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On algebras of finite Cohen–Macaulay type

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Abstract

We study Artin algebras Λ and commutative Noetherian complete local rings R in connection with the following decomposition property of Gorenstein-projective modules:

$$\text{any Gorenstein-projective module is a direct sum of finitely generated modules.} \tag{$\dagger$}$$

We show that the class of algebras Λ enjoying (†) coincides with the class of virtually Gorenstein algebras of finite Cohen–Macaulay type, introduced in Beligiannis and Reiten (2007) [27], Beligiannis (2005) [24]. Thus we solve the problem stated in Chen (2008) [33]. This is proved by characterizing when a resolving subcategory is of finite representation type in terms of decomposition properties of its closure under filtered colimits, thus generalizing a classical result of Auslander (1976) [9] and Ringel and Tachikawa (1974) [63]. In the commutative case, if R admits a non-free finitely generated Gorenstein-projective module, then we show that R is of finite Cohen–Macaulay type iff R is Gorenstein and satisfies (†). We also generalize a result of Yoshino (2005) [68] by characterizing when finitely generated modules without extensions with the ring are Gorenstein-projective. Finally we study the (stable) relative Auslander algebra of a virtually Gorenstein algebra of finite Cohen–Macaulay type and, under the presence of a cluster tilting object, we give descriptions of the stable category of Gorenstein-projective modules in terms of the cluster category associated to the quiver of the stable relative Auslander algebra. In this setting we show that the cluster category is invariant under derived equivalences.

MSC: 16E65; 16G50; 16G60; 18G25; 18E30; 13H10

Keywords: Gorenstein-projective modules; Artin algebras and local rings of finite Cohen–Macaulay type; Virtually Gorenstein algebras; Auslander algebras; Resolving subcategories; Approximations; Cluster tilting objects

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Contents

1. Introduction

Finitely generated modules of Gorenstein dimension zero over a Noetherian ring R were introduced by Maurice Auslander in the mid-sixties, see [6,11], as a natural generalization of finite projective modules in order to provide finer homological invariants of Noetherian rings. Since then they found important applications in commutative algebra, algebraic geometry, singularity theory and relative homological algebra. Later on the influential works of Auslander and Buchweitz [12,31] gave a more categorical approach to the study of modules of Gorenstein dimension zero.

In general the size and the homological complexity of the category $\mathsf{Gproj}\,R$ of finitely generated modules of Gorenstein dimension zero, also called, depending on the setting, Gorenstein-projective, totally reflexive, or maximal Cohen–Macaulay, modules, measure how far the ring R is from being Gorenstein. A particularly nice feature of $\mathsf{Gproj}\,R$ is that its stable version $\underline{\mathsf{Gproj}}\,R$ modulo projectives is triangulated, so stable homological phenomena concerning the ring take place in $\underline{\mathsf{Gproj}}\,R$. For instance $\underline{\mathsf{Gproj}}\,R$ admits a full triangulated embedding into the triangulated category of singularities of R, in Orlov's sense [57], which is an equivalence if and only if R is Gorenstein. On the other hand it was a discovery of many people that also representation-theoretic properties of the category $\mathsf{Gproj}\,R$ have important consequences for the structural shape of the ring, see Yoshino's book [67] for more details. For instance if the Noetherian ring R is commutative, complete, local and Gorenstein, then R is a simple singularity provided that the category $\mathsf{Gproj}\,R$ is of finite representation type, i.e. the set of isomorphism classes of its indecomposable objects is finite. In this case R is said to be of *finite Cohen–Macaulay type*, finite CM-type for short. In many cases also the converse holds. For instance in the recent paper [35], Christensen, Piepmeyer, Striuli and Takahashi proved that if a commutative Noetherian local ring R is of finite CM-type and admits a non-free finitely generated module of Gorenstein dimension zero, then R is Gorenstein and an isolated singularity, which is a simple hypersurface singularity if in addition R is complete.

In the non-commutative setting, Enochs and Jenda in [37] generalized the notion of finitely generated modules of Gorenstein dimension zero over a Noetherian ring in order to cover, not necessarily finitely generated, modules over any ring R, under the name Gorenstein-projective modules; the corresponding full subcategory is henceforth denoted by $\mathsf{GProj}\,R$. Enochs and his collaborators studied extensively relative homological algebra based on Gorenstein-projective modules, see the book [38] for more details and Christensen's book [34] for a hyperhomological approach. In the context of Artin algebras, Auslander and Reiten [14,15] and Happel [39], see also [31,27], studied finitely generated Gorenstein-projective modules and they established strong connections with cotilting theory and the shape of the singularity category. The results in the commutative setting suggest that there is a strong link between representation-theoretic properties of the category $\mathsf{Gproj}\, R$ and global structural properties of the ring R, especially when R is Gorenstein. In this connection the class of virtually Gorenstein Artin algebras was introduced in [27], and studied extensively in [24,25], as a convenient common generalization of Gorenstein algebras and algebras of finite representation type. Note that Λ is virtually Gorenstein if and only if any Gorenstein-projective module is a filtered colimit of finitely generated Gorenstein-projective modules, so in this case one has a nice control of the objects of $\mathsf{GProj}\, R$ in terms of the objects of $\mathsf{Gproj}\, R$.

Our aim in this paper is to study representation-theoretic aspects of Gorenstein-projective modules. In particular we are mainly interested in the class of Artin algebras and/or commutative Noetherian complete local rings of finite CM-type, and the structure of the category of Gorenstein-projective modules and how they are built from finitely generated ones.

The organization and the main results of the paper are as follows.

In Section 2, for completeness and reader's convenience, we collect several preliminary notions and results that will be useful throughout the paper and we fix notation. In Section 3, generalizing and extending classical results of Auslander [9], Ringel and Tachikawa [63], and others, we give several characterizations of when a resolving subcategory of the category $\mathsf{mod}\text{-}\Lambda$ of finitely generated modules over an Artin algebra Λ is of finite representation type in terms of decomposition properties of its closure under filtered colimits, see Theorem 3.1 and Corollary 3.5. We also point-out some consequences concerning submodule categories, Weyl modules over quasi-hereditary algebras and torsion pairs.

In Section 4, by applying the results of the previous section, we present a host of characterizations, more precisely 13 equivalent conditions, of when an Artin algebra enjoys the decomposition property (†): any Gorenstein-projective module is a direct sum of finitely generated modules; see Theorem 4.10 and Corollary 4.11. In particular we prove that this class of algebras coincides with the class of virtually Gorenstein algebras of finite CM-type. Since virtually Gorenstein algebras contain properly the Gorenstein algebras, we obtain as a direct consequence an extension of the main result of Chen [33], and at the same time we give a complete solution to the problem/conjecture stated in [33], see Corollary 4.13. We also characterize arbitrary Artin algebras of finite CM-type, see Proposition 4.18. In the commutative case, we characterize the commutative Noetherian complete local rings of finite CM-type admitting a non-free finitely generated Gorenstein-projective module as the Gorenstein rings satisfying (†); see Theorem 4.20. As a consequence any commutative Noetherian complete local Gorenstein ring satisfying (†) is a simple singularity.

Due to their importance in the homological or representation-theoretic structure of an Artin algebra, it is useful to have convenient descriptions of the category $\mathsf{Gproj}\, \Lambda$. In this connection Jorgensen and Sega [44] and Yoshino [68] studied the problem of when a finitely generated module without non-zero extensions with the ring, i.e. a *stable* module, is Gorenstein-projective. By the results of [44] this is not always true for certain commutative Noetherian local rings $(R, \mathfrak{m}, k)$. On the other hand Yoshino proved that all stable R-modules are Gorenstein-projective provided that R is Henselian and the full subcategory ${}^{\perp}R$ of all stable modules is of finite representation type, see [68]. Recall that for a full subcategory $\mathcal{X} \subseteq \mathsf{mod}\text{-}R$, ${}^{\perp}\mathcal{X}$ denotes the left Ext-orthogonal subcategory of $\mathcal{X}$ consisting of all modules $A \in \mathsf{mod}\text{-}R$ such that $\mathsf{Ext}^n_R(A, X) = 0$, $\forall X \in \mathcal{X}$, $\forall n \geqslant 1$. The right Ext-orthogonal subcategory $\mathcal{X}^{\perp}$ of $\mathcal{X}$ is defined in a dual way. Recently Takahashi [65] proved a far reaching generalization: if ${}^{\perp}R$ contains a non-projective module and k admits a right ${}^{\perp}R$-approximation, then R is Gorenstein and then ${}^{\perp}R = \mathsf{Gproj}\, R$. In Section 5 we extend and generalize Yoshino's results in the non-commutative setting, in fact to a large class of abelian categories. In particular we give a complete answer to the question of when all stable modules over a virtually Gorenstein algebra Λ, or more generally over an algebra for which $\mathsf{Gproj}\,\Lambda$ is contravariantly finite, are Gorenstein-projective, i.e. when ${}^{\perp}\Lambda = \mathsf{Gproj}\,\Lambda$. In fact the main result of Section 5 shows that this happens if and only if the category ${}^{\perp}\Lambda \cap (\mathsf{Gproj}\,\Lambda)^{\perp}$ is of finite representation type (in that case it coincides with the full subcategory of finite projective modules), see Theorem 5.2. This has some consequences for the, still open, Gorenstein Symmetry Conjecture, (GSC) for short, see [27,18,24], which predicts that any Artin algebra with finite one-sided self-injective dimension is Gorenstein. We show that (GSC) admits the following representation-theoretic formulation: an Artin algebra Λ with finite right self-injective dimension is Gorenstein if and only if $\mathsf{Gproj}\,\Lambda$ is contravariantly finite and the full subcategory ${}^{\perp}\Lambda \cap (\mathsf{Gproj}\,\Lambda)^{\perp}$ of $\mathsf{mod}\text{-}\Lambda$ is of finite representation type. On the other hand we show that if R is a commutative Noetherian complete local Gorenstein ring, then all infinitely generated stable modules are Gorenstein-projective if and only if R is Artinian, so the infinitely generated versions of the above results rarely hold in the commutative setting.

In Section 6, motivated by Auslander's homological theory of representation-finite Artin algebras Λ, see [7], we consider relative Auslander algebras $\mathsf{A}(\mathcal{X})$, defined as endomorphism rings of representation generators, i.e. modules containing as direct summands all types of indecomposable modules, of contravariantly finite resolving subcategories $\mathcal{X} \subseteq \mathsf{mod}\text{-}\Lambda$. Recall that Auslander proved that $\mathsf{A}(\mathcal{X})$ has global dimension at most 2, when $\mathcal{X} = \mathsf{mod}\text{-}\Lambda$ for an Artin algebra Λ of finite representation type. In Section 6 we compute the global dimension of $\mathsf{A}(\mathcal{X})$ in terms of the resolution dimension of $\mathsf{mod}\text{-}\Lambda$ with respect to $\mathcal{X}$. In particular any natural number or infinity can occur. As a consequence we characterize Gorenstein abelian categories of finite Cohen–Macaulay type, thus generalizing and, at the same time, giving a simple proof to the main result of [56], see Corollary 6.13. Then we concentrate on relative, resp. stable, Auslander algebras $\mathsf{A}(\mathsf{Gproj}\,\Lambda)$, resp. $\mathsf{A}(\underline{\mathsf{Gproj}}\,\Lambda)$, of Artin algebras Λ of finite CM-type which we call (stable) Cohen–Macaulay Auslander algebras. We show that the global dimension of $\mathsf{A}(\mathsf{Gproj}\,\Lambda)$ is bounded by the Gorenstein dimension $\mathsf{G\text{-}dim}\,\Lambda$ of Λ, so it is finite if and only if Λ is Gorenstein, whereas $\mathsf{A}(\underline{\mathsf{Gproj}}\,\Lambda)$ is always self-injective. As a consequence $\max\{2, \mathsf{G\text{-}dim}\,\Lambda\}$ is an upper bound for the dimension, in the sense of Rouquier [64], of the derived category $\mathbf{D}^b(\mathsf{mod}\text{-}\Lambda)$ of Λ. In the commutative case we show that the global dimension of the Cohen–Macaulay Auslander algebra $\mathsf{A}(\mathsf{Gproj}\,R)$ of a commutative Noetherian complete local ring R of finite CM-type is at most $\max\{2, \dim R\}$, and exactly $\dim R$ if $\dim R \geqslant 2$, so this number is an upper bound of the dimension of $\mathbf{D}^b(\mathsf{mod}\text{-}R)$. Finally we give an application to the periodicity of Hochschild (co)homology of the stable Cohen–Macaulay Auslander algebra $\mathsf{A}(\underline{\mathsf{Gproj}}\,\Lambda)$ which is based on results of Buchweitz [32].

The final Section 7 of the paper is devoted to the study of finitely generated Gorenstein-projective rigid modules, i.e. modules without self-extensions, or cluster tilting Gorenstein-projective modules, in the sense of Keller and Reiten [46] or Iyama [41], over an Artin algebra Λ in connection with the property that Λ is of finite CM-type. We show that Cohen–Macaulay finiteness of Λ is intimately related to the representation finiteness of the stable endomorphism ring $\underline{\mathsf{End}}_{\Lambda}(T)$ of a Gorenstein-projective rigid module T, and in case T is cluster tilting the two finiteness conditions are equivalent. As a consequence we show that if $\underline{\mathsf{End}}_{\Lambda}(T)$ is of finite CM-type, then the representation dimension $\mathsf{rep.dim}\,\underline{\mathsf{End}}_{\Lambda}(T)$ of $\underline{\mathsf{End}}_{\Lambda}(T)$, in the sense of Auslander [7], is at most 3, and is exactly 3 if and only if Λ is of infinite CM-type. Finally by using that the stable triangulated category $\underline{\mathsf{Gproj}}\,\Lambda$ of finitely generated Gorenstein-projective modules over a virtually Gorenstein algebra Λ admits Serre duality [24] and recent results of Amiot [1] and Keller and Reiten [46,47], we give descriptions of $\underline{\mathsf{Gproj}}\,\Lambda$, in case the latter is 2-Calabi–Yau in the sense of [46] and $\underline{\mathsf{End}}_{\Lambda}(T)$ has finite global dimension, in terms of the cluster category $\mathscr{C}_Q$ associated to the quiver Q of $\underline{\mathsf{End}}_{\Lambda}(T)$. In particular we show that, under the above assumptions, the cluster category $\mathscr{C}_Q$ is invariant under derived equivalences.

2. Preliminaries

In this section we fix notation and, for completeness and reader's convenience, recall some well-known notions and collect several preliminary results which will be used throughout the paper.

2.1. *Locally finitely presented categories*

Let $\mathscr{A}$ be an additive category with filtered colimits. An object X in $\mathscr{A}$ is called **finitely presented** if the functor $\mathscr{A}(X,-):\mathscr{A}\longrightarrow\mathfrak{Ab}$ commutes with filtered colimits. We denote by $\mathsf{fp}\,\mathscr{A}$ the full subcategory of $\mathscr{A}$ consisting of the finitely presented objects. If $\mathfrak{U}$ is a full subcategory of $\mathscr{A}$ we denote by $\varinjlim\mathfrak{U}$ the closure of $\mathfrak{U}$ in $\mathscr{A}$ under filtered colimits, that is $\varinjlim\mathfrak{U}$ is the full subcategory of $\mathscr{A}$ consisting of all filtered colimits of objects from $\mathfrak{U}$. We recall that $\mathscr{A}$ is called **locally finitely presented** if $\mathsf{fp}\,\mathscr{A}$ is skeletally small and $\mathscr{A}=\varinjlim\mathsf{fp}\,\mathscr{A}$.

