}}(\mathbf{P}_\Lambda)$ and $\bar{\mathfrak{I}}_\Lambda^{\prec\infty}$ and further gives a description of the left Ext-orthogonal subcategory $\underline{{}^{\perp}\,\mathrm{CM}(\mathbf{P}_\Lambda)}$.

Theorem 5.5.

(1) *The Auslander–Reiten operators $\tau^{\pm}$ induce quasi-inverse stable equivalences*

$$\tau^+ : {}^{\perp}\underline{\mathrm{CM}}(\mathbf{P}_\Lambda) \underset{\approx}{\overset{\approx}{\rightleftarrows}} \bar{\mathfrak{I}}_\Lambda^{\prec\infty} : \tau^-$$

which make the following diagram of right triangulated categories commutative:

$$\begin{tikzcd}
0 \arrow[r] & {}^{\perp}\underline{\mathrm{CM}}(\mathbf{P}_\Lambda) \arrow[r] \arrow[d, "\tau^+"', "\approx"] & \underline{\mathrm{Mod}}\text{-}\Lambda \arrow[r, "\mathbf{L}_{\mathrm{CM}}"] \arrow[d, "\tau^+"', "\approx"] & \underline{\mathrm{CM}}(\mathbf{P}_\Lambda) \arrow[r] \arrow[d, "\tau^+"', "\approx"] & 0 \\
0 \arrow[r] & \bar{\mathfrak{I}}_\Lambda^{\prec\infty} \arrow[r] & \overline{\mathrm{Mod}}\text{-}\Lambda \arrow[r, "\mathbf{L}_{\mathrm{CoCM}}"] & \overline{\mathrm{CoCM}}(\mathbf{I}_\Lambda) \arrow[r] & 0
\end{tikzcd}$$

(2) *The Auslander–Reiten operators $\tau^{\pm}$ induce quasi-inverse stable equivalences*

$$\tau^+ : \underline{{}^{\perp}\,\mathrm{CM}(\mathbf{P}_\Lambda)} \underset{\approx}{\overset{\approx}{\rightleftarrows}} \overline{\mathfrak{I}_\Lambda^{\prec\infty} \cap \Sigma^{\infty}(\mathrm{Mod}\text{-}\Lambda)} : \tau^- \;.$$

Proof. (1) Let $\underline{A}$ be in ${}^{\perp}\underline{\mathrm{CM}}(\mathbf{P}_\Lambda)$, i.e., $\mathbf{L}_{\mathrm{CM}}(\underline{A}) = 0$. Let $A \to P^A \to \Sigma_{\mathbf{P}}(A) \to 0$ be exact where $f : A \to P^A$ is the minimal left projective approximation of A. If $\alpha : A \to X$ is a map, where X is Cohen–Macaulay, then α factors through a projective module and therefore α factors through f. We infer that f is a left $\mathrm{CM}(\mathbf{P}_\Lambda)$-approximation of A. Applying N^+ to this sequence, we get an exact sequence $0 \to \mathsf{N}^+(A) \to \mathsf{N}^+(P^A) \to \mathsf{N}^+(\Sigma_{\mathbf{P}}(A)) \to 0$, which, by Lemma 5.4, is a special left $\mathrm{CoCM}(\mathbf{I}_\Lambda)$-approximation of $\mathsf{N}^+(A)$. In particular $\mathsf{N}^+(\Sigma_{\mathbf{P}}(A))$ lies in $\mathfrak{I}_\Lambda^{\prec\infty}$ and therefore $\mathsf{N}^+(A)$ lies in $\mathfrak{I}_\Lambda^{\prec\infty}$ since $\mathfrak{I}_\Lambda^{\prec\infty}$ is resolving. If $P_1 \to P_0 \to A \to 0$ is a projective presentation of A, then the exact sequence $0 \to \tau^+(A) \to \mathsf{N}^+(P_1) \to \mathsf{N}^+(P_0) \to \mathsf{N}^+(A) \to 0$ shows that $\tau^+(A)$ lies in $\mathfrak{I}_\Lambda^{\prec\infty}$. On the other hand if W lies in $\mathfrak{I}_\Lambda^{\prec\infty}$, let $\tau^-(W) \to X^{\tau^-(W)}$ be a left $\mathrm{CM}(\mathbf{P}_\Lambda)$-approximation of $\tau^-(W)$. Then the map $\tau^+\tau^-(\overline{W}) \xrightarrow{\cong} \overline{W} \to \tau^+(\underline{X}^{\tau^-(W)})$ is zero in $\overline{\mathrm{Mod}}\text{-}\Lambda$ since $\overline{W}$ lies in

$\bar{\mathfrak{I}}_{\Lambda}^{\prec\infty}$ and $\tau^{+}(X^{\tau^{-}(W)})$ is CoCohen–Macaulay. It follows that the map $\tau^{-}(W) \to X^{\tau^{-}(W)}$ is zero in $\underline{\mathrm{Mod}}$-Λ. Hence $\tau^{-}(\overline{W})$ lies in ${}^{\perp}\underline{\mathrm{CM}}(\mathbf{P}_{\Lambda})$ and consequently the stable equivalence $\tau^{+} : \underline{\mathrm{Mod}}\text{-}\Lambda \xrightarrow{\approx} \overline{\mathrm{Mod}}\text{-}\Lambda$ restricts to an equivalence $\tau^{+} : {}^{\perp}\underline{\mathrm{CM}}(\mathbf{P}_{\Lambda}) \xrightarrow{\approx} \bar{\mathfrak{I}}_{\Lambda}^{\prec\infty}$ with quasi-inverse $\tau^{-} : \bar{\mathfrak{I}}_{\Lambda}^{\prec\infty} \xrightarrow{\approx} {}^{\perp}\underline{\mathrm{CM}}(\mathbf{P}_{\Lambda})$.

(2) Let A be in ${}^{\perp}\underline{\mathrm{CM}(\mathbf{P}_{\Lambda})}$, i.e., $\mathrm{Ext}^{n}_{\Lambda}(A, X) = 0$, $\forall n \geqslant 1$, $\forall X \in \mathrm{CM}(\mathbf{P}_{\Lambda})$. In particular $A \in {}^{\perp}\Lambda$ and therefore $\mathrm{L}_{n}\mathsf{N}^{+}(A) = 0$, $\forall n \geqslant 1$. Then clearly $\underline{\mathrm{Hom}}_{\Lambda}(\Omega^{n}(\underline{A}), \underline{\mathrm{CM}}(\mathbf{P}_{\Lambda})) = 0$, and therefore $\Omega^{n}(\underline{A}) \in {}^{\perp}\underline{\mathrm{CM}}(\mathbf{P}_{\Lambda})$, $\forall n \geqslant 1$. Then by part (1) it follows that $\tau^{+}(\Omega^{n}(\underline{A}))$ lies in $\bar{\mathfrak{I}}_{\Lambda}^{\prec\infty}$, $\forall n \geqslant 1$. Let

$$\cdots \to P^{-2} \xrightarrow{f^{-1}} P^{-1} \xrightarrow{f^{0}} A \to 0$$

be a projective resolution of A. Using that $A \in {}^{\perp}\Lambda$, it follows that we have an exact sequence $\cdots \to \mathsf{N}^{+}(P^{-2}) \to \mathsf{N}^{+}(P^{-1}) \to \mathsf{N}^{+}(A) \to 0$ such that $\mathrm{Ker}\,\mathsf{N}^{+}(f^{-n}) \cong \tau^{+}(\Omega^{n-1}(A))$, $\forall n \geqslant 1$. Since $\mathfrak{I}_{\Lambda}^{\prec\infty}$ is coresolving, this implies that $\tau^{+}(A)$ lies in $\mathfrak{I}_{\Lambda}^{\prec\infty} \cap \Sigma^{\infty}(\mathrm{Mod}\text{-}\Lambda)$. Therefore the stable equivalence $\tau^{+} : \underline{\mathrm{Mod}}\text{-}\Lambda \to \overline{\mathrm{Mod}}\text{-}\Lambda$ restricts to a fully faithful functor

$$\tau^{+} : \underline{{}^{\perp}\mathrm{CM}(\mathbf{P}_{\Lambda})} \hookrightarrow \overline{\mathfrak{I}_{\Lambda}^{\prec\infty} \cap \Sigma^{\infty}(\mathrm{Mod}\text{-}\Lambda)}.$$

Now let B be in $\mathfrak{I}_{\Lambda}^{\prec\infty} \cap \Sigma^{\infty}(\mathrm{Mod}\text{-}\Lambda)$ and let $\cdots \to I^{-2} \to I^{-1} \to B \to 0$ be an exact sequence where the I^{-s} are injective. Then the sequence $\cdots \to \mathsf{N}^{-}(I^{-2}) \to \mathsf{N}^{-}(I^{-1}) \to \mathsf{N}^{-}(B) \to 0$ is a projective resolution of $\mathsf{N}^{+}(B)$. Applying N^{+} to this resolution and using that $\mathsf{N}^{+}\mathsf{N}^{-}(I^{t}) \cong I^{t}$, we infer that $\mathsf{N}^{+}\mathsf{N}^{-}(B) \cong B$. Hence for any Cohen–Macaulay module X, $\mathrm{Ext}^{n}_{\Lambda}(\mathsf{N}^{+}(B), X)$ is the nth cohomology of the complex $\mathrm{Hom}_{\Lambda}(\mathsf{N}^{-}(I^{*}), X)$. Since $X \cong \mathsf{N}^{-}\mathsf{N}^{+}(X)$, by the above argument it follows that $\mathrm{Ext}^{n}_{\Lambda}(\mathsf{N}^{-}(B), X)$ is the nth cohomology of the complex $\mathrm{Hom}_{\Lambda}(I^{*}, \mathsf{N}^{+}(X))$ which is clearly zero since B lies in $\mathfrak{I}_{\Lambda}^{\prec\infty}$ and $\mathsf{N}^{+}(X)$ is CoCohen–Macaulay. We infer that $\mathrm{Ext}^{n}_{\Lambda}(\mathsf{N}^{-}(B), X) = 0$, $\forall n \geqslant 1$ and therefore $\mathsf{N}^{-}(B) \in {}^{\perp}\mathrm{CM}(\mathbf{P}_{\Lambda})$. Now let $0 \to B \to I^{0} \to I^{1} \to \Sigma^{2}(B) \to 0$ be the start of a minimal injective resolution of B. Then we have an exact sequence $0 \to \mathsf{N}^{-}(B) \to \mathsf{N}^{-}(I^{0}) \to \mathsf{N}^{-}(I^{1}) \to \tau^{-}(B) \to 0$. Using that for any Cohen–Macaulay module X we have isomorphisms $\mathrm{Hom}_{\Lambda}(\mathsf{N}^{-}(I^{t}), X) \cong \mathrm{Hom}_{\Lambda}(I^{t}, \mathsf{N}^{+}(X))$, $t = 0, 1$, we infer easily the isomorphisms: $\mathrm{Ext}^{1}_{\Lambda}(\tau^{-}(B), X) \cong \mathrm{Ext}^{1}_{\Lambda}(\Sigma^{2}(B), \mathsf{N}^{+}(X))$ and $\mathrm{Ext}^{2}_{\Lambda}(\tau^{-}(B), X) \cong \mathrm{Ext}^{1}_{\Lambda}(\Sigma(B), \mathsf{N}^{+}(X))$. Since B lies in $\mathfrak{I}_{\Lambda}^{\prec\infty}$ and $\mathfrak{I}_{\Lambda}^{\prec\infty}$ is coresolving, it follows that $\Sigma^{t}(B)$ lies in $\mathfrak{I}_{\Lambda}^{\prec\infty}$. Then the above isomorphisms show that $\mathrm{Ext}^{1}_{\Lambda}(\tau^{-}(B), X) = 0 = \mathrm{Ext}^{2}_{\Lambda}(\tau^{-}(B), X)$ since $\mathsf{N}^{+}(X)$ is CoCohen–Macaulay. Also since $\mathsf{N}^{-}(B) = \Omega^{2}(\tau^{-}(B))$ and $\mathsf{N}^{-}(B)$ lies in ${}^{\perp}\mathrm{CM}(\mathbf{P}_{\Lambda})$, we infer that $\tau^{-}(B)$ lies in ${}^{\perp}\mathrm{CM}(\mathbf{P}_{\Lambda})$. Since $\tau^{+}\tau^{-}(\overline{B}) \cong \overline{B}$, we infer that τ^{+} is surjective on objects. $\square$

We now show that the exact sequence above behaves well with respect to stabilization. First recall that a sequence of triangulated categories and exact functors $0 \to \mathcal{B} \xrightarrow{G} \mathcal{A} \xrightarrow{F} \mathcal{C} \to 0$ is called **short exact** if G is the inclusion of a thick subcategory and F induces an equivalence $\mathcal{A}/\mathcal{B} \xrightarrow{\approx} \mathcal{C}$. It is called **localization**, respectively **colocalization**, **exact** if F admits a right, respectively left, adjoint. Then it is well known that the right, respectively left, adjoint of F is fully faithful and G admits a right, respectively left, adjoint. Consider the right exact reflections $\mathbf{L}_{\mathrm{CM}} : \underline{\mathrm{Mod}}\text{-}\Lambda \to \underline{\mathrm{CM}}(\mathbf{P}_{\Lambda})$ and

$\mathbf{L}_{\mathrm{CoCM}}:\overline{\mathrm{Mod}}\text{-}\Lambda \to \overline{\mathrm{CoCM}}(\mathbf{I}_\Lambda)$. By the universal property of right stabilizations, there exist unique exact functors $\mathbf{L}^*_{\mathrm{CM}}:\mathcal{T}_r(\underline{\mathrm{Mod}}\text{-}\Lambda) \to \underline{\mathrm{CM}}(\mathbf{P}_\Lambda)$ and $\mathbf{L}^*_{\mathrm{CoCM}}:\mathcal{T}_r(\overline{\mathrm{Mod}}\text{-}\Lambda) \to \overline{\mathrm{CoCM}}(\mathbf{I}_\Lambda)$ such that $\mathbf{L}^*_{\mathrm{CM}}\mathsf{P}_r = \mathbf{L}_{\mathrm{CM}}$ and $\mathbf{L}^*_{\mathrm{CoCM}}\mathsf{Q}_r = \mathbf{L}_{\mathrm{CoCM}}$. Note that by construction $\mathbf{L}^*_{\mathrm{CM}}(\underline{A}, n) = \Omega^{-n}\mathbf{L}_{\mathrm{CM}}(\underline{A})$ and $\mathbf{L}^*_{\mathrm{CoCM}}(\overline{A}, n) = \Sigma^n\mathbf{L}_{\mathrm{CoCM}}(\overline{A})$.

Proposition 5.6. *There exists a commutative diagram of localization short exact sequences*

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{T}_r({}^{\perp}\underline{\mathrm{CM}}(\mathbf{P}_\Lambda)) \arrow[r] \arrow[d, "\widetilde{\tau}^+{,}\widetilde{\mathsf{N}}^+"', "\approx"] & \mathcal{T}_r(\underline{\mathrm{Mod}}\text{-}\Lambda) \arrow[r, "\mathbf{L}^*_{\mathrm{CM}}"] \arrow[d, "\widetilde{\tau}^+{,}\widetilde{\mathsf{N}}^+"', "\approx"] & \underline{\mathrm{CM}}(\mathbf{P}_\Lambda) \arrow[r] \arrow[d, "\widetilde{\tau}^+{,}\widetilde{\mathsf{N}}^+"', "\approx"] & 0 \\
0 \arrow[r] & \mathcal{T}_r(\bar{\mathfrak{I}}^{\prec\infty}_\Lambda) \arrow[r] & \mathcal{T}_r(\overline{\mathrm{Mod}}\text{-}\Lambda) \arrow[r, "\mathbf{L}^*_{\mathrm{CoCM}}"] & \overline{\mathrm{CoCM}}(\mathbf{I}_\Lambda) \arrow[r] & 0
\end{tikzcd}$$

of triangulated categories, where the vertical functors are triangle equivalences.

Proof. By Proposition 3.4 and Theorem 5.5 it follows that the vertical arrows in the right square of the diagram are triangle equivalences. For any object $(\underline{A}, n)$ in $\mathcal{T}_r(\underline{\mathrm{Mod}}\text{-}\Lambda)$ we have isomorphisms:

$$\begin{aligned}
\mathsf{N}^+\mathbf{L}^*_{\mathrm{CM}}(\underline{A}, n) &= \mathsf{N}^+\Omega^{-n}\mathbf{L}_{\mathrm{CM}}(\underline{A}) = \Sigma^n\mathsf{N}^+\mathbf{L}_{\mathrm{CM}}(\underline{A}) = \Sigma^n\mathsf{N}^+\mathsf{N}^-\mathbf{L}_{\mathrm{CoCM}}\mathsf{N}^+(\underline{A}) \\
&\cong \Sigma^n\mathbf{L}_{\mathrm{CoCM}}\mathsf{N}^+(\underline{A}) = \mathbf{L}^*_{\mathrm{CoCM}}\big(\mathsf{N}^+(\underline{A}), n\big) = \mathbf{L}^*_{\mathrm{CoCM}}\widetilde{\mathsf{N}}^+(\underline{A}, n)
\end{aligned}$$

which show that there exists a natural isomorphism: $\mathsf{N}^+\mathbf{L}^*_{\mathrm{CM}} \xrightarrow{\cong} \mathbf{L}^*_{\mathrm{CoCM}}\widehat{\mathsf{N}}^+$ and therefore the right square in the above diagram commutes. By [22] we know that the lower sequence is a localization exact sequence of triangulated categories. Hence so is the upper sequence and the functors $\widehat{\tau}^+, \widehat{\mathsf{N}}^+$ induce the left vertical equivalences which clearly coincide with $\widetilde{\tau}^+, \widetilde{\mathsf{N}}^+$. The right adjoint of $\mathbf{L}^*_{\mathrm{CM}}$, respectively $\mathbf{L}^*_{\mathrm{CoCM}}$, is given by the unique exact extension of the inclusion functor $\mathbf{i}^*_{\mathrm{CM}}$, respectively $\mathbf{i}^*_{\mathrm{CoCM}}$. □

The above results admit dual versions which we include below for later use.

Proposition 5.7.

(1) *The inclusion* $\mathbf{i}_{\mathrm{CoCM}}:\overline{\mathrm{CoCM}}(\mathbf{I}_\Lambda) \hookrightarrow \overline{\mathrm{Mod}}\text{-}\Lambda$ *admits a left exact right adjoint* $\mathbf{R}_{\mathrm{CoCM}}:\overline{\mathrm{Mod}}\text{-}\Lambda \to \overline{\mathrm{CoCM}}(\mathbf{I}_\Lambda)$. *Moreover* $\mathbf{R}_{\mathrm{CoCM}} = \mathsf{N}^+\mathbf{R}_{\mathrm{CM}}\mathsf{N}^-$ *and* $\operatorname{Ker}\mathbf{R}_{\mathrm{CoCM}} = \overline{\mathrm{CoCM}}(\mathbf{I}_\Lambda)^{\perp}$.

(2) *The Auslander–Reiten operators* τ^- *and* τ^+ *induce quasi-inverse equivalences*

$$\tau^-:\overline{\mathrm{CoCM}}(\mathbf{I}_\Lambda)^{\perp} \underset{\approx}{\overset{\approx}{\rightleftarrows}} \underline{\mathfrak{P}}^{\prec\infty}_\Lambda:\tau^+ \quad \textit{and}$$

$$\tau^-:\overline{\mathrm{CoCM}(\mathbf{I}_\Lambda)^{\perp}} \underset{\approx}{\overset{\approx}{\rightleftarrows}} \underline{\mathfrak{P}^{\prec\infty}_\Lambda \cap \Omega^\infty(\mathrm{Mod}\text{-}\Lambda)}:\tau^+$$

which make the following diagram of left triangulated categories commutative:

$$\begin{tikzcd}
0 \arrow[r] & \overline{\mathrm{CoCM}}(\mathbf{I}_\Lambda)^\perp \arrow[r] \arrow[d, "\tau^-"', "\approx"] & \overline{\mathrm{Mod}}\text{-}\Lambda \arrow[r, "\mathbf{R}_{\mathrm{CoCM}}"] \arrow[d, "\tau^-"', "\approx"] & \overline{\mathrm{CoCM}}(\mathbf{I}_\Lambda) \arrow[r] \arrow[d, "\tau^-"', "\approx"] & 0 \\
0 \arrow[r] & \underline{\mathfrak{P}}_\Lambda^{\prec\propto} \arrow[r] & \underline{\mathrm{Mod}}\text{-}\Lambda \arrow[r, "\mathbf{R}_{\mathrm{CM}}"] & \underline{\mathrm{CM}}(\mathbf{P}_\Lambda) \arrow[r] & 0
\end{tikzcd}$$

This diagram extends to a colocalization exact sequence of triangulated categories:

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{T}_l(\overline{\mathrm{CoCM}}(\mathbf{I}_\Lambda)^\perp) \arrow[r] \arrow[d, "\widetilde{\tau}^-{,}\widetilde{\mathsf{N}}^-"', "\approx"] & \mathcal{T}_l(\overline{\mathrm{Mod}}\text{-}\Lambda) \arrow[r, "\mathbf{R}^*_{\mathrm{CoCM}}"] \arrow[d, "\widetilde{\tau}^-{,}\widetilde{\mathsf{N}}^-"', "\approx"] & \overline{\mathrm{CoCM}}(\mathbf{I}_\Lambda) \arrow[r] \arrow[d, "\widetilde{\tau}^-{,}\widetilde{\mathsf{N}}^-"', "\approx"] & 0 \\
0 \arrow[r] & \mathcal{T}_l(\underline{\mathfrak{P}}_\Lambda^{\prec\propto}) \arrow[r] & \mathcal{T}_l(\underline{\mathrm{Mod}}\text{-}\Lambda) \arrow[r, "\mathbf{R}^*_{\mathrm{CM}}"] & \underline{\mathrm{CM}}(\mathbf{P}_\Lambda) \arrow[r] & 0
\end{tikzcd}$$

Let $\mathcal{E}$ be any one of the categories $\mathbf{P}_\Lambda$, $\mathbf{I}_\Lambda$, $\mathrm{CM}(\mathbf{P}_\Lambda)$, $\mathrm{CoCM}(\mathbf{I}_\Lambda)$, $\mathfrak{P}_\Lambda^{\prec\propto}$, $\mathfrak{I}_\Lambda^{\prec\propto}$. Then $\mathcal{E}$ is an exact subcategory of Mod-Λ with split idempotents. Let $\mathbf{D}^{\mathrm{b}}(\mathcal{E})$ be the bounded derived category of $\mathcal{E}$ [48]. The following result shows that the exact inclusion $\mathcal{E} \hookrightarrow$ Mod-Λ extends to a full exact embedding $\mathbf{D}^{\mathrm{b}}(\mathcal{E}) \hookrightarrow \mathbf{D}^{\mathrm{b}}(\mathrm{Mod}\text{-}\Lambda)$ which fits nicely in an exact commutative diagram of triangulated categories. We state it only for $\mathcal{E} = \mathrm{CM}(\mathbf{P}_\Lambda)$, noting that there are similar exact commutative diagrams on the level of derived categories induced by the inclusions $\mathfrak{P}_\Lambda^{\prec\propto} \hookrightarrow \mathrm{Mod}\text{-}\Lambda \hookleftarrow \mathfrak{I}_\Lambda^{\prec\propto}$ and $\mathrm{CoCM}(\mathbf{I}_\Lambda) \hookrightarrow \mathrm{Mod}\text{-}\Lambda$. We leave to the reader to state the other versions.

Proposition 5.8. *The exact inclusion* $\mathbf{i}_{\mathrm{CM}} : \mathrm{CM}(\mathbf{P}_\Lambda) \hookrightarrow \mathrm{Mod}\text{-}\Lambda$ *extends to a fully faithful exact functor* $\mathbf{D}^{\mathrm{b}}(\mathbf{i}_{\mathrm{CM}}) : \mathbf{D}^{\mathrm{b}}(\mathrm{CM}(\mathbf{P}_\Lambda)) \to \mathbf{D}^{\mathrm{b}}(\mathrm{Mod}\text{-}\Lambda)$ *with strict image the complexes* $X^\bullet \in \mathbf{D}^{\mathrm{b}}(\mathrm{Mod}\text{-}\Lambda)$ *such that the canonical morphism* $\delta_{X^\bullet} : X^\bullet \to \mathbb{R}\mathrm{Hom}_\Lambda(\mathrm{D}(\Lambda), X^\bullet \otimes_\Lambda^{\mathbb{L}} \mathrm{D}(\Lambda))$ *is invertible. Moreover there exists an exact commutative diagram of triangulated categories*:

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \mathbf{D}^{\mathrm{b}}(\mathbf{P}_\Lambda) \arrow[r] \arrow[d, equal] & \mathbf{D}^{\mathrm{b}}(\mathrm{CM}(\mathbf{P}_\Lambda)) \arrow[r] \arrow[d, "\mathbf{D}^{\mathrm{b}}(\mathbf{i}_{\mathrm{CM}})"'] & \underline{\mathrm{CM}}(\mathbf{P}_\Lambda) \arrow[r] \arrow[d, "\mathsf{P}_l\mathbf{i}_{\mathrm{CM}}"'] & 0 \\
0 \arrow[r] & \mathbf{D}^{\mathrm{b}}(\mathbf{P}_\Lambda) \arrow[r] & \mathbf{D}^{\mathrm{b}}(\mathrm{Mod}\text{-}\Lambda) \arrow[r] \arrow[d] & \mathcal{T}_l(\underline{\mathrm{Mod}}\text{-}\Lambda) \arrow[r] \arrow[d] & 0 \\
 & & \mathcal{T}_l(\underline{\mathfrak{P}}_\Lambda^{\prec\propto}) \arrow[r, equal] \arrow[d] & \mathcal{T}_l(\underline{\mathfrak{P}}_\Lambda^{\prec\propto}) \arrow[d] & \\
 & & 0 & 0 &
\end{tikzcd}$$

Proof. Since $\mathrm{CM}(\mathbf{P}_\Lambda)$ is contravariantly finite resolving, it is not difficult to see (compare [28]) that $\mathbf{D}^{\mathrm{b}}(\mathbf{i}_{\mathrm{CM}})$ is fully faithful and clearly its strict image consists of all complexes A such that $A \cong X$ in $\mathbf{D}(\mathrm{Mod}\text{-}\Lambda)$ where X lies in $\mathcal{H}^{\mathrm{b}}(\mathrm{CM}(\mathbf{P}_\Lambda))$. Using the characterization of Cohen–Macaulay modules in Lemma 3.3 it follows easily that a bounded complex $X^\bullet$ lies in the strict image of $\mathbf{D}^{\mathrm{b}}(\mathbf{i}_{\mathrm{CM}})$ iff the canonical map $\delta_{X^\bullet} : X^\bullet \to \mathbb{R}\,\mathrm{Hom}_\Lambda(\mathrm{D}(\Lambda), X^\bullet \otimes^{\mathbb{L}}_\Lambda \mathrm{D}(\Lambda))$ is invertible, see also [30]. Then exactness and commutativity of the two upper squares follow from [18,40,56], and exactness and commutativity of the remaining part of the diagram follows from the localization sequence of Proposition 5.7 induced from the torsion pair $(\underline{\mathrm{CM}}(\mathbf{P}_\Lambda), \mathcal{T}_l(\underline{\mathfrak{P}}^{\prec\infty}_\Lambda))$ in $\mathcal{T}_l(\underline{\mathrm{Mod}}\text{-}\Lambda)$. $\square$

We have seen that the stabilization categories $\mathcal{T}_r(\underline{\mathrm{Mod}}\text{-}\Lambda)$ and $\mathcal{T}_r(\overline{\mathrm{Mod}}\text{-}\Lambda)$, respectively $\mathcal{T}_l(\underline{\mathrm{Mod}}\text{-}\Lambda)$ and $\mathcal{T}_l(\overline{\mathrm{Mod}}\text{-}\Lambda)$, are always triangle equivalent. We don't know if this holds for the stabilizations $\mathcal{T}_l(\underline{\mathrm{Mod}}\text{-}\Lambda)$ and $\mathcal{T}_r(\overline{\mathrm{Mod}}\text{-}\Lambda)$. We close this section with the following consequences of the exact (co)localization sequences constructed above which, besides several functorial characterizations of Gorensteinness, shows that in the Gorenstein case the stabilization categories can be realized in the stable module categories, and the stabilizations $\mathcal{T}_l(\underline{\mathrm{Mod}}\text{-}\Lambda)$ and $\mathcal{T}_r(\overline{\mathrm{Mod}}\text{-}\Lambda)$ are triangle equivalent.

Corollary 5.9. *The following are equivalent.*

(i) Λ *is Gorenstein.*
(ii) *The functor* $\mathbf{R}^*_{\mathrm{CM}} : \mathcal{T}_l(\underline{\mathrm{Mod}}\text{-}\Lambda) \to \underline{\mathrm{CM}}(\mathbf{P}_\Lambda)$ *is a triangle equivalence.*
(iii) *The functor* $\mathbf{L}^*_{\mathrm{CoCM}} : \mathcal{T}_r(\overline{\mathrm{Mod}}\text{-}\Lambda) \to \overline{\mathrm{CoCM}}(\mathbf{I}_\Lambda)$ *is a triangle equivalence.*
(iv) *The functor* $\mathbf{L}^*_{\mathrm{CM}} : \mathcal{T}_r(\underline{\mathrm{Mod}}\text{-}\Lambda) \to \underline{\mathrm{CM}}(\mathbf{P}_\Lambda)$ *is a triangle equivalence.*
(v) *The functor* $\mathbf{R}^*_{\mathrm{CoCM}} : \mathcal{T}_r(\overline{\mathrm{Mod}}\text{-}\Lambda) \to \overline{\mathrm{CoCM}}(\mathbf{I}_\Lambda)$ *is a triangle equivalence.*
(vi) *The functor* $\mathbf{D}^{\mathrm{b}}(\mathbf{i}_{\mathrm{CM}}) : \mathbf{D}^{\mathrm{b}}(\mathrm{CM}(\mathbf{P}_\Lambda)) \to \mathbf{D}^{\mathrm{b}}(\mathrm{Mod}\text{-}\Lambda)$ *is a triangle equivalence.*
(vii) *The functor* $\mathbf{D}^{\mathrm{b}}(\mathbf{i}_{\mathrm{CoCM}}) : \mathbf{D}^{\mathrm{b}}(\mathrm{CoCM}(\mathbf{I}_\Lambda)) \to \mathbf{D}^{\mathrm{b}}(\mathrm{Mod}\text{-}\Lambda)$ *is a triangle equivalence.*
(viii) *The functor* $\mathbf{i}^*_{\mathrm{CoCM}} \mathsf{N}^+ \mathbf{R}^*_{\mathrm{CM}} : \mathcal{T}_l(\underline{\mathrm{Mod}}\text{-}\Lambda) \to \mathcal{T}_r(\overline{\mathrm{Mod}}\text{-}\Lambda)$ *is a triangle equivalence.*
(ix) *The functor* $\mathbf{i}^*_{\mathrm{CM}} \mathsf{N}^- \mathbf{L}^*_{\mathrm{CoCM}} : \mathcal{T}_r(\overline{\mathrm{Mod}}\text{-}\Lambda) \to \mathcal{T}_l(\underline{\mathrm{Mod}}\text{-}\Lambda)$ *is a triangle equivalence.*
(x) *The exact inclusion* $\mathbf{P}_\Lambda \hookrightarrow \mathfrak{P}^{\prec\infty}_\Lambda$ *extends to a triangle equivalence* $\mathbf{D}^{\mathrm{b}}(\mathbf{P}_\Lambda) \xrightarrow{\approx} \mathbf{D}^{\mathrm{b}}(\mathfrak{P}^{\prec\infty}_\Lambda)$.
(xi) *The exact inclusion* $\mathbf{I}_\Lambda \hookrightarrow \mathfrak{I}^{\prec\infty}_\Lambda$ *extends to a triangle equivalence* $\mathbf{D}^{\mathrm{b}}(\mathbf{I}_\Lambda) \xrightarrow{\approx} \mathbf{D}^{\mathrm{b}}(\mathfrak{I}^{\prec\infty}_\Lambda)$.
(xii) $\forall A \in \mathrm{Mod}\text{-}\Lambda$ *with* $\underline{\mathrm{Hom}}_\Lambda(A, X) = 0$, $\forall X \in \mathrm{CM}(\mathbf{P}_\Lambda)$, $\exists n \geqslant 0$ *such that* $\Sigma^n_{\mathbf{P}}(A)$ *is projective.*
(xiii) $\forall B \in \mathrm{Mod}\text{-}\Lambda$ *with* $\overline{\mathrm{Hom}}_\Lambda(Z, B) = 0$, $\forall Z \in \mathrm{CoCM}(\mathbf{I}_\Lambda)$, $\exists m \geqslant 0$ *such that* $\Omega^m_{\mathbf{I}}(B)$ *is injective.*

Proof. By Remark 4.3, Propositions 5.6, 5.7 and Proposition 5.8 and its dual, any one of the conditions (ii), (v), (vi), (viii) and (x) is equivalent to

$$\mathcal{T}_l\big(\underline{\mathfrak{P}}^{\prec\infty}_\Lambda\big) = 0,$$

and any one of the conditions (iii), (iv), (vii), (ix), and (xi) is equivalent to

$$\mathcal{T}_r\big(\bar{\mathfrak{I}}_\Lambda^{\prec\propto}\big) = 0.$$

By Lemma 4.1, $\mathcal{T}_l(\underline{\mathfrak{P}}_\Lambda^{\prec\propto}) = 0$ is equivalent to

$$\mathfrak{P}_\Lambda^{\prec\propto} = \mathbf{P}_\Lambda^{<\infty}$$

and $\mathcal{T}_r(\bar{\mathfrak{I}}_\Lambda^{\prec\propto}) = 0$ is equivalent to

$$\mathfrak{I}_\Lambda^{\prec\propto} = \mathbf{I}_\Lambda^{<\infty}.$$

These last conditions are equivalent to (i) by Proposition 3.10. Finally the equivalences (i) $\Leftrightarrow$ (xii) and (i) $\Leftrightarrow$ (xiii) follow from Propositions 5.6, 5.7 and the fact that the assertions in (xii) and (xiii) are equivalent reformulations of the conditions $\mathcal{T}_r({}^{\perp}\underline{\mathrm{CM}}(\mathbf{P}_\Lambda)) = 0$ and $\mathcal{T}_l(\overline{\mathrm{CoCM}}(\mathbf{I}_\Lambda)^{\perp}) = 0$ respectively. □

6. Compact generators and pure-injective cogenerators

Compact and pure-injective objects play an important role in the investigation of torsion pairs in triangulated categories in connection with several finiteness conditions, see [22,42] for details. In this section we study the analogous situation in the stable module category of an Artin algebra in connection with the (Co)Cohen–Macaulay (co)torsion pairs.