For instance let $\mathscr{C}$ be a skeletally small additive category with split idempotents. Then the category $\mathsf{Mod}\text{-}\mathscr{C}$ of contravariant additive functors $\mathscr{C}^{\mathsf{op}}\longrightarrow\mathfrak{Ab}$ is locally finitely presented and $\mathsf{fp}\,\mathsf{Mod}\text{-}\mathscr{C}=\mathsf{mod}\text{-}\mathscr{C}$ is the full subcategory of coherent functors over $\mathscr{C}$. Recall that an additive functor $F:\mathscr{C}\longrightarrow\mathfrak{Ab}$ is called **coherent** [5], if there exists an exact sequence $\mathscr{C}(-,X)\longrightarrow\mathscr{C}(-,Y)\longrightarrow F\longrightarrow 0$. The categories $\mathscr{C}\text{-}\mathsf{Mod}$ and $\mathscr{C}\text{-}\mathsf{mod}$ are defined by $\mathscr{C}\text{-}\mathsf{Mod}=\mathsf{Mod}\text{-}\mathscr{C}^{\mathsf{op}}$, and $\mathscr{C}\text{-}\mathsf{mod}=\mathsf{mod}\text{-}\mathscr{C}^{\mathsf{op}}$ respectively. Note that $\mathsf{mod}\text{-}\mathscr{C}$ is abelian if and only if $\mathscr{C}$ has weak kernels. Recall that $\mathscr{C}$ has weak kernels if any map $X_2\longrightarrow X_3$ in $\mathscr{C}$ can be extended to a diagram $X_1\longrightarrow X_2\longrightarrow X_3$ in $\mathscr{C}$ such that the induced sequence of functors $\mathscr{C}(-,X_1)\longrightarrow\mathscr{C}(-,X_2)\longrightarrow\mathscr{C}(-,X_3)$ is exact; in this case the map $X_1\longrightarrow X_2$ is called a **weak kernel** of $X_2\longrightarrow X_3$. *Weak cokernels* are defined dually. Note that a locally finitely presented category $\mathscr{A}$ has products iff $\mathsf{fp}\,\mathscr{A}$ has weak cokernels and $\mathscr{A}$ is abelian iff $\mathscr{A}$ is a Grothendieck category, see [36]. In this case $\mathscr{A}$ is called **locally finite** if any finitely presented object has finite Jordan–Hölder length, equivalently $\mathsf{fp}\,\mathscr{A}$ is Artinian and Noetherian.

2.1.1. *Representation categories, purity and the Ziegler spectrum*

Let $\mathscr{A}=\varinjlim\mathsf{fp}\,\mathscr{A}$ be a locally finitely presented additive category. The **representation category** $\mathfrak{L}(\mathscr{A})$ of $\mathscr{A}$ is defined to be the functor category $\mathfrak{L}(\mathscr{A}):=\mathsf{Mod}\text{-}\,\mathsf{fp}\,\mathscr{A}$. It is well known that $\mathfrak{L}(\mathscr{A})$ reflects important representation-theoretic properties of $\mathscr{A}$ via the **representation functor**

$$\mathsf{H}:\mathscr{A}\longrightarrow\mathfrak{L}(\mathscr{A}),\qquad A\longmapsto\mathsf{H}(A)=\mathscr{A}(-,A)|_{\mathsf{fp}\,\mathscr{A}}$$

which induces an equivalence $\mathsf{H}:\mathscr{A}\xrightarrow{\approx}\mathsf{Flat}(\mathsf{fp}\,\mathscr{A})$, where for an additive category $\mathscr{C}$, $\mathsf{Flat}\,\mathscr{C}$ denotes the full subcategory of $\mathsf{Mod}\text{-}\mathscr{C}$ consisting of all flat functors; recall that $F:\mathscr{C}^{\mathsf{op}}\longrightarrow\mathfrak{Ab}$ is called **flat** if F is a filtered colimit of representable functors. Assume now that $\mathscr{A}$ has products. A sequence $(*):0\longrightarrow A\longrightarrow B\longrightarrow C\longrightarrow 0$ in $\mathscr{A}$ is called **pure-exact** if the induced sequence $0 \longrightarrow \mathsf{H}(A) \longrightarrow \mathsf{H}(B) \longrightarrow \mathsf{H}(C) \longrightarrow 0$ is exact in $\mathcal{L}(\mathscr{A})$. An object E in $\mathscr{A}$ is called **pure-projective**, resp. **pure-injective**, if for any pure-exact sequence $(*)$ as above, the induced sequence $0 \longrightarrow \mathscr{A}(E, A) \longrightarrow \mathscr{A}(E, B) \longrightarrow \mathscr{A}(E, C) \longrightarrow 0$, resp. $0 \longrightarrow \mathscr{A}(C, E) \longrightarrow \mathscr{A}(B, E) \longrightarrow \mathscr{A}(A, E) \longrightarrow 0$ is exact. It is easy to see that the full subcategory of $\mathscr{A}$ consisting of the pure-projective objects coincides with $\mathsf{Add}(\mathsf{fp}\,\mathscr{A})$. Recall that if $\mathcal{X}$ is a full subcategory of $\mathscr{A}$, then $\mathsf{Add}\,\mathcal{X}$, resp. $\mathsf{add}\,\mathcal{X}$, denotes the full subcategory of $\mathscr{A}$ consisting of the direct summands of all small, resp. finite, coproducts of objects of $\mathcal{X}$. The locally finitely presented category $\mathscr{A}$ is called **pure-semisimple** if any pure-exact sequence in $\mathscr{A}$ splits. Equivalently $\mathscr{A} = \mathsf{Add}(\mathsf{fp}\,\mathscr{A})$, or the representation category $\mathcal{L}(\mathscr{A})$ is **perfect**, i.e. any flat functor is projective.

We denote by $\mathbf{Zg}(\mathscr{A})$ the collection of the isomorphism classes of indecomposable pure-injective objects of $\mathscr{A}$. It is well known that $\mathbf{Zg}(\mathscr{A})$ is a small set and admits a natural topology, the **Ziegler topology**, defined as follows. Let Φ be a collection of maps between finitely presented objects. An object E is called *Φ-injective* if for any map $\phi : X \longrightarrow Y$ in Φ, the induced map $\mathscr{A}(\phi, E)$ is surjective. We denote by $\mathsf{Inj}(\Phi)$ the collection of Φ-injective objects of $\mathscr{A}$. Then the subsets of $\mathbf{Zg}(\mathscr{A})$ of the form $\mathbf{U}_\Phi := \mathbf{Zg}(\mathscr{A}) \cap \mathsf{Inj}(\Phi)$ constitute the closed sets of the Ziegler topology of $\mathbf{Zg}(\mathscr{A})$, i.e. a subset $\mathbf{U}$ of $\mathbf{Zg}(\mathscr{A})$ is Ziegler-closed if $\mathbf{U} = \mathbf{U}_\Phi$ for some collection of maps Φ in $\mathsf{fp}(\mathscr{A})$. A subset $\mathbf{C}$ of $\mathbf{Zg}(\mathscr{A})$ is Ziegler-open if its complement $\mathbf{Zg}(\mathscr{A}) \setminus \mathbf{C}$ is Ziegler-closed. We refer to [50] for more details.

2.2. *Contravariantly finite subcategories, filtrations, and cotorsion pairs*

Let $\mathscr{A}$ be an additive category. If $\mathcal{V}$ is a full subcategory of $\mathscr{A}$, then a map $f : A \longrightarrow B$ in $\mathscr{A}$ is called $\mathcal{V}$-**epic** if the map $\mathscr{A}(\mathcal{V}, f) : \mathscr{A}(\mathcal{V}, A) \longrightarrow \mathscr{A}(\mathcal{V}, B)$ is surjective. $\mathcal{V}$ is called **contravariantly finite** if there exists a $\mathcal{V}$-epic $f_A : V_A \longrightarrow A$ with V_A in $\mathcal{V}$. In this case f_A, or V_A, is called a **right** $\mathcal{V}$-**approximation** of A. *Covariantly finite* subcategories, *$\mathcal{V}$-monics* and *left $\mathcal{V}$-approximations* are defined dually. $\mathcal{V}$ is called *functorially finite* if it is both contravariantly and covariantly finite. A map $f : V \longrightarrow A$ is called *right minimal* if any endomorphism α of V such that $\alpha \circ f = f$ is an automorphism. A right minimal right $\mathcal{V}$-approximation is called a *minimal right $\mathcal{V}$-approximation*. *Left minimal* maps and *minimal left approximations* are defined dually. The objects of the **stable category** $\mathscr{A}/\mathcal{V}$ of $\mathscr{A}$ modulo $\mathcal{V}$ are the objects of $\mathscr{A}$; the morphism spaces are defined by $\mathscr{A}/\mathcal{V}(A, B) := \mathscr{A}(A, B)/\mathscr{A}_\mathcal{V}(A, B)$ where $\mathscr{A}_\mathcal{V}(A, B)$ is the subgroup of $\mathscr{A}(A, B)$ consisting of all maps factorizing through an object from $\mathcal{V}$. We have the natural projection functor $\pi : \mathscr{A} \longrightarrow \mathscr{A}/\mathcal{V}$, $\pi(A) = \underline{A}$ and $\pi(f) := \underline{f}$, which is universal for additive functors out of $\mathscr{A}$ killing the objects of $\mathcal{V}$.

Now let $\mathscr{A}$ be abelian. We denote by $\mathsf{Proj}\,\mathscr{A}$, resp. $\mathsf{Inj}\,\mathscr{A}$, the full subcategory of the projective, resp. injective, objects of $\mathscr{A}$. Let $\mathcal{U}$ be a full subcategory of $\mathscr{A}$. The stable categories $\mathcal{U}/\mathsf{Proj}\,\mathscr{A}$ and $\mathcal{U}/\mathsf{Inj}\,\mathscr{A}$ are denoted by $\underline{\mathcal{U}}$ and $\overline{\mathcal{U}}$ respectively. $\mathcal{U}$ is called **resolving** if $\mathcal{U}$ contains the projectives and is closed under extensions and kernels of epimorphisms. *Coresolving* subcategories are defined in a dual way. Typical examples of (co)resolving subcategories are the **left** and **right** Ext-**orthogonal** subcategories ${}^\perp\mathcal{U}$ and $\mathcal{U}^\perp$ of $\mathcal{U}$ defined as follows:

$$
{}^\perp\mathcal{U} := \{ A \in \mathscr{A} \mid \mathsf{Ext}^n_\mathscr{A}(A, U) = 0, \ \forall n \geqslant 1, \ \forall U \in \mathcal{U} \}
$$

$$
\mathcal{U}^\perp := \{ A \in \mathscr{A} \mid \mathsf{Ext}^n_\mathscr{A}(U, A) = 0, \ \forall n \geqslant 1, \ \forall U \in \mathcal{U} \}
$$

Definition 2.1. A pair $(\mathcal{X}, \mathcal{Y})$ of full subcategories of $\mathscr{A}$ is called a **cotorsion pair** if $\mathsf{Ext}^n(\mathcal{X}, \mathcal{Y}) = 0$, $\forall n \geqslant 1$, and for any object $A \in \mathscr{A}$, there exist short exact sequences

$$0 \longrightarrow Y_A \xrightarrow{g_A} X_A \xrightarrow{f_A} A \longrightarrow 0 \quad \text{and} \quad 0 \longrightarrow A \xrightarrow{g^A} Y^A \xrightarrow{f^A} X^A \longrightarrow 0 \tag{2.1}$$

where $X^A, X_A \in \mathcal{X}$ and $Y_A, Y^A \in \mathcal{Y}$. The **heart** of the cotorsion pair $(\mathcal{X}, \mathcal{Y})$ is the full subcategory $\mathcal{X} \cap \mathcal{Y}$.

Cotorsion pairs in the above sense are also known in the literature as complete hereditary cotorsion pairs. It follows easily that if $(\mathcal{X}, \mathcal{Y})$ is a cotorsion pair in $\mathscr{A}$, then $\mathcal{X}$ is contravariantly finite resolving and $\mathcal{X}^{\perp} = \mathcal{Y}$, and $\mathcal{Y}$ is covariantly coresolving and $\mathcal{X} = {}^{\perp}\mathcal{Y}$. Moreover the heart $\omega = \mathcal{X} \cap \mathcal{Y}$ is an Ext-injective cogenerator of $\mathcal{X}$ and an Ext-projective generator of $\mathcal{Y}$. Recall that a full subcategory ω of $\mathcal{X}$ is called an **Ext-injective cogenerator** of $\mathcal{X}$ provided that for any object $X \in \mathcal{X}$, there exists a short exact sequence $0 \longrightarrow X \longrightarrow T \longrightarrow X' \longrightarrow 0$, where T lies in ω, X' lies in $\mathcal{X}$, and $\mathsf{Ext}^1_{\mathscr{A}}(\mathcal{X}, T) = 0$. *Ext-projective generators* are defined dually. A cotorsion pair $(\mathcal{X}, \mathcal{Y})$ in $\mathscr{A}$ is called **projective** if $\mathcal{X} \cap \mathcal{Y} = \mathsf{Proj}\,\mathscr{A}$. Note that if $\mathscr{A}$ has enough projectives and $(\mathcal{X}, \mathcal{Y})$ is a projective cotorsion pair in $\mathscr{A}$, then the stable category $\underline{\mathcal{X}}$ is triangulated and the subcategory $\mathcal{Y}$ is *thick* in the sense that $\mathcal{Y}$ is closed under extensions, direct summands, kernels of epimorphisms and cokernels of monomorphisms. Moreover the canonical map $\mathsf{Ext}^n_{\mathscr{A}}(X, A) \longrightarrow \underline{\mathscr{A}}(\Omega^n X, A)$ is invertible, $\forall n \geqslant 1$, $\forall X \in \mathcal{X}$, $\forall A \in \mathscr{A}$. Here Ω is the Heller's loop space functor on the stable category $\underline{\mathscr{A}}$. *Injective* cotorsion pairs are defined dually; we refer to [27] for more details. If the abelian category $\mathscr{A}$ is cocomplete, then a cotorsion pair $(\mathcal{X}, \mathcal{Y})$ in $\mathscr{A}$ is called **smashing** if $\mathcal{Y}$ is closed under all small coproducts. If $\mathscr{A}$ is in addition locally finitely presented, then the cotorsion pair $(\mathcal{X}, \mathcal{Y})$ is called **of finite type** if there is a subset $\mathcal{S}$ of $\mathsf{fp}\,\mathscr{A}$ such that $\mathcal{Y} = \mathcal{S}^{\perp}$.

If $\mathcal{X}$ is a full subcategory of $\mathscr{A}$, we denote by $\mathsf{res.dim}_{\mathcal{X}} A$ the *$\mathcal{X}$-resolution dimension* of $A \in \mathscr{A}$ which is defined as the minimal number n such that there exists an exact sequence $0 \longrightarrow X_n \longrightarrow X_{n-1} \longrightarrow \cdots \longrightarrow X_0 \longrightarrow A \longrightarrow 0$, where the X_i lie in $\mathcal{X}$. If no such number n exists, then we set $\mathsf{res.dim}_{\mathcal{X}} A = \infty$. The **$\mathcal{X}$-resolution dimension** $\mathsf{res.dim}_{\mathcal{X}}\,\mathscr{A}$ of $\mathscr{A}$ is defined by $\mathsf{res.dim}_{\mathcal{X}}\,\mathscr{A} = \sup\{\mathsf{res.dim}_{\mathcal{X}} A \mid A \in \mathscr{A}\}$. The *coresolution dimensions* $\mathsf{cores.dim}_{\mathcal{X}} A$ and $\mathsf{cores.dim}_{\mathcal{X}}\,\mathscr{A}$ are defined dually. Finally we denote by $\mathsf{Filt}\,\mathcal{X}$ the full subcategory of $\mathscr{A}$ consisting of all direct summands of objects A admitting a finite filtration

$$0 = A_{n+1} \subseteq A_n \subseteq A_{n-1} \subseteq \cdots \subseteq A_2 \subseteq A_1 \subseteq A_0 = A \tag{2.2}$$

such that the subquotients A_i / A_{i+1}, for $i = 0, 1, \ldots, n$, lie in $\mathcal{X}$.

2.3. Compactly generated triangulated categories

Let $\mathcal{T}$ be a triangulated category which admits all small coproducts. An object T in $\mathcal{T}$ is called **compact** if the functor $\mathcal{T}(T, -) : \mathcal{T} \longrightarrow \mathfrak{Ab}$ preserves all small coproducts. We denote by $\mathcal{T}^{\mathsf{cpt}}$ the full subcategory of $\mathcal{T}$ consisting of all compact objects. It is easy to see that $\mathcal{T}^{\mathsf{cpt}}$ is a *thick* subcategory of $\mathcal{T}$, that is, $\mathcal{T}^{\mathsf{cpt}}$ is closed under direct summands and if two of the objects in a triangle $X \longrightarrow Y \longrightarrow Z \longrightarrow X[1]$ in $\mathcal{T}$ lie in $\mathcal{T}^{\mathsf{cpt}}$, then so does the third. The triangulated category $\mathcal{T}$ is called **compactly generated** if $\mathcal{T}^{\mathsf{cpt}}$ is skeletally small and *generates* $\mathcal{T}$ in the following sense: an object X in $\mathcal{T}$ is zero provided that $\mathcal{T}(C, X) = 0$, for any compact object $C \in \mathcal{T}^{\mathsf{cpt}}$.

2.3.1. Representation categories

Let $\mathcal{T}$ be a compactly generated triangulated category. Then clearly the category $\mathsf{Flat}(\mathcal{T}^{\mathsf{cpt}})$ of flat contravariant functors $\{\mathcal{T}^{\mathsf{cpt}}\}^{\mathsf{op}} \longrightarrow \mathfrak{Ab}$, which coincides with the category of cohomological functors over $\mathcal{T}^{\mathsf{cpt}}$, has products. We call $\mathsf{Mod}\text{-}\mathcal{T}^{\mathsf{cpt}} = \mathcal{L}(\mathsf{Flat}(\mathcal{T}^{\mathsf{cpt}}))$ the **representation category** of $\mathcal{T}$ and we denote it by $\mathcal{L}(\mathcal{T}) := \mathsf{Mod}\text{-}\mathcal{T}^{\mathsf{cpt}}$. In contrast to the case of locally finitely presented categories, the representation functor

$$\mathsf{H}: \mathcal{T} \longrightarrow \mathcal{L}(\mathcal{T}) = \mathsf{Mod}\text{-}\mathcal{T}^{\mathsf{cpt}}, \quad \mathsf{H}(A) = \mathcal{T}(-, A)|_{\mathcal{T}^{\mathsf{cpt}}}$$

is in general not fully faithful, due to the presence of **phantom maps**, i.e. maps $A \longrightarrow B$ in $\mathcal{T}$ such that the induced map $\mathcal{T}(X, A) \longrightarrow \mathcal{T}(X, B)$ is zero for any compact object X. In fact H is faithful if and only if H induces an equivalence $\mathsf{H}: \mathcal{T} \xrightarrow{\approx} \mathsf{Flat}(\mathcal{T}^{\mathsf{cpt}})$ if and only if $\mathcal{T}$ is **phantomless**, i.e. there are no non-zero phantom maps in $\mathcal{T}$; equivalently $\mathcal{T}$ is a (pure-semisimple) locally finitely presented category. However H is homological, has image in $\mathsf{Flat}(\mathcal{T}^{\mathsf{cpt}})$, and detects zero objects: $\mathsf{H}(A) = 0$ if and only if $A = 0$. In general for the representation functor $\mathsf{H}: \mathcal{T} \longrightarrow \mathsf{Flat}(\mathcal{T}^{\mathsf{cpt}})$ we have, in general non-reversible, implications: H is faithful $\Rightarrow$ H is full $\Rightarrow$ H is surjective on objects, see [21] for details.

2.3.2. Purity and the Ziegler spectrum

A triangle $(T): A \longrightarrow B \longrightarrow C \longrightarrow A[1]$ in $\mathcal{T}$ is called **pure** if the induced sequence $0 \longrightarrow \mathsf{H}(A) \longrightarrow \mathsf{H}(B) \longrightarrow \mathsf{H}(C) \longrightarrow 0$ is exact in $\mathcal{L}(\mathcal{T})$. An object E in $\mathcal{T}$ is called **pure-projective**, resp. **pure-injective**, if for any pure-triangle (T) as above, the induced sequence $0 \longrightarrow \mathcal{T}(E, A) \longrightarrow \mathcal{T}(E, B) \longrightarrow \mathcal{T}(E, C) \longrightarrow 0$, resp. $0 \longrightarrow \mathcal{T}(C, E) \longrightarrow \mathcal{T}(B, E) \longrightarrow \mathcal{T}(A, E) \longrightarrow 0$ is exact. Then $\mathcal{T}$ is called **pure-semisimple** if any pure-triangle splits. This happens if and only if any object of $\mathcal{T}$ is pure-projective, i.e. $\mathcal{T} = \mathsf{Add}\,\mathcal{T}^{\mathsf{cpt}}$, if and only if any object of $\mathcal{T}$ is pure-injective if and only if $\mathcal{T}$ is phantomless. Note that the collection $\mathbf{Zg}(\mathcal{T})$ of isoclasses of indecomposable pure-injective objects of $\mathcal{T}$ form a small set, called the *Ziegler spectrum*, and admits a natural topology, the Ziegler topology, as in the case of locally finitely presented categories. We refer to [23] for more details.

Conventions and notations. If Λ is a ring, we denote by $\mathsf{Mod}\text{-}\Lambda$ the category of right Λ-modules. Left Λ-modules are treated as right Λ^{op}-modules. The category of finitely presented right Λ-modules is denoted by $\mathsf{mod}\text{-}\Lambda$. The category of projective, resp. injective, modules is denoted by $\mathsf{Proj}\,\Lambda$, resp. $\mathsf{Inj}\,\Lambda$. The stable category of $\mathsf{Mod}\text{-}\Lambda$, resp. $\mathsf{mod}\text{-}\Lambda$, modulo projectives is denoted by $\underline{\mathsf{Mod}}\text{-}\Lambda$, resp. $\underline{\mathsf{mod}}\text{-}\Lambda$, and the stable category of $\mathsf{Mod}\text{-}\Lambda$, resp. $\mathsf{mod}\text{-}\Lambda$, modulo injectives is denoted by $\overline{\mathsf{Mod}}\text{-}\Lambda$, resp. $\overline{\mathsf{mod}}\text{-}\Lambda$. If $\mathcal{X}$ is a subcategory of $\mathsf{Mod}\text{-}\Lambda$, we use the notations $\underline{\mathcal{X}}$ and $\overline{\mathcal{X}}$ for the stable categories of $\mathcal{X}$ modulo projectives and injectives respectively. Moreover we set $\mathcal{X}^{\mathsf{fin}} := \mathcal{X} \cap \mathsf{mod}\text{-}\Lambda$. In particular we set $(\mathsf{Proj}\,\Lambda)^{\mathsf{fin}} = \mathsf{proj}\,\Lambda$ and $(\mathsf{Inj}\,\Lambda)^{\mathsf{fin}} = \mathsf{inj}\,\Lambda$. If $\mathcal{X} \subseteq \mathsf{mod}\text{-}\Lambda$, then we denote by ${}^{\perp}\mathcal{X}$ the full subcategory $\{A \in \mathsf{Mod}\text{-}\Lambda \mid \mathsf{Ext}^n_{\Lambda}(A, X) = 0, \forall X \in \mathcal{X}\}$, and dually for $\mathcal{X}^{\perp}$. Then ${}^{\perp}\mathcal{X} = ({}^{\perp}\mathcal{X})^{\mathsf{fin}} = {}^{\perp}\mathcal{X} \cap \mathsf{mod}\text{-}\Lambda$. If Λ is an Artin R-algebra over a commutative Artin ring R, we denote by D the usual duality of Artin algebras which is given by $\mathsf{Hom}_R(-, E): \mathsf{Mod}\text{-}\Lambda \longrightarrow \mathsf{Mod}\text{-}\Lambda^{\mathsf{op}}$ where E is the injective envelope of $R/\mathsf{rad}(R)$. The Jacobson radical of Λ is denoted by $\mathfrak{r}$. For all unexplained notions and results concerning the representation theory of Artin algebras we refer to the book [18].

The additive categories considered in this paper are assumed to admit finite direct sums and all full subcategories are closed under isomorphisms. The composition of morphisms in a given category is meant in the diagrammatic order: the composition of $f : A \longrightarrow B$ and $g : B \longrightarrow C$ is denoted by $f \circ g$.

3. Contravariantly finite resolving subcategories

Recall that an additive category $\mathscr{A}$ is called a *Krull–Schmidt* category if any object of $\mathscr{A}$ is a finite coproduct of indecomposable objects and any indecomposable object has local endomorphism ring. Assume that $\mathscr{A}$ is a skeletally small Krull–Schmidt category. We denote by $\mathsf{Ind}\,\mathscr{A}$ the set of isoclasses of indecomposable objects of $\mathscr{A}$. We say that $\mathscr{A}$ is of *finite representation type* if the set $\mathsf{Ind}\,\mathscr{A}$ is finite. In this case $\mathscr{A}$ admits a *representation generator*, i.e. an object T such that $\mathscr{A} = \mathsf{add}\,T$. If $F : \mathscr{A} \longrightarrow \mathfrak{Ab}$ is a covariant or contravariant additive functor, then the *support* of F is defined by $\mathsf{Supp}\,F = \{A \in \mathsf{Ind}\,\mathscr{A} \mid F(A) \neq 0\}$.

Let Λ be an Artin algebra. Our aim in this section is to prove the following result which gives characterizations for a contravariantly finite resolving subcategory $\mathcal{X}$ of $\mathsf{mod}\text{-}\Lambda$ to be of finite representation type in terms of decomposition properties of the closure of $\mathcal{X}$ under filtered colimits.

Theorem 3.1. *Let $\mathcal{X}$ be a contravariantly finite resolving subcategory of $\mathsf{mod}\text{-}\Lambda$, and let $\mathscr{A} = \varinjlim \mathcal{X}$. Then the following statements are equivalent.*

(i) *$\mathcal{X}$ is of finite representation type.*
(ii) *Any module in $\mathscr{A}$ is a direct sum of finitely generated modules.*
(iii) *The representation category $\mathfrak{L}(\mathscr{A}) = \mathsf{Mod}\text{-}\mathcal{X}$ is locally finite.*
(iv) *The functor $(-, X_{\Lambda/\mathfrak{r}}) \in \mathsf{Mod}\text{-}\mathcal{X}$ has finite length.*
(v) *Any indecomposable module in $\mathscr{A}$ is finitely generated.*
(vi) *Any module in $\mathscr{A}$ is a direct sum of indecomposable modules.*
(vii) *$\mathscr{A}$ is equivalent to the category of projective modules over an Artin algebra.*
(viii) *$\mathcal{X}$ is equivalent to the category of finitely generated projective modules over an Artin algebra.*

Proof. Consider the representation functor $\mathsf{H} : \mathscr{A} \longrightarrow \mathfrak{L}(\mathscr{A}) = \mathsf{Mod}\text{-}\mathcal{X}$, $\mathsf{H}(A) = \mathsf{Hom}_\Lambda(-, A)|_{\mathcal{X}}$, which induces an equivalence $\mathscr{A} \approx \mathsf{Flat}(\mathcal{X})$. By a result of Krause and Solberg [54], the category $\mathcal{X}$ is also covariantly finite. It follows that $\mathcal{X}$ is a functorially finite extension closed subcategory of $\mathsf{mod}\text{-}\Lambda$. Therefore $\mathcal{X}$ is a dualizing R-subvariety of $\mathsf{mod}\text{-}\Lambda$ in the sense of [16]. In particular $\mathcal{X}$ has left and right almost split maps, in the sense of [16], and there exists an equivalence $(\mathsf{mod}\text{-}\mathcal{X}^{\mathsf{op}})^{\mathsf{op}} \approx \mathsf{mod}\text{-}\mathcal{X}$, see [13].

(i) $\Rightarrow$ (ii) Assuming (i), let T be a representation generator of $\mathcal{X}$, i.e. $\mathcal{X} = \mathsf{add}\,T$. If $\Gamma := \mathsf{End}_\Lambda(T)^{\mathsf{op}}$, then $\mathfrak{L}(\mathscr{A}) = \mathsf{Mod}\text{-}\mathcal{X} \approx \mathsf{Mod}\text{-}\Gamma$. Since Γ is an Artin algebra we have $\mathscr{A} = \varinjlim \mathcal{X} \approx \mathsf{Flat}(\mathsf{proj}\,\Gamma) \approx \mathsf{Proj}\,\Gamma$ and therefore any module in $\mathscr{A}$ is a direct sum of (indecomposable) finitely generated modules.

(ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) By hypothesis the locally finitely presented category $\mathscr{A} = \varinjlim \mathcal{X}$ is pure-semisimple. By the several object version of the well-known result of Bass, see [36], this implies that for any chain

$$X_0 \xrightarrow{f_0} X_1 \xrightarrow{f_1} X_2 \longrightarrow \cdots \longrightarrow X_n \xrightarrow{f_n} X_{n+1} \longrightarrow \cdots \tag{3.1}$$

of non-isomorphisms between indecomposable modules in $\mathcal{X}$, the composition $f_0 \circ f_1 \circ \cdots \circ f_m$ is zero for large m. We claim that any $0 \neq F \in \mathcal{X}\text{-}\mathsf{Mod}$ has a simple subfunctor. Assuming that this is not the case, we construct a chain of maps as in (3.1) as follows. Since $F \neq 0$, there exists an indecomposable module X_0 in $\mathcal{X}$ such that $F(X_0) \neq 0$. This means that there is a non-zero map $\alpha_0 : (X_0, -) \longrightarrow F$; in particular the functor $F_0 = \mathsf{Im}\, \alpha_0$ is non-zero. Let $\alpha_0 = \varepsilon_0 \circ i_0 : (X_0, -) \twoheadrightarrow F_0 \rightarrowtail F$ be the canonical factorization of α_0. By hypothesis F_0 is not simple and therefore F_0 contains a proper non-zero subfunctor $\mu_0 : F_1 \rightarrowtail F_0$. Since F_1 is non-zero, there exists an indecomposable module X_1 in $\mathcal{X}$ and a non-zero map $\alpha_1 : (X_1, -) \longrightarrow F_1$. Since $(X_1, -)$ is a projective functor, the composition $\alpha_1 \circ \mu_0 : (X_1, -) \longrightarrow F_1 \longrightarrow F_0$ factors through the epimorphism $\varepsilon_0 : (X_0, -) \twoheadrightarrow F_0$, i.e. $(f_0, -) \circ \varepsilon_0 = \alpha_1 \circ \mu_0$ for some map $f_0 : X_0 \longrightarrow X_1$. Observe that f_0 is non-zero since $\alpha_1 \neq 0$. Let $F_2 = \mathsf{Im}\, \alpha_1$ and let $\alpha_1 = \varepsilon_1 \circ i_1 : (X_1, -) \twoheadrightarrow F_2 \rightarrowtail F_1$ be the canonical factorization of α_1. Since the non-zero functor F_2 is a subfunctor of F, by hypothesis it is not simple. Hence F_2 contains a non-zero proper subfunctor $\mu_1 : F_3 \rightarrowtail F_2$. Then as above there exists an indecomposable module X_2 in $\mathcal{X}$ and a non-zero map $\alpha_2 : (X_2, -) \longrightarrow F_3$. Since the functor $(X_2, -)$ is projective, the map $\alpha_2 \circ \mu_1$ factors through ε_1: $(f_1, -) \circ \varepsilon_1 = \alpha_2 \circ \mu_1$ for some map $f_1 : X_1 \longrightarrow X_2$. Observe that $(f_0 \circ f_1, -) \circ \alpha_0 = (f_1, -) \circ (f_0, -) \circ \alpha_0 = \alpha_2 \circ \mu_1 \circ i_1 \circ \mu_0 \circ i_0$. Then $f_0 \circ f_1 \neq 0$ since α_2 is non-zero and $\mu_1 \circ i_1 \circ \mu_0 \circ i_0$ is a composition of monics. Pictorially then we have the following diagram:

$$
\begin{tikzcd}
\cdots & (X_3,-) \arrow[rr, dashed, "{(f_2,-)}"] \arrow[dd, "\alpha_3"] & & (X_2,-) \arrow[rr, dashed, "{(f_1,-)}"] \arrow[dl, two heads, "\varepsilon_2"] \arrow[dd, "\alpha_2"] & & (X_1,-) \arrow[rr, dashed, "{(f_0,-)}"] \arrow[dl, two heads, "\varepsilon_1"] \arrow[dd, "\alpha_1"] & & (X_0,-) \arrow[dl, two heads, "\varepsilon_0"] \arrow[dd, "\alpha_0"] \\
\cdots & & F_4 \arrow[dr, hook, "i_2"] & & F_2 \arrow[dr, hook, "i_1"] & & F_0 \arrow[dr, hook, "i_0"] & \\
\cdots & F_5 \arrow[ur, hook, "\mu_2"] & & F_3 \arrow[ur, hook, "\mu_1"] & & F_1 \arrow[ur, hook, "\mu_0"] & & F
\end{tikzcd}
$$