Let $\mathcal{A}$ be an additive category. A set $\mathcal{X}$ of objects of $\mathcal{A}$ is called a **generating set** if $\mathcal{C}(X, A) = 0$, $\forall X \in \mathcal{X}$, implies that $A = 0$. Now let $\mathcal{C}$ be a pretriangulated category which admits all small coproducts. We say that $\mathcal{C}$ is **compactly generated** if $\mathcal{C}$ admits a set $\mathcal{X}$ of objects, which without loss of generality we may assume that is closed under Σ, consisting of compact generators. It is easy to see that if any monomorphism in $\mathcal{C}$ splits, then a set of compact objects $\mathcal{X}$ in $\mathcal{C}$ which is closed under Σ is a generating set iff the objects in $\mathcal{X}$ collectively reflect isomorphisms.

In the sequel we shall need the following easy and rather well-known result (compare [51]).

Lemma 6.1. *Let $(F, G) : \mathcal{C} \to \mathcal{D}$ be an adjoint pair of functors between additive categories. If G preserves coproducts, then F preserves compact objects. Assume that $\mathcal{C}$ is pretriangulated where the loop functor Ω preserves coproducts and further $\mathcal{C}$ admits a set of compact generators. If any monomorphism in $\mathcal{C}$ splits and F preserves compact objects, then G preserves coproducts.*

As a direct consequence we have the following.

Corollary 6.2. *The reflection functors $\mathbf{L}_{\mathrm{CM}} : \underline{\mathrm{Mod}}\text{-}\Lambda \to \underline{\mathrm{CM}}(\mathbf{P}_\Lambda)$ and $\mathbf{L}_{\mathrm{CoCM}} : \overline{\mathrm{Mod}}\text{-}\Lambda \to \overline{\mathrm{CoCM}}(\mathbf{I}_\Lambda)$ preserve compact objects and generators.*

Proof. Since $\mathrm{CM}(\mathbf{P}_\Lambda)$ and $\mathrm{CoCM}(\mathbf{I}_\Lambda)$ are closed under coproducts, the inclusion functors $\mathbf{i}_{\mathrm{CM}}$ and $\mathbf{i}_{\mathrm{CoCM}}$ preserve coproducts. Hence by Lemma 6.1 their left adjoints $\mathbf{L}_{\mathrm{CM}}$ and $\mathbf{L}_{\mathrm{CoCM}}$ preserve compact objects, and clearly they preserve generators. □

To proceed further we need a description of the compact objects of the stable module categories.

Lemma 6.3. *The pretriangulated categories* $\underline{\mathrm{Mod}}\text{-}\Lambda$ *and* $\overline{\mathrm{Mod}}\text{-}\Lambda$ *are compactly generated and*:

$$(\underline{\mathrm{Mod}}\text{-}\Lambda)^{\mathrm{b}} = \underline{\mathrm{mod}}\text{-}\Lambda \quad \textit{and} \quad \big(\overline{\mathrm{Mod}}\text{-}\Lambda\big)^{\mathrm{b}} = \overline{\mathrm{mod}}\text{-}\Lambda.$$

Proof. The first assertion follows from [19]. Clearly $\underline{\mathrm{mod}}\text{-}\Lambda \subseteq (\underline{\mathrm{Mod}}\text{-}\Lambda)^{\mathrm{b}}$. Let $\underline{T}$ be a compact object in $\underline{\mathrm{Mod}}\text{-}\Lambda$. Clearly we may assume that T has no non-zero projective summands. Let $T/\mathrm{Rad}(T) = \coprod_{i\in I} S_i$ be the top of T and let $\varepsilon : T \to \coprod_{i\in I} S_i$ be the canonical epimorphism. Consider the following commutative diagram, where the involved maps are the natural ones:

$$\begin{tikzcd}
\coprod_{i\in I} \mathrm{Hom}_\Lambda(T, S_i) \arrow[r, "\alpha"] \arrow[d, "\gamma"'] & \mathrm{Hom}_\Lambda(T, \coprod_{i\in I} S_i) \arrow[d, "\beta"'] \\
\coprod_{i\in I} \underline{\mathrm{Hom}}_\Lambda(T, S_i) \arrow[r, "\delta"] & \underline{\mathrm{Hom}}_\Lambda(T, \coprod_{i\in I} S_i)
\end{tikzcd}$$

Observe that α is monic, γ and β are epics and, by hypothesis, δ is invertible. Let $f : T \to \coprod_{i\in I} S_i$ be a map which factorizes through a projective module. Let

$$f := \kappa \circ \lambda : T \xrightarrow{\kappa} A \xrightarrow{\lambda} \coprod_{i\in I} S_i$$

be the canonical factorization of f. Then A is semisimple as a submodule of the semisimple module $\coprod_{i\in I} S_i$ and therefore λ is split monic. This implies easily that the map κ factorizes through a projective module, say as $\kappa = g \circ h : T \xrightarrow{g} P \xrightarrow{h} A$ where h is the projective cover of A. Since κ is epic and h is essential, it follows that g is epic and therefore P is a direct summand of T. Since T admits no non-zero projective summands, we infer that f is zero. Hence the map β is invertible and therefore so is α. It follows from this that the canonical map $\varepsilon : T \to \coprod_{i\in I} S_i$ factors through a finite subcoproduct $\coprod_{j\in J} S_j$ where $J \subseteq I$ with $|J| < \infty$. Since ε is epic, we infer that $\coprod_{j\in J} S_j = \coprod_{i\in I} S_i$ and therefore $I = J$ is finite. This implies that T is finitely generated. Hence $(\underline{\mathrm{Mod}}\text{-}\Lambda)^{\mathrm{b}} \subseteq \underline{\mathrm{mod}}\text{-}\Lambda$. Using the Auslander–Reiten equivalences $\tau^{\pm}$, we infer directly that

$$\big(\overline{\mathrm{Mod}}\text{-}\Lambda\big)^{\mathrm{b}} = \overline{\mathrm{mod}}\text{-}\Lambda. \qquad \square$$

For compactly generated triangulated categories there is defined a theory of purity which parallels the well-known theory of purity of modules. Let $\mathcal{T}$ be a compactly generated triangulated category. A triangle $(T) : A \to B \to C \to \Sigma(A)$ in $\mathcal{T}$ is called **pure** if the sequence $0 \to \mathcal{T}(X, A) \to \mathcal{T}(X, B) \to \mathcal{T}(X, C) \to 0$ is exact for any compact object X in $\mathcal{T}$. And an object E is called **pure-injective** if the sequence $0 \to \mathcal{T}(C, E) \to \mathcal{T}(B, E) \to \mathcal{T}(A, E) \to 0$ is exact for any pure triangle (T) as above. By [35,42] a module E in Mod-Λ, respectively an object E in $\mathcal{T}$, is pure-injective, iff for any index set I the summation map $\coprod_I E \to E$ factors through the canonical map $\coprod_I E \to \prod_I E$. We refer to [16,42] for more details concerning purity in triangulated categories.

In the sequel we shall need the following observation which shows that (Co)Cohen–Macaulay pure-injective objects and modules are closely related.

Lemma 6.4. *A Cohen–Macaulay module X, respectively CoCohen–Macaulay module Z, is pure-injective in* Mod-Λ *iff the object $\underline{X}$, respectively $\overline{Z}$, is pure-injective in* $\underline{\mathrm{CM}}(\mathbf{P}_\Lambda)$, *respectively* $\overline{\mathrm{CoCM}}(\mathbf{I}_\Lambda)$.

Proof. If X is pure-injective, then so is $\underline{X}$ since $\mathrm{CM}(\mathbf{P}_\Lambda)$ is closed under products and coproducts and the functor $\pi : \mathrm{CM}(\mathbf{P}_\Lambda) \to \underline{\mathrm{CM}}(\mathbf{P}_\Lambda)$ preserves products and coproducts. If $\underline{X}$ is pure-injective in $\underline{\mathrm{CM}}(\mathbf{P}_\Lambda)$, then let $\varepsilon : \coprod_I X \to X$ be the summation map and $\mu : \coprod_I X \to \prod_I X$ the canonical pure-monomorphism. Then there exists a map $\alpha : \prod_I X \to X$ such that $\mu \circ \alpha - \varepsilon : \coprod_I X \to X$ factors through a projective module P, say as $\mu \circ \alpha - \varepsilon = \coprod_I X \xrightarrow{\kappa} P \xrightarrow{\lambda} X$. Since P is pure-injective and μ is pure-mono, there exists a map $\rho : \coprod_I X \to P$ such that $\mu \circ \rho = \kappa$, and then $\varepsilon = \mu \circ (\alpha + \rho \circ \lambda)$. Hence X is pure-injective. The case of CoCohen–Macaulay modules is similar. $\square$

If $\mathcal{T}$ is a triangulated category and $\mathcal{V}$ is a family of objects of $\mathcal{T}$, then we denote by $\mathsf{thick}(\mathcal{V})$ the thick subcategory generated by $\mathcal{V}$, that is, the smallest full triangulated subcategory of $\mathcal{T}$ which is closed under direct summands and contains $\mathcal{V}$. We shall need the following result.

Lemma 6.5 (Neeman [49]). *Let $\mathcal{T}$ be a triangulated category which admits all small coproducts, and let $\mathcal{V}$ be a set of compact generators of $\mathcal{T}$. Then* $\mathsf{thick}(\mathcal{V}) = \mathcal{T}^{\mathrm{b}}$.

From now on we use the following notation (as before $\mathfrak{r}$ denotes the radical of Λ):

- $f_\mathfrak{r} : X_\mathfrak{r} \to \mathfrak{r}$ is the minimal right $\mathrm{CM}(\mathbf{P}_\Lambda)$-approximation of $\mathfrak{r}$ of Λ and $Y_\mathfrak{r} = \mathrm{Ker}(f_\mathfrak{r})$.
- $f_{\Lambda/\mathfrak{r}} : X_{\Lambda/\mathfrak{r}} \to \Lambda/\mathfrak{r}$ is the minimal right $\mathrm{CM}(\mathbf{P}_\Lambda)$-approximation of $\Lambda/\mathfrak{r}$ and $Y_{\Lambda/\mathfrak{r}} = \mathrm{Ker}(f_{\Lambda/\mathfrak{r}})$. Note that, up to projective summands, $X_\mathfrak{r} = \Omega(X_{\Lambda/\mathfrak{r}})$ and $\Sigma_\mathbf{P}(X_\mathfrak{r}) = X_{\Lambda/\mathfrak{r}}$.
- $g^{\Lambda/\mathfrak{r}} : \Lambda/\mathfrak{r} \to Z^{\Lambda/\mathfrak{r}}$ is a special $\mathrm{CoCM}(\mathbf{I}_\Lambda)$-approximation of $\Lambda/\mathfrak{r}$ and $W^{\Lambda/\mathfrak{r}} = \mathrm{Coker}(g^{\Lambda/\mathfrak{r}})$.
- $h^{\Lambda/\mathfrak{r}} : \Lambda/\mathfrak{r} \to X^{\Lambda/\mathfrak{r}}$ a left $\mathrm{CM}(\mathbf{P}_\Lambda)$-approximation of $\Lambda/\mathfrak{r}$.

We now show that the stable triangulated categories $\underline{\mathrm{CM}}(\mathbf{P}_\Lambda)$ and $\overline{\mathrm{CoCM}}(\mathbf{I}_\Lambda)$ are monogenic, that is, they admit a single compact generator. Moreover we determine pure-injective cogenerators.

Theorem 6.6. *The categories* $\underline{\mathrm{CM}}(\mathbf{P}_\Lambda)$ *and* $\overline{\mathrm{CoCM}}(\mathbf{I}_\Lambda)$ *are compactly generated by the sets* $\{\underline{X}^T \mid T \in \underline{\mathrm{mod}}\text{-}\Lambda\}$ *and* $\{\overline{Z}^T \mid T \in \overline{\mathrm{mod}}\text{-}\Lambda\}$ *respectively. Moreover we have the following.*

(i) $\overline{Z}^{\Lambda/\mathbf{r}}$ *is a compact generator of the category* $\overline{\mathrm{CoCM}}(\mathbf{I}_\Lambda)$, *hence*:

$$\overline{\mathrm{CoCM}}(\mathbf{I}_\Lambda)^{\mathrm{b}} = \mathsf{thick}\big(\overline{Z}^{\Lambda/\mathbf{r}}\big).$$

(ii) $\underline{X}^{\Lambda/\mathbf{r}}$ *is a compact generator of the category* $\underline{\mathrm{CM}}(\mathbf{P}_\Lambda)$, *hence*:

$$\underline{\mathrm{CM}}(\mathbf{P}_\Lambda)^{\mathrm{b}} = \mathsf{thick}\big(\underline{X}^{\Lambda/\mathbf{r}}\big).$$

(iii) $\mathfrak{I}_\Lambda^{\prec\propto} = {}^{\perp}\mathsf{N}^+(X_{\mathfrak{r}})$ *and the object* $\mathsf{N}^+(\underline{X}^{\Lambda/\mathfrak{r}})$ *is a compact generator of* $\overline{\mathrm{CoCM}}(\mathbf{I}_\Lambda)$. *Moreover* $\mathsf{N}^+(X_{\mathfrak{r}})$ *is pure-injective and* $\mathsf{N}^+(\underline{X}_{\mathfrak{r}})$ *is a pure-injective cogenerator of* $\overline{\mathrm{CoCM}}(\mathbf{I}_\Lambda)$.

(iv) $\mathfrak{P}_\Lambda^{\prec\propto} = \mathsf{N}^-(Z^{\Lambda/\mathfrak{r}})^{\perp}$ *and the object* $\mathsf{N}^-(\overline{Z}^{\Lambda/\mathfrak{r}})$ *is a compact generator of* $\underline{\mathrm{CM}}(\mathbf{P}_\Lambda)$. *Moreover* $X_{\mathfrak{r}}$ *is pure-injective and* $\underline{X}_{\mathbf{r}}$ *is a pure-injective cogenerator of* $\underline{\mathrm{CM}}(\mathbf{P}_\Lambda)$.

Proof. The first assertion follows from Lemma 6.3 and Corollary 6.2. Since the object $\mathbf{L}_{\mathrm{CoCM}}(\overline{\Lambda/\mathbf{r}}) = \overline{Z}^{\Lambda/\mathbf{r}}$ is compact in $\overline{\mathrm{CoCM}}(\mathbf{I}_\Lambda)$, for any CoCohen–Macaulay module Z, and $n \geqslant 0$, we have:

$$\overline{\mathrm{Hom}}_\Lambda\big(\overline{Z}^{\Lambda/\mathbf{r}}, \Sigma^n\big(\overline{Z}\big)\big) \cong \overline{\mathrm{Hom}}_\Lambda\big(\overline{\Lambda/\mathbf{r}}, \Sigma^n\big(\overline{Z}\big)\big) \cong \mathrm{Ext}^n(\Lambda/\mathbf{r}, Z).$$

If $\overline{\mathrm{Hom}}_\Lambda(\overline{Z}^{\Lambda/\mathbf{r}}, \Sigma^n(\overline{Z})) = 0$, $\forall n \in \mathbb{Z}$, it follows that $\mathrm{Ext}^n(\Lambda/\mathbf{r}, Z) = 0$, $\forall n \geqslant 1$. By induction on the length of a finitely generated module, this implies that $\mathrm{Ext}^n(T, Z) = 0$, $\forall n \geqslant 1$, $\forall T \in \mathrm{mod}\text{-}\Lambda$. Then clearly Z is injective, i.e., $\overline{Z} = 0$. We infer that $\overline{Z}^{\Lambda/\mathbf{r}}$ is a compact generator of $\overline{\mathrm{CoCM}}(\mathbf{I}_\Lambda)$ and therefore $\overline{\mathrm{CoCM}}(\mathbf{I}_\Lambda)^{\mathrm{b}} = \mathsf{thick}(\overline{Z}^{\Lambda/\mathbf{r}})$ by Lemma 6.5. Hence (i) holds and part (ii) is dual.

(iii) Let X be in $\mathrm{CM}(\mathbf{P}_\Lambda)$ and assume that $X \in {}^{\perp}X_{\mathfrak{r}}$. Then clearly X lies in ${}^{\perp}\mathfrak{r}$. Applying the functor $\mathrm{Hom}_\Lambda(X, -)$ to the exact sequence $0 \to \mathfrak{r} \to \Lambda \to \Lambda/\mathfrak{r} \to 0$ and using that X is Cohen–Macaulay, we infer that $X \in {}^{\perp}(\Lambda/\mathfrak{r})$. By induction on the length of a finitely generated module, this implies that $\mathrm{Ext}^n_\Lambda(X, T) = 0$, $\forall n \geqslant 1$, $\forall T \in \mathrm{mod}\text{-}\Lambda$. Then $\mathrm{Ext}_n^\Lambda(X, \mathrm{D}(S)) = 0$, $\forall n \geqslant 1$, $\forall S \in \mathrm{mod}\text{-}\Lambda^{\mathrm{op}}$. Using the duality isomorphism $\mathrm{Ext}^n_\Lambda(A, \mathrm{D}(B)) \cong \mathrm{D}\,\mathrm{Tor}_n^\Lambda(A, B)$, we infer that $\mathrm{Tor}_n^\Lambda(X, S) = 0$, $\forall n \geqslant 1$, $\forall S \in \mathrm{mod}\text{-}\Lambda^{\mathrm{op}}$. This clearly implies that X is projective. Consequently $\mathrm{CM}(\mathbf{P}_\Lambda) \cap {}^{\perp}X_{\mathfrak{r}} = \mathbf{P}_\Lambda$. Now let A be a module in ${}^{\perp}\mathsf{N}^+(X_{\mathfrak{r}})$, i.e., $\mathrm{Ext}^n_\Lambda(A, \mathsf{N}^+(X_{\mathfrak{r}})) = 0$, $\forall n \geqslant 1$. By Lemma 5.1 we have $\mathrm{Ext}^n_\Lambda(\mathsf{N}^-(Z^A), X_{\mathfrak{r}}) = 0$, $\forall n \geqslant 1$, and therefore $\mathsf{N}^-(Z^A)$ is projective, or equivalently Z^A is injective. It follows that $\mathbf{L}_{\mathrm{CoCM}}(\overline{A}) = 0$ and consequently $\overline{A} \in \overline{\mathfrak{I}}_\Lambda^{\prec\propto}$, i.e., A lies in $\mathfrak{I}_\Lambda^{\prec\propto}$. Hence ${}^{\perp}\mathsf{N}^+(X_{\mathfrak{r}}) \subseteq \mathfrak{I}_\Lambda^{\prec\propto}$ and therefore ${}^{\perp}\mathsf{N}^+(X_{\mathfrak{r}}) = \mathfrak{I}_\Lambda^{\prec\propto}$ since $\mathsf{N}^+(X_{\mathfrak{r}})$ is CoCohen–Macaulay. In particular ${}^{\perp}\mathsf{N}^+(X_{\mathfrak{r}}) \cap \mathrm{CoCM}(\mathbf{I}_\Lambda) = \mathbf{I}_\Lambda$ and this implies that the object $\mathsf{N}^+(\underline{X}_{\mathfrak{r}})$ is a cogenerator of $\overline{\mathrm{CoCM}}(\mathbf{I}_\Lambda)$. Finally since, by Proposition 3.8, we have $\mathrm{D}^2\,\mathrm{CM}(\mathbf{P}_\Lambda) \subseteq \mathrm{CM}(\mathbf{P}_\Lambda)$, it follows that $X_{\mathfrak{r}}$ is pure-injective by [46, Theorem 2.6]. This implies that $\underline{X}_{\mathfrak{r}}$ is pure-injective in $\underline{\mathrm{CM}}(\mathbf{P}_\Lambda)$. Since $\mathsf{N}^+ : \underline{\mathrm{CM}}(\mathbf{P}_\Lambda) \to \overline{\mathrm{CoCM}}(\mathbf{I}_\Lambda)$ is an equivalence, we infer that $\mathsf{N}^+(\underline{X}_{\mathfrak{r}})$ is pure-injective in $\overline{\mathrm{CoCM}}(\mathbf{I}_\Lambda)$.

(iv) The proof is similar to the proof of (iii) and is left to the reader. $\square$

Remark 6.7. Pure-injectivity of $\mathsf{N}^+(\underline{X}_{\mathfrak{r}})$, and therefore of $X_{\mathfrak{r}}$ and $\underline{X}_{\mathfrak{r}}$, follows from Brown representability. Indeed let as before $\mathbb{E}$ be the injective envelope of $R/\operatorname{Rad}(R)$. Then, by [22], the Brown–Comenetz dual $\mathsf{D}_{\mathbb{E}}(\overline{Z}^{\Lambda/\mathfrak{r}})$ in $\overline{\operatorname{CoCM}}(\mathbf{I}_\Lambda)$ of the compact generator $\overline{Z}^{\Lambda/\mathfrak{r}}$ is a pure-injective cogenerator of $\overline{\operatorname{CoCM}}(\mathbf{I}_\Lambda)$. Recall that the object $\mathsf{D}_{\mathbb{E}}(\overline{Z}^{\Lambda/\mathfrak{r}})$ is uniquely defined up to isomorphism by the following natural isomorphism which is a consequence of Brown representability [50]:

$$\mathrm{D}\overline{\operatorname{Hom}}_\Lambda\big(\overline{Z}^{\Lambda/\mathfrak{r}}, -\big)\big|_{\overline{\operatorname{CoCM}}(\mathbf{I}_\Lambda)} \cong \overline{\operatorname{Hom}}_\Lambda\big(-, \mathsf{D}_{\mathbb{E}}\big(\overline{Z}^{\Lambda/\mathfrak{r}}\big)\big)\big|_{\overline{\operatorname{CoCM}}(\mathbf{I}_\Lambda)}.$$

Using Auslander–Reiten's formula $\mathrm{D}\operatorname{Ext}^1_\Lambda(T, A) \cong \overline{\operatorname{Hom}}_\Lambda(\overline{A}, \mathrm{DTr}(\underline{T}))$ and the easily established fact that $\mathsf{N}^-\,\mathrm{DTr}(T) \cong \Omega^2(T)$, $\forall T \in \operatorname{mod-}\Lambda$ and $\forall A \in \operatorname{Mod-}\Lambda$, we have isomorphisms, $\forall Z \in \operatorname{CoCM}(\mathbf{I}_\Lambda)$:

$$\begin{aligned}
\mathrm{D}\overline{\operatorname{Hom}}_\Lambda\big(\overline{Z}^{\Lambda/\mathfrak{r}}, \overline{Z}\big) &= \mathrm{D}\overline{\operatorname{Hom}}_\Lambda\big(\mathbf{L}_{\operatorname{CoCM}}\big(\overline{\Lambda/\mathfrak{r}}\big), \overline{Z}\big) \cong \mathrm{D}\overline{\operatorname{Hom}}_\Lambda\big(\overline{\Lambda/\mathfrak{r}}, \overline{Z}\big)\\
&\cong \mathrm{D}\overline{\operatorname{Hom}}_\Lambda\big(\overline{\Lambda/\mathfrak{r}}, \Sigma\Omega_{\mathbf{I}}\big(\overline{Z}\big)\big) \cong \mathrm{D}\operatorname{Ext}^1_\Lambda\big(\Lambda/\mathfrak{r}, \Omega_{\mathbf{I}}(Z)\big)\\
&\cong \overline{\operatorname{Hom}}_\Lambda\big(\Omega_{\mathbf{I}}\big(\overline{Z}\big), \mathrm{DTr}(\underline{\Lambda/\mathfrak{r}})\big) \xrightarrow{\cong} \overline{\operatorname{Hom}}_\Lambda\big(\Omega_{\mathbf{I}}\big(\overline{Z}\big), \mathbf{R}_{\operatorname{CoCM}}\,\mathrm{DTr}(\underline{\Lambda/\mathfrak{r}})\big)\\
&\cong \overline{\operatorname{Hom}}_\Lambda\big(\overline{Z}, \Sigma\mathbf{R}_{\operatorname{CoCM}}\,\mathrm{DTr}(\underline{\Lambda/\mathfrak{r}})\big)\\
&\cong \overline{\operatorname{Hom}}_\Lambda\big(\overline{Z}, \Sigma\mathsf{N}^+\mathbf{R}_{\operatorname{CM}}\mathsf{N}^-\,\mathrm{DTr}(\underline{\Lambda/\mathfrak{r}})\big) \cong \overline{\operatorname{Hom}}_\Lambda\big(\overline{Z}, \Sigma\mathsf{N}^+\mathbf{R}_{\operatorname{CM}}\big(\Omega(\underline{\mathfrak{r}})\big)\big)\\
&\cong \overline{\operatorname{Hom}}_\Lambda\big(\overline{Z}, \Sigma\mathsf{N}^+(\underline{X}_{\Omega(\mathfrak{r})})\big) \cong \overline{\operatorname{Hom}}_\Lambda\big(\overline{Z}, \mathsf{N}^+\Sigma_{\mathbf{P}}(\underline{X}_{\Omega(\mathfrak{r}))})\big)\\
&\cong \overline{\operatorname{Hom}}_\Lambda\big(\overline{Z}, \mathsf{N}^+(\underline{X}_{\mathfrak{r}})\big).
\end{aligned}$$

So $\mathsf{N}^+(\underline{X}_{\mathfrak{r}}) \cong \mathsf{D}_{\mathbb{E}}(\overline{Z}^{\Lambda/\mathfrak{r}})$ and therefore $\mathsf{N}^+(\underline{X}_{\mathfrak{r}})$ is a pure-injective cogenerator of $\overline{\operatorname{CoCM}}(\mathbf{I}_\Lambda)$. Then $\underline{X}_{\mathfrak{r}} \cong \mathsf{N}^-\mathsf{N}^+(\underline{X}_{\mathfrak{r}})$ is pure-injective cogenerator of $\underline{\operatorname{CM}}(\mathbf{P}_\Lambda)$, and $X_{\mathfrak{r}}$ is pure-injective by Lemma 6.4.

We have seen in Proposition 3.8 that any Λ-module admits a minimal right $\operatorname{CM}(\mathbf{P}_\Lambda)$-approximation and a minimal left $\mathfrak{P}^{\prec\infty}_\Lambda$-approximation. As a consequence of Theorem 6.6 we have the following.

Corollary 6.8. *The subcategory $\mathfrak{I}^{\prec\infty}_\Lambda$ of* $\operatorname{Mod-}\Lambda$ *consisting of all modules of virtually finite injective dimension is closed under filtered colimits, pure submodules and pure factor modules. Moreover any module admits a minimal right $\mathfrak{I}^{\prec\infty}_\Lambda$-approximation and a minimal left* $\operatorname{CoCM}(\mathbf{I}_\Lambda)$*-approximation.*

Proof. By Theorem 6.6 we have $\mathfrak{I}^{\prec\infty}_\Lambda = {}^\perp\mathsf{N}^+(X_{\mathfrak{r}})$ and the module $\mathsf{N}^+(X_{\mathfrak{r}})$ is pure-injective. Then the assertions follow from [46, Lemmas 4.1 and 4.2] and [59]. □

We say that a cotorsion pair $(\mathcal{A}, \mathcal{B})$ is **generated**, respectively **cogenerated**, by a class of modules $\mathcal{V}$ if $\mathcal{A} = {}^\perp\mathcal{V}$, respectively $\mathcal{B} = \mathcal{V}^\perp$. In this terminology Theorem 6.6 says that the cotorsion pair $(\mathfrak{I}^{\prec\infty}_\Lambda, \operatorname{CoCM}(\mathbf{I}_\Lambda))$ is generated by the CoCohen–Macaulay module $\mathsf{N}^+(X_{\mathfrak{r}})$ and the cotorsion pair $(\operatorname{CM}(\mathbf{P}_\Lambda), \mathfrak{P}^{\prec\infty}_\Lambda)$ is cogenerated by the Cohen–Macaulay module $\mathsf{N}^{-}(Z^{\Lambda/\mathfrak{r}})$. The following result, which will be useful in the sequel, shows that the cotorsion pair $(\mathrm{CM}(\mathbf{P}_\Lambda), \mathfrak{P}_\Lambda^{\prec\infty})$ is generated by a pure-injective module of virtually finite projective dimension. For a subcategory $\mathcal{U}$ of Mod-Λ, we denote by $\mathrm{Filt}(\mathcal{U})$ the full subcategory of Mod-Λ consisting of direct summands of modules U which admit a finite filtration $0 = U_{t+1} \subseteq U_t \subseteq \cdots \subseteq U_1 \subseteq U_0 = U$ where each quotient U_i/U_{i+1} lies in $\mathcal{U}$.

Proposition 6.9.

(i) $\mathrm{CM}(\mathbf{P}_\Lambda) = {}^{\perp}Y_{\Lambda/\mathfrak{r}} = \mathrm{Filt}[\mathrm{Prod}(X_{\Lambda/\mathfrak{r}})]$.
(ii) $\mathfrak{P}_\Lambda^{\prec\infty} = [\mathrm{Prod}(X_{\Lambda/\mathfrak{r}})]^{\perp}$ *and* $\underline{\mathrm{CM}}(\mathbf{P}_\Lambda) = \mathsf{thick}[\mathrm{Prod}(\underline{X}_{\Lambda/\mathfrak{r}})]$.
(iii) $\forall n \geqslant 0$, *there exists a cotorsion pair* $(\mathrm{CM}(\mathbf{P}_\Lambda)^{\leqslant n}, \mathfrak{P}_\Lambda^{\preccurlyeq n})$ *in* Mod-Λ, *where*

$$\mathrm{CM}(\mathbf{P}_\Lambda)^{\leqslant n} = {}^{\perp}\big(\Sigma^n(Y_{\Lambda/\mathfrak{r}})\big) = \mathrm{Filt}\big[\mathrm{Prod}\big(X^n_{\Lambda/\mathfrak{r}}\big)\big]$$

is definable and $X^n_{\Lambda/\mathfrak{r}}$ *is the minimal right* $\mathrm{CM}(\mathbf{P}_\Lambda)^{\leqslant n}$*-approximation of* $\Lambda/\mathfrak{r}$.
(iv) $\forall n \geqslant 0$: $\mathrm{CM}(\mathbf{P}_\Lambda)^{\leqslant n} \cap \mathfrak{P}_\Lambda^{\preccurlyeq n} = \mathbf{P}_\Lambda^{\leqslant n} \cap (\mathbf{P}_\Lambda^{\leqslant n})^{\perp}$.

Proof. (i) and (ii). Since $\mathrm{CM}(\mathbf{P}_\Lambda)$ is a contravariantly finite resolving definable subcategory of Mod-Λ the assertions in (i) are consequences of [46, Theorem 3.1]. Clearly $\mathrm{Prod}(X_{\Lambda/\mathfrak{r}}) \subseteq \mathrm{CM}(\mathbf{P}_\Lambda)$ since the latter is closed under products. Hence $\mathfrak{P}_\Lambda^{\prec\infty} = \mathrm{CM}(\mathbf{P}_\Lambda)^{\perp} \subseteq [\mathrm{Prod}(X_{\Lambda/\mathfrak{r}})]^{\perp}$. If A lies in $[\mathrm{Prod}(X_{\Lambda/\mathfrak{r}})]^{\perp}$, then it follows easily by induction that $A \in (\mathrm{Filt}[\mathrm{Prod}(X_{\Lambda/\mathfrak{r}})])^{\perp} = \mathrm{CM}(\mathbf{P}_\Lambda)^{\perp} = \mathfrak{P}_\Lambda^{\prec\infty}$. Also (i) implies that the stable category $\underline{\mathrm{CM}}(\mathbf{P}_\Lambda)$ is the thick closure of $\mathrm{Prod}(\underline{X}_{\Lambda/\mathfrak{r}})$.

(iii) If $n = 0$ then the assertion follows from (i). If $n \geqslant 1$ and A lies in $\mathrm{CM}(\mathbf{P}_\Lambda)^{\leqslant n}$, then as in Proposition 3.9 we have $\Omega^n(A) \in \mathrm{CM}(\mathbf{P}_\Lambda)$. Then by (i) it follows directly that A lies in ${}^{\perp}(\Sigma^n Y_{\Lambda/\mathfrak{r}})$ and therefore $\mathrm{CM}(\mathbf{P}_\Lambda)^{\leqslant n} \subseteq {}^{\perp}(\Sigma^n Y_{\Lambda/\mathfrak{r}})$. Now let A be in ${}^{\perp}(\Sigma^n Y_{\Lambda/\mathfrak{r}})$ and consider the approximation sequence $0 \to A \to Y^A \to X^A \to 0$ from which we obtain directly that Y^A lies in ${}^{\perp}(\Sigma^n Y_{\Lambda/\mathfrak{r}})$. This implies that $\Omega^n(Y^A) \in {}^{\perp}(Y_{\Lambda/\mathfrak{r}})$ and therefore $\Omega^n(Y^A) = \Omega^{n-1}(\Omega(Y^A)) = \Omega^{n-1}(Y_A)$ is Cohen–Macaulay. Since $\Omega^{n-1}(Y_A)$ lies in $\mathfrak{P}_\Lambda^{\prec\infty}$, it follows that $\Omega^{n-1}(Y_A)$ is projective, i.e., $\mathrm{pd}\, Y_A \leqslant n-1$. Then from the approximation sequence $0 \to Y_A \to X_A \to A \to 0$ we get that $\mathrm{res.dim}_{\mathrm{CM}}\, A \leqslant n$. Hence $\mathrm{CM}(\mathbf{P}_\Lambda)^{\leqslant n} = {}^{\perp}(\Sigma^n Y_{\Lambda/\mathfrak{r}})$, $\forall n \geqslant 0$. Since $\Sigma^n(Y_{\Lambda/\mathfrak{r}})$ is pure-injective and $\mathrm{CM}(\mathbf{P}_\Lambda)^{\leqslant n}$ is closed under products, by [45, Corollary 4.5] we infer that $(\mathrm{CM}(\mathbf{P}_\Lambda)^{\leqslant n}, \mathfrak{P}_\Lambda^{\preccurlyeq n})$ is a cotorsion pair in Mod-Λ, where $\mathfrak{P}_\Lambda^{\preccurlyeq n} := (\mathrm{CM}(\mathbf{P}_\Lambda)^{\leqslant n})^{\perp}$, and $\mathrm{CM}(\mathbf{P}_\Lambda)^{\leqslant n}$ is definable. In particular $\mathrm{CM}(\mathbf{P}_\Lambda)^{\leqslant n}$ is closed under filtered colimits and therefore there exists a minimal right $\mathrm{CM}(\mathbf{P}_\Lambda)^{\leqslant n}$-approximation $0 \to Y^n_{\Lambda/\mathfrak{r}} \to X^n_{\Lambda/\mathfrak{r}} \to \Lambda/\mathfrak{r} \to 0$. Then $\mathrm{CM}(\mathbf{P}_\Lambda)^{\leqslant n} = \mathrm{Filt}[\mathrm{Prod}(X^n_{\Lambda/\mathfrak{r}})]$ by [46, Theorem 3.1].