Continuing in this way we produce a chain of maps between indecomposable modules in $\mathcal{X}$ as in (3.1) with non-zero composition $f_0 \circ f_1 \circ \cdots \circ f_n : X_0 \longrightarrow X_{n+1}$, $\forall n \geqslant 0$. This contradiction shows that any non-zero functor $\mathcal{X} \longrightarrow \mathfrak{Ab}$ contains a simple subfunctor. On the other hand since $\mathcal{X}$ has left and right almost split maps, by a well-known result of Auslander [8] it follows that the functor category $\mathcal{X}$-Mod is locally finite, i.e. the coherent functors $\mathcal{X}$-mod form a length category. Then the existence of a duality $(\mathcal{X}\text{-}\mathsf{mod})^{\mathsf{op}} \approx \mathsf{mod}\text{-}\mathcal{X}$ implies that mod-$\mathcal{X}$ is a length category, i.e. Mod-$\mathcal{X}$ is also locally finite. If this holds, then any representable functor, in particular $(-, X_{\Lambda/\mathfrak{r}})$, is of finite length.

(iv) $\Rightarrow$ (i) By a result of Auslander–Reiten [14] we have $\mathcal{X} = \mathsf{Filt}(X_{\Lambda/\mathfrak{r}})$, that is, $\mathcal{X}$ consists of the direct summands of modules A admitting a finite filtration

$$0 = A_{t+1} \subseteq A_t \subseteq A_{t-1} \subseteq \cdots \subseteq A_2 \subseteq A_1 \subseteq A_0 = A \tag{3.2}$$

such that the subquotients A_i / A_{i+1}, for $0 \leqslant i \leqslant t$, lie in the set $\mathcal{X}(\mathcal{S}) = \{X_{S_1}, X_{S_2}, \dots, X_{S_m}\}$, where $\mathcal{S} = \{S_1, S_2, \dots, S_m\}$ is the set of isoclasses of the simple Λ-modules and $X_{S_i} \longrightarrow S_i$ is the minimal right $\mathcal{X}$-approximation of S_i. Note that $\mathsf{add}\, \mathcal{X}(\mathcal{S}) = \mathsf{add}(X_{\Lambda/\mathfrak{r}})$. By induction on the length t of the filtration (3.2) it follows directly that $A \in \mathsf{Supp}(-, X_{\Lambda/\mathfrak{r}})$ and consequently $\mathsf{Ind}(\mathcal{X}) \subseteq \mathsf{Supp}(-, X_{\Lambda/\mathfrak{r}})$. Hence

$$\mathsf{Ind}(\mathcal{X}) = \mathsf{Supp}(-, X_{\Lambda/\mathfrak{r}}) \tag{3.3}$$

Since the functor $(-, X_{\Lambda/\mathfrak{r}})$ has finite length, there exists a composition series in $\mathsf{Mod}\text{-}\mathcal{X}$:

$$0 = F_{n+1} \subseteq F_n \subseteq F_{n-1} \subseteq \cdots \subseteq F_2 \subseteq F_1 \subseteq F_0 = (-, X_{\Lambda/\mathfrak{r}}) \tag{3.4}$$

where each subquotient $\mathbb{S}_i := F_i/F_{i+1}$ is simple. It follows that $|\mathsf{Supp}\,\mathbb{S}_i| < \infty$, $0 \leqslant i \leqslant n$, since if X_1, X_2 are indecomposable objects in $\mathcal{X}$ with $\mathbb{S}_i(X_1) \neq 0 \neq \mathbb{S}_i(X_2)$, for some i, then $X_1 \cong X_2$. Since clearly

$$\mathsf{Supp}(-, X_{\Lambda/\mathfrak{r}}) = \bigcup_{i=0}^{n} \mathsf{Supp}\,\mathbb{S}_i \tag{3.5}$$

we have $\mathsf{Supp}(-, X_{\Lambda/\mathfrak{r}}) < \infty$. Then (3.3), shows that $|\mathsf{Ind}\,\mathcal{X}| < \infty$, so $\mathcal{X}$ is of finite representation type.

(ii) $\Leftrightarrow$ (v) Clearly (ii) $\Rightarrow$ (v). Assume that any indecomposable module in $\mathscr{A}$ is finitely generated. Consider the Ziegler spectra $\mathbf{Zg}(\mathscr{A})$ and $\mathbf{Zg}(\mathsf{Mod}\text{-}\Lambda)$ of the locally finitely presented categories $\mathscr{A} = \varinjlim \mathcal{X}$ and $\mathsf{Mod}\text{-}\Lambda = \varinjlim \mathsf{mod}\text{-}\Lambda$. Then $\mathbf{Zg}(\mathscr{A}) = \mathsf{Ind}(\mathcal{X})$ since any indecomposable module in $\mathscr{A}$ is finitely generated and any finitely generated module over an Artin algebra is pure-injective in $\mathsf{Mod}\text{-}\Lambda$. On the other hand since $\mathcal{X}$ has left almost split morphisms, there exists a left almost split morphism $\phi : X \longrightarrow Y$ in $\mathcal{X}$, for any point X in $\mathsf{Ind}(\mathcal{X})$. If F lies in $\mathsf{Ind}(\mathcal{X}) \setminus \{X\}$ and $\alpha : X \longrightarrow F$ is a map, then α factors through ϕ since otherwise α would be split monic implying that $X \cong F$ which is not the case. Hence $\mathsf{Ind}(\mathcal{X}) \setminus \{X\}$ consists of ϕ-injectives, i.e. $\mathsf{Ind}(\mathcal{X}) \setminus \{X\}$ is closed and therefore $\{X\}$ is Ziegler-open. This means that any point X in $\mathbf{Zg}(\mathscr{A})$ is isolated in the Ziegler topology of $\mathbf{Zg}(\mathscr{A})$. We claim that $\mathbf{Zg}(\mathscr{A})$ is a quasi-compact space, i.e. any open cover of $\mathbf{Zg}(\mathscr{A})$ admits a finite subcover. So assume that $\mathbf{Zg}(\mathscr{A}) = \bigcup_{i \in I} \mathbf{V}_i$ where each $\mathbf{V}_i$ is Ziegler-open subset. Then $\mathbf{V}_i$ is of the form $\mathbf{V}_i = \{G \in \mathbf{Zg}(\mathscr{A}) \mid \text{the map } \mathsf{Hom}_\Lambda(Y, G) \longrightarrow \mathsf{Hom}_\Lambda(X, G) \text{ is not surjective}\}$ for some collection of maps $X \longrightarrow Y$ between modules in $\mathcal{X}$. However each $\mathbf{V}_i$ is also a Ziegler-open in $\mathbf{Zg}(\mathsf{Mod}\text{-}\Lambda)$ since pure-injective modules in $\mathscr{A}$ remain pure-injective in $\mathsf{Mod}\text{-}\Lambda$ and $\mathcal{X}$ consists of finitely generated modules. Since the space $\mathbf{Zg}(\mathsf{Mod}\text{-}\Lambda)$ is quasi-compact, see [50], we infer that there exists a finite subcover: $\mathbf{Zg}(\mathscr{A}) = \bigcup_{i \in J} \mathbf{V}_i$ for some finite subset J of I. It follows that $\mathbf{Zg}(\mathscr{A})$ is quasi-compact. Then the open cover $\mathbf{Zg}(\mathscr{A}) = \bigcup_{X \in \mathsf{Ind}(\mathcal{X})} \{X\}$ admits a finite subcover $\mathbf{Zg}(\mathscr{A}) = \bigcup_{i=1}^{n} \{X_i\}$. This means that $\mathsf{Ind}(\mathcal{X}) = \{X_1, X_2, \ldots, X_n\}$, i.e. $\mathcal{X}$ is of finite representation type.

(i) $\Leftrightarrow$ (vi) Taking into account that (i) is equivalent to (ii) and the fact that $\mathcal{X}$ is a Krull–Schmidt category, we infer that (i) implies (vi). If (vi) holds then, by [50, Corollary 2.7], $\mathscr{A}$ is pure-semisimple and then as in the proof of the direction (ii) $\Rightarrow$ (iii) we deduce that $\mathcal{X}$ is of finite representation type.

Finally the equivalences (i) $\Leftrightarrow$ (vii) $\Leftrightarrow$ (viii) are straightforward. □

Remark 3.2. (i) There is a dual version of Theorem 3.1 concerning coresolving covariantly finite subcategories of $\mathsf{mod}\text{-}\Lambda$. We leave its formulation to the reader.

(ii) Let Λ be connected and let $\mathcal{X}$ be a contravariantly finite resolving subcategory of $\mathsf{mod}\text{-}\Lambda$. Using Harada–Sai's Lemma as in [59] it follows that $\mathcal{X}$ is of finite representation type if and only if the (relative) Auslander–Reiten quiver $\Gamma(\mathcal{X})$, defined by using irreducible maps in $\mathcal{X}$, contains a component whose indecomposable modules are of bounded length.

Subcategories of $\mathsf{Mod}\text{-}\Lambda$ of the form $\varinjlim \mathcal{X}$, where $\mathcal{X}$ is a covariantly finite subcategory of $\mathsf{mod}\text{-}\Lambda$, are important examples of *definable* subcategories of $\mathsf{Mod}\text{-}\Lambda$, that is, subcategories closed under filtered colimits, products and pure submodules. On the other hand if $\mathscr{A}$ is a coresolving definable subcategory of $\mathsf{Mod}\text{-}\Lambda$ and any module in $\mathscr{A}$ is a direct sum of finitely generated modules, then clearly $\mathscr{A} = \varinjlim \mathcal{X}$ where $\mathcal{X}$ is a coresolving subcategory of $\mathsf{mod}\text{-}\Lambda$ and therefore $\mathscr{A}$ is locally finitely presented. Thus as a consequence of Theorem 3.1 we have the following.

Corollary 3.3. *For a coresolving definable subcategory $\mathscr{A}$ of $\mathsf{Mod}\text{-}\Lambda$ the following are equivalent.*

(i) *Any module in $\mathscr{A}$ is a direct sum of finitely generated modules.*
(ii) *$\mathscr{A} = \varinjlim \mathcal{Y}$, where $\mathcal{Y} \subseteq \mathsf{mod}\text{-}\Lambda$ is a coresolving subcategory of finite representation type.*

Corollary 3.4. *Let $(\mathscr{A}, \mathscr{B})$ be a cotorsion pair in $\mathsf{Mod}\text{-}\Lambda$. Then the following are equivalent.*

(i) *$(\mathscr{A}, \mathscr{B})$ is of finite type and the subcategory $\mathscr{A}^{\mathsf{fin}} := \mathscr{A} \cap \mathsf{mod}\text{-}\Lambda$ is of finite representation type.*
(ii) *$\mathscr{A}^{\mathsf{fin}}$ is contravariantly finite and any module in $\mathscr{A}$ is a direct sum of finitely generated modules.*

Proof. If (i) holds, then since $\mathscr{A}^{\mathsf{fin}}$ is contravariantly finite and $(\mathscr{A}, \mathscr{B})$ is of finite type, we have $\mathscr{A} = \varinjlim \mathscr{A}^{\mathsf{fin}}$ by [3, Theorem 5.3]. Since $\mathscr{A}^{\mathsf{fin}}$ is resolving, the assertion in (ii) follows from Theorem 3.1. If (ii) holds, then $\mathscr{A} = \varinjlim \mathscr{A}^{\mathsf{fin}} = \mathsf{Add}\, \mathscr{A}^{\mathsf{fin}}$. This implies easily that $\mathscr{B} = (\mathscr{A}^{\mathsf{fin}})^{\perp}$ so the cotorsion pair $(\mathscr{A}, \mathscr{B})$ is of finite type. Since $\mathscr{A}^{\mathsf{fin}}$ is contravariantly finite, $\mathscr{A}^{\mathsf{fin}}$ is representation finite by Theorem 3.1. $\square$

Recall that if $\mathscr{C}$ is a class of modules over a ring R, then a module X is called $\mathscr{C}$-**filtered** if there exists an ordinal τ and an increasing chain $\{X_\alpha \mid \alpha < \tau\}$ of submodules of X such that $X_0 = X$, $X_\alpha = \bigcup_{\beta<\alpha} X_\beta$ for each limit ordinal $\alpha < \tau$, $X = \bigcup_{\alpha<\tau} X_\alpha$, and each subquotient $X_{\alpha+1}/X_\alpha$ is isomorphic to a module in $\mathscr{C}$, for any $\alpha + 1 < \tau$. Note that if $\mathcal{X} \subseteq \mathsf{mod}\text{-}R$, then the category $\mathfrak{F}(\mathcal{X})$ of $\mathcal{X}$-filtered modules is contained in $\varinjlim \mathcal{X}$; this inclusion is an equality if $\mathfrak{F}(\mathcal{X})$ is closed under filtered colimits.

Corollary 3.5. *For a resolving subcategory $\mathcal{X}$ of $\mathsf{mod}\text{-}\Lambda$, the following are equivalent.*

(i) *$\mathcal{X}$ is of finite representation type.*
(ii) *$\mathcal{X}$ is contravariantly finite and any module in $\varinjlim \mathcal{X}$ is a direct sum of finitely generated modules.*
(iii) *$\mathcal{X}$ is contravariantly finite and any $\mathcal{X}$-filtered module is a direct sum of finitely generated modules.*

Proof. (i) $\Leftrightarrow$ (ii) follows from Theorem 3.1. By [2], the subcategory $\mathcal{X}$ cogenerates a cotorsion pair of finite type $(\mathscr{A}, \mathscr{B})$ in $\mathsf{Mod}\text{-}\Lambda$, where $\mathscr{B} = \mathcal{X}^{\perp}$ and $\mathscr{A} = {}^{\perp}\mathscr{B}$, and moreover we have $\mathscr{A}^{\mathsf{fin}} = \mathcal{X}$ and $\mathscr{A}$ consists of all direct summands of $\mathcal{X}$-filtered modules. Then (i) $\Leftrightarrow$ (iii) follows from Corollary 3.4. $\square$

Example 3.6. Let T be a (possibly infinitely generated) tilting module over Λ, see for instance [3]. Then $(\mathcal{X}, T^{\perp})$ is a cotorsion pair of finite type in $\mathsf{Mod}\text{-}\Lambda$, where $\mathcal{X} = {}^{\perp}(T^{\perp})$, see [19]. Hence $\mathcal{X}^{\mathsf{fin}}$ is of finite representation type if and only if $\mathcal{X}^{\mathsf{fin}}$ is contravariantly finite and any module in $\mathcal{X}$ is a direct sum of finitely generated modules. If T is finitely generated, then any module in $T^{\perp}$ is a direct sum of indecomposable modules if and only if $T^{\perp}$ is of finite representation type and $T^{\perp} = \varinjlim T^{\perp}$. Similar remarks apply for finitely generated cotilting modules.

Applying Theorem 3.1 to $\mathcal{X} = \mathsf{mod}\text{-}\Lambda$ we have the following version of the classical result, due to Auslander and Ringel–Tachikawa, characterizing Artin algebras of finite representation type.

Corollary 3.7. *(See [9,8,63].) For an Artin algebra Λ the following are equivalent.*

(i) *Λ is of finite representation type.*
(ii) *Any Λ-module is a direct sum of finitely generated modules.*
(iii) *Any Λ-module is a direct sum of indecomposable modules.*
(iv) *Any indecomposable Λ-module is finitely generated.*
(v) *The category $\mathsf{Mod}\text{-}(\mathsf{mod}\text{-}\Lambda)$ is locally finite.*
(vi) *The functor $(-, \Lambda/\mathfrak{r}) \in \mathsf{Mod}\text{-}(\mathsf{mod}\text{-}\Lambda)$ has finite length.*
(vii) *$\mathsf{Mod}\text{-}\Lambda$ is equivalent to the category of projective modules over an Artin algebra.*
(viii) *$\mathsf{mod}\text{-}\Lambda$ is equivalent to the category of finitely generated projective modules over an Artin algebra.*

If $\mathscr{E}$ is an exact category, let $\mathsf{K}_0(\mathscr{E})$ be the Grothendieck group of $\mathscr{E}$ and $\mathsf{K}_0(\mathscr{E}, \oplus)$ be the Grothendieck group of $\mathscr{E}$ endowed with the split exact structure; e.g. if $\mathcal{X} \subseteq \mathsf{mod}\text{-}\Lambda$ is resolving, then $\mathcal{X}$, $\mathcal{X}^{\perp}$ and $\mathcal{X} \cap \mathcal{X}^{\perp}$ are exact subcategories of $\mathsf{mod}\text{-}\Lambda$ since they are closed under extensions. Note that $\mathsf{K}_0(\mathcal{X} \cap \mathcal{X}^{\perp}) = \mathsf{K}_0(\mathcal{X} \cap \mathcal{X}^{\perp}, \oplus)$ since $\mathcal{X} \cap \mathcal{X}^{\perp}$ is Ext-orthogonal. Working as in [24, Theorem 8.9, Corollary 8.10], we have the following.

Proposition 3.8. *Let $\mathcal{X}$ be a contravariantly resolving subcategory of* $\mathsf{mod}\text{-}\Lambda$.

(i) *There is a decomposition $\mathsf{K}_0(\mathcal{X}) \oplus \mathsf{K}_0(\mathcal{X}^{\perp}) \cong \mathsf{K}_0(\mathcal{X} \cap \mathcal{X}^{\perp}) \oplus \mathsf{K}_0(\mathsf{mod}\text{-}\Lambda)$.*
(ii) *The following are equivalent.*
  (a) *$\mathcal{X}$ is of finite representation type.*
  (b) *The set $\{[X_1] - [X_2] + [X_3]\} \cup \{[X_{\mathfrak{r}P}] - [P]\}$ is a free basis of $\mathsf{K}_0(\mathcal{X}, \oplus)$, where $0 \longrightarrow X_1 \longrightarrow X_2 \longrightarrow X_3 \longrightarrow 0$ is an Auslander–Reiten sequence in $\mathcal{X}$ and $X_{\mathfrak{r}P}$ is the minimal right $\mathcal{X}$-approximation of $\mathfrak{r}P$, for any indecomposable projective module P.*

3.1. Weyl modules and quasi-hereditary algebras

We consider modules having good filtrations which arise in the study of highest weight categories. Let Λ be a quasi-hereditary algebra. We denote by Δ the Weyl modules and by ∇ the induced modules, see [58] for more details. Let $\mathsf{Filt}\,\mathsf{Add}\,\Delta$, resp. $\mathsf{Filt}\,\Delta$, be the full subcategory of $\mathsf{Mod}\text{-}\Lambda$, resp. $\mathsf{mod}\text{-}\Lambda$, consisting of the modules with Weyl filtration. Dually let $\mathsf{Filt}\,\mathsf{Add}\,\nabla$, resp. $\mathsf{Filt}\,\nabla$, the full subcategory of $\mathsf{Mod}\text{-}\Lambda$, resp. $\mathsf{mod}\text{-}\Lambda$ consisting of the modules with good filtration. Recall from [59] that Λ is called *Δ-finite*, resp. *∇-finite*, if the full subcategory of finitely generated Weyl, resp. good, modules is of finite representation type. By results of Ringel [58] it follows that $\mathsf{Filt}\,\nabla$ is a contravariantly finite resolving subcategory of $\mathsf{mod}\text{-}\Lambda$ and by results of Krause and Solberg [54] we have $\mathsf{Filt}\,\mathsf{Add}\,\Delta = \varinjlim \mathsf{Filt}\,\Delta$ and $\mathsf{Filt}\,\mathsf{Add}\,\nabla = \varinjlim \mathsf{Filt}\,\nabla$. Hence by Theorem 3.1 and Corollary 3.3 we have the following consequence.

Corollary 3.9. *Let Λ be a quasi-hereditary algebra. Then Λ is Δ-finite, resp. ∇-finite, if and only if any Weyl, resp. good, module is a direct sum of finitely generated Weyl, resp. good, modules.*

3.2. Submodule categories

For an abelian category $\mathscr{A}$, we denote by $\mathsf{proj}^{\leqslant 1}\,\mathscr{A}$ the full subcategory of $\mathscr{A}$ consisting of all objects A with $\mathsf{pd}\,A \leqslant 1$, and by $\mathsf{Sub}(\mathscr{A})$ the *subobject category* of $\mathscr{A}$; the objects of $\mathsf{Sub}(\mathscr{A})$ are monomorphisms $\mu : A \rightarrowtail B$ in $\mathscr{A}$ and a morphism in $\mathsf{Sub}(\mathscr{A})$ between $\mu : A \rightarrowtail B$ and $\mu' : A' \rightarrowtail B'$ is given by a pair of morphisms $g : A \longrightarrow A'$ and $f : B \longrightarrow B'$ such that $\mu \circ f = g \circ \mu'$.

If Λ is an Artin algebra of finite representation type, then we denote by $\mathsf{A}(\Lambda)$ the *Auslander algebra* of Λ, i.e. $\mathsf{A}(\Lambda) = \mathsf{End}_\Lambda(T)^{\mathsf{op}}$, where T is the direct sum of the isoclasses of indecomposable finitely generated modules.

Proposition 3.10. *If Λ is an Artin algebra, then there is an equivalence*

$$\mathsf{Sub}(\mathsf{mod}\text{-}\Lambda)/\mathsf{mod}\text{-}\Lambda \xrightarrow{\approx} \mathsf{proj}^{\leqslant 1}\,\mathsf{mod}\text{-}(\mathsf{mod}\text{-}\Lambda) \tag{3.6}$$

where we identify $\mathsf{mod}\text{-}\Lambda$ with the full subcategory $\mathfrak{U}$ of $\mathsf{Sub}(\mathsf{mod}\text{-}\Lambda)$ consisting of the isomorphisms. Moreover the following statements are equivalent.

(i) $\mathsf{Sub}(\mathsf{mod}\text{-}\Lambda)$ *is of finite representation type.*
(ii) *Any monomorphism in* $\mathsf{Mod}\text{-}\Lambda$ *is a direct sum of monomorphisms in* $\mathsf{mod}\text{-}\Lambda$.
(iii) *Any indecomposable object of* $\mathsf{Sub}(\mathsf{Mod}\text{-}\Lambda)$ *lies in* $\mathsf{Sub}(\mathsf{mod}\text{-}\Lambda)$.
(iv) $\mathsf{proj}^{\leqslant 1}\,\mathsf{mod}\text{-}(\mathsf{mod}\text{-}\Lambda)$ *is of finite representation type.*
(v) Λ *and* $\mathsf{proj}^{\leqslant 1}\,\mathsf{A}(\Lambda)$ *are of finite representation type.*

Proof. Let $\mathsf{Y} : \mathsf{mod}\text{-}\Lambda \longrightarrow \mathsf{mod}\text{-}(\mathsf{mod}\text{-}\Lambda)$, $A \longmapsto \mathsf{Y}(A) = \mathsf{Hom}_\Lambda(-, A)$, be the Yoneda embedding which induces an equivalence between $\mathsf{mod}\text{-}\Lambda$ and $\mathsf{proj}\,\mathsf{mod}\text{-}(\mathsf{mod}\text{-}\Lambda)$. Define a functor $\mathsf{H} : \mathsf{Sub}(\mathsf{mod}\text{-}\Lambda) \longrightarrow \mathsf{mod}\text{-}(\mathsf{mod}\text{-}\Lambda)$ by $\mathsf{H}(A' \rightarrowtail A) = \mathsf{Coker}(\mathsf{Y}(A') \longrightarrow \mathsf{Y}(A))$. Clearly the essential image of H coincides with the full subcategory $\mathsf{proj}^{\leqslant 1}\,\mathsf{mod}\text{-}(\mathsf{mod}\text{-}\Lambda)$. It is easy to see that H induces a full functor $\mathsf{H} : \mathsf{Sub}(\mathsf{mod}\text{-}\Lambda) \longrightarrow \mathsf{proj}^{\leqslant 1}\,\mathsf{mod}\text{-}(\mathsf{mod}\text{-}\Lambda)$ which is surjective on objects and moreover a map in $\mathsf{Sub}(\mathsf{mod}\text{-}\Lambda)$ is killed by H iff it factorizes through an object of the form $(X \xrightarrow{\cong} X)$, i.e. $\mathsf{Ker}\,\mathsf{H} = \mathfrak{U}$. Since the categories $\mathsf{mod}\text{-}\Lambda$ and $\mathfrak{U}$ are equivalent via the functor $A \longmapsto (A \xrightarrow{=} A)$, it follows that H induces the desired equivalence (3.6).

As in [61] we identify the morphism category of $\mathsf{Mod}\text{-}\Lambda$ with the module category over the triangular matrix algebra $T_2(\Lambda) := \left(\begin{smallmatrix}\Lambda & \Lambda\\ 0 & \Lambda\end{smallmatrix}\right)$. Under this identification $\mathsf{Sub}(\mathsf{mod}\text{-}\Lambda)$ is a full subcategory of $\mathsf{mod}\text{-}T_2(\Lambda)$ and $\mathsf{Sub}(\mathsf{Mod}\text{-}\Lambda)$ is a full subcategory of $\mathsf{Mod}\text{-}T_2(\Lambda)$, and both contain the projectives and are closed under extensions and submodules. Since filtered colimits are exact, it follows that $\varinjlim \mathsf{Sub}(\mathsf{mod}\text{-}\Lambda) \subseteq \mathsf{Sub}(\mathsf{Mod}\text{-}\Lambda)$. On the other hand it is well known that any monomorphism $f : A \rightarrowtail B$ in $\mathsf{Mod}\text{-}\Lambda$ can be written as a filtered colimit of monomorphisms between finitely generated Λ-modules, see [49]. Therefore $\mathsf{Sub}(\mathsf{Mod}\text{-}\Lambda) = \varinjlim \mathsf{Sub}(\mathsf{mod}\text{-}\Lambda)$. Since

$\mathsf{Sub}(\mathsf{mod}\text{-}\Lambda)$ is contravariantly finite in $\mathsf{mod}\text{-}T_2(\Lambda)$, see [61], the first three conditions are equivalent by Theorem 3.1.

(i) $\Leftrightarrow$ (iv) If $\mathsf{Sub}(\mathsf{mod}\text{-}\Lambda)$ is of finite representation type, then the equivalence (3.6) shows that so is the full subcategory $\mathsf{proj}^{\leqslant 1}\,\mathsf{mod}\text{-}(\mathsf{mod}\text{-}\Lambda)$. Conversely if this holds, then the full subcategory $\mathcal{U} \approx \mathsf{mod}\text{-}\Lambda \approx \mathsf{proj}\,\mathsf{mod}\text{-}(\mathsf{mod}\text{-}\Lambda) \subseteq \mathsf{proj}^{\leqslant 1}\,\mathsf{mod}\text{-}(\mathsf{mod}\text{-}\Lambda)$ is of finite representation type, i.e. Λ is of finite representation type, and the equivalence (3.6) shows that $\mathsf{Sub}(\mathsf{mod}\text{-}\Lambda)$ is of finite representation type. Finally (iv) $\Leftrightarrow$ (v) follows from the fact that $\mathsf{mod}\text{-}(\mathsf{mod}\text{-}\Lambda) \approx \mathsf{mod}\text{-}\mathsf{A}(\Lambda)$, provided that Λ is of finite representation type. $\square$

Example 3.11. For $n \geqslant 1$, let $\Lambda_n = k[t]/(t^n)$. By results of Ringel and Schmidmeier [62], $\mathsf{Sub}(\mathsf{mod}\text{-}\Lambda_n)$ is of finite representation type iff $n \leqslant 5$. It is well known that the Auslander algebra $\mathsf{A}(\Lambda_n)$ of Λ_n is quasi-hereditary and the Weyl modules coincide with the modules with projective dimension $\leqslant 1$. So by the above results, $\mathsf{Sub}(\mathsf{mod}\text{-}\Lambda_n)$ is of finite representation type iff $\mathsf{A}(\Lambda_n)$ is Δ-finite iff any monomorphism in $\mathsf{Mod}\text{-}\Lambda_n$ is a direct sum of monomorphisms in $\mathsf{mod}\text{-}\Lambda_n$ iff $n \leqslant 5$.

3.3. Torsion pairs

Let Λ be an Artin algebra. We consider full subcategories $\mathcal{F}$ of $\mathsf{mod}\text{-}\Lambda$ which are closed under extensions and subobjects. Clearly any such subcategory forms the torsion-free part $\mathcal{F}$ of a torsion pair $(\mathcal{T}, \mathcal{F})$ in $\mathsf{mod}\text{-}\Lambda$, where $\mathcal{T} = \{A \in \mathsf{mod}\text{-}\Lambda \mid \mathsf{Hom}_\Lambda(A, \mathcal{F}) = 0\}$. For instance, since for an algebra Λ of finite representation type we have $\mathsf{gl.dim}\,\mathsf{A}(\Lambda) \leqslant 2$, we can take $\mathcal{F}$ to be the full subcategory $\mathsf{proj}^{\leqslant 1}\,\mathsf{A}(\Lambda)$ considered above.

Proposition 3.12. *Let $\mathcal{F}$ be a full subcategory of $\mathsf{mod}\text{-}\Lambda$ which contains Λ and is closed under extensions and submodules. Then the following are equivalent.*

- (i) *$\mathcal{F}$ is of finite representation type.*
- (ii) *$\mathcal{F}$ is contravariantly finite and any module in $\varinjlim \mathcal{F}$ is a direct sum of finitely generated modules.*
- (iii) *$\mathcal{F}$ is contravariantly finite and any indecomposable module in $\varinjlim \mathcal{F}$ is finitely generated.*
- (iv) *$\mathcal{F} = {}^{\perp}T$, where T is a finitely generated cotilting module with $\mathsf{id}\,T \leqslant 1$, and any, resp. indecomposable, module in ${}^{\perp}T$ is a direct sum of finitely generated modules, resp. finitely generated.*

Proof. Clearly $\mathcal{F}$ is resolving so the first three conditions are equivalent by Theorem 3.1, Corollary 3.5.