(iii) Since $\mathrm{CM}(\mathbf{P}_\Lambda) \subseteq \mathrm{CM}(\mathbf{P}_\Lambda)^{\leqslant n}$, we have $\mathfrak{P}_\Lambda^{\preccurlyeq n} \subseteq \mathfrak{P}_\Lambda^{\prec\infty}$. Then by Proposition 3.9(i) we have

$$\mathrm{CM}(\mathbf{P}_\Lambda)^{\leqslant n} \cap \mathfrak{P}_\Lambda^{\preccurlyeq n} \subseteq \mathrm{CM}(\mathbf{P}_\Lambda)^{\leqslant n} \cap \mathfrak{P}_\Lambda^{\prec\infty} = \mathbf{P}_\Lambda^{\leqslant n}.$$

Since $\mathbf{P}_\Lambda^{\leqslant n} \subseteq \mathrm{CM}(\mathbf{P}_\Lambda)^{\leqslant n}$, we infer that $\mathrm{CM}(\mathbf{P}_\Lambda)^{\leqslant n} \cap \mathfrak{P}_\Lambda^{\preccurlyeq n} \subseteq \mathbf{P}_\Lambda^{\leqslant n} \cap (\mathbf{P}_\Lambda^{\leqslant n})^{\perp}$. Conversely if A lies in $\mathbf{P}_\Lambda^{\leqslant n} \cap (\mathbf{P}_\Lambda^{\leqslant n})^{\perp}$, then $A \in \mathrm{CM}(\mathbf{P}_\Lambda)^{\leqslant n}$. Let $0 \to A \to Y^A_{\leqslant n} \to X^A_{\leqslant n} \to 0$ be a left $\mathfrak{P}_\Lambda^{\preccurlyeq n}$-approximation sequence for A. Then clearly $X^A_{\leqslant n}$ lies in $\mathrm{CM}(\mathbf{P}_\Lambda)^{\leqslant n} \cap \mathfrak{P}_\Lambda^{\prec\infty}$ which is equal to $\mathbf{P}_\Lambda^{\leqslant n}$ by Proposition 3.9. Therefore the above sequence splits and A lies in $\mathrm{CM}(\mathbf{P}_\Lambda)^{\leqslant n} \cap \mathfrak{P}_\Lambda^{\preccurlyeq n}$. We conclude that $\mathrm{CM}(\mathbf{P}_\Lambda)^{\leqslant n} \cap \mathfrak{P}_\Lambda^{\preccurlyeq n} = \mathbf{P}_\Lambda^{\leqslant n} \cap (\mathbf{P}_\Lambda^{\leqslant n})^{\perp}$. □

We have the following consequence which generalizes [46, Corollary 2.7].

Corollary 6.10. *For an Artin algebra Λ, the following conditions are equivalent.*

(i) $\mathrm{FPD}(\Lambda) < \infty$.
(ii) $\widehat{\mathrm{CM}(\mathbf{P}_\Lambda)}$ *is contravariantly finite.*

If (i) *holds, then we have a cotorsion pair* $(\widehat{\mathrm{CM}(\mathbf{P}_\Lambda)}, \mathfrak{P}^{\lesssim\infty})$ *in* Mod-Λ *where* $\widehat{\mathrm{CM}(\mathbf{P}_\Lambda)}$ *is definable and* $\mathfrak{P}^{\lesssim\infty} := \widehat{\mathrm{CM}(\mathbf{P}_\Lambda)}^{\perp}$. *Moreover we have*: $\widehat{\mathrm{CM}(\mathbf{P}_\Lambda)} \cap \mathfrak{P}^{\lesssim\infty} = \mathbf{P}_\Lambda^{<\infty} \cap (\mathbf{P}_\Lambda^{<\infty})^{\perp}$.

Proof. If $\mathrm{FPD}(\Lambda) := d < \infty$, then 3.9(vi) implies that $\widehat{\mathrm{CM}(\mathbf{P}_\Lambda)} = \mathrm{CM}(\mathbf{P}_\Lambda)^{\leqslant d}$. Hence $\widehat{\mathrm{CM}(\mathbf{P}_\Lambda)}$ is contravariantly finite by Proposition 6.9. Conversely if (ii) holds, then $\widehat{\mathrm{CM}(\mathbf{P}_\Lambda)}$ is closed under coproducts and this clearly implies that $\sup\{\text{res.dim}_{\mathrm{CM}} A \mid \text{res.dim}_{\mathrm{CM}} A < \infty\} < \infty$. Then by Proposition 3.9(vi) we infer that $\mathrm{FPD}(\Lambda) < \infty$. The last assertion follows from Proposition 6.9. □

Remark 6.11. By the above results the least cotorsion pair and the greatest cotorsion pair in Mod-Λ are connected via the following chain of cotorsion pairs (the first one follows from [1]):

$$(\mathbf{P}_\Lambda, \text{Mod-}\Lambda) \preccurlyeq \left(\mathbf{P}_\Lambda^{\leqslant 1}, \left(\mathbf{P}_\Lambda^{\leqslant 1}\right)^{\perp}\right) \preccurlyeq \cdots \preccurlyeq \left(\mathbf{P}_\Lambda^{\leqslant n}, \left(\mathbf{P}_\Lambda^{\leqslant n}\right)^{\perp}\right) \preccurlyeq \cdots \preccurlyeq (\text{Mod-}\Lambda, \mathbf{I}_\Lambda),$$

$$\begin{aligned}(\mathbf{P}_\Lambda, \text{Mod-}\Lambda) &\preccurlyeq \cdots \preccurlyeq \left(\mathbf{P}_\Lambda^{\leqslant n}, \left(\mathbf{P}_\Lambda^{\leqslant n}\right)^{\perp}\right) \preccurlyeq \left(\mathrm{CM}(\mathbf{P}_\Lambda)^{\leqslant n}, \mathfrak{P}_\Lambda^{\preccurlyeq n}\right)\\ &\preccurlyeq \cdots \preccurlyeq (\text{Mod-}\Lambda, \mathbf{I}_\Lambda), \quad \forall n \geqslant 0,\\ (\mathbf{P}_\Lambda, \text{Mod-}\Lambda) &\preccurlyeq \left(\mathrm{CM}(\mathbf{P}_\Lambda), \mathfrak{P}_\Lambda^{\prec\infty}\right) \preccurlyeq \left(\mathrm{CM}(\mathbf{P}_\Lambda)^{\leqslant 1}, \mathfrak{P}_\Lambda^{\preccurlyeq 1}\right) \preccurlyeq \cdots \preccurlyeq \left(\mathrm{CM}(\mathbf{P}_\Lambda)^{\leqslant n}, \mathfrak{P}_\Lambda^{\preccurlyeq n}\right)\\ &\preccurlyeq \cdots \preccurlyeq (\text{Mod-}\Lambda, \mathbf{I}_\Lambda)\end{aligned}$$

and moreover: $\mathrm{CM}(\mathbf{P}_\Lambda)^{\leqslant n} \cap \mathfrak{P}_\Lambda^{\preccurlyeq n} = \mathbf{P}_\Lambda^{\leqslant n} \cap (\mathbf{P}_\Lambda^{\leqslant n})^{\perp}$, $\forall n \geqslant 0$. If $\mathrm{FPD}(\Lambda) < \infty$, then all these cotorsion pairs are contained in the cotorsion pair $(\widehat{\mathrm{CM}(\mathbf{P}_\Lambda)}, \mathfrak{P}^{\lesssim\infty})$, except possibly of $(\text{Mod-}\Lambda, \mathbf{I}_\Lambda)$. This raises the question of when $\widehat{\mathrm{CM}(\mathbf{P}_\Lambda)} = \mathbf{P}_\Lambda^{<\infty}$, see [14, page 9]. This is equivalent to ask when $\mathrm{CM}(\mathbf{P}_\Lambda) = \mathbf{P}_\Lambda$, i.e., when the above chains of cotorsion pairs coincide. Indeed if $\widehat{\mathrm{CM}(\mathbf{P}_\Lambda)} = \mathbf{P}_\Lambda^{<\infty}$, then $\widehat{\mathrm{CM}(\mathbf{P}_\Lambda)} \cap {}^{\perp}\Lambda = \mathbf{P}_\Lambda^{<\infty} \cap {}^{\perp}\Lambda$. Then by Proposition 3.9 we have $\mathrm{CM}(\mathbf{P}_\Lambda) = \mathbf{P}_\Lambda$. The converse is clear. Certainly $\mathrm{CM}(\mathbf{P}_\Lambda) = \mathbf{P}_\Lambda$ if $\text{gl.dim}\,\Lambda < \infty$ but we don't know if there are additional algebras. By [22], $\mathrm{CM}(\mathbf{P}_\Lambda) = \mathbf{P}_\Lambda$

iff the generalized Tate–Vogel Cohomology groups $\widehat{\mathrm{Ext}}^n_{(\mathrm{CM}(\mathbf{P}_\Lambda),\mathfrak{P}_\Lambda^{\prec\infty})}$, in the sense of [22], vanish for $n \geqslant 1$. Note that any tilting cotorsion pair $(\mathcal{X}, \mathcal{Y})$, i.e., $\mathcal{Y} = T^\perp$ for a tilting module T, necessarily satisfies $(\mathcal{X}, \mathcal{Y}) \preccurlyeq (\mathbf{P}_\Lambda^{\leqslant n}, (\mathbf{P}_\Lambda^{\leqslant n})^\perp)$, for $n = \mathrm{pd}\, T$, see [57].

Let Γ, respectively Δ, be the DG-algebra of graded stable endomorphisms of $\underline{X}^{\Lambda/\mathbf{r}}$, respectively $\overline{Z}^{\Lambda/\mathbf{r}}$. Note that $\Gamma_n = \mathrm{Ext}_\Lambda^{-n}(\Lambda/\mathbf{r}, X^{\Lambda/\mathbf{r}})$ for $n < 0$ and $\Gamma_n = \underline{\mathrm{Hom}}_\Lambda(\Lambda/\mathbf{r}, \Omega^n X^{\Lambda/\mathbf{r}})$ for $n \geqslant 0$, and $\Delta_n = \mathrm{Ext}_\Lambda^n(\Lambda/\mathbf{r}, Z^{\Lambda/\mathbf{r}})$ for $n > 0$ and $\Delta_n = \overline{\mathrm{Hom}}_\Lambda(\Lambda/\mathbf{r}, \Sigma^{-n} X^{\Lambda/\mathbf{r}})$ for $n \leqslant 0$. We let $\mathbf{D}(\mathsf{DG}\,\mathrm{Mod}\text{-}\Gamma)$, respectively $\mathbf{D}(\mathsf{DG}\,\mathrm{Mod}\text{-}\Delta)$, be the unbounded derived category of the DG-algebra Γ, respectively Δ. Since $\mathrm{CM}(\mathbf{P}_\Lambda)$ and $\mathrm{CoCM}(\mathbf{I}_\Lambda)$ are Frobenius exact categories such that their stable categories are compactly generated, by Keller's Morita Theorem for stable categories [38] we deduce the following.

Corollary 6.12. *There are triangle equivalences*:

$$\underline{\mathrm{CM}}(\mathbf{P}_\Lambda) \approx \mathbf{D}(\mathsf{DG}\,\mathrm{Mod}\text{-}\Gamma) \approx \mathbf{D}(\mathsf{DG}\,\mathrm{Mod}\text{-}\Delta) \approx \overline{\mathrm{CoCM}}(\mathbf{I}_\Lambda).$$

We have seen that if $\underline{E}$ is pure-injective in $\underline{\mathrm{CM}}(\mathbf{P}_\Lambda)$, then the module E is pure-injective. In contrast to this nice behavior of pure-injective modules and objects, the situation for compact objects is not so well behaved. That is, whereas $\underline{X}$ is compact in $\underline{\mathrm{CM}}(\mathbf{P}_\Lambda)$ for any compact (= finitely generated) Cohen–Macaulay module X, we don't know if any compact object in $\underline{\mathrm{CM}}(\mathbf{P}_\Lambda)$ is induced by a Cohen–Macaulay module which is compact in Mod-Λ, i.e., finitely generated. This leads to the investigation of when the Cohen–Macaulay torsion pair is of finite type discussed in the next section.

7. (Co)Cohen–Macaulay torsion pairs of (co)finite type

In this section we study finiteness conditions on the Cohen–Macaulay torsion pairs. It should be mentioned that torsion pairs of finite type, in the sense of [22], play an important role in stable homotopy theory and more generally in compactly generated triangulated categories. For instance they are in bijection with smashing subcategories and are involved in the Telescope Conjecture and the classification of thick subcategories of compact objects, see [22,42,44] for details. More precisely we are interested in finding necessary and sufficient conditions ensuring that the (Co)Cohen–Macaulay torsion pairs are of finite or cofinite type in the sense of the following definition which generalizes (and is inspired by) the notion of smashing subcategories of a triangulated category.

Definition 7.1. Let $\mathcal{C}$ be a pretriangulated category which admits all small (co)products and let $(\mathcal{X}, \mathcal{Y})$ be a torsion pair in $\mathcal{C}$. The torsion pair $(\mathcal{X}, \mathcal{Y})$ is said to be of **finite**, respectively **cofinite**, **type**, if the torsion-free class $\mathcal{Y}$, respectively the torsion class $\mathcal{X}$, is closed under all small coproducts, respectively products.

As in the triangulated case we have the following characterization of torsion pairs of (co)finite type. The proof is identical to the proof of the triangulated case, see for instance [22].

Lemma 7.2. *Let $\mathcal{C}$ be a pretriangulated category which admits all small products and coproducts, and assume that in the adjoint pair (Σ, Ω), the loop functor Ω preserves coproducts and the suspension functor Σ preserves products. If $(\mathcal{X}, \mathcal{Y})$ is a torsion pair in $\mathcal{C}$, then the following are equivalent*:

(i) $(\mathcal{X}, \mathcal{Y})$ *is of finite type, respectively cofinite type.*
(ii) *The coreflection functor $\mathbf{R}_{\mathcal{X}} : \mathcal{C} \to \mathcal{X}$ preserves coproducts, respectively the reflection functor $\mathbf{L}_{\mathcal{Y}} : \mathcal{C} \to \mathcal{Y}$ preserves products.*

If $(\mathcal{X}, \mathcal{Y})$ is of finite type, then the reflection functor $\mathbf{L}_{\mathcal{Y}} : \mathcal{C} \to \mathcal{Y}$ and the inclusion functor $\mathbf{i}_{\mathcal{X}} : \mathcal{X} \hookrightarrow \mathcal{C}$, preserve compact objects.

Corollary 7.3. *The torsion pair $(\underline{\mathrm{CM}}(\mathbf{P}_\Lambda), \underline{\mathfrak{P}}_\Lambda^{\prec\propto})$ in $\underline{\mathrm{Mod}}$-Λ is of cofinite type, and the torsion pair $(\bar{\mathfrak{I}}_\Lambda^{\prec\propto}, \overline{\mathrm{CoCM}}(\mathbf{I}_\Lambda))$ in $\overline{\mathrm{Mod}}$-Λ is of finite type. In particular the reflection functor $\mathbf{L}_{\mathrm{CoCM}} : \overline{\mathrm{Mod}}$-$\Lambda \to \overline{\mathrm{CoCM}}(\mathbf{I}_\Lambda)$ and the inclusion functor $\mathbf{i}_{\mathfrak{I}} : \bar{\mathfrak{I}}_\Lambda^{\prec\propto} \hookrightarrow \overline{\mathrm{Mod}}$-$\Lambda$ preserve compact objects.*

Proof. Since the pretriangulated categories $\underline{\mathrm{Mod}}$-Λ and $\overline{\mathrm{Mod}}$-Λ satisfy the assumptions of Lemma 7.2 and $\overline{\mathrm{CoCM}}(\mathbf{I}_\Lambda)$, respectively $\underline{\mathrm{CM}}(\mathbf{P}_\Lambda)$, is closed under coproducts in $\overline{\mathrm{Mod}}$-Λ, respectively products in $\underline{\mathrm{Mod}}$-Λ, the assertions follow from Lemma 7.2. □

Recall that a full subcategory $\mathcal{U}$ of an abelian category is called **thick** [45] if $\mathcal{U}$ is closed under extensions, kernels of epimorphisms and cokernels of monomorphisms. For instance $\mathfrak{I}_\Lambda^{\prec\propto}$ and $\mathfrak{P}_\Lambda^{\prec\propto}$ are thick subcategories of Mod-Λ and $(\mathfrak{I}_\Lambda^{\prec\propto})^{\mathrm{fin}}$ and $(\mathfrak{P}_\Lambda^{\prec\propto})^{\mathrm{fin}}$ are thick subcategories of mod-Λ since, by Theorem 3.5, they are resolving and coresolving.

Lemma 7.4. *Let $\mathcal{D}$ be a thick subcategory of* Mod-Λ *which is closed under products and coproducts. Then $\mathcal{D}$ is definable iff $\mathcal{D}$ is closed under pure subobjects iff $\mathcal{D}$ is closed under pure quotients.*

Proof. Clearly if $\mathcal{D}$ is definable, then $\mathcal{D}$ is closed under pure submodules and pure quotients. If $\mathcal{D}$ is closed under pure subobjects and $\{D_i \mid i \in I\}$ is a filtered system of modules from $\mathcal{D}$, then the pure extension $0 \to K \to \coprod_{i \in I} D_i \to \varinjlim D_i \to 0$ and the fact that $\mathcal{D}$ is thick show that $\varinjlim D_i$ lies in $\mathcal{D}$ and therefore $\mathcal{D}$ is definable. If $\mathcal{D}$ is closed under pure quotients, by the above argument $\mathcal{D}$ is closed under filtered colimits. Since $\mathcal{D}$ is thick, $\mathcal{D}$ is closed under pure submodules and so it is definable. □

The following characterizes when the CoCohen–Macaulay torsion pair is of cofinite type.

Proposition 7.5. *The following are equivalent.*

(i) *The torsion pair $(\bar{\mathfrak{I}}_\Lambda^{\prec\propto}, \overline{\mathrm{CoCM}}(\mathbf{I}_\Lambda))$ is of cofinite type.*
(ii) $\mathfrak{I}_\Lambda^{\prec\propto}$ *is closed under products.*

(iii) *If* $\{A_i\}_{i\in I}$ *is set of modules and* $A_i \to Z_i$ *are special left CoCohen–Macaulay approximations, then so is* $\prod_{i\in I} A_i \to \prod_{i\in I} Z_i$*. Equivalently* ${}^{\perp}\underline{\mathrm{CM}}(\mathbf{P}_\Lambda)$ *is closed under products in* $\underline{\mathrm{Mod}}$-Λ.
(iv) $\mathfrak{I}_\Lambda^{\prec\infty}$ *is definable.*

If (i) *holds, then* $(\bar{\mathfrak{I}}_\Lambda^{\prec\infty})^{\mathrm{b}} = \overline{(\mathfrak{I}_\Lambda^{\prec\infty})^{\mathrm{fin}}}$.

Proof. Clearly (i) ⇔ (ii), and (ii) ⇔ (iii) follows from Lemma 7.2, Theorem 5.5 and the definition of the reflection functor $\mathbf{L}_{\mathrm{CoCM}} : \overline{\mathrm{Mod}}$-$\Lambda \to \overline{\mathrm{CoCM}}(\mathbf{I}_\Lambda)$. Obviously (iv) ⇒ (ii) and the implication (ii) ⇒ (iv) follows from Corollary 6.8. The last assertion follows from Lemma 7.2 and Corollary 7.3. □

Corollary 7.6. *If the minimal right* $\underline{\mathrm{CM}}(\mathbf{P}_\Lambda)$*-approximation* $X_{\mathfrak{r}}$ *of the radical* $\mathfrak{r}$ *of* Λ *is finitely generated, then the torsion pair* $(\bar{\mathfrak{I}}_\Lambda^{\prec\infty}, \overline{\mathrm{CoCM}}(\mathbf{I}_\Lambda))$ *is of cofinite type.*

Proof. If $X_{\mathfrak{r}}$ is finitely generated, then so is $\mathsf{N}^+(X_{\mathfrak{r}})$. Using Auslander–Reiten's formula we have:

$$\begin{aligned}\mathrm{Ext}^n_\Lambda\big(-, \mathsf{N}^+(X_{\mathfrak{r}})\big) &\cong \overline{\mathrm{Hom}}_\Lambda\big(-, \Sigma^n \mathsf{N}^+(\underline{X}_{\mathfrak{r}})\big)\\ &\cong \mathrm{D}\underline{\mathrm{Hom}}_\Lambda\big(\mathrm{TrD}\,\Sigma^{n+1}\mathsf{N}^+(\underline{X}_{\mathfrak{r}}), -\big), \quad \forall n \geqslant 1.\end{aligned}$$

This implies that $\mathfrak{I}_\Lambda^{\prec\infty} = {}^{\perp}\mathsf{N}^+(X_{\mathfrak{r}})$ is closed under products. □

Now we turn our attention to the question of when the Cohen–Macaulay torsion pair is of finite type. We begin with the following useful characterization.

Lemma 7.7. *The following are equivalent.*

(i) $\underline{\mathrm{CM}}(\mathbf{P}_\Lambda)^{\mathrm{b}} = \underline{\mathrm{CM}}(\Lambda)$.
(ii) $\overline{\mathrm{CoCM}}(\mathbf{I}_\Lambda)^{\mathrm{b}} = \overline{\mathrm{CoCM}}(\mathrm{D}(\Lambda))$.
(iii) *The torsion pair* $(\underline{\mathrm{CM}}(\mathbf{P}_\Lambda), \underline{\mathfrak{P}}_\Lambda^{\prec\infty})$ *is of finite type.*

Proof. The equivalence (i) ⇔ (ii) follows from the fact that the adjoint pair $(\mathsf{N}^+, \mathsf{N}^-)$ induces equivalences between $\underline{\mathrm{CM}}(\mathbf{P}_\Lambda)$ and $\overline{\mathrm{CoCM}}(\mathbf{I}_\Lambda)$, hence between $\underline{\mathrm{CM}}(\mathbf{P}_\Lambda)^{\mathrm{b}}$ and $\overline{\mathrm{CoCM}}(\mathbf{I}_\Lambda)^{\mathrm{b}}$, and between $\underline{\mathrm{CM}}(\Lambda)$ and $\overline{\mathrm{CoCM}}(\mathrm{D}(\Lambda))$. By Lemmas 6.1 and 7.2 we have (iii) ⇒ (i). If (i) holds, then the inclusion $\mathbf{i}_{\mathrm{CM}}$ preserves compactness and therefore its right adjoint $\mathbf{R}_{\mathrm{CM}}$ preserves coproducts by Lemma 6.1. Hence the torsion pair $(\underline{\mathrm{CM}}(\mathbf{P}_\Lambda), \mathfrak{P}_\Lambda^{\prec\infty})$ is of finite type by Lemma 7.2. □

Proposition 7.8. *The following are equivalent.*

(i) $(\underline{\mathrm{CM}}(\mathbf{P}_\Lambda), \underline{\mathfrak{P}}_\Lambda^{\prec\infty})$ *is of finite type.*
(ii) $\mathfrak{P}_\Lambda^{\prec\infty}$ *is closed under coproducts.*

(iii) *If $\{A_i\}_{i\in I}$ is set of modules and $X_i \to A_i$ are special Cohen–Macaulay approximations, then so is $\coprod_{i\in I} X_i \to \coprod_{i\in I} A_i$. Equivalently $\overline{\mathrm{CoCM}}(\mathbf{I}_\Lambda)^\perp$ is closed under coproducts in $\overline{\mathrm{Mod}}$-Λ.*
(iv) $\mathfrak{P}_\Lambda^{\prec\propto}$ *is definable.*

Proof. Observe that $\mathfrak{P}_\Lambda^{\prec\propto}$ is closed under coproducts in Mod-Λ iff $\underline{\mathfrak{P}}_\Lambda^{\prec\propto}$ is closed under coproducts in $\underline{\mathrm{Mod}}$-Λ iff the torsion pair $(\underline{\mathrm{CM}}(\mathbf{P}_\Lambda), \underline{\mathfrak{P}}_\Lambda^{\prec\propto})$ is of finite type. Hence (i) $\Leftrightarrow$ (ii) and by Proposition 5.7 and the fact that the first part of (iii) is a reformulation of the fact that $\mathbf{R}_{\mathrm{CM}}$ preserves coproducts, we have (ii) $\Leftrightarrow$ (iii). Clearly (iv) $\Rightarrow$ (ii) and it remains to show that (ii) $\Rightarrow$ (iv). By Lemma 7.4 it suffices to show that $\mathfrak{P}_\Lambda^{\prec\propto}$ is closed under pure quotients. Let $0 \to A \to Y \to B \to 0$ be a pure extension in Mod-Λ with Y in $\mathfrak{P}_\Lambda^{\prec\propto}$ and let X be a finitely generated Cohen–Macaulay module. Since finitely generated modules are pure-projective, we have the following exact commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Hom}_\Lambda(X,A) \arrow[r] \arrow[d, "\varpi_{X,A}"'] & \mathrm{Hom}_\Lambda(X,Y) \arrow[r] \arrow[d, "\varpi_{X,Y}"'] & \mathrm{Hom}_\Lambda(X,B) \arrow[r] \arrow[d, "\varpi_{X,B}"'] & 0 \\
\cdots \arrow[r] & \underline{\mathrm{Hom}}_\Lambda(\underline{X},\underline{A}) \arrow[r] & \underline{\mathrm{Hom}}_\Lambda(\underline{X},\underline{Y}) \arrow[r] & \underline{\mathrm{Hom}}_\Lambda(\underline{X},\underline{B}) &
\end{tikzcd}$$

where the vertical arrows are the canonical epimorphisms. From the above diagram it follows that $\underline{\mathrm{Hom}}_\Lambda(\underline{X},\underline{B}) = 0$, hence $\underline{\mathrm{Hom}}_\Lambda(\underline{X}, \mathbf{R}_{\mathrm{CM}}(\underline{B})) = 0$ for any module X in $\mathrm{CM}(\mathbf{P}_\Lambda) \cap \text{mod-}\Lambda = \mathrm{CM}(\Lambda)$. Since, by Lemma 7.7, $\underline{\mathrm{CM}}(\Lambda) = \underline{\mathrm{CM}}(\mathbf{P}_\Lambda)^{\mathrm{b}}$ generates $\underline{\mathrm{CM}}(\mathbf{P}_\Lambda)$, we infer that $\mathbf{R}_{\mathrm{CM}}(\underline{B}) = 0$ and therefore $\underline{B}$ lies in $\mathfrak{P}_\Lambda^{\prec\propto}$. Consequently $\mathfrak{P}_\Lambda^{\prec\propto}$ is closed under pure quotients, i.e., $\mathfrak{P}_\Lambda^{\prec\propto}$ is definable. $\square$

Corollary 7.9. *If the minimal right* $\mathrm{CM}(\mathbf{P}_\Lambda)$*-approximation $X_{\Lambda/\mathfrak{r}}$ of $\Lambda/\mathfrak{r}$ is finitely generated, equivalently the minimal left $\mathfrak{P}_\Lambda^{\prec\propto}$-approximation $Y^{\Lambda/\mathfrak{r}}$ of $\Lambda/\mathfrak{r}$ is finitely generated, then the torsion pair $(\underline{\mathrm{CM}}(\mathbf{P}_\Lambda), \underline{\mathfrak{P}}_\Lambda^{\prec\propto})$ is of finite type. In this case $\mathfrak{P}_\Lambda^{\prec\propto} = (X_{\Lambda/\mathfrak{r}})^\perp$ and $\underline{X}_{\Lambda/\mathfrak{r}}$ is a compact generator and a pure-injective cogenerator of $\underline{\mathrm{CM}}(\mathbf{P}_\Lambda)$.*

Proof. Clearly $X_{\Lambda/\mathfrak{r}}$ is finitely generated iff so is $Y^{\Lambda/\mathfrak{r}}$. If this holds, then $\mathrm{Add}(X_{\Lambda/\mathfrak{r}}) = \mathrm{Prod}(X_{\Lambda/\mathfrak{r}})$ and therefore by Proposition 6.9 we have

$$\mathfrak{P}_\Lambda^{\prec\propto} = \left[\mathrm{Prod}(X_{\Lambda/\mathfrak{r}})\right]^\perp = \mathrm{Add}(X_{\Lambda/\mathfrak{r}})^\perp = (X_{\Lambda/\mathfrak{r}})^\perp.$$

This implies that $\mathfrak{P}_\Lambda^{\prec\propto}$ is closed under coproducts and therefore, by Proposition 7.8, the torsion pair $(\underline{\mathrm{CM}}(\mathbf{P}_\Lambda), \underline{\mathfrak{P}}_\Lambda^{\prec\propto})$ is of finite type. Since $X_{\Lambda/\mathbf{r}}$ is finitely generated, the object $\underline{X}_{\Lambda/\mathbf{r}}$ is compact in $\underline{\mathrm{CM}}(\mathbf{P}_\Lambda)$. Let $\underline{X}$ be in $\underline{\mathrm{CM}}(\mathbf{P}_\Lambda)$ and assume that $\underline{\mathrm{Hom}}_\Lambda(\Omega^n(\underline{X}_{\Lambda/\mathfrak{r}}), \underline{X}) = 0$, $\forall n \in \mathbb{Z}$. Since for $n \geqslant 1$, we have

$$\underline{\mathrm{Hom}}_\Lambda\big(\Omega^n(\underline{X}_{\Lambda/\mathfrak{r}}), \underline{X}\big) \cong \mathrm{Ext}^n(X_{\Lambda/\mathfrak{r}}, X),$$

it follows that $X \in \mathfrak{P}_\Lambda^{\prec\infty}$. Then $\underline{X} = 0$ since X is Cohen–Macaulay. We conclude that $\underline{X}_{\Lambda/\mathfrak{r}}$ is a compact generator of $\underline{\mathrm{CM}}(\mathbf{P}_\Lambda)$. Finally as in the proof of Theorem 6.6 it follows that $\underline{X}_{\Lambda/\mathfrak{r}}$ is a pure-injective cogenerator generator of $\underline{\mathrm{CM}}(\mathbf{P}_\Lambda)$. $\square$

Let $\mathcal{A}$ be an additive category with filtered colimits. If $\mathcal{U}$ is a full subcategory of $\mathcal{A}$, then we denote by $\varinjlim \mathcal{U}$ the full subcategory of $\mathcal{A}$ consisting of all filtered colimits of objects from $\mathcal{U}$. An object X in $\mathcal{A}$ is called *finitely presented* if the functor $\mathcal{A}(X,-): \mathcal{A} \to \mathcal{A}b$ preserves filtered colimits. The full subcategory of finitely presented objects of $\mathcal{A}$ is denoted by $\mathrm{fp}\,\mathcal{A}$. Following [29] we say that an additive category $\mathcal{A}$ is *locally finitely presented* if $\mathcal{A}$ has filtered colimits, $\mathrm{fp}\,\mathcal{A}$ is skeletally small and $\mathcal{A} = \varinjlim \mathrm{fp}\,\mathcal{A}$. The basic properties of the categories of the form $\varinjlim \mathcal{U}$ are described in the following well-known result.

Lemma 7.10 [29,47]. *Let $\mathcal{U}$ be a full subcategory of* mod-Λ. *Then $\varinjlim \mathcal{U}$ is locally finitely presented and closed under filtered colimits in* Mod-Λ. *Moreover $\varinjlim \mathcal{U} \cap \text{mod-}\Lambda = \mathrm{add}\,\mathcal{U} = \mathrm{fp}\varinjlim \mathcal{U}$ and a module A lies in $\varinjlim \mathcal{U}$ iff for any finitely presented module T, any map $T \to A$ factors through a module in $\mathcal{U}$. Finally $\varinjlim \mathcal{U}$ is closed under products in* Mod-Λ *iff* $\mathrm{add}\,\mathcal{U}$ *is covariantly finite in* mod-Λ.

To proceed further we shall need the following basic result of Krause–Solberg.

Lemma 7.11 [45]. *Let $\mathcal{F}$ be a resolving, respectively coresolving, subcategory of* mod-Λ. *If $\mathcal{F}$ is contravariantly, respectively covariantly, finite, then $\mathcal{F}$ is covariantly, respectively contravariantly, finite. In particular a thick subcategory which contains the projective and the injective modules is contravariantly finite iff it is covariantly finite.*

Finally we need to recall some results from [22]. First we introduce some terminology. We call a cotorsion pair $(\mathcal{X}, \mathcal{Y})$ in Mod-Λ **projective**, respectively **injective**, if $\mathcal{X} \cap \mathcal{Y} = \mathbf{P}_\Lambda$, respectively $\mathcal{X} \cap \mathcal{Y} = \mathbf{I}_\Lambda$. For instance the cotorsion pair $(\mathrm{CM}(\mathbf{P}_\Lambda), \mathfrak{P}_\Lambda^{\prec\infty})$ is projective and the cotorsion pair $(\mathfrak{I}_\Lambda^{\prec\infty}, \mathrm{CoCM}(\mathbf{I}_\Lambda))$ is injective. Following [22] we say that a triple of full subcategories $(\mathcal{X}, \mathcal{Y}, \mathcal{Z})$ in an abelian category $\mathcal{A}$ is a **cotorsion triple** if $(\mathcal{X}, \mathcal{Y})$ and $(\mathcal{Y}, \mathcal{Z})$ are cotorsion pairs. In this case the cotorsion pair $(\mathcal{X}, \mathcal{Y})$ is projective and the cotorsion pair $(\mathcal{Y}, \mathcal{Z})$ is injective [22]. If $\mathcal{A}$ is Frobenius, then $(\mathcal{X}, \mathcal{Y})$ is projective iff it is injective iff $\mathcal{Y}$ is closed under kernels of epics iff $\mathcal{X}$ is closed under cokernels of monics.

Theorem 7.12 [22].

(1) *Let $(\mathcal{X}, \mathcal{Y})$ be a cotorsion pair in* Mod-Λ *or in* mod-Λ. *Then the following conditions are equivalent.*
   (i) *$(\mathcal{X}, \mathcal{Y})$ is a projective cotorsion pair.*
   (ii) *$\mathcal{Y}$ is resolving, in which case $\mathcal{X}$ consists of Cohen–Macaulay modules.*
   (iii) *The stable category $\mathcal{X}/\mathcal{X} \cap \mathcal{Y}$ is triangulated.*
(2) *If $\mathcal{Y}$ is a functorially finite resolving and coresolving subcategory of* mod-Λ, *then there exists a cotorsion triple $(\mathcal{X}, \mathcal{Y}, \mathcal{Z})$ in* mod-Λ *with* $\mathcal{X} \subseteq \mathrm{CM}(\Lambda)$ *and* $\mathcal{Z} \subseteq$

$\mathrm{CoCM}(\mathrm{D}(\Lambda))$. *Moreover the Nakayama functors* $\mathsf{N}^{\pm}$ *induce quasi-inverse equivalences* $(\mathsf{N}^{+}, \mathsf{N}^{-}) : \mathcal{X} \xrightarrow{\approx} \mathcal{Z}$, *and the Auslander–Reiten operators* $\tau^{\pm}$ *induce triangle equivalences* $(\tau^{+}, \tau^{-}) : \underline{\mathcal{X}} \xrightarrow{\approx} \overline{\mathcal{Z}}$.

Now we can prove the following basic result which gives characterizations of when the Cohen–Macaulay torsion pair is of finite type in terms of finitely generated modules. This result will play a crucial role in the study of virtually Gorenstein algebras in the next section.