(i) $\Rightarrow$ (iv) Since $\mathcal{F}$ is contravariantly finite, there exists a cotorsion pair $(\mathcal{F}, \mathcal{R})$ in $\mathsf{mod}\text{-}\Lambda$, see [14], and clearly $\mathsf{res.dim}_{\mathcal{F}}\,\mathsf{mod}\text{-}\Lambda \leqslant 1$. Hence by [14] there exists a cotilting module T in $\mathsf{mod}\text{-}\Lambda$ with $\mathsf{id}\,T \leqslant 1$ such that ${}^{\perp}T = \mathcal{F}$. It is not difficult to see that $\varinjlim \mathcal{F} = {}^{\perp}T$, and then (iv) follows form (ii) and (iii).

(iv) $\Rightarrow$ (i) The hypothesis implies that we have a cotorsion pair $(\varinjlim \mathcal{F}, \mathcal{Y})$ in $\mathsf{Mod}\text{-}\Lambda$ and $(\varinjlim \mathcal{F})^{\mathsf{fin}} = \mathcal{F} = {}^{\perp}T$ is contravariantly finite in $\mathsf{mod}\text{-}\Lambda$. Then $\mathcal{F}$ is of finite representation type by Corollary 3.4. $\square$

Subcategories of the form ${}^{\perp}T$, where T is a finitely generated cotilting module with $\mathsf{id}_{\Lambda} T \leqslant 1$, are closed under submodules, extensions, filtered colimits and contain Λ. Thus the following gives a converse to Proposition 3.12.

Corollary 3.13. *Let $\mathscr{A}$ be a full subcategory of* Mod-Λ *which contains Λ and is closed under extensions, submodules and filtered colimits. Then the following are equivalent:*

(i) *$\mathscr{A}^{\mathsf{fin}}$ is of finite representation type.*
(ii) *Any module in $\mathscr{A}$ is a direct sum of indecomposable modules.*
(iii) *Any indecomposable module in $\mathscr{A}$ is finitely generated.*

If (i) *holds, then $\mathscr{A} = {}^{\perp}T$, where T is a finitely generated* 1*-cotilting module and $\mathscr{A}$ is definable.*

Proof. By [30] it follows that $\mathscr{A} = \varinjlim \mathscr{A}^{\mathsf{fin}}$ and clearly $\mathscr{A}^{\mathsf{fin}}$ is resolving. Then the assertions follow from Theorem 3.1 and Proposition 3.12. $\square$

4. Artin algebras and local rings of finite CM-type

In this section we apply the results of the previous section to the study of Gorenstein-projective modules over an Artin algebra. In particular we give several characterizations of virtually Gorenstein Artin algebras of finite Cohen–Macaulay type in terms of decomposition properties of the category of Gorenstein-projective modules. We also prove analogous results in the commutative setting.

4.1. Gorenstein-projective modules and virtually Gorenstein algebras

Let $\mathscr{A}$ be an abelian category. An acyclic complex of projective, resp. injective, objects of $\mathscr{A}$

$$X^{\bullet} = \cdots \longrightarrow X^{i-1} \longrightarrow X^{i} \longrightarrow X^{i+1} \longrightarrow \cdots$$

is called **totally acyclic**, if $\mathsf{Hom}_{\mathscr{A}}(X^{\bullet}, P)$, resp. $\mathsf{Hom}_{\mathscr{A}}(I, X^{\bullet})$, is acyclic, $\forall P \in \mathsf{Proj}\,\mathscr{A}$, resp. $\forall I \in \mathsf{Inj}\,\mathscr{A}$.

The following classes of objects have been introduced by Auslander and Bridger [11] in the context of finitely generated modules over a Noetherian ring, and by Enochs and Jenda [37] in the context of arbitrary modules over any ring.

Definition 4.1. Let $\mathscr{A}$ be an abelian category. An object $A \in \mathscr{A}$ is called:

(i) **Gorenstein-projective** if it is of the form $A = \mathsf{Coker}(X^{-1} \longrightarrow X^{0})$ for some totally acyclic complex $X^{\bullet}$ of projective objects.
(ii) **Gorenstein-injective** if it is of the from $A = \mathsf{Ker}(X^{0} \longrightarrow X^{1})$ for some totally acyclic complex $X^{\bullet}$ of injective objects.

Remark 4.2. Depending on the setting, finitely generated Gorenstein-projective modules appear in the literature under various names, for instance: totally reflexive modules [44], modules of G-dimension zero [11,68] or (maximal) Cohen–Macaulay modules [15,27].

From now on we fix an Artin algebra Λ. We denote by $\mathsf{GProj}\,\Lambda$ the full subcategory of $\mathsf{Mod}\text{-}\Lambda$ which is formed by all Gorenstein-projective Λ-modules, and $\mathsf{GInj}\,\Lambda$ denotes the full subcategory which is formed by all Gorenstein-injective Λ-modules. In [24], it is shown that $\mathsf{GProj}\,\Lambda$ is a definable subcategory of $\mathsf{Mod}\text{-}\Lambda$ and it is the largest resolving subcategory of $\mathsf{Mod}\text{-}\Lambda$ which admits the full subcategory $\mathsf{Proj}\,\Lambda$ of projective modules as an Ext-injective cogenerator. Moreover there are cotorsion pairs

$$\big(\mathsf{GProj}\,\Lambda, (\mathsf{GProj}\,\Lambda)^{\perp}\big) \quad \text{and} \quad \big({}^{\perp}(\mathsf{GInj}\,\Lambda), \mathsf{GInj}\,\Lambda\big)$$

for $\mathsf{Mod}\text{-}\Lambda$ satisfying: $\mathsf{GProj}\,\Lambda \cap (\mathsf{GProj}\,\Lambda)^{\perp} = \mathsf{Proj}\,\Lambda$ and ${}^{\perp}(\mathsf{GInj}\,\Lambda) \cap \mathsf{GInj}\,\Lambda = \mathsf{Inj}\,\Lambda$. Throughout we shall use the category $\mathsf{Gproj}\,\Lambda$ of finitely generated Gorenstein-projective modules and the category $\mathsf{Ginj}\,\Lambda$ of finitely Gorenstein-injective modules defined as follows:

$$\mathsf{Gproj}\,\Lambda := \mathsf{GProj}\,\Lambda \cap \mathsf{mod}\text{-}\Lambda \quad \text{and} \quad \mathsf{Ginj}\,\Lambda := \mathsf{GInj}\,\Lambda \cap \mathsf{mod}\text{-}\Lambda$$

Remark 4.3. Note that the Λ-dual functor $\mathsf{d} := \mathsf{Hom}_{\Lambda}(-,\Lambda) : \mathsf{mod}\text{-}\Lambda \longrightarrow \mathsf{mod}\text{-}\Lambda^{\mathsf{op}}$ induces a duality

$$\mathsf{d} : (\mathsf{Gproj}\,\Lambda)^{\mathsf{op}} \xrightarrow{\approx} \mathsf{Gproj}\,\Lambda^{\mathsf{op}}$$

Moreover the adjoint pair $(\mathsf{N}^{+} := - \otimes_{\Lambda} \mathsf{D}\Lambda,\ \mathsf{N}^{-} := \mathsf{Hom}_{\Lambda}(\mathsf{D}\Lambda, -)) : \mathsf{Mod}\text{-}\Lambda \leftrightarrows \mathsf{Mod}\text{-}\Lambda$ induces equivalences

$$\big(\mathsf{N}^{+}, \mathsf{N}^{-}\big) : \mathsf{GProj}\,\Lambda \xrightarrow{\approx} \mathsf{GInj}\,\Lambda \quad \text{and} \quad \big(\mathsf{N}^{+}, \mathsf{N}^{-}\big) : \mathsf{Gproj}\,\Lambda \xrightarrow{\approx} \mathsf{Ginj}\,\Lambda$$

The stable categories $\underline{\mathsf{GProj}}\,\Lambda$ and $\overline{\mathsf{GInj}}\,\Lambda$ are compactly generated triangulated categories, the stable categories $\underline{\mathsf{Gproj}}\,\Lambda$ and $\overline{\mathsf{Ginj}}\,\Lambda$ are triangulated subcategories of $\underline{\mathsf{GProj}}\,\Lambda$ and $\overline{\mathsf{GInj}}\,\Lambda$ respectively consisting of compact objects, and the adjoint pair of Nakayama functors induce triangulated equivalences

$$\big(\mathsf{N}^{+}, \mathsf{N}^{-}\big) : \underline{\mathsf{GProj}}\,\Lambda \xrightarrow{\approx} \overline{\mathsf{GInj}}\,\Lambda \quad \text{and} \quad \big(\mathsf{N}^{+}, \mathsf{N}^{-}\big) : \underline{\mathsf{Gproj}}\,\Lambda \xrightarrow{\approx} \overline{\mathsf{Ginj}}\,\Lambda$$

It is not difficult to see that $(\mathsf{Gproj}\,\Lambda)^{\perp} = {}^{\perp}(\mathsf{Ginj}\,\Lambda)$ [24]. However there are Artin algebras for which $(\mathsf{GProj}\,\Lambda)^{\perp} \neq {}^{\perp}(\mathsf{GInj}\,\Lambda)$ [25]. To remedy this pathology, the following class of algebras was introduced in [27] as a common generalization of Gorenstein algebras and algebras of finite representation type.

Definition 4.4. An Artin algebra Λ is called **virtually Gorenstein** if:

$$(\mathsf{GProj}\,\Lambda)^{\perp} = {}^{\perp}(\mathsf{GInj}\,\Lambda)$$

Example 4.5. The following list shows that the class of virtually Gorenstein algebras is rather large, see [24] for more details and [25] for an example of an Artin algebra which is not virtually Gorenstein.

1. Gorenstein algebras are virtually Gorenstein.
   Recall that an Artin algebra Λ is called **Gorenstein** if $\mathsf{id}_{\Lambda}\Lambda < \infty$ and $\mathsf{id}\,\Lambda_{\Lambda} < \infty$; in this case it is known that $\mathsf{id}_{\Lambda}\Lambda = \mathsf{id}\,\Lambda_{\Lambda}$. Important classes of Gorenstein algebras are the algebras of finite global dimension and the self-injective algebras. In the first case we have $\mathsf{GProj}\,\Lambda = \mathsf{Proj}\,\Lambda$ and in the second case we have $\mathsf{GProj}\,\Lambda = \mathsf{Mod}\text{-}\Lambda$.
2. Any Artin algebra of finite representation type is virtually Gorenstein.
3. Let Λ be an Artin algebra and assume that any Gorenstein-projective module is projective, i.e. $\mathsf{GProj}\,\Lambda = \mathsf{Proj}\,\Lambda$. Then Λ is virtually Gorenstein.
4. Let Λ be an Artin algebra which is derived equivalent or stably equivalent to a virtually Gorenstein algebra, e.g. to an algebra of finite representation type. Then Λ is virtually Gorenstein.

We refer to [24] for an extensive discussion of virtually Gorenstein algebras. In particular we need in the sequel the following properties enjoyed by the class of virtually Gorenstein algebras, see [24,25]:

Remark 4.6. Let Λ be an Artin algebra.

1. Λ is virtually Gorenstein if and only if $\mathsf{GProj}\,\Lambda = \varinjlim \mathsf{Gproj}\,\Lambda$. In this case $\mathsf{GProj}\,\Lambda$ is a locally finitely presented additive category with products and $(\mathsf{GProj}\,\Lambda)^{\perp} = \varinjlim(\mathsf{Gproj}\,\Lambda)^{\perp}$.
2. Λ is virtually Gorenstein if and only if the cotorsion pair $(\mathsf{GProj}\,\Lambda, (\mathsf{GProj}\,\Lambda)^{\perp})$ is smashing.
3. Λ is virtually Gorenstein if and only if $(\mathsf{GProj}\,\Lambda)^{\perp} \cap \mathsf{mod}\text{-}\Lambda$ is contravariantly, or equivalently covariantly, finite, in $\mathsf{mod}\text{-}\Lambda$. In this case $(\mathsf{GProj}\,\Lambda)^{\perp} \cap \mathsf{mod}\text{-}\Lambda = (\mathsf{Gproj}\,\Lambda)^{\perp}$, see 4 below.
4. Λ is virtually Gorenstein if and only if $(\mathsf{Gproj}\,\Lambda)^{\perp} \subseteq (\mathsf{GProj}\,\Lambda)^{\perp}$. Indeed if the last inclusion holds, then since we always have $(\mathsf{GProj}\,\Lambda)^{\perp} \subseteq (\mathsf{Gproj}\,\Lambda)^{\perp}$, it follows that $(\mathsf{GProj}\,\Lambda)^{\perp} = (\mathsf{Gproj}\,\Lambda)^{\perp}$. This implies that $(\mathsf{GProj}\,\Lambda)^{\perp}$ is closed under coproducts, i.e. the cotorsion pair $(\mathsf{GProj}\,\Lambda, (\mathsf{GProj}\,\Lambda)^{\perp})$ is smashing. Therefore by 2, Λ is virtually Gorenstein. If this holds, then by [24] we have $(\mathsf{GProj}\,\Lambda)^{\perp} = X_{\Lambda/\mathfrak{r}}^{\perp}$ and the minimal right $\mathsf{GProj}\,\Lambda$-approximation $X_{\Lambda/\mathfrak{r}}$ of $\Lambda/\mathfrak{r}$ is finitely generated. This clearly implies that $(\mathsf{Gproj}\,\Lambda)^{\perp} \subseteq X_{\Lambda/\mathfrak{r}}^{\perp} = (\mathsf{GProj}\,\Lambda)^{\perp}$.
5. If Λ is virtually Gorenstein, then $\mathsf{Gproj}\,\Lambda$ is functorially finite in $\mathsf{mod}\text{-}\Lambda$; in particular $\mathsf{Gproj}\,\Lambda$ has Auslander–Reiten sequences, and $\mathsf{Gproj}\,\Lambda = \mathsf{Filt}(X_{\Lambda/\mathfrak{r}})$ and $\mathsf{GProj}\,\Lambda = \mathsf{Filt}(\mathsf{Add}\,X_{\Lambda/\mathfrak{r}})$.
6. Λ is virtually Gorenstein if and only if Λ^{op} is virtually Gorenstein.
7. $\underline{\mathsf{GProj}}\,\Lambda$ admits a single compact generator $\underline{X}_{\Lambda/\mathfrak{r}}$. Moreover Λ is virtually Gorenstein iff $(\underline{\mathsf{GProj}}\,\Lambda)^{\mathsf{cpt}} = \underline{\mathsf{Gproj}}\,\Lambda$, i.e. the compact objects of $\underline{\mathsf{GProj}}\,\Lambda$ are coming from the finitely generated Gorenstein-projective modules. In this case $\underline{\mathsf{Gproj}}\,\Lambda$ has Auslander–Reiten triangles and admits a Serre functor.

We have the following characterization of virtually Gorenstein algebras which will be useful later.