Theorem 7.13. *For an Artin algebra* Λ *the following are equivalent.*

(i) *The torsion pair* $(\underline{\mathrm{CM}}(\mathbf{P}_{\Lambda}), \underline{\mathfrak{P}}_{\Lambda}^{\prec\propto})$ *in* $\underline{\mathrm{Mod}}$-Λ *is of finite type.*
(ii) *Any finitely generated module admits a special left CoCohen–Macaulay approximation which is finitely generated.*
(iii) *There exists a cotorsion triple* $(\mathrm{CM}(\Lambda), (\mathfrak{P}_{\Lambda}^{\prec\propto})^{\mathrm{fin}} = (\mathfrak{I}_{\Lambda}^{\prec\propto})^{\mathrm{fin}}, \mathrm{CoCM}(\mathrm{D}(\Lambda)))$ *in* mod-Λ.
(iv) *The full subcategory* $(\mathfrak{I}_{\Lambda}^{\prec\propto})^{\mathrm{fin}}$ *of finitely generated modules of virtually finite injective dimension is covariantly finite in* mod-Λ.

Proof. (i) $\Rightarrow$ (ii). Since the torsion pair $(\bar{\mathfrak{I}}_{\Lambda}^{\prec\propto}, \overline{\mathrm{CoCM}}(\mathbf{I}_{\Lambda}))$ in $\overline{\mathrm{Mod}}$-Λ is of finite type, by Corollary 6.2, the reflection functor $\mathbf{L}_{\mathrm{CoCM}} : \overline{\mathrm{Mod}}\text{-}\Lambda \to \overline{\mathrm{CoCM}}(\mathbf{I}_{\Lambda})$ preserves compact objects. Therefore for any T in mod-Λ the reflection $\mathbf{L}_{\mathrm{CoCM}}(\overline{T})$ of $\overline{T}$ in $\overline{\mathrm{CoCM}}(\mathbf{I}_{\Lambda})$ is compact in $\overline{\mathrm{CoCM}}(\mathbf{I}_{\Lambda})$. By Lemma 7.7, this implies that the left CoCohen–Macaulay approximation of T is finitely generated. Hence for any T in mod-Λ, there exists a special left CoCohen–Macaulay approximation sequence $0 \to T \to Z^{T} \to W^{T} \to 0$ in Mod-Λ, where Z^{T}, hence also W^{T}, is finitely generated. That is, Z^{T} lies in $\mathrm{CoCM}(\mathrm{D}(\Lambda))$ and W^{T} lies in $(\mathfrak{I}_{\Lambda}^{\prec\propto})^{\mathrm{fin}}$.

(ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv). The assumption in (ii) implies that the coresolving subcategory $\mathrm{CoCM}(\mathrm{D}(\Lambda))$ is covariantly finite in mod-Λ and we have a cotorsion pair $((\mathfrak{I}_{\Lambda}^{\prec\propto})^{\mathrm{fin}}, \mathrm{CoCM}(\mathrm{D}(\Lambda))$ in mod-Λ. Then Lemma 7.11 implies that $(\mathfrak{I}_{\Lambda}^{\prec\propto})^{\mathrm{fin}}$ is covariantly finite since it is resolving. By Theorem 7.12 the thick subcategory $(\mathfrak{I}_{\Lambda}^{\prec\propto})^{\mathrm{fin}}$ induces a cotorsion triple $(\mathcal{X}, (\mathfrak{I}_{\Lambda}^{\prec\propto})^{\mathrm{fin}}, \mathrm{CoCM}(\mathrm{D}(\Lambda)))$ in mod-Λ with $\mathcal{X} = {}^{\perp}[(\mathfrak{I}_{\Lambda}^{\prec\propto})^{\mathrm{fin}}]$, where the operation "${}^{\perp}$" is performed in mod-$\Lambda$. Also by Theorem 7.12 we have $\mathcal{X} \subseteq \mathrm{CM}(\Lambda)$ and $\mathrm{DTr}(\underline{X})$ lies in $\overline{\mathrm{CoCM}}(\mathrm{D}(\Lambda))$ for any X in $\underline{\mathrm{CM}}(\Lambda)$. Therefore $\overline{\mathrm{Hom}}_{\Lambda}(W, \mathrm{DTr}(\underline{X})) = 0$, $\forall W \in (\mathfrak{I}_{\Lambda}^{\prec\propto})^{\mathrm{fin}}$. Auslander–Reiten formula $\overline{\mathrm{Hom}}_{\Lambda}(W, \mathrm{DTr}(\underline{X})) \cong \mathrm{D}\,\mathrm{Ext}_{\Lambda}^{1}(X, W)$ and the fact that $(\mathfrak{I}_{\Lambda}^{\prec\propto})^{\mathrm{fin}}$ is coresolving shows that X lies in ${}^{\perp}[(\mathfrak{I}_{\Lambda}^{\prec\propto})^{\mathrm{fin}}] = \mathcal{X}$. We infer that $\mathcal{X} = \mathrm{CM}(\Lambda)$.

(iv) $\Rightarrow$ (i). By 7.11 we deduce the existence of a cotorsion triple $(\mathcal{X}, (\mathfrak{I}_{\Lambda}^{\prec\propto})^{\mathrm{fin}}, \mathcal{Z})$ in mod-Λ. Using [45, Theorem 2.4], it follows directly that this cotorsion triple extends to a cotorsion triple $(\varinjlim \mathcal{X}, \varinjlim (\mathfrak{I}_{\Lambda}^{\prec\propto})^{\mathrm{fin}}, \varinjlim \mathcal{Z})$ in Mod-Λ. Since $(\mathfrak{I}_{\Lambda}^{\prec\propto})^{\mathrm{fin}} \subseteq \mathfrak{I}_{\Lambda}^{\prec\propto}$ and, by Corollary 6.8, the latter is closed under filtered colimits, it follows that $\varinjlim (\mathfrak{I}_{\Lambda}^{\prec\propto})^{\mathrm{fin}} \subseteq \mathfrak{I}_{\Lambda}^{\prec\propto}$. Then $\mathrm{CoCM}(\mathbf{I}_{\Lambda}) = (\mathfrak{I}_{\Lambda}^{\prec\propto})^{\perp} \subseteq (\varinjlim (\mathfrak{I}_{\Lambda}^{\prec\propto})^{\mathrm{fin}})^{\perp} = \varinjlim \mathcal{Z}$. Now since $(\mathfrak{I}_{\Lambda}^{\prec\propto})^{\mathrm{fin}}$ is thick and covariantly finite, by [45, Corollary 3.6], it follows that $\varinjlim (\mathfrak{I}_{\Lambda}^{\prec\propto})^{\mathrm{fin}}$ is thick. Then by Theorem 7.12 and its dual we infer that $\varinjlim \mathcal{X}$ consists of Cohen–Macaulay modules and $\varinjlim \mathcal{Z}$

consists of CoCohen–Macaulay modules, i.e., $\varinjlim \mathcal{X} \subseteq \mathrm{CM}(\mathbf{P}_\Lambda)$ and $\varinjlim \mathcal{Z} \subseteq \mathrm{CoCM}(\mathbf{I}_\Lambda)$. In particular $\varinjlim \mathcal{Z} = \mathrm{CoCM}(\mathbf{I}_\Lambda)$ and therefore $\mathfrak{I}_\Lambda^{\prec\propto} = {}^\perp\,\mathrm{CoCM}(\mathbf{I}_\Lambda) = \varinjlim(\mathfrak{I}_\Lambda^{\prec\propto})^{\mathrm{fin}}$. Since the Nakayama functors N^+ and N^- preserve filtered colimits and induce an equivalence between $\mathrm{CM}(\mathbf{P}_\Lambda)$ and $\mathrm{CoCM}(\mathbf{I}_\Lambda)$, we infer that $\varinjlim \mathcal{X} = \mathrm{CM}(\mathbf{P}_\Lambda)$. Hence $\mathfrak{P}_\Lambda^{\prec\propto} = \mathrm{CM}(\mathbf{P}_\Lambda)^\perp = \varinjlim(\mathfrak{I}_\Lambda^{\prec\propto})^{\mathrm{fin}} = \mathfrak{I}_\Lambda^{\prec\propto}$ and this implies that $\mathfrak{P}_\Lambda^{\prec\propto}$ is closed under coproducts. Then by Proposition 7.8 we infer that the torsion pair $(\underline{\mathrm{CM}}(\mathbf{P}_\Lambda), \underline{\mathfrak{P}}_\Lambda^{\prec\propto})$ is of finite type. □

Now we can give the converse to Corollary 7.9.

Corollary 7.14. *The following are equivalent.*

(i) *The torsion pair $(\underline{\mathrm{CM}}(\mathbf{P}_\Lambda), \underline{\mathfrak{P}}_\Lambda^{\prec\propto})$ in $\underline{\mathrm{Mod}}$-Λ is of finite type.*
(ii) *The minimal right $\mathrm{CM}(\mathbf{P}_\Lambda)$-approximation $X_{\Lambda/\mathfrak{r}}$ of $\Lambda/\mathfrak{r}$ is finitely generated.*

Proof. By Corollary 7.9 it suffices to prove that (i) ⇒ (ii). From the proof of Theorem 7.13 it follows that $\mathfrak{P}_\Lambda^{\prec\propto}$ is definable and $\mathfrak{P}_\Lambda^{\prec\propto} = \varinjlim(\mathfrak{P}_\Lambda^{\prec\propto})^{\mathrm{fin}}$. Then, by [43, Theorem 3.12], there exists a special left $\mathfrak{P}_\Lambda^{\prec\propto}$-approximation sequence $0 \to \Lambda/\mathfrak{r} \to Y^{\Lambda/\mathfrak{r}} \to X^{\Lambda/\mathfrak{r}} \to 0$ where $Y^{\Lambda/\mathfrak{r}}$, hence $X^{\Lambda/\mathfrak{r}}$, is finitely generated. Then $X_{\Lambda/\mathfrak{r}}$ is finitely generated, since, by [22], we have $\Omega(X^{\Lambda/\mathfrak{r}}) \cong X_{\Lambda/\mathfrak{r}}$. □

Corollary 7.15. *If the Cohen–Macaulay torsion pair $(\underline{\mathrm{CM}}(\mathbf{P}_\Lambda), \underline{\mathfrak{P}}_\Lambda^{\prec\propto})$ is of finite type, then the CoCohen–Macaulay torsion pair $(\overline{\mathfrak{I}}_\Lambda^{\prec\propto}, \overline{\mathrm{CoCM}}(\mathbf{I}_\Lambda))$ is of cofinite type.*

Proof. By Corollary 7.14, the module $X_{\Lambda/\mathfrak{r}}$ is finitely generated. Since $X_\mathfrak{r} = \Omega(X_{\Lambda/\mathfrak{r}})$, it follows that the module $X_\mathfrak{r}$ is finitely generated. Then the assertion follows from Corollary 7.6. □

8. Virtually Gorenstein algebras

In this section we investigate in detail the class of virtually Gorenstein algebras, introduced in [22], which provides a natural generalization of Gorenstein algebras and algebras of finite representation type, in connection with finiteness conditions on the Cohen–Macaulay (co)torsion pairs.

For a study of the connections between relative homological algebra, closed model structures in the sense of Quillen and virtually Gorenstein algebras we refer to [22].

Definition 8.1 [22]. An Artin algebra Λ is called **virtually Gorenstein** if the full subcategory of modules of virtually finite projective dimension coincides with the full subcategory of modules of virtually finite injective dimension, i.e., $\mathfrak{P}_\Lambda^{\prec\propto} = \mathfrak{I}_\Lambda^{\prec\propto}$.

Since for a Gorenstein algebra Λ we have $\mathfrak{P}_\Lambda^{\prec\propto} = \mathbf{P}_\Lambda^{<\infty} = \mathbf{I}_\Lambda^{<\infty} = \mathfrak{I}_\Lambda^{\prec\propto}$, it follows that Gorenstein algebras, in particular self-injective and algebras of finite global dimension, are virtually Gorenstein. Trivially Λ is virtually Gorenstein provided that $\mathrm{CM}(\mathbf{P}_\Lambda) = \mathbf{P}_\Lambda$

or equivalently $\mathrm{CoCM}(\mathbf{I}_\Lambda) = \mathbf{I}_\Lambda$. This follows from Remark 3.7 since the condition $\mathrm{CM}(\mathbf{P}_\Lambda) = \mathbf{P}_\Lambda$ implies that $\mathfrak{P}_\Lambda^{\prec\infty} = \text{Mod-}\Lambda = \mathfrak{I}_\Lambda^{\prec\infty}$.

8.1. Characterizations of virtually Gorenstein algebras

If Λ is virtually Gorenstein, then the equation $\mathfrak{P}_\Lambda^{\prec\infty} = \mathfrak{I}_\Lambda^{\prec\infty}$ implies that the Cohen–Macaulay torsion pair $(\underline{\mathrm{CM}}(\mathbf{P}_\Lambda), \underline{\mathfrak{P}}_\Lambda^{\prec\infty})$ in $\underline{\mathrm{Mod}}\text{-}\Lambda$ is of finite type. Our previous results enable us to show that this condition, as well as a host of other equivalent conditions, characterize the class of virtually Gorenstein algebras.

Theorem 8.2. *For an Artin algebra Λ, the following are equivalent.*

(i) *Λ is virtually Gorenstein.*
(ii) *The Cohen–Macaulay torsion pair $(\underline{\mathrm{CM}}(\mathbf{P}_\Lambda), \underline{\mathfrak{P}}_\Lambda^{\prec\infty})$ is of finite type.*
(iii) $\overline{\mathrm{CoCM}}(\mathbf{I}_\Lambda)^{\mathrm{b}} = \overline{\mathrm{CoCM}}(\mathrm{D}(\Lambda))$.
(iv) $\underline{\mathrm{CM}}(\mathbf{P}_\Lambda)^{\mathrm{b}} = \underline{\mathrm{CM}}(\Lambda)$.
(v) *The subcategory $\mathfrak{P}_\Lambda^{\prec\infty}$ of modules with virtual finite projective dimension is definable.*
(vi) *The subcategory $\mathfrak{I}_\Lambda^{\prec\infty}$ of modules with virtual finite injective dimension is definable and any finitely generated module admits a finitely generated $\mathfrak{I}_\Lambda^{\prec\infty}$-approximation.*
(vii) *The subcategory $(\mathfrak{I}_\Lambda^{\prec\infty})^{\mathrm{fin}}$ is contravariantly, equivalently covariantly, finite in* mod-Λ.
(viii) *The subcategory $(\mathfrak{P}_\Lambda^{\prec\infty})^{\mathrm{fin}}$ is covariantly, equivalently contravariantly, finite in* mod-Λ.
(ix) *The subcategory* $\mathrm{CM}(\Lambda)$ *is contravariantly finite in* mod-Λ *and* $\mathrm{CM}(\Lambda)^\perp \subseteq (\mathfrak{P}_\Lambda^{\prec\infty})^{\mathrm{fin}}$.
(x) *The subcategory* $\mathrm{CoCM}(\mathrm{D}(\Lambda))$ *is covariantly finite in* mod-Λ *and* ${}^\perp\mathrm{CoCM}(\mathrm{D}(\Lambda)) \subseteq (\mathfrak{I}_\Lambda^{\prec\infty})^{\mathrm{fin}}$.
(xi) *The category* $\mathrm{CM}(\mathbf{P}_\Lambda)$ *is locally finitely presented and*
$$\mathrm{fp}\,\mathrm{CM}(\mathbf{P}_\Lambda) = \mathrm{CM}(\Lambda).$$
(xii) *The category* $\mathrm{CoCM}(\mathbf{I}_\Lambda)$ *is locally finitely presented and*
$$\mathrm{fp}\,\mathrm{CoCM}(\mathbf{I}_\Lambda) = \mathrm{CoCM}\big(\mathrm{D}(\Lambda)\big).$$
(xiii) *If $0 \to A \to B \to C \to 0$ is exact in* Mod-Λ *or* mod-Λ, *then the following are equivalent*:
  (a) *Any map $X \to C$ where X is Cohen–Macaulay factors through $B \to C$.*
  (b) *Any map $A \to Z$ where Z is CoCohen–Macaulay factors through $A \to B$.*

If Λ is virtually Gorenstein, then the torsion pair $(\bar{\mathfrak{I}}_\Lambda^{\prec\infty}, \overline{\mathrm{CoCM}}(\mathbf{I}_\Lambda))$ is of cofinite type and we have:

- *A cotorsion triple* $(\mathrm{CM}(\mathbf{P}_\Lambda), \mathfrak{P}_\Lambda^{\prec\alpha} = \mathfrak{I}_\Lambda^{\prec\alpha}, \mathrm{CoCM}(\mathbf{I}_\Lambda))$ *in* Mod-Λ *where all the involved categories are functorially finite and definable.*
- *A cotorsion triple* $(\mathrm{CM}(\Lambda), (\mathfrak{P}_\Lambda^{\prec\alpha})^{\mathrm{fin}} = (\mathfrak{I}_\Lambda^{\prec\alpha})^{\mathrm{fin}}, \mathrm{CoCM}(\mathrm{D}(\Lambda)))$ *in* mod-Λ *where all the involved categories are functorially finite.*
- *A torsion pair* $(\underline{\mathrm{CM}}(\Lambda), (\underline{\mathfrak{P}}_\Lambda^{\prec\alpha})^{\mathrm{b}})$ *in* $\underline{\mathrm{mod}}$-Λ*, where*

$$\underline{\left(\mathfrak{P}_\Lambda^{\prec\alpha}\right)^{\mathrm{fin}}} = \left(\underline{\mathfrak{P}}_\Lambda^{\prec\alpha}\right)^{\mathrm{b}}.$$

- *A torsion pair* $((\bar{\mathfrak{I}}_\Lambda^{\prec\alpha})^{\mathrm{b}}, \overline{\mathrm{CoCM}}(\mathrm{D}(\Lambda)))$ *in* $\overline{\mathrm{mod}}$-Λ*, where* $\overline{(\mathfrak{I}_\Lambda^{\prec\alpha})^{\mathrm{fin}}} = (\bar{\mathfrak{I}}_\Lambda^{\prec\alpha})^{\mathrm{b}}$.
- *The categories* $\mathrm{CM}(\mathbf{P}_\Lambda)$, $\mathfrak{P}_\Lambda^{\prec\alpha} = \mathfrak{I}_\Lambda^{\prec\alpha}$*, and* $\mathrm{CoCM}(\mathbf{I}_\Lambda)$ *are locally finitely presented and*:

$$\mathrm{CM}(\mathbf{P}_\Lambda) = \varinjlim \mathrm{CM}(\Lambda), \qquad \mathfrak{P}_\Lambda^{\prec\alpha} = \varinjlim\left[\left(\mathfrak{P}_\Lambda^{\prec\alpha}\right)^{\mathrm{fin}}\right] = \varinjlim\left[\left(\mathfrak{I}_\Lambda^{\prec\alpha}\right)^{\mathrm{fin}}\right] = \mathfrak{I}_\Lambda^{\prec\alpha},$$
$$\mathrm{CoCM}(\mathbf{I}_\Lambda) = \varinjlim \mathrm{CoCM}\big(\mathrm{D}(\Lambda)\big).$$

Proof. The equivalences (ii) $\Leftrightarrow$ (iii) $\Leftrightarrow$ (iv) $\Leftrightarrow$ (v) follow from Lemma 7.7 and Proposition 7.8. The equivalence (i) $\Leftrightarrow$ (xiii) follows from [22]. Also the equivalences (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iv) $\Leftrightarrow$ (vii) and the fact that (ii) implies (viii), (ix) (x), (xi) and (xii) follow from Theorem 7.13 and its proof. We first show that each of the conditions in (viii), (ix) and (x) implies that Λ is virtually Gorenstein. Assume first that $(\mathfrak{P}_\Lambda^{\prec\alpha})^{\mathrm{fin}}$ is covariantly finite in mod-Λ. Then, by [9], there exists a cotorsion pair $(\mathcal{A}, (\mathfrak{P}_\Lambda^{\prec\alpha})^{\mathrm{fin}})$ in mod-Λ and then by Theorem 7.12 we have $\mathcal{A} \subseteq \mathrm{CM}(\Lambda)$. Let $(E): 0 \to Y^*_{\Lambda/\mathfrak{r}} \to X^*_{\Lambda/\mathfrak{r}} \to \Lambda/\mathfrak{r} \to 0$ be exact where the map $f_{\Lambda/\mathfrak{r}}: X^*_{\Lambda/\mathfrak{r}} \to \Lambda/\mathfrak{r}$ is the minimal right $\mathcal{A}$-approximation of $\Lambda/\mathfrak{r}$ in mod-Λ. Then $Y^*_{\Lambda/\mathfrak{r}}$ lies in $(\mathfrak{P}_\Lambda^{\prec\alpha})^{\mathrm{fin}}$ since any minimal right $\mathcal{A}$-approximation is special. If $\alpha: X \to \Lambda/\mathfrak{r}$ is a map where X lies in $\mathrm{CM}(\mathbf{P}_\Lambda)$, then the pull-back of α along (E) splits since $Y^*_{\Lambda/\mathfrak{r}}$ lies in $\mathfrak{P}_\Lambda^{\prec\alpha}$. This implies that α factors through $f_{\Lambda/\mathfrak{r}}$ and therefore the latter is the minimal right $\mathrm{CM}(\mathbf{P}_\Lambda)$-approximation of $\Lambda/\mathfrak{r}$. Since $X^*_{\Lambda/\mathfrak{r}}$ is finitely generated, Corollary 7.9 implies that the torsion pair $(\underline{\mathrm{CM}}(\mathbf{P}_\Lambda), \underline{\mathfrak{P}}_\Lambda^{\prec\alpha})$ is of finite type. Next we show that (ix) implies (viii), leaving the proof of the implication (x) $\Rightarrow$ (vii) to the reader since it is completely dual. So let $\mathrm{CM}(\Lambda)$ be contravariantly finite and $\mathrm{CM}(\Lambda)^\perp \subseteq (\mathfrak{P}_\Lambda^{\prec\alpha})^{\mathrm{fin}}$. Also let $(\mathrm{CM}(\Lambda), \mathcal{B})$ be the induced cotorsion pair in mod-Λ. If Y lies in $(\mathfrak{P}_\Lambda^{\prec\alpha})^{\mathrm{fin}}$, then the left $\mathcal{B}$-approximation sequence $0 \to Y \to B^Y \to X^Y \to 0$ of Y splits since X^Y lies in $\mathrm{CM}(\Lambda)$. Hence Y lies in $\mathcal{B}$ and therefore $(\mathfrak{P}_\Lambda^{\prec\alpha})^{\mathrm{fin}} \subseteq \mathcal{B}$. Then by hypothesis we have $(\mathfrak{P}_\Lambda^{\prec\alpha})^{\mathrm{fin}} = \mathcal{B}$ and consequently $(\mathfrak{P}_\Lambda^{\prec\alpha})^{\mathrm{fin}}$ is covariantly finite. Since $\mathrm{CM}(\mathbf{P}_\Lambda)$ and $\mathrm{CoCM}(\mathbf{I}_\Lambda)$ are equivalent via the Nakayama functors, it is clear that (xi) is equivalent to (xii). If (xii) holds, then clearly $\mathrm{CoCM}(\mathbf{I}_\Lambda) = \varinjlim \mathrm{CoCM}(\mathrm{D}(\Lambda))$. Since, by Proposition 3.8, the former is definable, it follows by [43, Theorem 3.12] that any finitely generated module admits a finitely generated left $\mathrm{CoCM}(\mathbf{I}_\Lambda)$-approximation. Then Theorem 7.13 implies that condition (ii) holds. The remaining assertions follow from Theorem 7.13 and its proof. □

Combining Proposition 6.9, Corollary 7.14 and Theorem 8.2 we have the following consequence which gives a characterization of virtually Gorenstein algebras in terms of finite generation and filtration properties of the approximation modules $X_{\Lambda/\mathfrak{r}}$, $Y^{\Lambda/\mathfrak{r}}$, $W_{\Lambda/\mathfrak{r}}$ and $Z^{\Lambda/\mathfrak{r}}$.

Corollary 8.3. *For an Artin algebra the following are equivalent.*

(i) *Λ is virtually Gorenstein.*
(ii) *The minimal right $\mathrm{CM}(\mathbf{P}_\Lambda)$-approximation $X_{\Lambda/\mathfrak{r}}$ of $\Lambda/\mathfrak{r}$ is finitely generated.*
(iii) *The minimal left $\mathfrak{P}_\Lambda^{\prec\propto}$-approximation $Y^{\Lambda/\mathfrak{r}}$ of $\Lambda/\mathfrak{r}$ is finitely generated.*
(iv) *The minimal right $\mathfrak{I}_\Lambda^{\prec\propto}$-approximation $W_{\Lambda/\mathfrak{r}}$ of $\Lambda/\mathfrak{r}$ is finitely generated.*
(v) *The minimal left $\mathrm{CoCM}(\mathbf{I}_\Lambda)$-approximation $Z^{\Lambda/\mathfrak{r}}$ of $\Lambda/\mathfrak{r}$ is finitely generated.*

If Λ is virtually Gorenstein, then we have the following.

(i) $\mathrm{CM}(\Lambda) = \mathrm{Filt}(\mathrm{add}\, X_{\Lambda/\mathfrak{r}})$ *and* $\mathrm{CM}(\mathbf{P}_\Lambda) = {}^{\perp}Y_{\Lambda/\mathfrak{r}} = \mathrm{Filt}(\mathrm{Add}\, X_{\Lambda/\mathfrak{r}})$. *Moreover* $\underline{\mathrm{CM}}(\mathbf{P}_\Lambda) = \mathsf{thick}(\mathrm{Add}\, \underline{X}_{\Lambda/\mathfrak{r}})$ *and* $\underline{\mathrm{CM}}(\Lambda) = \mathsf{thick}(\underline{X}_{\Lambda/\mathfrak{r}})$.
(ii) $(\mathfrak{P}_\Lambda^{\prec\propto})^{\mathrm{fin}} = \mathrm{Filt}(\mathrm{add}\, W_{\Lambda/\mathfrak{r}}) = \mathrm{Filt}(\mathrm{add}\, Y^{\Lambda/\mathfrak{r}}) = (\mathfrak{I}_\Lambda^{\prec\propto})^{\mathrm{fin}}$ *and* $\mathfrak{I}_\Lambda^{\prec\propto} = \mathrm{Filt}(\mathrm{Add}\, Y^{\Lambda/\mathfrak{r}}) = \mathrm{Filt}(\mathrm{Add}\, W_{\Lambda/\mathfrak{r}}) = {}^{\perp}Z^{\Lambda/\mathfrak{r}} = (X_{\Lambda/\mathfrak{r}})^{\perp}$.
(iii) $\mathrm{CoCM}(\mathrm{D}(\Lambda)) = \mathrm{Filt}(\mathrm{add}\, Z^{\Lambda/\mathfrak{r}})$ *and* $\mathrm{CoCM}(\mathbf{I}_\Lambda) = (W^{\Lambda/\mathfrak{r}})^{\perp} = \mathrm{Filt}(\mathrm{Add}\, Z^{\Lambda/\mathfrak{r}})$. *Moreover* $\overline{\mathrm{CoCM}}(\mathbf{I}_\Lambda) = \mathsf{thick}(\mathrm{Add}\, \overline{Z}^{\Lambda/\mathfrak{r}})$ *and* $\overline{\mathrm{CoCM}}(\mathrm{D}(\Lambda)) = \mathsf{thick}(\overline{Z}^{\Lambda/\mathfrak{r}})$.

Example 8.4. (1) By Theorem 8.2 it follows that any Artin algebra of finite representation type is virtually Gorenstein, since clearly representation finiteness implies $\underline{\mathrm{CM}}(\mathbf{P}_\Lambda)^{\mathrm{b}} = \underline{\mathrm{CM}}(\Lambda)$.

(2) We say that Λ is of **finite Cohen–Macaulay type** if the full subcategory $\mathrm{CM}(\Lambda)$ of finitely generated Cohen–Macaulay modules is of finite representation type. In this case if $X^{\dagger}$ is a representation generator of $\mathrm{CM}(\Lambda)$, that is, $X^{\dagger}$ is such that $\mathrm{add}\, X^{\dagger} = \mathrm{CM}(\Lambda)$, then since $\mathrm{CM}(\Lambda)$ is a Frobenius category, by [27] we infer that the stable endomorphism ring $\mathrm{End}(\underline{X}^{\dagger})$ is a self-injective Artin algebra. It follows easily from this that if $\underline{X}$ is compact in $\underline{\mathrm{CM}}(\mathbf{P}_\Lambda)$, then $\underline{X}$ is a direct summand of a finite coproduct of copies of $\underline{X}^{\dagger}$. Hence $\underline{\mathrm{CM}}(\mathbf{P}_\Lambda)^{\mathrm{b}} = \underline{\mathrm{CM}}(\Lambda)$. We conclude that any Artin algebra of finite Cohen–Macaulay type is virtually Gorenstein.

Combining Corollary 8.3 and Theorem 8.2, we have the following description of Gorensteinness.

Corollary 8.5. *The following are equivalent.*

(i) *Λ is Gorenstein.*
(ii) *$\mathfrak{P}_\Lambda^{\prec\propto}$ is closed under coproducts and the minimal right $\mathfrak{P}_\Lambda^{\prec\propto}$-approximation of any simple module has finite projective dimension.*
(iii) *$\mathfrak{P}_\Lambda^{\prec\propto}$ is closed under coproducts and $\mathrm{pd}\, Y_{\Lambda/\mathbf{r}} < \infty$, where $Y_{\Lambda/\mathbf{r}}$ is the minimal right $\mathfrak{P}_\Lambda^{\prec\propto}$-approximation of $\Lambda/\mathbf{r}$.*
(iv) *$\mathfrak{I}_\Lambda^{\prec\propto}$ is closed under products and the minimal left $\mathfrak{I}_\Lambda^{\prec\propto}$-approximation of any simple module is finitely generated with finite injective dimension.*

(v) $\mathfrak{I}_\Lambda^{\prec\propto}$ *is closed under products and* $\operatorname{id} W^{\Lambda/\mathbf{r}} < \infty$, *where* $W^{\Lambda/\mathbf{r}}$ *is the minimal left* $\mathfrak{I}_\Lambda^{\prec\propto}$*-approximation of* $\Lambda/\mathbf{r}$ *which is finitely generated.*

Proof. (ii) $\Rightarrow$ (i). Since $\mathfrak{P}_\Lambda^{\prec\propto}$ is closed under coproducts, by Theorem 8.2, the resolving subcategory $(\mathfrak{P}_\Lambda^{\prec\propto})^{\text{fin}}$ of mod-Λ is contravariantly finite. Hence by [9],

$$\sup\bigl\{\operatorname{pd} Y \mid Y \in \bigl(\mathfrak{P}_\Lambda^{\prec\propto}\bigr)^{\text{fin}}\bigr\} = \max\{\operatorname{pd} Y_S \mid S \text{ is simple}\}$$

where Y_S is the minimal right $(\mathfrak{P}_\Lambda^{\prec\propto})^{\text{fin}}$-approximation of S. By hypothesis $\sup\{\operatorname{pd} Y \mid Y \in (\mathfrak{P}_\Lambda^{\prec\propto})^{\text{fin}}\} < \infty$ and therefore $(\mathfrak{P}_\Lambda^{\prec\propto})^{\text{fin}} \subseteq \mathcal{P}_\Lambda^{<\infty}$. Since we always have $\mathcal{P}_\Lambda^{<\infty} \subseteq (\mathfrak{P}_\Lambda^{\prec\propto})^{\text{fin}}$, we infer that $(\mathfrak{P}_\Lambda^{\prec\propto})^{\text{fin}} = \mathcal{P}_\Lambda^{<\infty}$ and this implies that $\widehat{\mathrm{CM}(\Lambda)} = \text{mod-}\Lambda$. Hence Λ is Gorenstein.

(i) $\Rightarrow$ (ii). If Λ is Gorenstein, then the assertions in (ii) follow from the fact that $(\mathfrak{P}_\Lambda^{\prec\propto})^{\text{fin}} = \mathcal{P}_\Lambda^{<\infty}$ and $\mathfrak{P}_\Lambda^{\prec\propto} = \mathbf{P}_\Lambda^{<\infty}$ is closed under coproducts since $\operatorname{FPD}(\Lambda) < \infty$ by Proposition 3.10. Using Theorem 8.2, the proof of the other equivalences is similar and is left to the reader. □

8.2. Symmetry for virtually Gorenstein algebras

It is well known that Λ is Gorenstein or representation finite iff so is Λ^{op}. Generalizing this fact we shall show that virtual Gorensteinness is left-right symmetric. First we need the following preliminary result.

Lemma 8.6.

(1) *The duality* D *induces adjoint on the right pairs of functors*

$$\mathrm{D} : \mathrm{CM}(\mathbf{P}_\Lambda) \leftrightarrows \mathrm{CoCM}(\mathbf{I}_{\Lambda^{\text{op}}}) : \mathrm{D} \quad \textit{and} \quad \mathrm{D} : \mathrm{CoCM}(\mathbf{I}_\Lambda) \leftrightarrows \mathrm{CM}(\mathbf{P}_{\Lambda^{\text{op}}}) : \mathrm{D},$$

and equivalences of categories

$$\mathrm{D} : \mathrm{CM}(\Lambda)^{\text{op}} \xrightarrow{\approx} \mathrm{CoCM}\bigl(\mathrm{D}\bigl(\Lambda^{\text{op}}\bigr)\bigr) \quad \textit{and} \quad \mathrm{D} : \mathrm{CoCM}\bigl(\mathrm{D}(\Lambda)\bigr)^{\text{op}} \xrightarrow{\approx} \mathrm{CM}\bigl(\Lambda^{\text{op}}\bigr).$$

(2) *The duality* D *induces equivalences*:

$$\mathrm{D} : \bigl[\bigl(\mathfrak{P}_\Lambda^{\prec\propto}\bigr)^{\text{fin}}\bigr]^{\text{op}} \xrightarrow{\approx} \bigl[\mathfrak{I}_{\Lambda^{\text{op}}}^{\prec\propto}\bigr]^{\text{fin}} \quad \textit{and} \quad \mathrm{D} : \bigl[\bigl(\mathfrak{I}_\Lambda^{\prec\propto}\bigr)^{\text{fin}}\bigr]^{\text{op}} \xrightarrow{\approx} \bigl[\mathfrak{P}_{\Lambda^{\text{op}}}^{\prec\propto}\bigr]^{\text{fin}}.$$

Proof. Part (1) is easy and is left to the reader. To prove (2) let Y be in $(\mathfrak{P}_\Lambda^{\prec\propto})^{\text{fin}}$ and let $(E) : 0 \to Z \to C \to \mathrm{D}(Y) \to 0$ be an extension in Mod-Λ^{op} where Z is CoCohen–Macaulay. Since Y is finitely generated, dualizing (E) we get an extension $\mathrm{D}(E) : 0 \to Y \to \mathrm{D}(C) \to \mathrm{D}(Z) \to 0$ in Mod-Λ where $\mathrm{D}(Z)$ is Cohen–Macaulay by part (1). Since Y lies in $\mathfrak{P}_\Lambda^{\prec\propto}$, the extension $\mathrm{D}(E)$ splits. It is well known and easy to see that this implies that the extension (E) is pure. Since $\mathrm{D}(Y)$ is finitely generated we infer that (E) splits. It follows that $\operatorname{Ext}^1_\Lambda(\mathrm{D}(Y), Z) = 0$ for any CoCohen–Macaulay Λ^{op}-module Z and therefore

$\mathrm{D}(Y)$ lies in ${}^{\perp}\mathrm{CoCM}(\mathbf{I}_{\Lambda^{\mathrm{op}}}) = \mathfrak{I}^{\prec\infty}_{\Lambda^{\mathrm{op}}}$. Similarly $\mathrm{D}(W)$ lies in $(\mathfrak{P}^{\prec\infty}_{\Lambda})^{\mathrm{fin}}$ for any Λ-module W in $[\mathfrak{I}^{\prec\infty}_{\Lambda^{\mathrm{op}}}]^{\mathrm{b}}$. We conclude that D gives a duality between $(\mathfrak{P}^{\prec\infty}_{\Lambda})^{\mathrm{fin}}$ and $(\mathfrak{I}^{\prec\infty}_{\Lambda^{\mathrm{op}}})^{\mathrm{fin}}$. The second equivalence is proved similarly. □

Theorem 8.7. *Λ is virtually Gorenstein iff Λ^{op} is virtually Gorenstein.*

Proof. Let Λ be virtually Gorenstein. By Theorem 8.2 it follows that we have a cotorsion triple $(\mathrm{CM}(\Lambda), (\mathfrak{P}^{\prec\infty}_{\Lambda})^{\mathrm{b}} = (\mathfrak{I}^{\prec\infty}_{\Lambda})^{\mathrm{b}}, \mathrm{CoCM}(\mathrm{D}(\Lambda)))$ in mod-Λ and then Lemma 8.6 implies the existence of a cotorsion triple

$$\big(\mathrm{CM}\big(\Lambda^{\mathrm{op}}\big), \mathrm{D}\big[\big(\mathfrak{I}^{\prec\infty}_{\Lambda}\big)^{\mathrm{fin}}\big] = \mathrm{D}\big[\big(\mathfrak{P}^{\prec\infty}_{\Lambda}\big)^{\mathrm{fin}}\big], \mathrm{CoCM}\big(\mathrm{D}\big(\Lambda^{\mathrm{op}}\big)\big)\big)$$

in mod-Λ^{op}. By [45] the last cotorsion triple induces a cotorsion triple

$$\Big(\varinjlim \mathrm{CM}\big(\Lambda^{\mathrm{op}}\big), \varinjlim \mathrm{D}\big[\big(\mathfrak{I}^{\prec\infty}_{\Lambda}\big)^{\mathrm{fin}}\big] = \varinjlim \mathrm{D}\big[\big(\mathfrak{P}^{\prec\infty}_{\Lambda}\big)^{\mathrm{fin}}\big], \varinjlim \mathrm{CoCM}\big(\mathrm{D}\big(\Lambda^{\mathrm{op}}\big)\big)\Big)$$

in Mod-Λ^{op}. By Lemma 8.6 this cotorsion triple is equal to

$$\Big(\varinjlim \mathrm{CM}\big(\Lambda^{\mathrm{op}}\big), \varinjlim\big(\mathfrak{P}^{\prec\infty}_{\Lambda^{\mathrm{op}}}\big)^{\mathrm{fin}} = \varinjlim\big(\mathfrak{I}^{\prec\infty}_{\Lambda^{\mathrm{op}}}\big)^{\mathrm{fin}}, \varinjlim \mathrm{CoCM}\big(\mathrm{D}\big(\Lambda^{\mathrm{op}}\big)\big)\Big).$$

It follows that $\varinjlim(\mathfrak{I}^{\prec\infty}_{\Lambda^{\mathrm{op}}})^{\mathrm{fin}}$ is closed under products in Mod-Λ^{op} and therefore, by Lemma 7.10, $(\mathfrak{I}^{\prec\infty}_{\Lambda^{\mathrm{op}}})^{\mathrm{fin}}$ is covariantly finite in mod-Λ^{op}. Then Λ^{op} is virtually Gorenstein by Theorem 8.2. □

8.3. Grothendieck Groups and Auslander–Reiten sequences/triangles

Gorensteinness has several nice consequences for Grothendieck groups and Auslander–Reiten theory, see [11]. In this subsection we show that this continues to hold for virtually Gorenstein algebras.

We begin with the following generalization of a result of Auslander–Reiten [11] from Gorenstein algebras to virtually Gorenstein algebras.

Proposition 8.8. *Let Λ be a virtually Gorenstein algebra.*

(i) *The subcategories* $\mathrm{CM}(\Lambda)$, $\mathrm{CoCM}(\mathrm{D}(\Lambda))$ *and* $(\mathfrak{P}^{\prec\infty}_{\Lambda})^{\mathrm{fin}} = (\mathfrak{I}^{\prec\infty}_{\Lambda})^{\mathrm{fin}}$ *have Auslander–Reiten sequences.*
(ii) *The triangulated categories* $\underline{\mathrm{CM}}(\Lambda) = \underline{\mathrm{CM}}(\mathbf{P}_{\Lambda})^{\mathrm{b}}$ *and* $\overline{\mathrm{CoCM}}(\mathrm{D}(\Lambda)) = \overline{\mathrm{CoCM}}(\mathbf{I}_{\Lambda})^{\mathrm{b}}$ *have Auslander–Reiten triangles which remain such in* $\underline{\mathrm{CM}}(\mathbf{P}_{\Lambda})$ *and* $\overline{\mathrm{CoCM}}(\mathbf{I}_{\Lambda})$ *respectively.*
(iii) *The triangulated categories* $\underline{\mathrm{CM}}(\mathbf{P}_{\Lambda})^{\mathrm{b}}$ *and* $\overline{\mathrm{CoCM}}(\mathbf{I}_{\Lambda})^{\mathrm{b}}$ *admit a Serre functor which is given by* $\Sigma_{\mathbf{P}}\mathbf{R}_{\mathrm{CM}}\,\mathrm{DTr}$ *and* $\Omega_{\mathbf{I}}\mathbf{L}_{\mathrm{CoCM}}\,\mathrm{TrD}$ *respectively.*

Proof. Part (i) follows from the fact that, by Theorem 8.2, the involved subcategories are functorially finite, see [7]. If $X \in \mathrm{CM}(\Lambda)$, then, using Auslander–Reiten formula, we have the following isomorphisms:

$$\begin{aligned}
\mathrm{D}\underline{\mathrm{Hom}}_\Lambda(X, X') &\xrightarrow{\cong} \mathrm{Ext}^1_\Lambda(X', \mathrm{DTr}(X)) \xrightarrow{\cong} \underline{\mathrm{Hom}}_\Lambda(\Omega(X'), \mathrm{DTr}(X)) \\
&\xrightarrow{\cong} \underline{\mathrm{Hom}}_\Lambda(\Omega(X'), \mathbf{R}_{\mathrm{CM}}\,\mathrm{DTr}(X)) \\
&\xrightarrow{\cong} \underline{\mathrm{Hom}}_\Lambda(X', \Sigma_{\mathbf{P}}\mathbf{R}_{\mathrm{CM}}\,\mathrm{DTr}(X)), \quad \forall X' \in \mathrm{CM}(\Lambda).
\end{aligned}$$

Since right $\mathrm{CM}(\mathbf{P}_\Lambda)$-approximations of finitely generated modules are finitely generated, it follows that the functor $\mathrm{D}\underline{\mathrm{Hom}}_\Lambda(X, -) : \underline{\mathrm{CM}}(\Lambda)^{\mathrm{op}} \to \mathcal{A}b$ is representable by the object $\Sigma_{\mathbf{P}}\mathbf{R}_{\mathrm{CM}}\,\mathrm{DTr}(\underline{X})$. Dually if Z lies in $\mathrm{CoCM}(\mathrm{D}(\Lambda))$, then the functor $\mathrm{D}\overline{\mathrm{Hom}}_\Lambda(-, Z) : \overline{\mathrm{CoCM}}(\mathrm{D}(\Lambda)) \to \mathcal{A}b$ is representable by the object $\Omega_{\mathbf{I}}\mathbf{L}_{\mathrm{CoCM}}\,\mathrm{TrD}(\overline{Z})$ Consequently the functor $\mathrm{D}\underline{\mathrm{Hom}}_\Lambda(-, X) : \underline{\mathrm{CM}}(\Lambda) \to \mathcal{A}b$ is representable by the object $\mathrm{TrD}\,\Omega_{\mathbf{I}}\mathbf{L}_{\mathrm{CoCM}}(\overline{X})$ and the functor $\mathrm{D}\overline{\mathrm{Hom}}_\Lambda(Z, -) : \overline{\mathrm{CoCM}}(\mathrm{D}(\Lambda))^{\mathrm{op}} \to \mathcal{A}b$ is representable by the object $\mathrm{DTr}\,\Sigma_{\mathbf{P}}\mathbf{R}_{\mathrm{CM}}(\underline{Z})$. Hence $\mathsf{S}_{\mathrm{CM}} := \Sigma_{\mathbf{P}}\mathbf{R}_{\mathrm{CM}}\,\mathrm{DTr} : \underline{\mathrm{CM}}(\Lambda) \to \underline{\mathrm{CM}}(\Lambda)$, respectively $\mathsf{S}_{\mathrm{CoCM}} := \Omega_{\mathbf{I}}\mathbf{L}_{\mathrm{CoCM}}\,\mathrm{TrD} : \overline{\mathrm{CoCM}}(\mathrm{D}(\Lambda)) \to \overline{\mathrm{CoCM}}(\mathrm{D}(\Lambda))$, is a Serre functor in $\underline{\mathrm{CM}}(\Lambda)$, respectively $\overline{\mathrm{CoCM}}(\mathrm{D}(\Lambda))$. Then by [53], $\underline{\mathrm{CM}}(\Lambda)$, respectively $\overline{\mathrm{CoCM}}(\mathrm{D}(\Lambda))$, has Auslander–Reiten triangles which, by [20], remain such in $\underline{\mathrm{CM}}(\mathbf{P}_\Lambda)$, respectively $\overline{\mathrm{CoCM}}(\mathbf{I}_\Lambda)$, since any compact object in $\underline{\mathrm{CM}}(\mathbf{P}_\Lambda)$, respectively $\overline{\mathrm{CoCM}}(\mathbf{I}_\Lambda)$, is pure-injective by Corollary 7.9. □

If $\tau^{\pm}_{\mathrm{CM}}$, respectively $\tau^{\pm}_{\mathrm{CoCM}}$, denotes the Auslander–Reiten translations in $\underline{\mathrm{CM}}(\Lambda)$, respectively $\overline{\mathrm{CoCM}}(\mathrm{D}(\Lambda))$, then since $\Omega_{\mathbf{I}}|_{\overline{\mathrm{mod}}\text{-}\Lambda} = \mathrm{DTr}\,\Omega\,\mathrm{TrD}$ and $\Sigma_{\mathbf{P}}|_{\underline{\mathrm{mod}}\text{-}\Lambda} = \mathrm{Tr}\,\Omega\,\mathrm{Tr}$, Proposition 8.8 implies that:

$$\tau^{+}_{\mathrm{CM}} = \mathbf{R}_{\mathrm{CM}}\,\mathrm{DTr}, \qquad \tau^{-}_{\mathrm{CM}} = \mathrm{TrD}\,\mathbf{L}_{\mathrm{CoCM}} \qquad \text{and}$$
$$\tau^{+}_{\mathrm{CoCM}} = \mathbf{L}_{\mathrm{CoCM}}\,\mathrm{TrD}, \qquad \tau^{-}_{\mathrm{CoCM}} = \mathrm{DTr}\,\mathbf{R}_{\mathrm{CM}}.$$

Let $\mathcal{U}$ be any one of the exact subcategories of mod-Λ: $\mathrm{CM}(\Lambda)$, $\mathrm{CoCM}(\mathrm{D}(\Lambda))$, $(\mathfrak{P}^{\prec\propto}_\Lambda)^{\mathrm{fin}}$, $(\mathfrak{I}^{\prec\propto}_\Lambda)^{\mathrm{fin}}$. We denote by $K_0(\mathcal{U})$ the 0th Quillen's K-group of $\mathcal{U}$. The following result shows that the group $K_0(\mathcal{U})$ is free provided that Λ is virtually Gorenstein.

Theorem 8.9. *If Λ is virtually Gorenstein, then there exist isomorphisms*:

$$K_0(\mathrm{CM}(\Lambda)) \amalg K_0((\mathfrak{P}^{\prec\propto}_\Lambda)^{\mathrm{fin}}) \cong K_0(\mathcal{P}_\Lambda) \amalg K_0(\text{mod-}\Lambda),$$
$$K_0(\mathrm{CoCM}(\mathrm{D}(\Lambda))) \amalg K_0((\mathfrak{I}^{\prec\propto}_\Lambda)^{\mathrm{fin}}) \cong K_0(\mathcal{I}_\Lambda) \amalg K_0(\text{mod-}\Lambda),$$
$$K_0(\underline{\mathrm{CM}}(\Lambda)) \amalg K_0((\underline{\mathfrak{P}}^{\prec\propto}_\Lambda)^{\mathrm{fin}}) \cong K_0(\underline{\mathrm{mod}}\text{-}\Lambda),$$
$$K_0(\overline{\mathrm{CoCM}}(\mathrm{D}(\Lambda))) \amalg K_0((\bar{\mathfrak{I}}^{\prec\propto}_\Lambda)^{\mathrm{fin}}) \cong K_0(\overline{\mathrm{mod}}\text{-}\Lambda).$$

Proof. By Theorem 8.2 we have a cotorsion pair $(\mathrm{CM}(\Lambda), (\mathfrak{P}^{\prec\infty}_{\Lambda})^{\mathrm{fin}})$ in mod-Λ with $\mathrm{CM}(\Lambda) \cap (\mathfrak{P}^{\prec\infty}_{\Lambda})^{\mathrm{fin}} = \mathcal{P}_{\Lambda}$. Let $\mathsf{i} : K_0(\mathcal{P}_{\Lambda}) \to K_0(\mathrm{CM}(\Lambda))$ and $\mathsf{j} : K_0(\mathcal{P}_{\Lambda}) \to K_0((\mathfrak{P}^{\prec\infty}_{\Lambda})^{\mathrm{fin}})$ be the natural maps defined by $\mathsf{i}([P]) = [P]$ and $\mathsf{j}([P]) = [P]$. Also let $\alpha : K_0(\mathrm{CM}(\Lambda)) \to K_0(\text{mod-}\Lambda) \leftarrow K_0((\mathfrak{P}^{\prec\infty}_{\Lambda})^{\mathrm{fin}}) : \beta$ be defined by $\alpha([X]) = [X]$ and $\beta([Y]) = [Y]$. We claim that there exists a cocartesian square

$$\begin{tikzcd} K_0(\mathcal{P}_{\Lambda}) \arrow[r, "\mathsf{j}"] \arrow[d, "\mathsf{i}"'] & K_0((\mathfrak{P}^{\prec\infty}_{\Lambda})^{\mathrm{fin}}) \arrow[d, "\beta"] \\ K_0(\mathrm{CM}(\Lambda)) \arrow[r, "\alpha"] & K_0(\text{mod-}\Lambda) \end{tikzcd} \tag{†}$$

in $\mathcal{A}b$. Clearly the above diagram commutes. Let G be an abelian group and let $\zeta : K_0((\mathfrak{P}^{\prec\infty}_{\Lambda})^{\mathrm{fin}}) \to G$ and $\vartheta : K_0(\mathrm{CM}(\Lambda)) \to G$ be group homomorphisms, such that $\mathsf{j} \circ \zeta = \mathsf{i} \circ \theta$. We define a group homomorphism $\eta : K_0(\text{mod-}\Lambda) \to G$ as follows. Let A be in mod-Λ and consider the exact sequence $0 \to Y_A \to X_A \to A \to 0$ where the map $X_A \to A$ is the minimal right Cohen–Macaulay approximation of A. We set $\eta'(A) = \theta([X_A]) - \zeta([Y_A])$. If $0 \to \widetilde{Y}_A \to \widetilde{X}_A \to A \to 0$ is exact where the map $\widetilde{X}_A \to A$ is a right Cohen–Macaulay approximation of A, then it is easy to see that there exists a projective module P such that $\widetilde{X}_A \cong X_A \oplus P$ and $\widetilde{Y}_A \cong Y_A \oplus P$. Then $\vartheta([\widetilde{X}_A]) = \vartheta([X_A]) + \vartheta([P])$ and $\zeta([\widetilde{Y}_A]) = \zeta([Y_A]) + \zeta([P])$ in G and therefore $\vartheta(\widetilde{X}_A) - \zeta(\widetilde{Y}_A) = \vartheta([X_A]) - \zeta([Y_A]) - (\vartheta([P]) - \zeta([P]))$. Since $\vartheta([P]) - \zeta([P]) = \vartheta(\mathsf{i}([P])) - \zeta(\mathsf{j}([P])) = 0$, it follows that the assignment $A \mapsto \eta'(A) = \theta([X_A]) - \zeta([Y_A])$ is independent of the right Cohen–Macaulay approximations and gives a well defined map on the set of isoclasses of mod-Λ. If $0 \to A \to B \to C \to 0$ is a short exact sequence in mod-Λ, then by [9, Proposition 3.6] it follows that there exist exact sequences $0 \to X_A \to X_B \to X_C \to 0$ in $\mathrm{CM}(\Lambda)$ and $0 \to Y_A \to Y_B \to Y_C \to 0$ in $(\mathfrak{P}^{\prec\infty}_{\Lambda})^{\mathrm{fin}}$ and therefore we have: $\theta([X_B]) - \zeta([Y_B]) = \theta([X_A]) + \theta([X_C]) - (\zeta([Y_A]) + \zeta([Y_C])) = (\theta([X_A]) - \zeta([Y_A])) + (\theta([X_C]) - \zeta([Y_C]))$. It follows that $\eta'(A) = \eta'(B) + \eta'(C)$ and therefore there exists a unique group map $\eta : K_0(\text{mod-}\Lambda) \to G$ such that $\eta'(A) = \eta([A])$. Clearly $\alpha \circ \eta = \vartheta$. Let Y be in $(\mathfrak{P}^{\prec\infty}_{\Lambda})^{\mathrm{fin}}$. Then the minimal right Cohen–Macaulay approximation of Y is its projective cover $P \twoheadrightarrow Y$. Therefore $\eta(\beta([P]) = \eta([P]) = \vartheta([P]) - \zeta([\Omega(Y)]) = \vartheta\mathsf{i}([P]) - \zeta([\Omega(Y)]) = \zeta\mathsf{j}([P]) - \zeta([\Omega(Y)]) = \zeta([P] - [\Omega(Y)]) = \zeta([Y])$. Hence $\beta \circ \eta = \zeta$. If $\mu : K_0(\text{mod-}\Lambda) \to G$ is a group map such that $\alpha \circ \mu = \beta \circ \mu$, then for any module A we have $\mu([A]) = \mu([X_A] - [Y_A]) = \mu(\alpha([X_A]) - \beta([Y_A])) = \vartheta([X_A]) - \zeta([Y_A]) = \eta([A])$. Hence $\eta = \mu$. We infer that (†) is cocartesian and therefore we have a short exact sequence in $\mathcal{A}b$

$$K_0(\mathcal{P}_{\Lambda}) \xrightarrow{(\mathsf{i},-\mathsf{j})} K_0(\mathrm{CM}\,\Lambda)) \amalg K_0\big(\big(\mathfrak{P}^{\prec\infty}_{\Lambda}\big)^{\mathrm{b}}\big) \xrightarrow{\binom{\alpha}{\beta}} K_0(\text{mod-}\Lambda) \longrightarrow 0 \tag{††}$$

which induces a short exact sequence $0 \to H \to K_0(\mathcal{P}_{\Lambda}) \to G \to 0$ where $G := \operatorname{Ker}\binom{\alpha}{\beta}$. We shall show that $H = 0$. To this end we first show that the finitely generated abelian groups $K_0(\mathrm{CM}(\Lambda))$ and $K_0((\mathfrak{P}^{\prec\infty}_{\Lambda})^{\mathrm{fin}})$ both have rank $\geqslant n$, where $\{S_1, \ldots, S_n\}$ are the non-isomorphic simple Λ-modules. For $i = 1, \ldots, n$, let X_i, respectively Y_i, be the minimal right $\mathrm{CM}(\Lambda)$-, respectively $(\mathfrak{P}^{\prec\infty}_{\Lambda})^{\mathrm{fin}}$-, approximation of S_i. Let $\mathcal{A}$, respectively $\mathcal{B}$, be the full subcategory $\mathrm{CM}(\Lambda)$, respectively $(\mathfrak{P}_\Lambda^{\prec\infty})^{\mathrm{fin}}$, consisting of all modules which admit finite filtrations with factors in $\{X_i\}$, respectively $\{Y_i\}$. By a result of Auslander–Reiten [9], for any Cohen–Macaulay module X and any module Y in $(\mathfrak{P}_\Lambda^{\prec\infty})^{\mathrm{fin}}$, there exist projective modules P and Q such that $X \oplus P \in \mathcal{A}$ and $Y \oplus Q \in \mathcal{B}$. This means that the category $\mathcal{A}$, respectively $\mathcal{B}$, is cofinal in $\mathrm{CM}(\Lambda)$, respectively $(\mathfrak{P}_\Lambda^{\prec\infty})^{\mathrm{fin}}$. Therefore the canonical maps $K_0(\mathcal{A}) \to K_0(\mathrm{CM}(\Lambda))$ and $K_0(\mathcal{B}) \to K_0((\mathfrak{P}_\Lambda^{\prec\infty})^{\mathrm{fin}})$ are monomorphisms. It follows that $K_0(\mathrm{CM}(\Lambda)$ and $K_0((\mathfrak{P}_\Lambda^{\prec\infty})^{\mathrm{fin}})$ both have rank $\geqslant n$, since clearly $K_0(\mathcal{A})$, respectively $K_0(\mathcal{B})$, is free on the set $\{X_i\}$, respectively $\{Y_i\}$. Since $K_0(\text{mod-}\Lambda)$ is free of rank n, the rank of G is $\geqslant n$. In turn this implies that the rank of the free subgroup H of $K_0(\mathcal{P}_\Lambda)$ is zero, hence $H = 0$. Therefore the map $(\mathsf{i}, -\mathsf{j})$ in (††) is a monomorphism and then the freeness of $K_0(\text{mod-}\Lambda)$ implies the first isomorphism. The second isomorphism follows in a similar way by using the cotorsion pair $((\mathfrak{I}_\Lambda^{\prec\infty})^{\mathrm{fin}}, \mathrm{CoCM}(\mathrm{D}(\Lambda)))$ in mod-Λ with $\mathrm{CoCM}(\mathrm{D}(\Lambda)) \cap (\mathfrak{I}_\Lambda^{\prec\infty})^{\mathrm{fin}} = \mathcal{I}_\Lambda$. Finally the last two isomorphisms follow directly from first two or alternatively from [22, Corollary II.5.7]. □

We close this section with the following consequence of Proposition 8.8 and [20, Theorem 12.1].

Corollary 8.10. *The following are equivalent for an Artin algebra Λ.*

(i) *Λ is of finite Cohen–Macaulay type.*
(ii) *Λ is virtually Gorenstein and the set $\{[\underline{X}_1] - [\underline{X}_2] + [\underline{X}_3] \in K_0(\underline{\mathrm{CM}}(\Lambda), \oplus)\}$, where $\Omega(\underline{X}_3) \to \underline{X}_1 \to \underline{X}_2 \to \underline{X}_3$ is an Auslander–Reiten triangle in $\underline{\mathrm{CM}}(\Lambda)$, is a free basis of $K_0(\underline{\mathrm{CM}}(\Lambda), \oplus)$.*
(iii) *Λ is virtually Gorenstein and the set $\{[X_1] - [X_2] + [X_3]\} \cup \{[X_{\mathbf{r}P}] - [P]\} \subseteq K_0(\mathrm{CM}(\Lambda), \oplus)$, where $0 \to X_1 \to X_2 \to X_3 \to 0$ is an Auslander–Reiten sequence in $\mathrm{CM}(\Lambda)$ and $X_{\mathbf{r}P}$ is the minimal right Cohen–Macaulay approximation of $\mathbf{r}P$ for any indecomposable projective module P, is a free basis of $K_0(\mathrm{CM}(\Lambda), \oplus)$.*

8.4. Derived equivalences, stable equivalences of Morita type and virtually Gorenstein algebras

It is well known that Gorensteinness is preserved under derived equivalences, this is also a consequence of our next result. On the other hand there exist derived equivalent Artin algebras Λ and Γ such that Λ is representation finite but Γ is not, see [31]. The next result, which shows that virtual Gorensteinness is preserved under derived equivalences, implies that if Λ is representation finite, hence virtually Gorenstein, and derived equivalent to Γ, then Γ is virtually Gorenstein.

Theorem 8.11. *Let Λ and Γ be derived equivalent finite-dimensional k-algebras over a field k. Then there exist triangle equivalences*:

$$\underline{\mathrm{CM}}(\mathbf{P}_\Lambda) \xrightarrow{\approx} \underline{\mathrm{CM}}(\mathbf{P}_\Gamma) \quad \textit{and} \quad \overline{\mathrm{CoCM}}(\mathbf{I}_\Lambda) \xrightarrow{\approx} \overline{\mathrm{CoCM}}(\mathbf{I}_\Gamma).$$

Moreover:

(i) *Λ is virtually Gorenstein iff Γ is virtually Gorenstein.*
(ii) gl.dim $\Lambda < \infty$, *respectively Λ is Gorenstein, iff the same holds for Γ.*
(iii) *Λ is of finite Cohen–Macaulay type iff Γ is of finite Cohen–Macaulay type.*
(iv) *If Λ is representation-finite, then Γ is virtually Gorenstein of finite Cohen–Macaulay type.*

Proof. Let $F : \mathbf{D}(\text{Mod-}\Lambda) \xrightarrow{\approx} \mathbf{D}(\text{Mod-}\Gamma)$ be a triangle equivalence, which we may assume that it is standard [55], that is, F is given by a two-sided tilting complex. Assume first that Λ is virtually Gorenstein, in particular we have $\underline{\mathrm{CM}}(\Lambda) = \underline{\mathrm{CM}}(\mathbf{P}_\Lambda)^{\mathrm{b}}$. Since, by Theorem 8.2, there exists a cotorsion triple $(\mathrm{CM}(\Lambda), (\mathfrak{P}_\Lambda^{\prec\infty})^{\mathrm{fin}} = (\mathfrak{I}_\Lambda^{\prec\infty})^{\mathrm{fin}}, \mathrm{CoCM}(\mathrm{D}(\Lambda)))$ in mod-Λ, where all the involved categories are functorially finite, there exists a "finitely presented" version of the exact commutative diagram of Proposition 5.8, i.e., the same diagram but $\mathrm{CM}(\mathbf{P}_\Lambda)$ is replaced by $\mathrm{CM}(\Lambda)$, $\mathbf{P}_\Lambda$ and Mod-Λ are replaced by $\mathcal{P}_\Lambda$ and mod-Λ respectively, and $\mathfrak{P}_\Lambda^{\prec\infty}$ is replaced by $(\mathfrak{P}_\Lambda^{\prec\infty})^{\mathrm{fin}}$. Now it is well known that F restricts to a triangle equivalence between $\mathbf{D}^{\mathrm{b}}(\text{Mod-}\Lambda)$ and $\mathbf{D}^{\mathrm{b}}(\text{Mod-}\Gamma)$, between $\mathbf{D}^{\mathrm{b}}(\text{mod-}\Lambda)$ and $\mathbf{D}^{\mathrm{b}}(\text{mod-}\Gamma)$, between $\mathcal{H}^{\mathrm{b}}(\mathbf{P}_\Lambda)$ and $\mathcal{H}^{\mathrm{b}}(\mathbf{P}_\Gamma)$ and finally between $\mathcal{H}^{\mathrm{b}}(\mathcal{P}_\Lambda)$ and $\mathcal{H}^{\mathrm{b}}(\mathcal{P}_\Gamma)$ [54]. In particular by Remark 4.3, F induces a triangle equivalence between $\mathcal{T}_l(\underline{\text{Mod}}\text{-}\Lambda)$ and $\mathcal{T}_l(\underline{\text{Mod}}\text{-}\Gamma)$, and between $\mathcal{T}_l(\underline{\text{mod}}\text{-}\Lambda)$ and $\mathcal{T}_l(\underline{\text{mod}}\text{-}\Gamma)$. Also by [55] it follows that F commutes with the total derived functors $- \otimes_\Lambda^{\mathbb{L}} \mathrm{D}(\Lambda)$ and $- \otimes_\Gamma^{\mathbb{L}} \mathrm{D}(\Gamma)$. Using the characterization of the objects in the strict image of the fully faithful functor $\mathbf{D}^{\mathrm{b}}(\mathbf{i}_{\mathrm{CM}}) : \mathbf{D}^{\mathrm{b}}(\mathrm{CM}(\mathbf{P}_\Lambda)) \hookrightarrow \mathbf{D}^{\mathrm{b}}(\text{Mod-}\Lambda)$ in Proposition 5.8, it follows that F induces a triangle equivalence between $\mathbf{D}^{\mathrm{b}}(\mathrm{CM}(\mathbf{P}_\Lambda))$ and $\mathbf{D}^{\mathrm{b}}(\mathrm{CM}(\mathbf{P}_\Gamma))$. Using the finitely presented version of Proposition 5.8, it follows that this triangle equivalence restricts to a triangle equivalence between $\mathbf{D}^{\mathrm{b}}(\mathrm{CM}(\Lambda))$ and $\mathbf{D}^{\mathrm{b}}(\mathrm{CM}(\Gamma))$. Then from the exact commutative diagram of Proposition 5.8 and its finitely presented version, it follows that F induces a triangle equivalence between $\underline{\mathrm{CM}}(\mathbf{P}_\Lambda)$ and $\underline{\mathrm{CM}}(\mathbf{P}_\Gamma)$, hence between $\underline{\mathrm{CM}}(\mathbf{P}_\Lambda)^{\mathrm{b}}$ and $\underline{\mathrm{CM}}(\mathbf{P}_\Gamma)^{\mathrm{b}}$, which restricts to a triangle equivalence between $\underline{\mathrm{CM}}(\Lambda)$ and $\underline{\mathrm{CM}}(\Gamma)$. That is, we have the following commutative diagram where the horizontal arrows are inclusions and the vertical arrows are triangle equivalences induced by F:

$$\begin{tikzcd}
\mathcal{T}_l(\underline{\text{mod}}\text{-}\Lambda) \arrow[d, "F"', "\approx"] & \underline{\mathrm{CM}}(\Lambda) \arrow[l] \arrow[r] \arrow[d, "F"', "\approx"] & \underline{\mathrm{CM}}(\mathbf{P}_\Lambda)^{\mathrm{b}} \arrow[d, "F"', "\approx"] \\
\mathcal{T}_l(\underline{\text{mod}}\text{-}\Gamma) & \underline{\mathrm{CM}}(\Gamma) \arrow[l] \arrow[r] & \underline{\mathrm{CM}}(\mathbf{P}_\Gamma)^{\mathrm{b}}
\end{tikzcd}$$

Since $\underline{\mathrm{CM}}(\Lambda) = \underline{\mathrm{CM}}(\mathbf{P}_\Lambda)^{\mathrm{b}}$, it follows that $\underline{\mathrm{CM}}(\Gamma) = \underline{\mathrm{CM}}(\mathbf{P}_\Gamma)^{\mathrm{b}}$ and therefore Γ is virtually Gorenstein by Theorem 8.2. Since, by Corollary 5.9 and Lemma 4.1, an Artin algebra Δ is Gorenstein, respectively it holds gl.dim $\Delta < \infty$, iff $\underline{\mathrm{CM}}(\Delta) \approx \mathcal{T}_l(\underline{\text{mod}}\text{-}\Delta)$, respectively $\mathcal{T}_l(\underline{\text{mod}}\text{-}\Delta) = 0$, it follows that Λ is Gorenstein, respectively it holds gl.dim $\Lambda < \infty$ iff the same holds for Γ. Now part (iii) follows from the above analysis and Corollary 8.10, and part (iv) follows from (i) and (iii) and the fact that, by Example 8.4, representation finite algebras are virtually Gorenstein. □

Recall that a stable equivalence $F:\underline{\mathrm{Mod}}\text{-}\Lambda \xrightarrow{\approx} \underline{\mathrm{Mod}}\text{-}\Gamma$ between Artin algebras Λ and Γ is called a **stable equivalence of Morita type**, and then Λ and Γ are stably equivalent of Morita type, if there are bimodules ${}_\Lambda M_\Gamma$ and ${}_\Gamma N_\Lambda$ such that (α) ${}_\Lambda M$ and N_Λ are finitely generated projective Λ-modules and M_Γ and ${}_\Gamma N$ are finitely generated projective Γ-modules, (β) $M \otimes_\Gamma N \cong \Lambda \oplus P$ as Λ-bimodules where P is a projective Λ-bimodule, and $N \otimes_\Lambda M \cong \Gamma \oplus Q$ as Γ-bimodules where Q is a projective Γ-bimodule, and (γ) there exists a natural isomorphism $F\pi_\Lambda \xrightarrow{\cong} \pi_\Gamma(- \otimes_\Lambda M)$ of functors: $\mathrm{Mod}\text{-}\Lambda \to \underline{\mathrm{Mod}}\text{-}\Gamma$ ($\pi_\Lambda : \mathrm{Mod}\text{-}\Lambda \to \underline{\mathrm{Mod}}\text{-}\Lambda$ and $\pi_\Gamma : \mathrm{Mod}\text{-}\Gamma \to \underline{\mathrm{Mod}}\text{-}\Gamma$ are the canonical functors).

We have the following result which shows that (virtual) Gorensteinness is invariant under stable equivalences of Morita type.