Proposition 4.7. *For an Artin algebra Λ, the following statements are equivalent.*

(i) *Λ is virtually Gorenstein.*
(ii) (a) *$\mathsf{Gproj}\,\Lambda$ is contravariantly finite.*
  (b) *Any module in $\mathsf{GProj}\,\Lambda \cap \varinjlim(\mathsf{Gproj}\,\Lambda)^{\perp}$ is a direct sum of finitely generated modules.*

Proof. If Λ is virtually Gorenstein, then $\mathsf{Gproj}\,\Lambda$ is contravariantly finite by Remark 4.6.5, and (ii)(b) holds since $\mathsf{GProj}\,\Lambda \cap \varinjlim(\mathsf{Gproj}\,\Lambda)^{\perp} = \mathsf{GProj}\,\Lambda \cap (\mathsf{GProj}\,\Lambda)^{\perp} = \mathsf{Proj}\,\Lambda$ by Remark 4.6.1. Conversely if (ii) holds, then by [24, Corollary 9.6], (a) implies the existence of a cotorsion pair $(\varinjlim \mathsf{Gproj}\,\Lambda, \varinjlim(\mathsf{Gproj}\,\Lambda)^{\perp})$ in $\mathsf{Mod}\text{-}\Lambda$ and $\varinjlim \mathsf{Gproj}\,\Lambda \subseteq \mathsf{GProj}\,\Lambda$ since the latter is closed under filtered colimits. Let A be in $\mathsf{GProj}\,\Lambda \cap \varinjlim(\mathsf{Gproj}\,\Lambda)^{\perp}$. Then by (b), $A = \bigoplus_{i \in I} A_i$ where each A_i is finitely generated and clearly lies in $\mathsf{GProj}\,\Lambda \cap \varinjlim(\mathsf{Gproj}\,\Lambda)^{\perp} \cap \mathsf{mod}\text{-}\Lambda = \mathsf{Gproj}\,\Lambda \cap (\mathsf{Gproj}\,\Lambda)^{\perp}$ since both $\mathsf{GProj}\,\Lambda$ and $\varinjlim(\mathsf{Gproj}\,\Lambda)^{\perp}$ are closed under direct summands. However it is easy to see that $\mathsf{Gproj}\,\Lambda \cap (\mathsf{Gproj}\,\Lambda)^{\perp} = \mathsf{proj}\,\Lambda$ and then this implies that $\mathsf{GProj}\,\Lambda \cap \varinjlim(\mathsf{Gproj}\,\Lambda)^{\perp} = \mathsf{Add}\,\mathsf{proj}\,\Lambda = \mathsf{Proj}\,\Lambda$. Now let $A \in \mathsf{GProj}\,\Lambda$ and let $0 \longrightarrow A \longrightarrow Y^A \longrightarrow X^A \longrightarrow 0$ be an exact sequence, where Y^A is a left $\varinjlim(\mathsf{Gproj}\,\Lambda)^{\perp}$-approximation of A and $X^A \in \varinjlim \mathsf{Gproj}\,\Lambda$. Clearly Y^A lies in $\mathsf{GProj}\,\Lambda \cap \varinjlim(\mathsf{Gproj}\,\Lambda)^{\perp} = \mathsf{Proj}\,\Lambda$ and therefore A lies in $\varinjlim \mathsf{Gproj}\,\Lambda$, since the latter is resolving. We infer that $\mathsf{GProj}\,\Lambda = \varinjlim \mathsf{Gproj}\,\Lambda$ and therefore Λ is virtually Gorenstein by Remark 4.6.1. □

Corollary 4.8. *If* $\mathsf{Gproj}\,\Lambda = \mathsf{proj}\,\Lambda$*, then the following are equivalent.*

(i) *Λ is virtually Gorenstein.*
(ii) *Any module in* $\mathsf{GProj}\,\Lambda$ *is a direct sum of finitely generated modules.*

4.2. Artin algebras of finite CM-type

Our aim in this section is to characterize Artin algebras enjoying the property that any Gorenstein-projective module is a direct sum of finitely generated modules. This class of algebras, clearly contained in the class of virtually Gorenstein algebras, is intimately related to the class of Artin algebras of finite Cohen–Macaulay type, in the sense of the following definition.

Definition 4.9. An Artin algebra Λ is said to be of **finite Cohen–Macaulay type**, **finite CM-type** for short, if the full subcategory $\mathsf{Gproj}\,\Lambda$ of finitely generated Gorenstein-projective Λ-modules is of finite representation type.

Clearly an Artin algebra Λ is virtually Gorenstein of finite CM-type if Λ is of finite representation type or if $\mathsf{GProj}\,\Lambda = \mathsf{Proj}\,\Lambda$, e.g. if $\mathsf{gl.dim}\,\Lambda < \infty$. On the other hand if Λ is of infinite CM-type, then obviously $\mathsf{gl.dim}\,\Lambda = \infty$. Note that a self-injective algebra is of finite CM-type iff it is of finite representation type. Finally by Remark 4.3 it follows that Λ is of finite CM-type if and only if so is Λ^{op}.

The following result shows that the class of Artin algebras enjoying the property that any Gorenstein-projective module is a direct sum of finitely generated modules coincides with the class of virtually Gorenstein algebras of finite CM-type, and condition (vii) characterizes them in terms of an interesting factorization property. To avoid trivialities, we assume that $\mathsf{GProj}\,\Lambda \neq \mathsf{Proj}\,\Lambda$.

Theorem 4.10. *For an Artin algebra Λ the following are equivalent:*

(i) *Any Gorenstein-projective Λ-module is a direct sum of finitely generated modules.*
(ii) *Λ is virtually Gorenstein and any Gorenstein-projective Λ-module is a direct sum of indecomposable modules.*

(iii) *Λ is virtually Gorenstein and any indecomposable Gorenstein-projective module is finitely generated.*
(iv) *Λ is virtually Gorenstein of finite CM-type.*
(v) *Λ is virtually Gorenstein and the triangulated category $\underline{\mathsf{GProj}}\,\Lambda$ is phantomless.*
(vi) *Λ is virtually Gorenstein and the functor category $\mathsf{Mod}\text{-}\underline{\mathsf{Gproj}}\,\Lambda$ is Frobenius.*
(vii) *A map $f : A \longrightarrow B$ between Gorenstein-projective modules factors through a projective module if and only if for any finitely generated Gorenstein-projective module X and any map $\alpha : X \longrightarrow A$, the composition $\alpha \circ f : X \longrightarrow B$ factors through a projective module.*
(viii) *The above conditions with Λ replaced by Λ^{op}.*

Proof. The proof that the first four conditions are equivalent is a direct consequence of Theorem 3.1 and the fact that Λ is virtually Gorenstein if and only if $\mathsf{GProj}\,\Lambda = \varinjlim \mathsf{Gproj}\,\Lambda$, see 4.6.1. The equivalence (iv) $\Leftrightarrow$ (viii) follows from the fact that Λ is virtually Gorenstein if and only if so is Λ^{op}, see 4.6.4, and from the existence of an equivalence $(\mathsf{Gproj}\,\Lambda)^{\mathsf{op}} \approx \mathsf{Gproj}\,\Lambda^{\mathsf{op}}$, see Remark 4.3. If (i) holds, then Λ is virtually Gorenstein and therefore $(\underline{\mathsf{GProj}}\,\Lambda)^{\mathsf{cpt}} = \underline{\mathsf{Gproj}}\,\Lambda$. It follows from 4.6.7 that any object in $\underline{\mathsf{GProj}}\,\Lambda$ is a direct sum of compact objects and therefore by [21, Theorem 9.3], $\underline{\mathsf{GProj}}\,\Lambda$ is phantomless, so (v) holds. Using 4.6.6, the proof that (v) $\Leftrightarrow$ (vi) follows from [21, Proposition 9.2, Theorem 9.3]. We now show that (vi) implies (i). Since $\underline{\mathsf{GProj}}\,\Lambda$ is phantomless, by [21, Theorem 9.3], for any Gorenstein-projective module A there is a decomposition $\underline{A} = \bigoplus_{i \in I} \underline{X}_i$, where the X_i are finitely generated. This implies that there are projective modules P and Q such that $A \oplus P \cong \bigoplus_{i \in I} X_i \oplus Q$. Since Λ is virtually Gorenstein, by 4.6.1, the category $\mathsf{GProj}\,\Lambda$ is locally finitely presented and $\mathsf{GProj}\,\Lambda = \varinjlim \mathsf{Gproj}\,\Lambda$. Consider the representation functor $\mathsf{H} : \mathsf{GProj}\,\Lambda \longrightarrow \mathsf{Mod}\text{-}\mathsf{Gproj}\,\Lambda$. Then $\mathsf{H}(A) \oplus \mathsf{H}(P) \cong \mathsf{H}(\bigoplus_{i \in I} X_i) \oplus \mathsf{H}(Q) \cong \bigoplus_{i \in I} \mathsf{H}(X_i) \oplus \mathsf{H}(Q)$. Since the objects $\mathsf{H}(X_i)$ and $\mathsf{H}(Q)$ are projective, we infer that $\mathsf{H}(A)$ is a projective object of $\mathsf{Mod}\text{-}\mathsf{Gproj}\,\Lambda$. Since any flat functor in $\mathsf{Mod}\text{-}\mathsf{Gproj}\,\Lambda$ is of the form $\mathsf{H}(A)$, where $A \in \mathsf{GProj}\,\Lambda$, we infer that any flat functor is projective, i.e. the representation category $\mathsf{Mod}\text{-}\mathsf{Gproj}\,\Lambda$ is perfect. It is well known that in a perfect functor category any projective functor is a direct sum of finitely generated projective functors, see [43]. It follows that $\mathsf{H}(A) = \bigoplus_{j \in J} \mathsf{H}(X'_j)$, where the X'_j lie in $\mathsf{Gproj}\,\Lambda$. Since H is fully faithful we conclude that A is a direct sum of finitely generated modules and (i) holds. Finally we show that (v) $\Leftrightarrow$ (vii). Assuming (vii), let $\underline{f} : \underline{A} \longrightarrow \underline{B}$ be a map in $\underline{\mathsf{GProj}}\,\Lambda$ such that the induced map $\underline{\mathsf{Hom}}_\Lambda(X, A) \longrightarrow \underline{\mathsf{Hom}}_\Lambda(X, B)$ is zero for any finitely generated Gorenstein-projective module X. This means that for any map $X \longrightarrow A$, the composition $X \longrightarrow A \longrightarrow B$ factorizes through a projective module. Then condition (vii) implies f factorizes through a projective module, so $\underline{f} = 0$ in $\underline{\mathsf{GProj}}\,\Lambda$. Applying this to $A = B$ and $f = 1_A$, we have that $\underline{\mathsf{Hom}}(X, A) = 0$, $\forall X \in \mathsf{Gproj}\,\Lambda$, implies that $\underline{A} = 0$, so $\underline{\mathsf{Gproj}}\,\Lambda$ (compactly) generates $\underline{\mathsf{GProj}}\,\Lambda$. Then by [21, Theorem 9.3] we infer that $(\underline{\mathsf{GProj}}\,\Lambda)^{\mathsf{cpt}} = \underline{\mathsf{Gproj}}\,\Lambda$, so by 4.6.6, Λ is virtually Gorenstein, and any object of $\underline{\mathsf{GProj}}\,\Lambda$ is a direct sum of objects from $\underline{\mathsf{Gproj}}\,\Lambda$, i.e. $\underline{\mathsf{GProj}}\,\Lambda$ is phantomless and (v) holds. Conversely if $\underline{\mathsf{GProj}}\,\Lambda$ is phantomless, then (vii) holds since $\underline{\mathsf{Gproj}}\,\Lambda \subseteq (\underline{\mathsf{GProj}}\,\Lambda)^{\mathsf{cpt}}$. $\square$

Note that by the above proof, if Λ is virtually Gorenstein of finite CM-type and T is a representation generator of $\mathsf{Gproj}\,\Lambda$, then there are equivalences $\mathsf{GProj}\,\Lambda \approx \mathsf{Proj}\,\Gamma$ and $\mathsf{Gproj}\,\Lambda \approx \mathsf{proj}\,\Gamma$, where $\Gamma = \mathsf{End}_\Lambda(T)^{\mathsf{op}}$. Moreover let $\Delta = \underline{\mathsf{End}}_\Lambda(T)^{\mathsf{op}}$ be the stable endomorphism algebra of T. Since $\underline{\mathsf{Gproj}}\,\Lambda = \mathsf{add}\,\underline{T}$, there are equivalences $\underline{\mathsf{GProj}}\,\Lambda \approx \mathsf{Proj}\,\Delta$ and $\underline{\mathsf{Gproj}}\,\Lambda \approx \mathsf{proj}\,\Delta$. Since $\underline{\mathsf{GProj}}\,\Lambda$ is triangulated, by [21,32], Δ is a self-injective Artin algebra, and, using Theorem 3.1 and Proposition 3.8, we have the following consequence.

Corollary 4.11. *For an Artin algebra Λ, the following statements are equivalent.*

(i) *Λ is virtually Gorenstein of finite CM-type.*
(ii) $\mathsf{GProj}\,\Lambda$ *is equivalent to the category of projective modules over an Artin algebra.*
(iii) $\underline{\mathsf{GProj}}\,\Lambda$ *is equivalent to the category of projective modules over a self-injective Artin algebra.*
(iv) *Λ is virtually Gorenstein and* $\mathsf{Gproj}\,\Lambda$ *is equivalent to the category of finitely generated projective modules over an Artin algebra.*
(v) *Λ is virtually Gorenstein and* $\underline{\mathsf{Gproj}}\,\Lambda$ *is equivalent to the category of finitely generated projective modules over a self-injective Artin algebra.*
(vi) *Λ is virtually Gorenstein and the set $\{[X_1]-[X_2]+[X_3]\}\cup\{[X_{\mathfrak{r}P}]-[P]\}$ is a free basis of* $\mathsf{K}_0(\mathsf{Gproj}\,\Lambda,\oplus)$, *where $0\longrightarrow X_1\longrightarrow X_2\longrightarrow X_3\longrightarrow 0$ is an Auslander–Reiten sequence in* $\mathsf{Gproj}\,\Lambda$ *and $X_{\mathfrak{r}P}$ is the minimal right* $\mathsf{Gproj}\,\Lambda$*-approximation of $\mathfrak{r}P$, $\forall P\in\mathsf{Ind}(\mathsf{proj}\,\Lambda)$.*
(vii) *Λ is virtually Gorenstein and the set $\{[\underline{X}_1]-[\underline{X}_2]+[\underline{X}_3]\}$ is a free basis of* $\mathsf{K}_0(\underline{\mathsf{Gproj}}\,\Lambda,\oplus)$, *where $\underline{X}_1\longrightarrow\underline{X}_2\longrightarrow\underline{X}_3\longrightarrow\Omega^{-1}\underline{X}_1$ is an Auslander–Reiten triangle in* $\underline{\mathsf{Gproj}}\,\Lambda$.

Remark 4.12. By Remark 4.3, the equivalent conditions of Theorem 4.10 and Corollary 4.11 which characterize virtually Gorenstein algebras of finite CM-type, are also equivalent to the conditions resulting by replacing everywhere Gorenstein-projectives with Gorenstein-injectives and projectives with injectives.

Since Gorenstein algebras are virtually Gorenstein and, by [24, Theorem 11.4], virtually Gorenstein algebras with one-sided finite self-injective dimension are Gorenstein, we have as a corollary of Theorem 4.10, the following result which, extending the main result of Chen [33, Main Theorem], gives a characterization of Gorenstein algebras of finite CM-type.

Corollary 4.13. *For an Artin algebra Λ the following are equivalent.*

(i) *Λ is a Gorenstein algebra of finite CM-type.*
(ii) $\mathsf{id}\,\Lambda_\Lambda<\infty$ *and any Gorenstein-projective Λ-module is a direct sum of finitely generated modules.*
(iii) *Λ is virtually Gorenstein with* $\mathsf{id}\,\Lambda_\Lambda<\infty$ *and any indecomposable Gorenstein-projective is finitely generated.*
(iv) *The above conditions with Λ replaced by Λ^{op}.*

By [24] if Λ and Γ are Artin algebras which are derived equivalent or stably equivalent of Morita type, then Λ is virtually Gorenstein iff so is Γ; in this case the stable triangulated categories $\underline{\mathsf{Gproj}}\,\Lambda$ and $\underline{\mathsf{Gproj}}\,\Gamma$ are (triangle) equivalent. Since Artin algebras of finite representation type are virtually Gorenstein, we have the following consequence. Note that algebras derived equivalent to algebras of finite representation type may be of tame infinite, even wild, representation type.