Theorem 8.12. *Let ${}_\Lambda M_\Gamma$ and ${}_\Gamma N_\Lambda$ be bimodules inducing a stable equivalence of Morita type $F:\underline{\mathrm{Mod}}\text{-}\Lambda \xrightarrow{\approx} \underline{\mathrm{Mod}}\text{-}\Gamma$ between the Artin algebras Λ and Γ.*

- (i) *F induces a triangle equivalence $F:\underline{\mathrm{CM}}(\mathbf{P}_\Lambda) \xrightarrow{\approx} \underline{\mathrm{CM}}(\mathbf{P}_\Gamma)$.*
- (ii) *Λ is virtually Gorenstein iff Γ is virtually Gorenstein.*
- (iii) *Λ is Gorenstein iff Γ is Gorenstein.*

Proof. (i) Clearly it suffices to show that if X is a Cohen–Macaulay Λ-module, then $X \otimes_\Lambda M$ is a Cohen–Macaulay Γ-module. Let $(P^\bullet, d_{P^\bullet})$ be an acyclic complex of projective Λ-modules which remains exact after the application of $\mathrm{Hom}_\Lambda(-, P)$, where P is projective, and such that $\mathrm{Ker}\, d^0_{P^\bullet} = X$. Since the modules ${}_\Lambda M$ and M_Γ are projective, it follows that the functor $F_M := - \otimes_\Lambda M : \mathrm{Mod}\text{-}\Lambda \to \mathrm{Mod}\text{-}\Gamma$ is exact and preserves projectives. Therefore we have an acyclic complex $(F_M(P^\bullet), F_M(d_{P^\bullet}))$ of projective Γ-modules such that $\mathrm{Ker}\, F_M(d^0_{P^\bullet}) = F_M(X)$. Consequently $F_M(X)$ is Cohen–Macaulay provided that X lies in ${}^\perp\Lambda$ implies that $F_M(X)$ lies in ${}^\perp\Gamma$. Equivalently if X satisfies $\mathrm{Tor}^\Lambda_n(X, I) = 0$, $\forall n \geqslant 1$, for any injective Λ-module I, then $\mathrm{Tor}^\Gamma_n(F_M(X), J) = 0$, $\forall n \geqslant 1$, for any injective Γ-module J. Let $\cdots \to P^{-1} \to P^0 \to X \to 0$ be a projective resolution of $X \in {}^\perp\Lambda$. Then we have a projective resolution $\cdots \to F_M(P^{-1}) \to F_M(P^0) \to F_M(X) \to 0$ of $F_M(X)$ in $\mathrm{Mod}\text{-}\Gamma$ and therefore for any injective Γ^{op}-module J, $\mathrm{Tor}^\Gamma_n(F_M(X), J)$ is the n-th cohomology of the complex $\cdots \to P^{-1} \otimes_\Lambda M \otimes_\Gamma J \to P^0 \otimes_\Lambda M \otimes_\Gamma J \to 0$. It is easy to see that the functor ${}_M F := M \otimes_\Gamma - : \mathrm{Mod}\text{-}\Gamma^{\mathrm{op}} \to \mathrm{Mod}\text{-}\Lambda^{\mathrm{op}}$ preserves injectives. Hence $M \otimes_\Gamma J$ is an injective Λ^{op}-module and therefore $\mathrm{Tor}^\Gamma_n(F_M(X), J) \cong \mathrm{Tor}^\Lambda_n(X, {}_M F(J)) = 0$, $\forall n \geqslant 1$, since X lies in ${}^\perp\Lambda$ and ${}_M F(J)$ lies in $\mathbf{I}_{\Lambda^{\mathrm{op}}}$. We infer that the functor F_M preserves Cohen–Macaulay modules. Hence F induces a stable equivalence between $\underline{\mathrm{CM}}(\mathbf{P}_\Lambda)$ and $\underline{\mathrm{CM}}(\mathbf{P}_\Gamma)$ which is clearly a triangulated functor.

(ii) and (iii). Since the modules ${}_\Lambda M$, N_Λ, M_Γ and ${}_\Gamma N$ are finitely generated, it follows directly that the triangle equivalence of (i) restricts to a triangle equivalence $\underline{\mathrm{CM}}(\Lambda) \xrightarrow{\approx} \underline{\mathrm{CM}}(\Gamma)$. Then the assertion in (ii) follows by Theorem 8.2. If Λ is Gorenstein, then, by Proposition 3.10, any Λ-module lies in $\widehat{\mathrm{CM}(\mathbf{P}_\Lambda)}$. By (i) this easily implies that any Γ-modules is a direct summand of a module which admits a finite exact resolution by Cohen–Macaulay modules. Therefore Γ is Gorenstein. $\square$

Although the class of virtually Gorenstein algebras is rather large, we don't know of any example of an Artin algebra that is not virtually Gorenstein. We close this section with the following result which shows that "locally", i.e., at the finitely generated level, all Artin algebras are virtually Gorenstein.

Proposition 8.13. *For any Artin algebra Λ we have*:

$$\left(\mathfrak{P}_{\Lambda}^{\prec\propto}\right)^{\mathrm{fin}} = \left(\mathfrak{I}_{\Lambda}^{\prec\propto}\right)^{\mathrm{fin}}.$$

Proof. Let T be in $(\mathfrak{I}_{\Lambda}^{\prec\propto})^{\mathrm{fin}}$. By Proposition 5.5(1) we have that $\mathrm{TrD}(\overline{T})$ lies in ${}^{\perp}\underline{\mathrm{CM}}(\mathbf{P}_{\Lambda}) \cap \underline{\mathrm{mod}}\text{-}\Lambda$. Also by Proposition 3.8(vi) we have $\mathrm{Ext}_{\Lambda}^{n}(X, \mathrm{DTr}\,\mathrm{TrD}(T)) = 0$, $\forall n \geqslant 1$, $\forall X \in \mathrm{CM}(\mathbf{P}_{\Lambda})$. Hence T lies in $\mathfrak{P}_{\Lambda}^{\prec\propto} \cap \mathrm{mod}\text{-}\Lambda = (\mathfrak{P}_{\Lambda}^{\prec\propto})^{\mathrm{fin}}$. Therefore $(\mathfrak{I}_{\Lambda}^{\prec\propto})^{\mathrm{fin}} \subseteq (\mathfrak{P}_{\Lambda}^{\prec\propto})^{\mathrm{fin}}$. Similarly $(\mathfrak{P}_{\Lambda}^{\prec\propto})^{\mathrm{fin}} \subseteq (\mathfrak{I}_{\Lambda}^{\prec\propto})^{\mathrm{fin}}$. $\square$

9. Thick subcategories, (co)torsion pairs and virtually Gorenstein algebras

In this section we give relative versions of our previous results thus generalizing the situation of virtually Gorenstein algebras. More precisely we present methods for constructing thick subcategories, cotorsion pairs/triples, and torsion pairs of finite type. In addition we give bijections between certain cotorsion pairs in the module category and torsion pairs in the stable category which are of interest in connection with the Telescope Conjecture for stable categories discussed in Section 10.

9.1. Torsion pairs induced by Cohen–Macaulay modules

We have seen that $\mathfrak{I}_{\Lambda}^{\prec\propto}$ and $\mathfrak{P}_{\Lambda}^{\prec\propto}$ are thick resolving and coresolving subcategories of Mod-Λ. On the other hand it is easy to see that $\mathrm{CM}(\mathbf{P}_{\Lambda})$ is coresolving, respectively $\mathrm{CoCM}(\mathbf{I}_{\Lambda})$ is resolving, iff Λ is self-injective. The following result shows that $\mathrm{CM}(\mathbf{P}_{\Lambda})$ and $\mathrm{CoCM}(\mathbf{I}_{\Lambda})$ are not always thick.

Lemma 9.1. *For an Artin algebra Λ the following are equivalent.*

(i) $\mathrm{CM}(\mathbf{P}_{\Lambda})$, *respectively* $\mathrm{CoCM}(\mathbf{I}_{\Lambda})$, *is thick.*
(ii) $\widehat{\mathrm{CM}(\mathbf{P}_{\Lambda})} = \mathrm{CM}(\mathbf{P}_{\Lambda})$, *respectively* $\mathrm{Co}\widetilde{\mathrm{CM}(\mathbf{I}_{\Lambda})} = \mathrm{CoCM}(\mathbf{I}_{\Lambda})$.
(iii) $\mathrm{FPD}(\Lambda) = 0$, *respectively* $\mathrm{FID}(\Lambda) = 0$.

Proof. If $\mathrm{CM}(\mathbf{P}_{\Lambda})$ is thick, then clearly $\widehat{\mathrm{CM}(\mathbf{P}_{\Lambda})} = \mathrm{CM}(\mathbf{P}_{\Lambda})$. By Theorem 3.5, (ii) implies that $\mathbf{P}_{\Lambda}^{<\infty} = \mathbf{P}_{\Lambda}$ and therefore $\mathrm{FPD}(\Lambda) = 0$. If $\mathrm{FPD}(\Lambda) = 0$, then $\widehat{\mathrm{CM}(\mathbf{P}_{\Lambda})} = \mathrm{CM}(\mathbf{P}_{\Lambda})$ by Proposition 3.9. Hence $\mathrm{CM}(\mathbf{P}_{\Lambda})$ is thick since so is $\widehat{\mathrm{CM}(\mathbf{P}_{\Lambda})}$. CoCohen–Macaulay modules are treated dually. $\square$

Example 9.2. If Λ is a local Artin algebra, then it is easy to see that $\mathrm{FPD}(\Lambda) = 0 = \mathrm{FID}(\Lambda)$. Hence the subcategories $\mathrm{CM}(\mathbf{P}_{\Lambda})$ and $\mathrm{CoCM}(\mathbf{I}_{\Lambda})$ are thick. Since there exist Artin algebras Λ such that $\mathrm{FPD}(\Lambda) = 0$ and $\mathrm{FID}(\Lambda) \neq 0$, it may happen that $\mathrm{CM}(\mathbf{P}_{\Lambda})$ is thick but $\mathrm{CoCM}(\mathbf{I}_{\Lambda})$ is not thick.

However $\mathrm{CM}(\mathbf{P}_\Lambda)$ is projectively thick and $\mathrm{CoCM}(\mathbf{I}_\Lambda)$ is injectively thick in the following sense. A full subcategory $\mathcal{U}$ of an abelian category $\mathcal{C}$ is called **projectively**, respectively **injectively**, **thick**, if $\mathcal{U}$ is resolving, respectively coresolving, and closed under cokernels of $\mathcal{P}$-monics, respectively kernels of $\mathcal{I}$-epics, where $\mathcal{P}$, respectively $\mathcal{I}$, are the projectives, respectively injectives, of $\mathcal{C}$. It is easy to see that a full subcategory of $\mathcal{U}$ of $\mathrm{CM}(\mathbf{P}_\Lambda)$, respectively $\mathrm{CoCM}(\mathbf{I}_\Lambda)$, is projectively, respectively injectively, thick iff its image $\underline{\mathcal{U}}$, respectively $\overline{\mathcal{U}}$, in the stable category is a thick subcategory of $\underline{\mathrm{CM}}(\mathbf{P}_\Lambda)$, respectively $\overline{\mathrm{CoCM}}(\mathbf{I}_\Lambda)$. Also a cotorsion pair $(\mathcal{X},\mathcal{Y})$ is projective, respectively injective, iff $\mathcal{X}$, respectively $\mathcal{Y}$, is projectively, respectively injectively, thick and consists of Cohen–Macaulay, respectively CoCohen–Macaulay, modules.

Throughout we fix an Artin algebra Λ. In the sequel we shall need the following result from [22]. The corresponding result for injective cotorsion pairs is dual.

Theorem 9.3 [22]. *The map $\Phi:(\mathcal{X},\mathcal{Y})\mapsto(\underline{\mathcal{X}},\underline{\mathcal{Y}})$ gives a bijective correspondence between projective cotorsion pairs $(\mathcal{X},\mathcal{Y})$ in* $\mathrm{Mod}\text{-}\Lambda$ *and torsion pairs $(\underline{\mathcal{X}},\underline{\mathcal{Y}})$ in* $\underline{\mathrm{Mod}}\text{-}\Lambda$ *such that $\underline{\mathcal{X}}$ is triangulated.*

We are interested in cotorsion pairs induced by projectively thick subcategories consisting of finitely generated Cohen–Macaulay modules. In this connection we have the following result.

Theorem 9.4. *Let $\mathcal{F}$ be a projectively thick subcategory of* $\mathrm{CM}(\Lambda)$.

(i) *There exists a cotorsion triple $(\mathcal{X}_\mathcal{F},\mathcal{Y}_\mathcal{F},\mathcal{Z}_\mathcal{F})$ in* $\mathrm{Mod}\text{-}\Lambda$, *where*:

$$\mathcal{X}_\mathcal{F}=\varinjlim\mathcal{F},\qquad \mathcal{Y}_\mathcal{F}=\mathcal{F}^\perp={}^\perp\{\tau^+(\mathcal{F})\},\qquad \mathcal{Z}_\mathcal{F}=\mathcal{F}^{\perp\perp}=\{{}^\perp\{\tau^+(\mathcal{F})\}\}^\perp.$$

(ii) *$\mathcal{X}_\mathcal{F}\subseteq\mathrm{CM}(\mathbf{P}_\Lambda)$, $\mathcal{Z}_\mathcal{F}\subseteq\mathrm{CoCM}(\mathbf{I}_\Lambda)$, and $\mathcal{Y}_\mathcal{F}$ is thick and definable.*

(iii) *There exists a torsion pair $(\underline{\mathcal{X}}_\mathcal{F},\underline{\mathcal{Y}}_\mathcal{F})$ of finite type in* $\underline{\mathrm{Mod}}\text{-}\Lambda$ *where the triangulated category $\underline{\mathcal{X}}_\mathcal{F}$ is compactly generated, and a torsion pair $(\overline{\mathcal{Y}}_\mathcal{F},\overline{\mathcal{Z}}_\mathcal{F})$ of cofinite type in* $\overline{\mathrm{Mod}}\text{-}\Lambda$ *where the triangulated category $\overline{\mathcal{Z}}_\mathcal{F}$ is compactly generated.*

(iv) *The categories $\underline{\mathcal{X}}_\mathcal{F}$, $\overline{\mathcal{Z}}_\mathcal{F}$ are triangle equivalent. The Auslander–Reiten operator* DTr *induces a triangle equivalence* $\mathrm{DTr}:\underline{\mathcal{F}}\xrightarrow{\approx}\overline{\mathcal{Z}^{\mathrm{fin}}_\mathcal{F}}$ *with quasi-inverse induced by* TrD *and the Nakayama functor* N^+ *induces an equivalence* $\mathsf{N}^+:\mathcal{F}\xrightarrow{\approx}\mathcal{Z}^{\mathrm{fin}}_\mathcal{F}$ *with quasi-inverse the functor* N^-.

Proof. As already mentioned the stable category $\underline{\mathcal{F}}$ is a thick subcategory of $\underline{\mathrm{CM}}(\Lambda)$. Let $\mathcal{X}_\mathcal{F}$ be the full subcategory of $\mathrm{Mod}\text{-}\Lambda$ consisting of all modules A such that $\underline{A}$ lies in the localizing subcategory of $\underline{\mathrm{CM}}(\mathbf{P}_\Lambda)$ generated by $\underline{\mathcal{F}}$, which we denote by $\underline{\mathcal{X}}_\mathcal{F}$. Since $\underline{\mathcal{F}}$ consists of compact objects of $\underline{\mathrm{CM}}(\mathbf{P}_\Lambda)$, it follows that the inclusion $\mathbf{i}'_\mathcal{X}:\underline{\mathcal{X}}_\mathcal{F}\hookrightarrow\underline{\mathrm{CM}}(\mathbf{P}_\Lambda)$ admits a right adjoint $\mathbf{R}'_\mathcal{F}:\underline{\mathrm{CM}}(\mathbf{P}_\Lambda)\to\underline{\mathcal{X}}_\mathcal{F}$ which preserves coproducts [50]. Then the composition $\mathbf{R}_\mathcal{F}:=\mathbf{R}'_\mathcal{F}\mathbf{R}_{\mathrm{CM}}:\underline{\mathrm{Mod}}\text{-}\Lambda\to\underline{\mathcal{X}}_\mathcal{F}$ is a right adjoint of the inclusion $\mathbf{i}_\mathcal{F}:\underline{\mathcal{X}}_\mathcal{F}\hookrightarrow\underline{\mathrm{Mod}}\text{-}\Lambda$. Since $\underline{\mathcal{X}}_\mathcal{F}$ is a triangulated subcategory of $\underline{\mathrm{Mod}}\text{-}\Lambda$ it follows that $\mathcal{X}_\mathcal{F}$ is a resolving subcategory of $\mathrm{Mod}\text{-}\Lambda$ and admits $\mathbf{P}_\Lambda$ as an Ext-injective cogenerator. Therefore, by [22], we have a torsion pair $(\underline{\mathcal{X}_{\mathcal{F}}}, \underline{\mathcal{Y}_{\mathcal{F}}})$ in $\underline{\mathrm{Mod}}$-Λ, where $\underline{\mathcal{Y}_{\mathcal{F}}} = (\underline{\mathcal{X}_{\mathcal{F}}})^{\perp}$, and a cotorsion pair $(\mathcal{X}_{\mathcal{F}}, \mathcal{Y}_{\mathcal{F}})$ in Mod-Λ, where $\mathcal{Y}_{\mathcal{F}} = (\mathcal{X}_{\mathcal{F}})^{\perp}$. Since $(\underline{\mathcal{X}_{\mathcal{F}}})^{\mathrm{b}} = \underline{\mathcal{F}}$, it follows that the inclusion $\mathbf{i}_{\mathcal{F}}$ preserves compact objects and therefore by Lemmas 6.1 and 7.2 the torsion pair $(\underline{\mathcal{X}_{\mathcal{F}}}, \underline{\mathcal{Y}_{\mathcal{F}}})$ is of finite type. Since $\underline{\mathcal{X}_{\mathcal{F}}}$ is triangulated, by [22] it follows that $\mathcal{Y}_{\mathcal{F}}$ is thick. Next we show that $\mathcal{Y}_{\mathcal{F}} = \mathcal{F}^{\perp}$. Since $\mathcal{F} \subseteq \mathcal{X}_{\mathcal{F}}$ it follows that $\mathcal{Y}_{\mathcal{F}} = (\mathcal{X}_{\mathcal{F}})^{\perp} \subseteq \mathcal{F}^{\perp}$. Let A be in $\mathcal{F}^{\perp}$, i.e., $\mathrm{Ext}^n_{\Lambda}(\mathcal{F}, A) = 0$, $\forall n \geqslant 1$. By Proposition 3.8 we have

$$\underline{\mathrm{Hom}}_{\Lambda}\big(\Omega^n(\underline{F}), \underline{A}\big) \xrightarrow{\cong} \underline{\mathrm{Hom}}_{\Lambda}\big(\Omega^n(\underline{F}), \mathbf{R}_{\mathcal{F}}(\underline{A})\big) = 0, \quad \forall n \geqslant 1,\ \forall F \in \mathcal{F}.$$

Since $\underline{\mathcal{F}}$ is a thick generating subcategory of $\underline{\mathcal{X}_{\mathcal{F}}}$, we infer that $\mathbf{R}_{\mathcal{F}}(\underline{A}) = 0$ and therefore $\underline{A}$ lies in $\underline{\mathcal{Y}_{\mathcal{F}}}$, i.e., A lies in $\mathcal{Y}_{\mathcal{F}}$. Hence $\mathcal{F}^{\perp} = \mathcal{Y}_{\mathcal{F}}$. Since $\mathcal{F}$ consists of finitely generated modules, it follows from [43] that $\mathcal{Y}_{\mathcal{F}}$ is definable. Since it is also resolving, by [46] it follows that $\mathcal{Y}_{\mathcal{F}}$ is contravariantly finite and there exists a pure-injective module T such that $\mathcal{Y}_{\mathcal{F}} = {}^{\perp}T$, in fact T is the kernel of the minimal right $\mathcal{Y}_{\mathcal{F}}$-approximation $Y_{\Lambda/\mathfrak{r}} \to \Lambda/\mathfrak{r}$ of $\Lambda/\mathfrak{r}$. Since $\mathcal{F}^{\perp}$ is resolving, by [32], T generates a cotorsion pair $({}^{\perp}T, ({}^{\perp}T)^{\perp})$ in Mod-Λ. Setting $\mathcal{Z}_{\mathcal{F}} := ({}^{\perp}T)^{\perp}$, it follows that $\mathcal{Z}_{\mathcal{F}} \subseteq \mathrm{CoCM}(\mathbf{I}_{\Lambda})$ and therefore we have a cotorsion triple $(\mathcal{X}_{\mathcal{F}}, \mathcal{Y}_{\mathcal{F}}, \mathcal{Z}_{\mathcal{F}})$ in Mod-Λ. Moreover by [22] we have $\mathcal{Y}_{\mathcal{F}} \cap \mathcal{Z}_{\mathcal{F}} = \mathbf{I}_{\Lambda}$, the stable category $\overline{\mathcal{Z}_{\mathcal{F}}}$ is triangulated and triangle equivalent to $\underline{\mathcal{X}_{\mathcal{F}}}$. We now show that $\mathcal{X}_{\mathcal{F}} = \varinjlim \mathcal{F}$. By the results of Krause–Solberg [46], it follows that a module A lies in $\mathcal{Y}_{\mathcal{F}}$ iff A is a direct summand of a module Y which admits a finite filtration $0 = Y_0 \subseteq Y_1 \subseteq \cdots \subseteq Y_{t-1} \subseteq Y_t = Y$ where each quotient Y_k/Y_{k-1} lies in $\mathrm{Prod}(Y_{\Lambda/\mathfrak{r}})$. Using that $Y_{\Lambda/\mathfrak{r}}$ is pure-injective, it follows that

$$\mathrm{Ext}^n_{\Lambda}\big(\varinjlim \mathcal{F}, \mathrm{Prod}(Y_{\Lambda/\mathfrak{r}})\big) \xrightarrow{\cong} \mathrm{Prod}\,\mathrm{Ext}^n_{\Lambda}(\varinjlim \mathcal{F}, Y_{\Lambda/\mathfrak{r}}) \xrightarrow{\cong} \mathrm{Prod}\varinjlim \mathrm{Ext}^n_{\Lambda}(\mathcal{F}, Y_{\Lambda/\mathfrak{r}}) = 0.$$

Then by induction we have $\varinjlim \mathcal{F} \subseteq {}^{\perp}\mathcal{Y}_{\mathcal{F}} = \mathcal{X}_{\mathcal{F}}$. Finally by the results of Hügel–Trlifaj [2] it follows that $\varinjlim \mathcal{F} = {}^{\perp}\mathrm{PInj}(\mathcal{Y}_{\mathcal{F}})$, where $\mathrm{PInj}(\mathcal{Y}_{\mathcal{F}})$ denotes the class of pure-injective modules in $\mathcal{Y}_{\mathcal{F}}$. Since clearly $\mathcal{X}_{\mathcal{F}} \subseteq {}^{\perp}\mathrm{PInj}(\mathcal{Y}_{\mathcal{F}})$, we infer that $\mathcal{X}_{\mathcal{F}} \subseteq \varinjlim \mathcal{F}$. We conclude that $\mathcal{X}_{\mathcal{F}} = \varinjlim \mathcal{F}$. Now let F be in $\mathcal{F}$ and Z in $\mathcal{Z}^{\mathrm{fin}}_{\mathcal{F}}$. Then from the Auslander–Reiten formulas

$$\mathrm{Ext}^1_{\Lambda}\big(\mathcal{F}^{\perp}, \mathrm{DTr}(F)\big) \cong \mathrm{D}\underline{\mathrm{Hom}}_{\Lambda}\big(\underline{F}, \underline{\mathcal{F}^{\perp}}\big) \quad \text{and} \quad \mathrm{DExt}^1_{\Lambda}\big(\mathrm{TrD}(Z), \mathcal{F}^{\perp}\big) \cong \overline{\mathrm{Hom}}_{\Lambda}\big(\overline{\mathcal{F}^{\perp}}, \overline{Z}\big)$$

we infer that $\mathrm{DTr}(F)$ lies in $\mathcal{Z}^{\mathrm{fin}}_{\mathcal{F}}$ and $\mathrm{TrD}(Z)$ lies in $\mathcal{X}^{\mathrm{fin}}_{\mathcal{F}} = \mathcal{F}$. This implies that $\mathsf{N}^{+}(F)$ lies in $\mathcal{Z}^{\mathrm{fin}}_{\mathcal{F}}$ and $\mathsf{N}^{-}(Z)$ lies in $\mathcal{F}$. Then the assertions in (iv) follow directly from these observations. Now since the restriction $\Sigma_{\mathbf{P}}|_{\underline{\mathcal{F}}}$ is an equivalence, the isomorphisms

$$\begin{aligned}\mathrm{DExt}^n_{\Lambda}\big(A, \tau^{+}(\mathcal{F})\big) &\cong \mathrm{D}\overline{\mathrm{Hom}}_{\Lambda}\big(\overline{A}, \Sigma^n\tau^{+}(\underline{\mathcal{F}})\big) \cong \mathrm{D}\overline{\mathrm{Hom}}_{\Lambda}\big(\overline{A}, \tau^{+}\Sigma^n_{\mathbf{P}}(\underline{\mathcal{F}})\big)\\ &\cong \mathrm{Ext}^1_{\Lambda}\big(\Sigma^n_{\mathbf{P}}(\underline{\mathcal{F}}), A\big)\end{aligned}$$

show directly that $\mathcal{Y}_{\mathcal{F}} = {}^{\perp}\tau^{+}(\mathcal{F})$. The remaining assertions follow from the results of [22]. $\square$

As already stated in Lemma 7.11, Krause and Solberg proved in [45] that contravariantly finite resolving, respectively covariantly finite coresolving, subcategories of mod-Λ are covariantly, respectively contravariantly, finite. Notice that there exist covariantly finite subcategories $\mathcal{Y} \subseteq$ mod-Λ which are not contravariantly finite [34]. In [34] such a $\mathcal{Y}$ consists of modules of finite projective dimension. The following result, which gives a converse to the result of [45], shows that the above "pathology" is impossible for subcategories of Cohen–Macaulay modules.

Corollary 9.5. *Let $\mathcal{F}$ be a projectively thick subcategory of* $\mathrm{CM}(\Lambda)$ *and $\mathcal{H}$ an injectively thick subcategory of* $\mathrm{CoCM}(\mathrm{D}(\Lambda))$. *Then we have the following.*

(i) *If $\mathcal{F}$ is covariantly finite, then $\mathcal{F}$ is contravariantly finite.*
(ii) *If $\mathcal{H}$ is contravariantly finite, then $\mathcal{H}$ is covariantly finite.*

In particular $\mathrm{CM}(\Lambda)$, *respectively* $\mathrm{CoCM}(\mathrm{D}(\Lambda))$, *is contravariantly finite iff it is covariantly finite. If Λ is self-injective, then a thick subcategory of* mod-Λ *is contravariantly finite iff it is covariantly finite.*

Proof. Let $(\mathcal{X}_\mathcal{F}, \mathcal{Y}_\mathcal{F}, \mathcal{Z}_\mathcal{F})$ be the cotorsion triple in Mod-Λ constructed in Theorem 9.4. Then $\mathcal{X}_\mathcal{F} = \varinjlim \mathcal{F}$ and the adjoint pair $(\mathsf{N}^+, \mathsf{N}^-)$ induces an equivalence between $\mathcal{F} = \mathcal{X}_\mathcal{F}^{\mathrm{fin}}$ and $\mathcal{Z}_\mathcal{F}^{\mathrm{fin}}$. This implies that $\mathcal{Z}^{\mathrm{fin}}$ is covariantly finite; indeed if $T \in$ mod-Λ and $\mathsf{N}^+(T) \to X^{\mathsf{N}^+(T)}$ is a left $\mathcal{F}$-approximation, then it is easy to see that the composition $T \to \mathsf{N}^-\mathsf{N}^+(T) \to \mathsf{N}^-(X^{\mathsf{N}^+(T)})$ is a left $\mathcal{Z}_\mathcal{F}^{\mathrm{fin}}$-approximation of T. Since $\mathcal{Z}_\mathcal{F}^{\mathrm{fin}}$ is coresolving, Lemma 7.11 implies that $\mathcal{Z}_\mathcal{F}^{\mathrm{fin}}$ is contravariantly finite. Using again the adjoint pair $(\mathsf{N}^+, \mathsf{N}^-)$ we infer that $\mathcal{F}$ is contravariantly finite. Part (ii) follows by duality and the final assertion follows from the fact that $\mathrm{CM}(\Lambda) =$ mod-Λ if Λ is self-injective. $\square$

We continue with other consequences of Theorem 9.4.

Corollary 9.6. *The map $\phi : \mathcal{F} \mapsto (\mathcal{F}^{\perp\perp})^{\mathrm{fin}}$ gives a bijection between contravariantly* (*covariantly*) *finite projectively thick subcategories $\mathcal{F}$ of* $\mathrm{CM}(\Lambda)$ *and covariantly* (*contravariantly*) *finite injectively thick subcategories $\mathcal{H}$ of* $\mathrm{CoCM}(\mathrm{D}(\Lambda))$. *The inverse is given by $\psi : \mathcal{H} \mapsto ({}^{\perp\perp}\mathcal{H})^{\mathrm{fin}}$. Any such $\mathcal{F}$, respectively $\mathcal{H}$, induces a cotorsion triple $(\varinjlim \mathcal{F}, \varinjlim \mathcal{G}, \varinjlim \mathcal{H})$ in* Mod-Λ, *where $\mathcal{G} = (\mathcal{F}^{\perp})^{\mathrm{fin}} = ({}^{\perp}\mathcal{H})^{\mathrm{fin}}$.*

Proof. Let $(\mathcal{X}_\mathcal{F}, \mathcal{Y}_\mathcal{F}, \mathcal{Z}_\mathcal{F})$ be the cotorsion triple in Mod-Λ induced by $\mathcal{F}$ as in Theorem 9.4. Since $\mathcal{F}$ is contravariantly finite, by [22], there exists a cotorsion triple $(\mathcal{F}, \mathcal{G}, \mathcal{H})$ in mod-Λ, which, by [45], induces a cotorsion triple $(\varinjlim \mathcal{F}, \varinjlim \mathcal{G}, \varinjlim \mathcal{H})$ in Mod-Λ. Since $\mathcal{X}_\mathcal{F} = \varinjlim \mathcal{F}$, it follows that

$$\mathcal{Y}_\mathcal{F} = \mathcal{F}^{\perp} = \varinjlim \mathcal{G} \quad \text{and} \quad \mathcal{Z}_\mathcal{F} = \varinjlim \mathcal{H} = \mathcal{F}^{\perp\perp}.$$

Since $\varinjlim \mathcal{H}$ is closed under products, Lemma 7.10 implies that $(\mathcal{F}^{\perp\perp})^{\mathrm{fin}} = \mathcal{H}$ is covariantly finite and plainly $\mathcal{H}$ is injectively thick and consists of CoCohen–Macaulay modules. Clearly the map ϕ is a bijection with inverse ψ. $\square$

Let $\mathcal{C}$ be a pretriangulated, respectively abelian, category which admits all small (co)products. We say that a torsion, respectively cotorsion, pair $(\mathcal{X}, \mathcal{Y})$ in $\mathcal{C}$ is **perfect** if $(\mathcal{X}, \mathcal{Y})$ is both of finite and cofinite type. Combining Lemma 7.10, Theorems 9.3, 9.4 and Corollary 9.6, we have the following consequence.

Corollary 9.7. *If* $\mathcal{F} \subseteq \mathrm{CM}(\Lambda)$ *is projectively thick, then the torsion* (*cotorsion*) *pair* $(\underline{\mathcal{X}}_{\mathcal{F}}, \underline{\mathcal{Y}}_{\mathcal{F}})$ $((\mathcal{X}_{\mathcal{F}}, \mathcal{Y}_{\mathcal{F}}))$ *in* $\underline{\mathrm{Mod}}\text{-}\Lambda$ $(\mathrm{Mod}\text{-}\Lambda)$ *is perfect iff* $\mathcal{F}$ *is contravariantly or covariantly finite in* $\mathrm{mod}\text{-}\Lambda$.

Corollary 9.8. *The map* $\Phi : \mathcal{F} \mapsto (\underline{\mathcal{X}}_{\mathcal{F}}, \underline{\mathcal{Y}}_{\mathcal{F}})$ *gives a bijection between the class of projectively thick subcategories* $\mathcal{F}$ *of* $\mathrm{CM}(\Lambda)$ *and the class of torsion pairs* $(\underline{\mathcal{X}}, \underline{\mathcal{Y}})$ *of finite type in* $\underline{\mathrm{Mod}}\text{-}\Lambda$ *such that the torsion class* $\underline{\mathcal{X}}$ *is triangulated and compactly generated. Under this correspondence,* $\mathcal{F}$ *is contravariantly* (*covariantly*) *finite iff the torsion pair* $(\underline{\mathcal{X}}_{\mathcal{F}}, \underline{\mathcal{Y}}_{\mathcal{F}})$ *is perfect.*

Proof. Clearly the map Φ is injective. Let $(\underline{\mathcal{X}}, \underline{\mathcal{Y}})$ be a torsion pair of finite type in $\underline{\mathrm{Mod}}\text{-}\Lambda$ where $\underline{\mathcal{X}}$ is a compactly generated triangulated category. Then $\mathcal{X} \subseteq \mathrm{CM}(\mathbf{P}_{\Lambda})$ by Theorems 7.12 and 9.3. Also by Lemma 7.2 the finite type property implies that $\underline{\mathcal{X}}^{\mathrm{b}} \subseteq (\underline{\mathrm{Mod}}\text{-}\Lambda)^{\mathrm{b}} = \underline{\mathrm{mod}}\text{-}\Lambda$ and therefore $\underline{\mathcal{X}}^{\mathrm{b}} \subseteq \underline{\mathrm{CM}}(\Lambda)$. Finally compact generation implies that $\underline{\mathcal{X}} = \underline{\mathcal{X}}_{\mathcal{F}}$ where $\mathcal{F} = \{F \in \mathrm{mod}\text{-}\Lambda \mid \underline{F} \in \underline{\mathcal{X}}^{\mathrm{b}}\}$ is clearly a projectively thick subcategory of $\mathrm{CM}(\Lambda)$, since the subcategory $\underline{\mathcal{X}}^{\mathrm{b}}$ is thick. Hence $(\underline{\mathcal{X}}, \underline{\mathcal{Y}}) = (\underline{\mathcal{X}}_{\mathcal{F}}, \underline{\mathcal{Y}}_{\mathcal{F}})$ and therefore the map Φ is bijective. □

Summarizing the above results, we have the following consequence.

Corollary 9.9. *There exist bijective correspondences between*:

(i) *Contravariantly* (*covariantly*) *finite projectively thick subcategories* $\mathcal{F}$ *of* $\mathrm{CM}(\Lambda)$.
(ii) *Covariantly* (*contravariantly*) *finite injectively thick subcategories* $\mathcal{H}$ *of* $\mathrm{CoCM}(\mathrm{D}(\Lambda))$.
(iii) *Perfect projective cotorsion pairs* $(\mathcal{X}, \mathcal{Y})$ *in* $\mathrm{Mod}\text{-}\Lambda$ *such that* $\mathcal{X} = \varinjlim \mathcal{X}^{\mathrm{fin}}$.
(iv) *Perfect injective cotorsion pairs* $(\mathcal{W}, \mathcal{Z})$ *in* $\mathrm{Mod}\text{-}\Lambda$ *such that* $\mathcal{Z} = \varinjlim \mathcal{Z}^{\mathrm{fin}}$.
(v) *Perfect torsion pairs* $(\underline{\mathcal{X}}, \underline{\mathcal{Y}})$ *in* $\underline{\mathrm{Mod}}\text{-}\Lambda$ *such that* $\underline{\mathcal{X}}$ *is triangulated and compactly generated.*
(vi) *Perfect torsion pairs* $(\overline{\mathcal{W}}, \overline{\mathcal{Z}})$ *in* $\overline{\mathrm{Mod}}\text{-}\Lambda$ *such that* $\overline{\mathcal{Z}}$ *is triangulated and compactly generated.*
(vii) *Cotorsion triples* $(\mathcal{A}, \mathcal{B}, \mathcal{C})$ *in* $\mathrm{Mod}\text{-}\Lambda$ *such that* $\mathcal{A}$ *is closed under products and* $\mathcal{A} = \varinjlim \mathcal{A}^{\mathrm{fin}}$.
(viii) *Cotorsion triples* $(\mathcal{A}, \mathcal{B}, \mathcal{C})$ *in* $\mathrm{Mod}\text{-}\Lambda$ *such that* $\mathcal{C} = \varinjlim \mathcal{C}^{\mathrm{fin}}$.

We close this section with the following complement to Corollary 9.8 which generalizes a result of Krause–Solberg [45] who proved the following result for self-injective algebras by using functor categories.

Theorem 9.10. *Let Λ be an Artin algebra and let $(\mathcal{X}, \mathcal{Y})$ be a projective cotorsion pair in* Mod-Λ. *Then the following statements are equivalent.*

(i) *The torsion pair $(\underline{\mathcal{X}}, \underline{\mathcal{Y}})$ in $\underline{\mathrm{Mod}}$-Λ is of finite type and $\underline{\mathcal{X}}$ is compactly generated.*
(ii) $\mathcal{X} = \varinjlim \mathcal{X}^{\mathrm{fin}}$.

If (ii) *holds, then $\mathcal{Y}$ is definable and the map $(\mathcal{X}, \mathcal{Y}) \mapsto (\underline{\mathcal{X}}, \underline{\mathcal{Y}})$ gives a bijection between projective cotorsion pairs $(\mathcal{X}, \mathcal{Y})$ in* Mod-Λ *such that $\mathcal{X} = \varinjlim \mathcal{X}^{\mathrm{fin}}$, and torsion pairs of finite type $(\underline{\mathcal{A}}, \underline{\mathcal{B}})$ in $\underline{\mathrm{Mod}}$-Λ such that the torsion class $\underline{\mathcal{A}}$ is a compactly generated triangulated category.*

Proof. (i) $\Rightarrow$ (ii). The assumptions imply that $\underline{\mathcal{X}}$ is compactly generated by $\underline{\mathcal{X}}^{\mathrm{b}} \subseteq \underline{\mathrm{mod}}$-$\Lambda$. Then by Corollary 9.8 and Theorem 9.4 we infer that $\mathcal{X} = \varinjlim \mathcal{X}^{\mathrm{fin}}$.

(ii) $\Rightarrow$ (i). We first show that $\underline{\mathcal{X}}$ is compactly generated. Let A be a module in $\mathcal{X}$ such that $\underline{\mathrm{Hom}}_\Lambda(X, A) = 0$ for any $\underline{X} \in \underline{\mathcal{X}}^{\mathrm{fin}}$ and let $\alpha : T \to A$ be a map where T lies in mod-Λ. Since $A \in \mathcal{X} = \varinjlim \mathcal{X}^{\mathrm{fin}}$, α factors through a module X in $\mathcal{X}^{\mathrm{fin}}$. Since any map $X \to A$ factors through a projective module and X is finitely generated, it follows that any map $X \to A$ factors through a finitely generated projective module. This implies, by Lemma 7.10, that A lies in $\varinjlim \mathcal{P}_\Lambda = \mathbf{P}_\Lambda$. Hence $\underline{A} = 0$ and therefore $\underline{\mathcal{X}}^{\mathrm{fin}}$ generates $\underline{\mathcal{X}}$. Since $\underline{\mathcal{X}}^{\mathrm{fin}}$ is thick and consists of compact objects, we infer that $\underline{\mathcal{X}}^{\mathrm{fin}} = \underline{\mathcal{X}}^{\mathrm{b}}$ is a compact generating subcategory of $\underline{\mathcal{X}}$. Now let $\{Y_i\}_{i \in I}$ be a filtered system of modules in $\mathcal{Y}$. Then for any module X in $\mathcal{X}^{\mathrm{fin}}$ we have $\mathrm{Ext}^n_\Lambda(X, \varinjlim Y_i) \cong \varinjlim \mathrm{Ext}^n_\Lambda(X, Y_i) = 0$. Since $\underline{\mathcal{X}}^{\mathrm{b}}$ is triangulated, it follows that any module X in $\mathcal{X}^{\mathrm{fin}}$ is an arbitrary syzygy of a module X' in $\mathcal{X}^{\mathrm{b}}$. This implies that $\underline{\mathrm{Hom}}_\Lambda(X, \varinjlim Y_i) \cong \underline{\mathrm{Hom}}_\Lambda(\Omega(X'), \varinjlim Y_i) \cong \mathrm{Ext}^1_\Lambda(X', \varinjlim Y_i) = 0$. But $\underline{\mathrm{Hom}}_\Lambda(X, \varinjlim Y_i) \cong \underline{\mathrm{Hom}}_\Lambda(X, \mathbf{R}_{\mathcal{X}}(\varinjlim Y_i))$ and therefore $\underline{\mathrm{Hom}}_\Lambda(X, \mathbf{R}_{\mathcal{X}}(\varinjlim Y_i)) = 0$ for any object $\underline{X}$ in $\underline{\mathcal{X}}^{\mathrm{b}}$. Since $\underline{\mathcal{X}}^{\mathrm{b}}$ generates $\underline{\mathcal{X}}$, we infer that $\mathbf{R}_{\mathcal{X}}(\varinjlim Y_i) = 0$ in $\underline{\mathrm{Mod}}$-Λ and therefore $\varinjlim Y_i \in \underline{\mathcal{Y}}$, i.e., $\varinjlim Y_i$ lies in $\mathcal{Y}$. We conclude that $\mathcal{Y}$ is closed under filtered colimits, in particular $\underline{\mathcal{Y}}$ is closed under coproducts and therefore the torsion pair $(\underline{\mathcal{X}}, \underline{\mathcal{Y}})$ is of finite type. $\square$

10. The Telescope Conjecture for stable categories

The results of the previous section suggest to study further the question of when torsion pairs of finite type are generated by compact objects. This question is exactly the content of the Telescope Conjecture in case we work with a compactly generated triangulated category, see [42]. The latter conjecture is a generalization of the classical conjecture of Bousfield and Ravenel for the stable homotopy category of CW-complexes, see [26,52]. The Telescope Conjecture can be formulated more generally for pretriangulated categories and more concretely for stable categories.

Let $\mathcal{C}$ be an additive category which admits all small coproducts and let $\mathcal{U}$ be a functorially finite subcategory of $\mathcal{C}$ with the property that any $\mathcal{U}$-epic admits a kernel in $\mathcal{C}$ and any $\mathcal{U}$-monic admits a cokernel in $\mathcal{C}$. Then by [19] the stable category $\mathcal{C}/\mathcal{U}$ is pretriangulated. For instance we can choose $\mathcal{C}$ to be Mod-Λ where Λ is an Artin algebra and $\mathcal{U}$ a functorially finite subcategory.

Telescope Conjecture for stable categories. *Assume that the stable category $\mathcal{C}/\mathcal{U}$ is compactly generated as a right triangulated category. If $(\underline{\mathcal{X}}, \underline{\mathcal{Y}})$ is a torsion pair of finite type in $\mathcal{C}/\mathcal{U}$, then $\underline{\mathcal{X}}$ is generated by compact objects from $\mathcal{C}/\mathcal{U}$.*

Remark 10.1. Note that this conjecture is equivalent to the telescope conjecture for derived categories of rings and stable categories of Frobenius exact categories, for instance $\underline{\text{Mod}}$-Λ where Λ is a self-injective Artin algebra. In the first case choose $\mathcal{C}$ to be the homotopically projective complexes of Λ-modules in the sense of Keller [39] and $\mathcal{U}$ is the subcategory of contractible complexes. Then the stable category $\mathcal{C}/\mathcal{U}$ is equivalent to the unbounded derived category $\mathbf{D}(\text{Mod-}\Lambda)$.

We say that a torsion pair $(\mathcal{X}/\mathcal{U}, \mathcal{Y}/\mathcal{U})$ of finite type in $\mathcal{C}/\mathcal{U}$ **satisfies the Telescope Conjecture** if the stable category $\mathcal{X}/\mathcal{U}$ is generated by compact objects from $\mathcal{C}/\mathcal{U}$. Using this terminology, Theorem 9.10 can be formulated as follows.

Theorem 10.2. *Let Λ be an Artin algebra and let $(\mathcal{X}, \mathcal{Y})$ be a projective cotorsion pair in* Mod-Λ*. Assume that $\mathcal{Y}$ is closed under coproducts. Then the following are equivalent.*

(i) *The torsion pair $(\underline{\mathcal{X}}, \underline{\mathcal{Y}})$ in $\underline{\text{Mod}}$-Λ satisfies the Telescope Conjecture.*
(ii) $\mathcal{X} = \varinjlim \mathcal{X}^{\text{fin}}$.

If (ii) *holds, then $\mathcal{Y}$ is definable and the map $(\mathcal{X}, \mathcal{Y}) \mapsto (\underline{\mathcal{X}}, \underline{\mathcal{Y}})$ gives a bijection between projective cotorsion pairs $(\mathcal{X}, \mathcal{Y})$ in* Mod-Λ *such that $\mathcal{X} = \varinjlim \mathcal{X}^{\text{fin}}$, and torsion pairs of finite type $(\underline{\mathcal{A}}, \underline{\mathcal{B}})$ in $\underline{\text{Mod}}$-Λ satisfying the Telescope Conjecture and such that the torsion class $\underline{\mathcal{A}}$ is triangulated.*

10.1. Self-injective algebras

We recall that a full thick subcategory $\mathcal{L}$ of a triangulated category $\mathcal{T}$ is called **localizing** if the inclusion $\mathcal{L} \hookrightarrow \mathcal{T}$ admits a right adjoint, i.e., $\mathcal{L}$ is the torsion class of a torsion pair $(\mathcal{L}, \mathcal{L}^{\perp})$ in $\mathcal{T}$. And a localizing subcategory $\mathcal{L}$ is called **smashing** if the right adjoint of the inclusion $\mathcal{L} \hookrightarrow \mathcal{T}$ preserves coproducts, i.e., the torsion pair $(\mathcal{L}, \mathcal{L}^{\perp})$ is of finite type. As a consequence of Corollary 9.8 and Theorem 10.2 we have the following result of Krause–Solberg [45].

Corollary 10.3 [45]. *If Λ is a self-injective Artin algebra, then the following are equivalent.*

(i) *The stable category $\underline{\text{Mod}}$-Λ satisfies the Telescope Conjecture.*
(ii) *If $\underline{\mathcal{X}}$ is a smashing subcategory of $\underline{\text{Mod}}$-Λ, then $\mathcal{X} = \varinjlim \mathcal{X}^{\text{fin}}$.*

(iii) *If $\mathcal{X}$ is a thick subcategory of* Mod-Λ*, then* $\mathcal{X} = \varinjlim \mathcal{X}^{\mathrm{fin}}$ *provided that any module admits a special right $\mathcal{X}$-approximation and the class of special right $\mathcal{X}$-approximations is closed under all small coproducts.*

If Λ is a self-injective algebra and $(\mathcal{X}, \mathcal{Y}, \mathcal{Z})$ is a cotorsion triple in Mod-Λ, then the cotorsion pairs $(\mathcal{X}, \mathcal{Y})$ and $(\mathcal{Y}, \mathcal{Z})$ are projective–injective. Hence the stable categories $\underline{\mathcal{X}}$, $\underline{\mathcal{Y}}$ and $\underline{\mathcal{Z}}$ are triangulated. Moreover the torsion pair $(\underline{\mathcal{X}}, \underline{\mathcal{Y}})$ is of finite type and the torsion pair $(\underline{\mathcal{Y}}, \underline{\mathcal{Z}})$ is of cofinite type. We are interested in finding necessary and sufficient conditions ensuring that the torsion pair $(\underline{\mathcal{Y}}, \underline{\mathcal{Z}})$ is of finite type and the torsion pair $(\underline{\mathcal{X}}, \underline{\mathcal{Y}})$ is of cofinite type. To proceed further we need the following.

Lemma 10.4 [20, Corollary 5.15]. *Let $\mathcal{C}$ be a compactly generated triangulated category and let $(\mathcal{X}, \mathcal{Y})$ be a torsion pair in $\mathcal{C}$. Then we have the following.*

(i) *$(\mathcal{X}, \mathcal{Y})$ is of cofinite type iff there exists a torsion triple $(\mathcal{W}, \mathcal{X}, \mathcal{Y})$ in $\mathcal{C}$ iff the inclusion $\mathcal{X} \hookrightarrow \mathcal{C}$ admits a left adjoint. If this is the case, then $\mathcal{X}$ is compactly generated.*
(ii) *$(\mathcal{X}, \mathcal{Y})$ is of finite type iff there exists a torsion triple $(\mathcal{X}, \mathcal{Y}, \mathcal{Z})$ in $\mathcal{C}$ iff the inclusion $\mathcal{Y} \hookrightarrow \mathcal{C}$ admits a right adjoint. If this holds, then the torsion-free class $\mathcal{Y}$ is compactly generated.*
(iii) *If $(\mathcal{X}, \mathcal{Y})$ is of finite type, then $(\mathcal{X}, \mathcal{Y})$ satisfies the Telescope Conjecture iff the torsion class $\mathcal{X}$ is compactly generated.*
(iv) *$\mathcal{C}$ satisfies the Telescope Conjecture, if any torsion pair of finite type is of cofinite type.*

Theorem 10.5. *Let Λ be a self-injective algebra and let $(\mathcal{X}, \mathcal{Y}, \mathcal{Z})$ be a cotorsion triple in* Mod-Λ*. Then the following are equivalent.*

(i) *The torsion pair $(\underline{\mathcal{X}}, \underline{\mathcal{Y}})$ is of cofinite type, i.e., $\underline{\mathcal{X}}$ is closed under products in* Mod-Λ.
(ii) *The torsion pair $(\underline{\mathcal{Y}}, \underline{\mathcal{Z}})$ is of finite type, i.e., $\underline{\mathcal{Z}}$ is closed under coproducts in* Mod-Λ.
(iii) *$\mathcal{X}^{\mathrm{fin}}$ or $\mathcal{Y}^{\mathrm{fin}}$ or $\mathcal{Z}^{\mathrm{fin}}$ is contravariantly, or equivalently covariantly, finite in* mod-Λ.
(iv) *There exists a cotorsion triple $(\mathcal{X}^{\mathrm{fin}}, \mathcal{Y}^{\mathrm{fin}}, \mathcal{Z}^{\mathrm{fin}})$ in* mod-Λ.
(v) *There exists a torsion triple $(\underline{\mathcal{X}}^{\mathrm{b}}, \underline{\mathcal{Y}}^{\mathrm{b}}, \underline{\mathcal{Z}}^{\mathrm{b}})$ in* $\underline{\mathrm{mod}}$-Λ.

If one of the above conditions holds, then $\mathcal{X} = \varinjlim \mathcal{X}^{\mathrm{fin}}$, $\mathcal{Y} = \varinjlim \mathcal{Y}^{\mathrm{fin}}$ and $\mathcal{Z} = \varinjlim \mathcal{Z}^{\mathrm{fin}}$. Moreover the stable categories $\underline{\mathcal{X}}$, $\underline{\mathcal{Y}}$ and $\underline{\mathcal{Z}}$ are compactly generated and the maps

$$\mathcal{F} \mapsto \left(\mathcal{F}, \mathcal{F}^{\perp}, \mathcal{F}^{\perp\perp}\right) \quad \text{and} \quad (\mathcal{F}, \mathcal{G}, \mathcal{H}) \mapsto \left(\mathcal{X} := \varinjlim \mathcal{F},\ \mathcal{Y} := \varinjlim \mathcal{G},\ \mathcal{Z} := \varinjlim \mathcal{H}\right)$$

give bijections between contravariantly finite resolving subcategories $\mathcal{F}$ of mod-Λ*, cotorsion triples $(\mathcal{F}, \mathcal{G}, \mathcal{H})$ in* mod-Λ*, and cotorsion triples $(\mathcal{X}, \mathcal{Y}, \mathcal{Z})$ in* Mod-Λ *such that the torsion pair $(\underline{\mathcal{Y}}, \underline{\mathcal{Z}})$ is of finite type or equivalently the torsion pair $(\underline{\mathcal{X}}, \underline{\mathcal{Y}})$ is of cofinite type.*

Proof. (i) $\Rightarrow$ (ii). Assume that the torsion pair $(\underline{\mathcal{X}}, \underline{\mathcal{Y}})$ is of cofinite type. Then by Lemma 10.4 it follows that $\underline{\mathcal{X}}$ is compactly generated. By Theorem 9.4 and Corollary 9.8 we have $\mathcal{X} = \varinjlim \mathcal{F}$ where $\underline{\mathcal{F}} = \underline{\mathcal{X}}^{\mathrm{b}}$ and $\mathcal{Z} = \mathcal{F}^{\perp\perp}$. Since $\underline{\mathcal{X}}$ is closed under products in

$\underline{\mathrm{Mod}}$-Λ, the same holds for $\mathcal{X}$ in Mod-Λ. By Lemma 7.10 it follows that $\mathcal{F}$ is covariantly finite and then by Corollary 9.5 it follows that $\mathcal{F}$ is contravariantly finite. Hence there exists a cotorsion triple $(\mathcal{F}, \mathcal{G}, \mathcal{H})$ in mod-Λ and, as in the proof of Corollary 9.6, we infer that $\mathcal{Z} = \varinjlim \mathcal{H}$. Hence $\mathcal{Z}$ is closed under coproducts.

(ii) $\Rightarrow$ (i). If $\underline{\mathcal{Z}}$ is closed under coproducts, then by [22, Proposition IV.1.11] it follows that $\underline{\mathcal{X}}$ is compactly generated and we have a torsion pair $(\underline{\mathcal{X}}^{\mathrm{b}}, \underline{\mathcal{Y}}^{\mathrm{b}})$ in $\underline{\mathrm{mod}}$-Λ. This implies that $\mathcal{X}^{\mathrm{fin}}$ is contravariantly finite, hence covariantly finite by Corollary 9.5. Then, by Lemma 7.10, $\varinjlim \mathcal{X}^{\mathrm{fin}}$ is closed under products and the assertion follows since $\mathcal{X} = \varinjlim \mathcal{X}^{\mathrm{fin}}$ by Corollaries 9.6 and 9.7.

The equivalences (iii) $\Leftrightarrow$ (iv) and (i) $\Leftrightarrow$ (ii) follow from [22, Proposition IV.4.11 and Corollaries VI.4.10 and VI.4.11]. The remaining assertions follow from Corollary 9.8. □

We say that a pretriangulated category $\mathcal{C}$ with all small products and coproducts satisfies the **strong Telescope Conjecture** if any torsion pair of finite type $\mathcal{C}$ is of cofinite type. By part (iv) of Lemma 10.4 it follows that the strong Telescope Conjecture implies the Telescope Conjecture.

Theorem 10.6. *For a self-injective algebra Λ, the following are equivalent.*

- (i) *The stable category $\underline{\mathrm{Mod}}$-Λ satisfies the strong Telescope Conjecture.*
- (ii) *The stable category $\underline{\mathrm{Mod}}$-Λ satisfies the Telescope Conjecture and the map $\mathcal{F} \mapsto \varinjlim \mathcal{F}$ gives a bijection between the set* Thick *of thick subcategories of* mod-Λ *and the set* ThDef *of thick definable subcategories of* Mod-Λ.

Proof. (i) $\Rightarrow$ (ii). As already mentioned $\underline{\mathrm{Mod}}$-Λ satisfies the Telescope Conjecture. Let $\mathcal{F}$ be a thick subcategory of mod-Λ. Then by Theorem 9.4, we have a torsion pair of finite type $(\underline{\mathcal{X}}, \underline{\mathcal{Y}})$ in $\underline{\mathrm{Mod}}$-Λ, where $\mathcal{X} = \varinjlim \mathcal{F}$. By hypothesis, $\mathcal{X}$ is closed under products and therefore $\mathcal{F}$ is covariantly finite in mod-Λ by Lemma 7.10. Then $\varinjlim \mathcal{F}$ is definable by [43], and thick since the stable category $\overline{\varinjlim \mathcal{F}}$ is thick. Hence we have a map Thick $\mapsto$ ThDef, $\mathcal{F} \mapsto \varinjlim \mathcal{F}$ which is clearly injective. Now let $\mathcal{D}$ be a thick definable subcategory of Mod-Λ. Then by [46, Theorem 2.6 and Corollary 4.5] it follows that $\mathcal{D}$ is contravariantly finite. Since $\mathcal{D}$ is closed under filtered colimits, this implies that there exists a cotorsion pair $(\mathcal{D}, \mathcal{C})$ in Mod-Λ. Also thickness of $\mathcal{D}$ implies that $\underline{\mathcal{D}}$ is triangulated and therefore the cotorsion pair $(\mathcal{D}, \mathcal{C})$ is projective–injective by Theorem 9.3. We infer that $(\underline{\mathcal{D}}, \underline{\mathcal{C}})$ is a torsion pair in $\underline{\mathrm{Mod}}$-Λ which is clearly of cofinite type. Then, by Lemma 10.4, $\underline{\mathcal{D}}$ is compactly generated and there exists a torsion triple $(\underline{\mathcal{B}}, \underline{\mathcal{D}}, \underline{\mathcal{C}})$ in $\underline{\mathrm{Mod}}$-Λ. Since the torsion pair $(\underline{\mathcal{B}}, \underline{\mathcal{D}})$ is of finite type, by hypothesis, $\underline{\mathcal{B}}$ is closed under products. Then Theorem 10.5 implies that $\underline{\mathcal{C}}$ is closed under coproducts, i.e., the torsion pair $(\underline{\mathcal{D}}, \underline{\mathcal{C}})$ is of finite type. By Theorem 10.2 we infer that $\mathcal{D} = \varinjlim \mathcal{D}^{\mathrm{fin}}$ where the subcategory $\mathcal{D}^{\mathrm{fin}}$ is thick. Hence the map Thick $\mapsto$ ThDef is surjective.

(ii) $\Rightarrow$ (i). If $(\underline{\mathcal{X}}, \underline{\mathcal{Y}})$ is a torsion pair of finite type in $\underline{\mathrm{Mod}}$-Λ, then the conditions in (ii) imply that $\mathcal{X} = \varinjlim \mathcal{X}^{\mathrm{fin}}$. Since $\mathcal{X}^{\mathrm{fin}}$ is thick, we infer that $\mathcal{X}$ is definable. It follows that $\mathcal{X}$, hence $\underline{\mathcal{X}}$, is closed under products and therefore the torsion pair $(\underline{\mathcal{X}}, \underline{\mathcal{Y}})$ is of cofinite type. □

Let G a finite p-group, where p is a prime, and let k be a field. Also let $\mathcal{V}_G$ be the maximal ideal spectrum of the cohomology ring $H^*(G,k)$ of G. We let Var be the collection of all closed homogeneous subvarieties of $\mathcal{V}_G$ which are closed under subvarieties and finite unions. For $V \in \mathsf{Var}$, let $\mathcal{C}(V)$ be the full subcategory of $\underline{\mathrm{mod}}$-kG consisting of all modules M whose variety $V_G(M)$, in the sense of [23], is contained in V. Then by [23] and Theorem 10.6 we have the following consequence.

Corollary 10.7. *If* $\underline{\mathrm{Mod}}$-kG *satisfies the strong Telescope Conjecture, then the map* $\mathcal{V} \mapsto \varinjlim \mathcal{C}(\mathcal{V})$ *gives a bijection between subsets of* Var *and the class* ThDef *of thick definable subcategories of* Mod-kG.

10.2. Virtually Gorenstein algebras

In this section we study the Telescope Conjecture for the stable category of a virtually Gorenstein algebra. We begin with the following result which, in particular, shows that the class of projective cotorsion pairs of finite type in Mod-Λ is a set which is a complete lattice under the order relation $(\mathcal{X}_1,\mathcal{Y}_1) \prec (\mathcal{X}_2,\mathcal{Y}_2)$ iff $\mathcal{X}_1 \subseteq \mathcal{X}_2$. We say that a full triangulated subcategory $\mathcal{S}$ of a compactly generated triangulated category $\mathcal{T}$ is **definable** if $\mathcal{S}$ is closed under products and coproducts in $\mathcal{T}$.

Theorem 10.8. *Let Λ be a virtually Gorenstein algebra.*

(i) *The map* $\Phi : (\mathcal{X},\mathcal{Y}) \mapsto \underline{\mathcal{X}}$ *induces a bijection between the class* $\mathfrak{C}$ *of projective cotorsion pairs of finite type in* Mod-Λ *and the class* $\mathfrak{S}_{\mathrm{CM}}$ *of smashing subcategories of* $\underline{\mathrm{CM}}(\mathbf{P}_\Lambda)$.
(ii) *The map* $\Psi : (\mathcal{X},\mathcal{Y}) \mapsto \underline{\mathcal{Y}}_{\mathrm{CM}} := \underline{\mathrm{CM}}(\mathbf{P}_\Lambda) \cap \underline{\mathcal{Y}}$ *induces a bijection between* $\mathfrak{C}$ *and the class* $\mathfrak{D}_{\mathrm{CM}}$ *of definable compactly generated subcategories of* $\underline{\mathrm{CM}}(\mathbf{P}_\Lambda)$.

In particular the classes $\mathfrak{C}$, $\mathfrak{D}_{\mathrm{CM}}$ *and* $\mathfrak{S}_{\mathrm{CM}}$ *are sets which are complete lattices.*

Proof. (i) If $(\mathcal{X},\mathcal{Y})$ lies in $\mathfrak{C}$, then $(\underline{\mathcal{X}},\underline{\mathcal{Y}})$ is a torsion pair of finite type in $\underline{\mathrm{Mod}}$-Λ and $\underline{\mathcal{X}}$ is triangulated, in particular $\mathcal{X} \subseteq \mathrm{CM}(\mathbf{P}_\Lambda)$. It is easy to see that the functor $\mathbf{R}_{\mathcal{X}}\mathbf{i}_{\mathrm{CM}} : \underline{\mathrm{CM}}(\mathbf{P}_\Lambda) \to \underline{\mathcal{X}}$ is the coreflection of $\underline{\mathrm{CM}}(\mathbf{P}_\Lambda)$ in $\underline{\mathcal{X}}$ and preserves coproducts. Hence $(\underline{\mathcal{X}}, \underline{\mathcal{Y}} \cap \underline{\mathrm{CM}}(\mathbf{P}_\Lambda))$ is a torsion pair of finite type in $\underline{\mathrm{CM}}(\mathbf{P}_\Lambda)$, i.e., $\underline{\mathcal{X}}$ is a smashing subcategory of $\underline{\mathrm{CM}}(\mathbf{P}_\Lambda)$. If $(\underline{\mathcal{X}},\underline{\mathcal{Y}})$ is a torsion pair of finite type in $\underline{\mathrm{CM}}(\mathbf{P}_\Lambda)$, then the coreflection $\mathbf{R}_{\mathcal{X}} : \underline{\mathrm{CM}}(\mathbf{P}_\Lambda) \to \underline{\mathcal{X}}$ preserves coproducts and it is easy to see that the functor $\mathbf{R}_{\mathcal{X}}\mathbf{R}_{\mathrm{CM}} : \underline{\mathrm{Mod}}$-$\Lambda \to \underline{\mathcal{X}}$ is the coreflection of $\underline{\mathrm{Mod}}$-Λ in $\underline{\mathcal{X}}$. Therefore, by [22], we obtain a torsion pair $(\underline{\mathcal{X}},\underline{\mathcal{Y}}')$ in $\underline{\mathrm{Mod}}$-Λ and a projective cotorsion pair $(\mathcal{X},\mathcal{Y}')$ in Mod-Λ. Since Λ is virtually Gorenstein, $\mathbf{R}_{\mathcal{X}}\mathbf{R}_{\mathrm{CM}}$ preserves coproducts and therefore the torsion pair $(\underline{\mathcal{X}},\underline{\mathcal{Y}}')$ in $\underline{\mathrm{Mod}}$-Λ is of finite type, i.e., $\underline{\mathcal{Y}}'$, or equivalently $\mathcal{Y}'$, is closed under coproducts. Hence the cotorsion pair $(\mathcal{X},\mathcal{Y}')$ in Mod-Λ is of finite type. Clearly the map $\Phi : \mathfrak{C} \to \mathfrak{S}_{\mathrm{CM}}, (\mathcal{X},\mathcal{Y}) \mapsto \underline{\mathcal{X}}$ is a bijection.

(ii) If $(\mathcal{X},\mathcal{Y}) \in \mathfrak{C}$, then $\underline{\mathcal{Y}}_{\mathrm{CM}}$ is a triangulated subcategory of $\underline{\mathrm{CM}}(\mathbf{P}_\Lambda)$ since $\mathcal{Y}$ is thick. By Lemma 10.4, $\underline{\mathcal{Y}}_{\mathrm{CM}}$ is compactly generated and definable. If $\underline{\mathcal{Y}}'$ lies in $\mathfrak{D}_{\mathrm{CM}}$, then the inclusion $\underline{\mathcal{Y}}' \hookrightarrow \underline{\mathrm{CM}}(\mathbf{P}_\Lambda)$ admits a left adjoint [51]. Hence we have a torsion pair of finite type $(\underline{\mathcal{X}}, \underline{\mathcal{Y}}')$ in $\underline{\mathrm{CM}}(\mathbf{P}_\Lambda)$, i.e., $\underline{\mathcal{X}}$ is smashing in $\underline{\mathrm{CM}}(\mathbf{P}_\Lambda)$. Then by (i) there exists a projective cotorsion pair of finite type $(\mathcal{X}, \mathcal{Y})$ in Mod-Λ. Clearly $\Psi : \mathfrak{C} \to \mathfrak{D}_{\mathrm{CM}}, (\mathcal{X}, \mathcal{Y}) \mapsto \underline{\mathcal{Y}}_{\mathrm{CM}}$ is a bijection. The last assertion follows from [44]. □

The following comparison for the Telescope Conjecture follows from Theorems 9.3, 9.10 and 10.8.

Corollary 10.9. *Let Λ be a virtually Gorenstein algebra. Then the following are equivalent.*

(i) *The triangulated category $\underline{\mathrm{CM}}(\mathbf{P}_\Lambda)$ satisfies the Telescope Conjecture.*
(ii) $\underline{\mathrm{Mod}}$-Λ *satisfies the Telescope Conjecture for torsion pairs with triangulated torsion class.*
(iii) *For any projective cotorsion pair $(\mathcal{X}, \mathcal{Y})$ of finite type in* Mod-Λ, *we have $\mathcal{X} = \varinjlim \mathcal{X}^{\mathrm{fin}}$.*

If (i) *holds, then the map $\mathcal{F} \mapsto \varinjlim \mathcal{F}$, respectively $\mathcal{F} \mapsto \mathcal{F}^{\perp} \cap \underline{\mathrm{CM}}(\mathbf{P}_\Lambda)$, gives a bijection between the complete lattice of projectively thick subcategories of* $\mathrm{CM}(\Lambda)$ *and the complete lattice $\mathfrak{S}_{\mathrm{CM}}$, respectively $\mathfrak{D}_{\mathrm{CM}}$, of smashing, respectively definable and compactly generated, subcategories of $\underline{\mathrm{CM}}(\mathbf{P}_\Lambda)$.*

The following two results show that validity of the Telescope Conjecture for the stable category of a virtually Gorenstein algebra is invariant under of derived equivalences and stable equivalences of Morita type.

Theorem 10.10. *Let Λ, Γ be derived equivalent finite dimensional k-algebras over a field k. Assume that Λ is virtually Gorenstein. Then the stable category $\underline{\mathrm{Mod}}$-Λ satisfies the Telescope Conjecture for torsion pairs of finite type such that the torsion class is triangulated iff so does $\underline{\mathrm{Mod}}$-Γ.*

Proof. The algebra Γ is virtually Gorenstein by Theorem 8.11. Assuming that the condition holds for Γ, let $(\underline{\mathcal{X}}, \underline{\mathcal{Y}})$ be a torsion pair of finite type in $\underline{\mathrm{Mod}}$-Λ such that $\underline{\mathcal{X}}$ is triangulated. Then $\underline{\mathcal{X}} \subseteq \underline{\mathrm{CM}}(\mathbf{P}_\Lambda)$ and we obtain a torsion pair $(\underline{\mathcal{X}}, \underline{\mathcal{Y}}')$ in $\underline{\mathrm{CM}}(\mathbf{P}_\Lambda)$, where $\underline{\mathcal{Y}}' = \underline{\mathcal{Y}} \cap \underline{\mathrm{CM}}(\mathbf{P}_\Lambda)$, which clearly is of finite type. By Theorem 8.11, it follows that a given derived equivalence between Λ and Γ induces a triangle equivalence $F : \underline{\mathrm{CM}}(\mathbf{P}_\Lambda) \xrightarrow{\approx} \underline{\mathrm{CM}}(\mathbf{P}_\Gamma)$. Then $(\underline{\mathcal{W}}, \underline{\mathcal{Z}})$ is a torsion pair of finite type in $\underline{\mathrm{CM}}(\mathbf{P}_\Gamma)$, where $\underline{\mathcal{W}} := F(\underline{\mathcal{X}})$ and $\underline{\mathcal{Z}} := F(\underline{\mathcal{Y}}')$. As in the proof of Theorem 10.8, this torsion pair extends to a torsion pair $(\underline{\mathcal{W}}, \underline{\mathcal{Z}}')$ of finite type in $\underline{\mathrm{Mod}}$-Γ, where $\underline{\mathcal{W}}$ is triangulated. By hypothesis $\underline{\mathcal{W}}$ is compactly generated and therefore so is $\underline{\mathcal{X}}$. Hence the torsion pair $(\underline{\mathcal{X}}, \underline{\mathcal{Y}})$ in $\underline{\mathrm{Mod}}$-Λ satisfies the Telescope Conjecture. □

Using Theorem 8.13, a similar argument as in the proof of Theorem 10.10 implies the following.

Theorem 10.11. *Let Λ, Γ be Artin algebras which are stably equivalent of Morita type. Assume that Λ is virtually Gorenstein. Then the stable category $\underline{\mathrm{Mod}}$-Λ satisfies the Telescope Conjecture for torsion pairs of finite type such that the torsion class is triangulated iff so does $\underline{\mathrm{Mod}}$-Γ.*

Since a representation-finite algebra is virtually Gorenstein and obviously satisfies the Telescope Conjecture for torsion pairs with triangulated torsion class, we have the following consequence.

Corollary 10.12. *Let Λ be a finite dimensional k-algebra of finite representation type over a field k. If Γ is a k-algebra derived equivalent to Λ, then the stable category $\underline{\mathrm{Mod}}$-Γ satisfies the Telescope Conjecture for torsion pairs such that the torsion class is triangulated.*

If Λ is self-injective and $(\mathcal{X}, \mathcal{Y})$ is a cotorsion pair in Mod-Λ, then by a result of Krause–Solberg [45, Theorem 7.6] the subcategories $\mathcal{X}$ and $\mathcal{Y}$ are closed under filtered colimits. The following observation generalizes the result of Krause–Solberg from self-injective to virtually Gorenstein algebras.

Theorem 10.13. *Let Λ be a virtually Gorenstein algebra and let $(\mathcal{X}, \mathcal{Y})$ be a projective cotorsion pair in* Mod-Λ. *Then for the following statements*

(i) *the torsion pair $(\underline{\mathcal{X}}, \underline{\mathcal{Y}})$ in $\underline{\mathrm{Mod}}$-Λ is of finite type*;
(ii) *$\mathcal{X}$ and $\mathcal{Y} \cap \mathrm{CM}(\mathbf{P}_\Lambda)$ are closed under filtered colimits*;

we have (i) $\Rightarrow$ (ii). *If $\underline{\mathcal{X}}$ is compactly generated then they are equivalent.*

Proof. By Theorem 8.2 we have $\underline{\mathrm{CM}}(\mathbf{P}_\Lambda)^{\mathrm{b}} = \underline{\mathrm{CM}}(\Lambda)$. This enables us to consider the functors

$$\mathsf{H}_{\mathrm{CM}} : \underline{\mathrm{CM}}(\mathbf{P}_\Lambda) \to \mathrm{Mod}\text{-}\underline{\mathrm{CM}}(\Lambda), \quad \mathsf{H}_{\mathrm{CM}}(\underline{X}) = \mathrm{Hom}(-, \underline{X})|_{\underline{\mathrm{CM}}(\Lambda)},$$
$$\mathsf{T}_{\mathrm{CM}} : \mathrm{CM}(\mathbf{P}_\Lambda) \to \mathrm{Mod}\text{-}\,\mathrm{CM}(\Lambda^{\mathrm{op}})^{\mathrm{op}}, \quad \mathsf{T}_{\mathrm{CM}}(X) = X \otimes_\Lambda -|_{\mathrm{CM}(\Lambda^{\mathrm{op}})}$$

as in [45, Section 7]. It is easy to see that the arguments of Krause–Solberg work in our setting, if we replace Mod-Λ with $\mathrm{CM}(\mathbf{P}_\Lambda)$. We leave to the reader to fill in the details. □

11. Algebras with finite right self-injective dimension

In this section we study Artin algebras with finite right self-injective dimension. More precisely we analyze the consequences of the assumption $\mathrm{id}\,\Lambda_\Lambda < \infty$ on the structure of Cohen–Macaulay modules in connection with the virtual Gorensteinness property. In particular we are interested in finding conditions ensuring that Λ is Gorenstein provided that $\mathrm{id}\,\Lambda_\Lambda < \infty$. Our motivation here emerges from the well-known **Gorenstein Symmetry Conjecture**, (GSC) for short, see [13,22]:

(GSC) If $\operatorname{id}\Lambda_\Lambda < \infty$, then $\operatorname{id}{}_\Lambda\Lambda < \infty$, i.e., Λ is Gorenstein.

Recall that, by Lemma 5.4, the inclusion $\mathbf{i}_{\mathrm{CM}} : \underline{\mathrm{CM}}(\mathbf{P}_\Lambda) \hookrightarrow \underline{\mathrm{Mod}}\text{-}\Lambda$ admits a left adjoint $\mathbf{L}_{\mathrm{CM}} : \underline{\mathrm{Mod}}\text{-}\Lambda \to \underline{\mathrm{CM}}(\mathbf{P}_\Lambda)$ which preserves compact objects. We denote by $\mathbf{L}^{\mathrm{b}}_{\mathrm{CM}} : \underline{\mathrm{mod}}\text{-}\Lambda \to \underline{\mathrm{CM}}(\mathbf{P}_\Lambda)^{\mathrm{b}}$ the induced right exact functor. If $\operatorname{id}\Lambda_\Lambda < \infty$, there is an alternative description of $\mathbf{L}_{\mathrm{CM}}$ which will be useful later. We use the adjoint pair $(\underline{\mathsf{Sp}}, \underline{\mathsf{Z}}) : \underline{\mathrm{Mod}}\text{-}\Lambda \leftrightarrows \mathcal{H}_{\mathsf{Ac}}(\mathbf{P}_\Lambda)$ from 4.2. Note that if $\operatorname{id}\Lambda_\Lambda < \infty$, then, by Corollary 4.8, the costabilization functor $\underline{\mathsf{Z}} : \mathcal{H}_{\mathsf{Ac}}(\mathbf{P}_\Lambda) \to \underline{\mathrm{Mod}}\text{-}\Lambda$ is fully faithful and admits a factorization $\underline{\mathsf{Z}} = \mathbf{i}_{\mathrm{CM}}\underline{\mathsf{Z}}_{\mathrm{CM}}$, where $\underline{\mathsf{Z}}_{\mathrm{CM}} : \mathcal{H}_{\mathsf{Ac}}(\mathbf{P}_\Lambda) \to \underline{\mathrm{CM}}(\mathbf{P}_\Lambda)$ is a triangle equivalence with quasi-inverse the functor $\underline{\mathsf{Sp}\mathbf{i}_{\mathrm{CM}}}$. Recall from Section 4 that if $F : \underline{\mathrm{Mod}}\text{-}\Lambda \to \mathcal{T}$ is a right exact functor, where $\mathcal{T}$ is a triangulated category, then $F^* : \mathcal{T}_r(\underline{\mathrm{Mod}}\text{-}\Lambda) \to \mathcal{T}$ denotes the unique exact extension of F through the right projective stabilization functor $\mathsf{P}_r : \underline{\mathrm{Mod}}\text{-}\Lambda \to \mathcal{T}_r(\underline{\mathrm{Mod}}\text{-}\Lambda)$.

Lemma 11.1. *Assume that* $\operatorname{id}\Lambda_\Lambda < \infty$. *Then there exist natural isomorphisms of functors*

$$\mathbf{L}_{\mathrm{CM}} \cong \underline{\mathsf{Z}}_{\mathrm{CM}}\underline{\mathsf{Sp}} : \underline{\mathrm{Mod}}\text{-}\Lambda \to \underline{\mathrm{CM}}(\mathbf{P}_\Lambda) \quad \textit{and} \quad \mathbf{L}^*_{\mathrm{CM}} \cong \underline{\mathsf{Z}}_{\mathrm{CM}}\underline{\mathsf{Sp}}^* : \mathcal{T}_r(\underline{\mathrm{Mod}}\text{-}\Lambda) \to \underline{\mathrm{CM}}(\mathbf{P}_\Lambda).$$

Proof. Let A be in $\mathrm{Mod}\text{-}\Lambda$ and X in $\mathrm{CM}(\mathbf{P}_\Lambda)$. Setting $Q^\bullet := \underline{\mathsf{Sp}\mathbf{i}_{\mathrm{CM}}}(\underline{X}) \in \mathcal{H}_{\mathsf{Ac}}(\mathbf{P}_\Lambda)$, hence $\underline{\mathsf{Z}}_{\mathrm{CM}}(Q^\bullet) = \underline{X}$, and using that $\underline{\mathsf{Z}}_{\mathrm{CM}}$ is an equivalence, we have the following natural isomorphisms:

$$\begin{aligned}\underline{\mathrm{Hom}}_\Lambda\big(\underline{A}, \mathbf{i}_{\mathrm{CM}}(\underline{X})\big) &\cong \underline{\mathrm{Hom}}_\Lambda\big(\underline{A}, \mathbf{i}_{\mathrm{CM}}\underline{\mathsf{Z}}_{\mathrm{CM}}\big(Q^\bullet\big)\big) \cong \underline{\mathrm{Hom}}_\Lambda\big(\underline{A}, \underline{\mathsf{Z}}\big(Q^\bullet\big)\big) \cong \mathrm{Hom}\big(\underline{\mathsf{Sp}}\big(\underline{A}\big), Q^\bullet\big)\\ &\cong \underline{\mathrm{Hom}}_\Lambda\big(\underline{\mathsf{Z}}_{\mathrm{CM}}\underline{\mathsf{Sp}}(\underline{A}), \underline{\mathsf{Z}}_{\mathrm{CM}}\big(Q^\bullet\big)\big) \cong \underline{\mathrm{Hom}}_\Lambda\big(\underline{\mathsf{Z}}_{\mathrm{CM}}\underline{\mathsf{Sp}}(\underline{A}), \underline{X}\big)\end{aligned}$$

which show that the functor $\underline{\mathsf{Z}}_{\mathrm{CM}}\underline{\mathsf{Sp}}$ is the left adjoint of $\mathbf{i}_{\mathrm{CM}}$. Therefore $\mathbf{L}_{\mathrm{CM}} \cong \underline{\mathsf{Z}}_{\mathrm{CM}}\underline{\mathsf{Sp}}$. Since the functor $\underline{\mathsf{Z}}_{\mathrm{CM}} : \mathcal{H}_{\mathsf{Ac}}(\mathbf{P}_\Lambda) \to \underline{\mathrm{CM}}(\mathbf{P}_\Lambda)$ is exact, using that $\Omega^{-n}|_{\underline{\mathrm{CM}}(\mathbf{P}_\Lambda)} = \Sigma^n_{\mathbf{P}}|_{\underline{\mathrm{CM}}(\mathbf{P}_\Lambda)}$, $\forall n \geqslant 0$, we have the following isomorphisms, for any object $(\underline{A}, n) \in \mathcal{T}_r(\underline{\mathrm{Mod}}\text{-}\Lambda)$:

$$\begin{aligned}\mathbf{L}^*_{\mathrm{CM}}(\underline{A}, n) &\cong \Sigma^n_{\mathbf{P}}\mathbf{L}_{\mathrm{CM}}(\underline{A}) \cong \Sigma^n_{\mathbf{P}}\big(\underline{\mathsf{Z}}_{\mathrm{CM}}\big(\underline{\mathsf{Sp}}(\underline{A})\big)\big) \cong \underline{\mathsf{Z}}_{\mathrm{CM}}\big(\underline{\mathsf{Sp}}(\underline{A})[n]\big) \cong \underline{\mathsf{Z}}_{\mathrm{CM}}\big(\underline{\mathsf{Sp}}^*(\underline{A}, n)\big)\\ &\cong \underline{\mathsf{Z}}_{\mathrm{CM}}\underline{\mathsf{Sp}}^*(\underline{A}, n).\end{aligned}$$

Consequently we have a natural isomorphism of functors $\mathbf{L}^*_{\mathrm{CM}} \cong \underline{\mathsf{Z}}_{\mathrm{CM}}\underline{\mathsf{Sp}}^*$. □

As a direct consequence we have the following useful result.

Lemma 11.2. *Assume that* $\operatorname{id}\Lambda_\Lambda < \infty$. *Then the natural isomorphism of functors* $\mathbf{L}^*_{\mathrm{CM}} \cong \underline{\mathsf{Z}}_{\mathrm{CM}}\underline{\mathsf{Sp}}^* : \mathcal{T}_r(\underline{\mathrm{Mod}}\text{-}\Lambda) \to \underline{\mathrm{CM}}(\mathbf{P}_\Lambda)$ *induces a natural isomorphism of functors* $(\mathbf{L}^{\mathrm{b}}_{\mathrm{CM}})^* \cong \underline{\mathsf{Z}}^{\mathrm{b}}_{\mathrm{CM}}\underline{\mathsf{Sp}}^{\mathrm{b}} : \mathcal{T}_r(\underline{\mathrm{mod}}\text{-}\Lambda) \to \underline{\mathrm{CM}}(\mathbf{P}_\Lambda)^{\mathrm{b}}$. *Moreover* $(\mathbf{L}^{\mathrm{b}}_{\mathrm{CM}})^* \cong \mathbf{L}^*_{\mathrm{CM}}|_{\mathcal{T}_r(\underline{\mathrm{mod}}\text{-}\Lambda)}$ *and there exists a* *commutative diagram of triangulated categories where all the involved functors are triangle equivalences.*

$$\begin{array}{ccc} \mathcal{T}_r(\underline{\mathrm{mod}}\text{-}\Lambda) & \xrightarrow{(\mathbf{L}^{\mathrm{b}}_{\mathrm{CM}})^*} & \underline{\mathrm{CM}}(\mathbf{P}_\Lambda)^{\mathrm{b}} \\ {\scriptstyle \widehat{\mathsf{N}}^+}\Big\downarrow{\scriptstyle \approx} & & {\scriptstyle \mathsf{N}^+}\Big\downarrow{\scriptstyle \approx} \\ \mathcal{T}_r(\overline{\mathrm{mod}}\text{-}\Lambda) & \xrightarrow{(\mathbf{L}^{\mathrm{b}}_{\mathrm{CoCM}})^*} & \overline{\mathrm{CoCM}}(\mathbf{I}_\Lambda)^{\mathrm{b}}. \end{array}$$

Proof. Since the reflection functor $\mathbf{L}_{\mathrm{CM}} : \underline{\mathrm{Mod}}\text{-}\Lambda \to \underline{\mathrm{CM}}(\mathbf{P}_\Lambda)$ restricts to a right exact functor $\mathbf{L}_{\mathrm{CM}} : \underline{\mathrm{mod}}\text{-}\Lambda \to \underline{\mathrm{CM}}(\mathbf{P}_\Lambda)^{\mathrm{b}}$ it follows directly that

$$\left(\mathbf{L}^{\mathrm{b}}_{\mathrm{CM}}\right)^* \cong \mathbf{L}^*_{\mathrm{CM}}|_{\mathcal{T}_r(\underline{\mathrm{mod}}\text{-}\Lambda)} : \mathcal{T}_r(\underline{\mathrm{mod}}\text{-}\Lambda) \to \underline{\mathrm{CM}}(\mathbf{P}_\Lambda)^{\mathrm{b}}.$$

Hence by Lemma 11.1 we have $(\mathbf{L}^{\mathrm{b}}_{\mathrm{CM}})^* \cong (\underline{\mathsf{Z}}_{\mathrm{CM}}\underline{\mathsf{Sp}}^*)|_{\mathcal{T}_r(\underline{\mathrm{mod}}\text{-}\Lambda)}$. Since, by Proposition 4.5, $\underline{\mathsf{Sp}}^{\mathrm{b}} = \underline{\mathsf{Sp}}^*|_{\mathcal{T}_r(\underline{\mathrm{mod}}\text{-}\Lambda)}$ is a triangle equivalence and since, by Corollary 4.8, $\underline{\mathsf{Z}}^{\mathrm{b}}_{\mathrm{CM}} = \underline{\mathsf{Z}}_{\mathrm{CM}}|_{\mathcal{H}^{\mathrm{b}}_{\mathrm{Ac}}(\mathbf{P}_\Lambda)}$ is a triangle equivalence, we infer that

$$\left(\mathbf{L}^{\mathrm{b}}_{\mathrm{CM}}\right)^* \cong \underline{\mathsf{Z}}^{\mathrm{b}}_{\mathrm{CM}}\underline{\mathsf{Sp}}^{\mathrm{b}} : \mathcal{T}_r(\underline{\mathrm{mod}}\text{-}\Lambda) \xrightarrow{\approx} \underline{\mathrm{CM}}(\mathbf{P}_\Lambda)^{\mathrm{b}}$$

is a triangle equivalence. Finally since the right exact reflection functor $\mathbf{L}_{\mathrm{CoCM}} : \overline{\mathrm{Mod}}\text{-}\Lambda \to \overline{\mathrm{CoCM}}(\mathbf{I}_\Lambda)$ preserves compact objects, it induces a right exact functor $\mathbf{L}_{\mathrm{CoCM}} : \overline{\mathrm{mod}}\text{-}\Lambda \to \overline{\mathrm{CoCM}}(\mathbf{I}_\Lambda)^{\mathrm{b}}$. Then the commutativity of the diagram and the claim that the vertical arrows are triangle equivalences follow from Proposition 5.6. Since $(\mathbf{L}^{\mathrm{b}}_{\mathrm{CM}})^*$ is a triangle equivalence, so is $(\mathbf{L}^{\mathrm{b}}_{\mathrm{CoCM}})^*$. □

We know that in general it holds $\mathcal{I}^{<\infty}_\Lambda \subseteq (\mathfrak{I}^{\prec\propto}_\Lambda)^{\mathrm{fin}}$ and that the inclusion $\mathbf{I}^{<\infty}_\Lambda \subseteq \mathfrak{I}^{\prec\propto}_\Lambda$ is an equality iff Λ is Gorenstein. The next result gives several characterizations of the Artin algebras with finite right self-injective dimension in terms of properties of modules of (virtually) finite injective dimension. As a consequence the inclusion $\mathcal{I}^{<\infty}_\Lambda \subseteq (\mathfrak{I}^{\prec\propto}_\Lambda)^{\mathrm{fin}}$ is an equality if and only if $\mathrm{id}\,\Lambda_\Lambda < \infty$.

Theorem 11.3. *The following are equivalent.*

(i) $\mathrm{id}\,\Lambda_\Lambda < \infty$.
(ii) $(\mathfrak{I}^{\prec\propto}_\Lambda)^{\mathrm{fin}} = \mathcal{I}^{<\infty}_\Lambda$.
(iii) $(\mathfrak{P}^{\prec\propto}_{\Lambda^{\mathrm{op}}})^{\mathrm{fin}} = \mathcal{P}^{<\infty}_{\Lambda^{\mathrm{op}}}$.
(iv) $\mathfrak{I}^{\prec\propto}_\Lambda = \varinjlim \mathcal{I}^{<\infty}_\Lambda$.
(v) $\varinjlim (\mathfrak{P}^{\prec\propto}_{\Lambda^{\mathrm{op}}})^{\mathrm{fin}} = \varinjlim \mathcal{P}^{<\infty}_{\Lambda^{\mathrm{op}}}$.

In particular if $\mathrm{id}\,\Lambda_\Lambda < \infty$*, then:* $\mathbf{I}^{<\infty}_\Lambda = \varinjlim \mathcal{I}^{<\infty}_\Lambda \cap \widetilde{\mathrm{CoCM}(\mathbf{I}_\Lambda)}$.

Proof. Clearly (ii) $\Rightarrow$ (i) since Λ lies in $(\mathfrak{I}_\Lambda^{\prec\propto})^{\mathrm{fin}}$. If $\mathrm{id}\,\Lambda_\Lambda < \infty$, then consider the sequence

$$0 \longrightarrow \mathcal{T}_r(\bar{\mathfrak{I}}_\Lambda^{\prec\propto}) \longrightarrow \mathcal{T}_r(\overline{\mathrm{Mod}}\text{-}\Lambda) \xrightarrow{\mathbf{L}^*_{\mathrm{CoCM}}} \overline{\mathrm{CoCM}}(\mathbf{I}_\Lambda) \longrightarrow 0$$

which is localization exact by Proposition 5.6. By Lemma 11.2 we have a triangle equivalence $(\mathbf{L}^{\mathrm{b}}_{\mathrm{CoCM}})^* : \mathcal{T}_r(\overline{\mathrm{mod}}\text{-}\Lambda) \xrightarrow{\approx} \overline{\mathrm{CoCM}}(\mathbf{I}_\Lambda)^{\mathrm{b}}$. Since the functor $\mathbf{L}^*_{\mathrm{CoCM}}|_{\mathcal{T}_r(\overline{\mathrm{mod}}\text{-}\Lambda)}$ is isomorphic to $(\mathbf{L}^{\mathrm{b}}_{\mathrm{CoCM}})^*$, it follows that the kernel of $\mathbf{L}^*_{\mathrm{CoCM}}|_{\mathcal{T}_r(\overline{\mathrm{mod}}\text{-}\Lambda)}$, which is equivalent to $\mathcal{T}_r(\overline{(\mathfrak{I}_\Lambda^{\prec\propto})^{\mathrm{fin}}})$, is trivial. It follows that $\mathcal{T}_r(\overline{(\mathfrak{I}_\Lambda^{\prec\propto})^{\mathrm{fin}}}) = 0$ and therefore, by Lemma 4.1, $(\mathfrak{I}_\Lambda^{\prec\propto})^{\mathrm{fin}} = \mathcal{I}_\Lambda^{<\infty}$. Hence (i) $\Leftrightarrow$ (ii) and clearly (iii) $\Leftrightarrow$ (v). Also the implication (iv) $\Rightarrow$ (i) follows since $\Lambda \in (\mathfrak{I}_\Lambda^{\prec\propto})^{\mathrm{fin}}$ and the equivalence (ii) $\Leftrightarrow$ (iii) follows by using the duality D and Lemma 8.6. Assume now that (i) holds or equivalently $\varinjlim[(\mathfrak{I}_\Lambda^{\prec\propto})^{\mathrm{fin}}] = \varinjlim \mathcal{I}_\Lambda^{<\infty}$. By a result of Eklof–Trlifaj [32], the subcategory $\mathcal{I}_\Lambda^{<\infty}$ cogenerates a cotorsion pair $({}^\perp[(\mathcal{I}_\Lambda^{<\infty})^\perp], (\mathcal{I}_\Lambda^{<\infty})^\perp)$ in Mod-Λ. By Proposition 4.4 we have $\mathrm{D}(\Lambda)^\perp = \mathrm{CoCM}(\mathbf{I}_\Lambda)$ and by induction it is not difficult to see that $(\mathcal{I}_\Lambda^{<\infty})^\perp = \mathrm{D}(\Lambda)^\perp$. Consequently $\mathrm{CoCM}(\mathbf{I}_\Lambda) = (\mathcal{I}_\Lambda^{<\infty})^\perp$ and therefore $\mathfrak{I}_\Lambda^{\prec\propto} = {}^\perp[(\mathcal{I}_\Lambda^{<\infty})^\perp]$. Since Λ lies in $\mathcal{I}_\Lambda^{<\infty}$, by a result of Hügel–Trlifaj [2, Theorem 2.3], we have $\varinjlim {}^\perp[(\mathcal{I}_\Lambda^{<\infty})^\perp] \subseteq \varinjlim \mathcal{I}_\Lambda^{<\infty}$. Hence $\mathfrak{I}_\Lambda^{\prec\propto} \subseteq \varinjlim \mathcal{I}_\Lambda^{<\infty}$. Since, by Corollary 6.8, $\mathfrak{I}_\Lambda^{\prec\propto}$ is closed under filtered colimits and contains $\mathcal{I}_\Lambda^{<\infty}$, we conclude that $\varinjlim \mathcal{I}_\Lambda^{<\infty} = \mathfrak{I}_\Lambda^{\prec\propto}$, hence (iv) holds. The last assertion follows from Theorem 3.5. □

Now we can prove the following characterization of Gorensteinness.

Theorem 11.4. *For an Artin algebra Λ the following conditions are equivalent.*

(i) *Λ is Gorenstein.*
(ii) *Λ is virtually Gorenstein and* $\mathrm{id}\,\Lambda_\Lambda < \infty$.
(iii) *Λ is virtually Gorenstein and* $\mathrm{id}\,{}_\Lambda\Lambda < \infty$.
(iv) *Λ is virtually Gorenstein and* $\mathbf{I}_\Lambda^{<\infty} = \varinjlim(\mathfrak{I}_\Lambda^{\prec\propto})^{\mathrm{fin}}$.
(v) *Λ is virtually Gorenstein and* $\mathbf{P}_\Lambda^{<\infty} = \varinjlim(\mathfrak{P}_\Lambda^{\prec\propto})^{\mathrm{fin}}$.
(vi) $\mathfrak{I}_\Lambda^{\prec\propto} \subseteq \varinjlim \mathcal{I}_\Lambda^{<\infty}$ *and $\mathcal{I}_\Lambda^{<\infty}$ is covariantly finite, respectively* $\mathrm{fid}(\Lambda) < \infty$.
(vii) $\mathfrak{P}_\Lambda^{\prec\propto} = \varinjlim \mathcal{P}_\Lambda^{<\infty}$.
(viii) $\mathrm{id}\,\Lambda_\Lambda < \infty$ *and* $\mathfrak{I}_\Lambda^{\prec\propto} \subseteq \varinjlim \mathcal{P}_\Lambda^{<\infty}$.
(ix) $\mathrm{id}\,\Lambda_\Lambda < \infty$ *and* $\varinjlim \mathcal{I}_\Lambda^{<\infty} \subseteq \widetilde{\mathrm{CoCM}(\mathbf{I}_\Lambda)}$.
(x) $\mathrm{id}\,\Lambda_\Lambda < \infty$ *and the torsion pair* $(\bar{\mathfrak{I}}_\Lambda^{\prec\propto}, \overline{\mathrm{CoCM}}(\mathbf{I}_\Lambda))$ *is of cofinite type.*
(xi) $(\mathfrak{I}_\Lambda^{\prec\propto})^{\mathrm{fin}} = \mathcal{I}_\Lambda^{<\infty}$ *and* $(\mathfrak{P}_\Lambda^{\prec\propto})^{\mathrm{fin}} = \mathcal{P}_\Lambda^{<\infty}$.

Proof. Clearly (i) implies all the remaining assertions and (ii) is equivalent to (iii) since, by Theorem 8.7, virtual Gorensteinness is left–right symmetric. If (ii) holds, then by Theorem 11.3 we have $(\mathfrak{I}_\Lambda^{\prec\propto})^{\mathrm{fin}} = \mathcal{I}_\Lambda^{<\infty}$ and then by Theorem 8.2 we have a cotorsion pair $(\mathcal{I}_\Lambda^{<\infty}, \mathrm{CoCM}(\mathrm{D}(\Lambda))$ in mod-Λ. This implies that $\widetilde{\mathrm{CoCM}(\mathrm{D}(\Lambda))} = \text{mod-}\Lambda$ and therefore Λ is Gorenstein by [22]. Since Λ lies in $(\mathfrak{I}_\Lambda^{\prec\propto})^{\mathrm{fin}}$, it follows that (iv) implies (ii). Similarly since $\mathrm{D}(\Lambda)$ lies in $(\mathfrak{P}_\Lambda^{\prec\propto})^{\mathrm{fin}}$, equivalently Λ lies in $\mathfrak{I}_{\Lambda^{\mathrm{op}}}^{\prec\propto}$, it follows directly that (v) $\Rightarrow$ (iii). Assume now that $\mathfrak{I}_\Lambda^{\prec\propto} = \varinjlim \mathcal{I}_\Lambda^{<\infty}$ and therefore $(\mathfrak{I}_\Lambda^{\prec\propto})^{\mathrm{fin}} = \mathcal{I}_\Lambda^{<\infty}$. If $\mathcal{I}_\Lambda^{<\infty}$ is covariantly finite, then Λ is virtually Gorenstein by Theorem 8.2. Since Λ lies in $\mathfrak{I}_\Lambda^{\prec\propto}$ we have clearly $\mathrm{id}\,\Lambda_\Lambda < \infty$ and therefore condition (ii) holds. If $\mathrm{fid}(\Lambda) < \infty$, then the inclusion $\mathfrak{I}_\Lambda^{\prec\propto} \subseteq \varinjlim \mathcal{I}_\Lambda^{<\infty}$ implies that any module in $\mathfrak{I}_\Lambda^{\prec\propto}$ has finite injective dimension and therefore $\mathfrak{I}_\Lambda^{\prec\propto} = \mathbf{I}_\Lambda^{<\infty}$. Hence Λ is Gorenstein by Proposition 3.10. Now (vii) implies that $\mathfrak{P}_\Lambda^{\prec\propto}$ is closed under coproducts and $\mathrm{D}(\Lambda) \in \mathcal{P}_\Lambda^{<\infty}$. Therefore Λ is virtually Gorenstein and $\mathrm{id}\,_\Lambda\Lambda < \infty$, i.e., condition (iii) holds. The implications (viii), (ix) $\Rightarrow$ (i) follow by Proposition 8.13 and Theorem 11.3. Also the implication (xi) $\Rightarrow$ (i) is a direct consequence of Theorem 11.3. Finally if (x) holds, then $\mathfrak{I}_\Lambda^{\prec\propto}$ is closed under products. Hence by Theorem 11.3(iv) and Lemma 7.10 we have that $(\mathfrak{I}_\Lambda^{\prec\propto})^{\mathrm{fin}} = \mathcal{I}_\Lambda^{<\infty}$ is covariantly finite. Hence Λ is virtually Gorenstein and the assertion follows. □

The following consequence gives necessary and sufficient conditions for an Artin algebra Λ to be Gorenstein provided that Λ has finite right self-injective dimension.

Corollary 11.5. *If* $\mathrm{id}\,\Lambda_\Lambda < \infty$*, then the following are equivalent.*

- (i) *Λ is Gorenstein.*
- (ii) $(\mathfrak{I}_\Lambda^{\prec\propto})^{\mathrm{fin}}$ *is covariantly finite, equivalently contravariantly finite, in* mod-Λ.
- (iii) $(\mathfrak{P}_\Lambda^{\prec\propto})^{\mathrm{fin}}$ *is contravariantly finite, equivalently covariantly finite, in* mod-Λ.
- (iv) *The minimal right* $\mathrm{CM}(\mathbf{P}_\Lambda)$*-approximation $X_{\Lambda/\mathfrak{r}}$ of $\Lambda/\mathfrak{r}$ is finitely generated.*
- (v) *The minimal left $\mathfrak{P}_\Lambda^{\prec\propto}$-approximation $Y^{\Lambda/\mathfrak{r}}$ of $\Lambda/\mathfrak{r}$ is finitely generated.*
- (vi) *The minimal right $\mathfrak{I}_\Lambda^{\prec\propto}$-approximation $W_{\Lambda/\mathfrak{r}}$ of $\Lambda/\mathfrak{r}$ is finitely generated.*
- (vii) *The minimal left* $\mathrm{CoCM}(\mathbf{I}_\Lambda)$*–approximation $Z^{\Lambda/\mathfrak{r}}$ of $\Lambda/\mathfrak{r}$ is finitely generated.*

We have the following consequences of Theorems 11.3, 8.11 and 8.12 which show that the Gorenstein Symmetry Conjecture holds for algebras lying in the derived equivalence class or the stable equivalence class of Morita type of a virtually Gorenstein algebra.

Theorem 11.6. *Let Λ be an Artin algebra such that* $\mathrm{id}\,_\Lambda\Lambda < \infty$ *or* $\mathrm{id}\,\Lambda_\Lambda < \infty$*. If Λ is derived equivalent or stably equivalent of Morita type to a virtually Gorenstein algebra, then Λ is Gorenstein.*

Corollary 11.7. *Let Λ be an Artin algebra such that* $\mathrm{id}\,_\Lambda\Lambda < \infty$ *or* $\mathrm{id}\,\Lambda_\Lambda < \infty$*. If Λ is derived equivalent to an algebra of finite representation or Cohen–Macaulay type, then Λ is Gorenstein.*

Theorem 11.6 shows that the Gorenstein Symmetry Conjecture is equivalent to the following.

Conjecture 11.8. An Artin algebra with finite right (or left) self-injective dimension is derived equivalent to a virtually Gorenstein algebra.

We now give two additional conditions ensuring that the Gorenstein Symmetry Conjecture holds. First recall that, for $k \geqslant 0$, Λ is called *k-Gorenstein* if in the minimal injective resolution $0 \to \Lambda \to I^0 \to I^1 \to \cdots$ of Λ we have $\operatorname{pd} I^t \leqslant t$ for $1 \leqslant t < k$. Auslander–Reiten proved that being k-Gorenstein is left–right symmetric and in addition that Λ is Gorenstein if $\operatorname{id} \Lambda_\Lambda < \infty$ and Λ is k-Gorenstein for all $k \geqslant 0$, see [12]. In the next result we observe that it is sufficient to take Λ to be k-Gorenstein for $k = \operatorname{id} \Lambda_\Lambda$ and we also show that if $\operatorname{id} \Lambda_\Lambda < \infty$, then Λ is Gorenstein iff the compact objects of the costabilization $\mathcal{H}_{\mathsf{Ac}}(\mathbf{P}_\Lambda)$ of $\underline{\mathrm{Mod}}$-Λ coincide with the costabilization $\mathcal{H}_{\mathsf{Ac}}(\mathcal{P}_\Lambda)$ of $\underline{\mathrm{mod}}$-Λ.

Corollary 11.9. *Let Λ be an Artin algebra with* $\operatorname{id} \Lambda_\Lambda < \infty$. *Then Λ is Gorenstein if one of the following conditions hold:* (α) $\mathcal{H}^{\mathrm{b}}_{\mathsf{Ac}}(\mathbf{P}_\Lambda) = \mathcal{H}_{\mathsf{Ac}}(\mathcal{P}_\Lambda)$, ($\beta$) *$\Lambda$ is* $\operatorname{id} \Lambda_\Lambda$*-Gorenstein.*

Proof. (α) By Corollary 4.8 it follows that $\underline{\mathrm{CM}}(\mathbf{P}_\Lambda)^{\mathrm{b}} = \underline{\mathrm{CM}}(\Lambda)$, hence Λ is virtually Gorenstein. Then Λ is Gorenstein by Theorem 11.4. (β) By [12] there exists a finitely generated cotilting module T whose indecomposable summands are the indecomposable projective modules P with $\operatorname{id} P \leqslant d$ and $\Omega^d I$ where I is indecomposable injective with $\operatorname{pd} I > d$. Since $\operatorname{id} \Lambda_\Lambda < \infty$, Λ is a direct summand of T and since T is cotilting it follows that $\operatorname{pd} \mathrm{D}(\Lambda)_\Lambda < \infty$. Hence Λ is Gorenstein. □

We have seen in Theorem 6.6 that the cotorsion pair $(\mathfrak{I}_\Lambda^{\prec\infty}, \mathrm{CoCM}(\mathbf{I}_\Lambda))$ is generated by a CoCohen–Macaulay module and the cotorsion pair $(\mathrm{CM}(\mathbf{P}_\Lambda), \mathfrak{P}_\Lambda^{\prec\infty})$ is cogenerated by a Cohen–Macaulay module. The following consequence of our previous results characterizes Gorensteinness in terms of (co)generation properties of the (co)Cohen–Macaulay cotorsion pairs and (co)tilting modules.

Corollary 11.10. *Then following are equivalent.*

(i) *Λ is Gorenstein.*
(ii) $\operatorname{id} \Lambda_\Lambda < \infty$ *and the minimal left $\mathfrak{P}_\Lambda^{\prec\infty}$-approximation $Y^{\Lambda/\mathfrak{r}}$ of $\Lambda/\mathfrak{r}$ has* $\operatorname{id} Y^{\Lambda/\mathfrak{r}} < \infty$.
(iii) $\operatorname{id}_\Lambda \Lambda < \infty$ *and the minimal right $\mathfrak{I}_\Lambda^{\prec\infty}$-approximation $W_{\Lambda/\mathfrak{r}}$ of $\Lambda/\mathfrak{r}$ has* $\operatorname{pd} W_{\Lambda/\mathfrak{r}} < \infty$.
(iv) $(\mathrm{CM}(\mathbf{P}_\Lambda), \mathfrak{P}_\Lambda^{\prec\infty})$ *is generated by a module S with finite injective dimension.*
(v) $(\mathfrak{I}_\Lambda^{\prec\infty}, \mathrm{CoCM}(\mathbf{I}_\Lambda))$ *is cogenerated by a module T with finite projective dimension.*

In cases (iv)*, respectively* (v)*, we may choose the module S, respectively T, to be a* (*finitely generated*) *cotilting, respectively tilting, module and then* $\mathrm{Prod}(S) = \mathbf{P}_\Lambda = \mathrm{Prod}(\Lambda)$ *and* $\mathrm{Add}(T) = \mathbf{I}_\Lambda = \mathrm{Add}(\mathrm{D}(\Lambda))$.

Note. The results of this paper are extended to more general situations in [21]. There, instead of working with the (Co)Cohen–Macaulay (co)torsion pairs, we work with (co)torsion pairs induced by relative (Co)Cohen–Macaulay modules $\mathrm{CM}(T)$ and $\mathrm{CoCM}(S)$ in the sense of [22], where S and T are suitable (co)tilting modules in the sense of Wakamatsu [58].

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