Corollary 4.14. *Let Λ and Γ be Artin algebras which are derived equivalent or stably equivalent of Morita type. Then Λ is virtually Gorenstein of finite CM-type iff Γ is virtually Gorenstein of finite CM-type. In particular if Λ is an Artin algebra which is derived equivalent or stably equivalent of Morita type to an Artin algebra of finite representation type, then Λ is virtually Gorenstein of finite CM-type.*

Corollary 4.15. *Let Λ be an Artin algebra. Then the following are equivalent.*

(i) $(\mathsf{GProj}\,\Lambda)^{\perp} \cap \mathsf{mod}\text{-}\Lambda$ *is of finite representation type.*
(ii) ${}^{\perp}(\mathsf{GInj}\,\Lambda) \cap \mathsf{mod}\text{-}\Lambda$ *is of finite representation type.*
(iii) Λ *is virtually Gorenstein and any module in* $(\mathsf{GProj}\,\Lambda)^{\perp}$ *is a direct sum of finitely generated modules.*
(iv) Λ *is virtually Gorenstein and any module in* ${}^{\perp}(\mathsf{GInj}\,\Lambda)$ *is a direct sum of finitely generated modules.*

Proof. By [24] we have $\mathcal{X} := (\mathsf{GProj}\,\Lambda)^{\perp} \cap \mathsf{mod}\text{-}\Lambda = \mathsf{mod}\text{-}\Lambda \cap {}^{\perp}(\mathsf{GInj}\,\Lambda)$, so (i) $\Leftrightarrow$ (ii). Clearly (iii) $\Leftrightarrow$ (iv). If (i) holds, then $\mathcal{X}$ is contravariantly finite so, by 4.6.3, Λ is virtually Gorenstein. Since $\mathcal{X}$ is resolving, the assertion in (iii) follows from 4.6.1, 4.6.3, and Theorem 3.1. The implication (iii) $\Rightarrow$ (i) is similar. □

Remark 4.16. Let Λ be an Artin algebra and assume that $\mathsf{id}\,\Lambda_{\Lambda} < \infty$.

(i) Any module in ${}^{\perp}(\mathsf{GInj}\,\Lambda)$ is a direct sum of finitely generated modules if and only if the full subcategory $\mathsf{inj}^{<\infty}\,\Lambda$ of $\mathsf{mod}\text{-}\Lambda$, consisting of all modules with finite injective dimension, is of finite representation type. This follows from Theorem 3.1 and the fact that $\mathsf{id}\,\Lambda_{\Lambda} < \infty$ implies that $\mathsf{inj}^{<\infty}\,\Lambda$ is resolving and ${}^{\perp}(\mathsf{GInj}\,\Lambda) = \varinjlim \mathsf{inj}^{<\infty}\,\Lambda$, see [24, Theorem 11.3].
(ii) If Λ is virtually Gorenstein, then: any module in $(\mathsf{GProj}\,\Lambda)^{\perp}$ is a direct sum of finitely generated modules if and only if the full subcategory $\mathsf{proj}^{<\infty}\,\Lambda$ of $\mathsf{mod}\text{-}\Lambda$, consisting of all modules with finite projective dimension, is of finite representation type. This follows from Theorem 3.1 and the fact that virtual Gorensteinness and $\mathsf{id}\,\Lambda_{\Lambda} < \infty$ implies that Λ is Gorenstein and $(\mathsf{GProj}\,\Lambda)^{\perp} = \varinjlim \mathsf{proj}^{<\infty}\,\Lambda$, see [24, Theorem 11.4].

Example 4.17. Let Λ be a self-injective algebra. Then the following are equivalent.

(i) $\mathsf{Sub}(\mathsf{mod}\text{-}\Lambda)$ is of finite representation type.
(ii) $T_2(\Lambda) := \left(\begin{smallmatrix} \Lambda & \Lambda \\ 0 & \Lambda \end{smallmatrix}\right)$ is of finite CM-type.

It is well known that $T_2(\Lambda)$ is Gorenstein with $\mathsf{id}\,T_2(\Lambda) \leqslant 1$. This easily implies that $\mathsf{Gproj}\,T_2(\Lambda) = \mathsf{Sub}(\mathsf{proj}\,T_2(\Lambda))$. On the other hand it is not difficult to see that $\mathsf{Sub}(\mathsf{mod}\text{-}\Lambda) = \mathsf{Sub}(\mathsf{proj}\,T_2(\Lambda))$, hence $\mathsf{Sub}(\mathsf{mod}\text{-}\Lambda) = \mathsf{Gproj}\,T_2(\Lambda)$. It follows that the submodule category $\mathsf{Sub}(\mathsf{mod}\text{-}\Lambda)$ of $\mathsf{mod}\text{-}\Lambda$ is of finite representation type if and only if the Gorenstein algebra $T_2(\Lambda)$ is of finite CM-type. E.g. if $\Lambda_n = k[t]/(t^n)$, then by Example 3.11, $T_2(\Lambda_n)$ is of finite CM-type iff $n \leqslant 5$.

We close this subsection with a characterization of Artin algebras of finite CM-type. Recall that if $\mathcal{S}$ is a class of objects in a triangulated category $\mathcal{T}$, then the localizing subcategory $\mathsf{Loc}(\mathcal{S})$ of $\mathcal{T}$ generated by $\mathcal{S}$, is the smallest full triangulated subcategory of $\mathcal{T}$ which contains $\mathcal{S}$ and is closed under arbitrary direct sums.

Proposition 4.18. *For an Artin algebra Λ the following statements are equivalent.*

(i) Λ *is of finite CM-type.*
(ii) $\mathsf{Gproj}\,\Lambda$ *is contravariantly finite and any* $(\mathsf{Gproj}\,\Lambda)$*-filtered module is a direct sum of finitely generated modules.*

(iii) $\mathsf{Gproj}\,\Lambda$ *is contravariantly finite and any module in* $\varinjlim \mathsf{Gproj}\,\Lambda$ *is a direct sum of finitely generated modules.*
(iv) $\mathsf{Gproj}\,\Lambda$ *is contravariantly finite and the localizing subcategory of* $\underline{\mathsf{GProj}}\,\Lambda$ *generated by* $\underline{\mathsf{Gproj}}\,\Lambda$ *is phantomless.*
(v) $\mathsf{Gproj}\,\Lambda$ *is contravariantly finite and the module category* $\mathsf{Mod}\text{-}\underline{\mathsf{Gproj}}\,\Lambda$ *is Frobenius.*

Proof. The equivalence (i) $\Leftrightarrow$ (ii) follows from Corollary 3.5. By [24, Theorem 9.4] the localizing subcategory $\mathcal{L} = \mathsf{Loc}(\underline{\mathsf{Gproj}}\,\Lambda)$ of $\underline{\mathsf{GProj}}\,\Lambda$ generated by $\underline{\mathsf{Gproj}}\,\Lambda$ is compactly generated, coincides with the stable category $(\varinjlim \mathsf{Gproj}\,\Lambda)/\,\mathsf{Proj}\,\Lambda$, and we have $\{(\varinjlim \mathsf{Gproj}\,\Lambda)/\,\mathsf{Proj}\,\Lambda\}^{\mathsf{cpt}} = \underline{\mathsf{Gproj}}\,\Lambda$. If (i) holds and $\mathsf{Gproj}\,\Lambda = \mathsf{add}\,T$ for some $T \in \mathsf{Gproj}\,\Lambda$, then $\mathsf{Gproj}\,\Lambda$ is contravariantly finite and the representation category $\mathsf{Mod}\text{-}\underline{\mathsf{Gproj}}\,\Lambda$ of $\mathcal{L}$ is equivalent to the module category $\mathsf{Mod}\text{-}\,\underline{\mathsf{End}}_\Lambda(T)^{\mathsf{op}}$, hence it is locally finite. Since $\underline{\mathsf{End}}_\Lambda(T)^{\mathsf{op}}$ is self-injective, by [21, Theorem 9.3] $\mathcal{L}$ is phantomless. If (iv) holds, then any object of $\mathcal{L}$ is a direct sum of objects from $\underline{\mathsf{Gproj}}\,\Lambda$. This implies that any module in $\varinjlim \mathsf{Gproj}\,\Lambda$ is a direct sum of finitely generated modules, so (iii) holds. If (iii) holds, then Λ is of finite CM-type by Theorem 3.1. Finally the equivalence (iv) $\Leftrightarrow$ (v) follows from [21, Theorem 9.3]. $\square$

4.3. Projective cotorsion pairs

Recall that for any Artin algebra Λ we have a projective cotorsion pair $(\mathsf{GProj}\,\Lambda, (\mathsf{GProj}\,\Lambda)^{\perp})$ with heart $\mathsf{GProj}\,\Lambda \cap (\mathsf{GProj}\,\Lambda)^{\perp} = \mathsf{Proj}\,\Lambda$, and the stable category $\underline{\mathsf{GProj}}\,\Lambda$ is a compactly generated triangulated category. If $(\mathcal{X}, \mathcal{Y})$ is an arbitrary projective cotorsion pair in $\mathsf{Mod}\text{-}\Lambda$, then the stable category $\underline{\mathcal{X}} := \mathcal{X}/\,\mathsf{Proj}\,\Lambda$ is again triangulated, it admits all small coproducts, and $\mathcal{X} \subseteq \mathsf{GProj}\,\Lambda$. So $(\mathsf{GProj}\,\Lambda, (\mathsf{GProj}\,\Lambda)^{\perp})$ is the largest projective cotorsion pair in $\mathsf{Mod}\text{-}\Lambda$.

The following result generalizes Theorem 4.10 to arbitrary projective cotorsion pairs. Note that we may recover Theorem 4.10 by choosing below $(\mathcal{X}, \mathcal{Y})$ to be the cotorsion pair $(\mathsf{GProj}\,\Lambda, (\mathsf{GProj}\,\Lambda)^{\perp})$ and using 4.6.2. The dual version concerning injective cotorsion pairs is left to the reader.

Theorem 4.19. *If* $(\mathcal{X}, \mathcal{Y})$ *is a projective cotorsion pair in* $\mathsf{Mod}\text{-}\Lambda$ *then the following are equivalent.*

(i) *Any module in* $\mathcal{X}$ *is a direct sum of finitely generated modules.*
(ii) *The cotorsion pair* $(\mathcal{X}, \mathcal{Y})$ *is smashing and* $\mathcal{X}^{\mathsf{fin}}$ *is of finite representation type.*
(iii) *The cotorsion pair* $(\mathcal{X}, \mathcal{Y})$ *is smashing and any module in* $\mathcal{X}$ *is a direct sum of indecomposable modules.*
(iv) *The cotorsion pair* $(\mathcal{X}, \mathcal{Y})$ *is smashing and any indecomposable module in* $\mathcal{X}$ *is finitely generated.*
(v) *The cotorsion pair* $(\mathcal{X}, \mathcal{Y})$ *is smashing and the triangulated category* $\underline{\mathcal{X}}$ *is phantomless.*
(vi) *The cotorsion pair* $(\mathcal{X}, \mathcal{Y})$ *is smashing and the functor category* $\mathsf{Mod}\text{-}\underline{\mathcal{X}}^{\mathsf{fin}}$ *is Frobenius.*
(vii) $\mathcal{X}$ *is equivalent to the category of projective modules over an Artin algebra.*
(viii) $\underline{\mathcal{X}}$ *is equivalent to the category of projective modules over a self-injective Artin algebra.*
(ix) *The cotorsion pair* $(\mathcal{X}, \mathcal{Y})$ *is smashing and* $\mathcal{X}^{\mathsf{fin}}$ *is equivalent to the category of finitely generated projective modules over an Artin algebra.*
(x) *The cotorsion pair* $(\mathcal{X}, \mathcal{Y})$ *is smashing and* $\underline{\mathcal{X}}^{\mathsf{fin}}$ *is equivalent to the category of finitely generated projective modules over a self-injective Artin algebra.*

(xi) *A map $f : A \longrightarrow B$ between modules in $\mathcal{X}$ factors through a projective module if and only if for any module X in $\mathcal{X}^{\mathsf{fin}}$ and any map $\alpha : X \longrightarrow A$, the composition $\alpha \circ f : X \longrightarrow B$ factors through a projective module.*

Proof. (i) $\Rightarrow$ (ii) Let any module in $\mathcal{X}$ be a direct sum of finitely generated modules, i.e. $\mathcal{X} = \mathsf{Add}\,\mathcal{X}^{\mathsf{fin}}$. Let $\{Y_i\}_{i\in I}$ be a small set of modules in $\mathcal{Y}$. Then $\bigoplus_{i\in I} Y_i$ lies in $\mathcal{Y}$ if $\mathsf{Ext}^n_\Lambda(X, \bigoplus_{i\in I} Y_i) = 0$, $\forall n \geqslant 1$, $\forall X \in \mathcal{X}$. However X is a direct summand of a coproduct $\bigoplus_{j\in J} X_j$, where $X_j \in \mathcal{X}^{\mathsf{fin}}$, and $\mathsf{Ext}^n_\Lambda(X_j, \bigoplus_{i\in I} Y_i) = \bigoplus_{i\in I} \mathsf{Ext}^n_\Lambda(X_j, Y_i) = 0$, $\forall n \geqslant 1$, $\forall j \in J$, since X_j is finitely generated. Hence $\bigoplus_{i\in I} Y_i$ lies in $\mathcal{Y}$ and therefore $\mathcal{Y}$ is closed under all small coproducts, i.e. the cotorsion pair $(\mathcal{X}, \mathcal{Y})$ is smashing. Since $\underline{\mathcal{X}} = \mathsf{Add}\,\underline{\mathcal{X}}^{\mathsf{fin}}$, it follows by [21, Theorem 9.3] that the stable triangulated category $\underline{\mathcal{X}}$ is compactly generated. Then by [24, Theorem 9.10] we have $\mathcal{X} = \varinjlim \mathcal{X}^{\mathsf{fin}}$ and by Theorem 3.1, $\mathcal{X}^{\mathsf{fin}}$ is of finite representation type.

(ii) $\Rightarrow$ (i) Since the cotorsion pair $(\mathcal{X}, \mathcal{Y})$ is smashing, as in [24, Theorem 6.6] we infer that the stable triangulated category $\underline{\mathcal{X}}$ is compactly generated. Then by [24, Theorem 9.10] we have $\mathcal{X} = \varinjlim \mathcal{X}^{\mathsf{fin}}$. By Theorem 3.1 then any module in $\mathcal{X}$ is a direct sum of finitely generated modules.

The rest of proof is similar to the proof of Theorem 4.10 and is left to the reader. $\square$

4.4. Local rings of finite CM-type

Our aim in this subsection is to give characterizations of commutative Noetherian complete local rings $(R, \mathfrak{m}, k)$ of finite CM-type in terms of decomposition properties of the category of Gorenstein-projective modules. Note that if $\mathsf{GProj}\,R = \mathsf{Proj}\,R$, then $\mathsf{Gproj}\,R = \mathsf{proj}\,R$ and clearly R is of finite CM-type and any Gorenstein-projective R-module is a direct sum of finitely generated modules. So it suffices to treat the case in which R admits a non-free finitely generated Gorenstein-projective module. First recall that a module A over any ring S is called *Gorenstein-flat* if there exists an acyclic complex $F^\bullet$ of flat S-modules such that $A \cong \mathsf{Ker}(F^0 \longrightarrow F^1)$ and the complex $F^\bullet \otimes_S E$ is exact for any injective S^{op}-module E.

Theorem 4.20. *Let $(R, \mathfrak{m}, k)$ be a commutative Noetherian complete local ring. If $\mathsf{Gproj}\,R \neq \mathsf{proj}\,R$, then the following statements are equivalent.*

(i) *R is of finite CM-type.*
(ii) *R is Gorenstein and any Gorenstein-projective R-module is a direct sum of finitely generated modules.*
(iii) *$\mathsf{Gproj}\,R$ is contravariantly finite in mod-R and any $(\mathsf{Gproj}\,R)$-filtered module is a direct sum of finitely generated modules.*
(iv) *Any Gorenstein-flat module is a filtered colimit of finitely generated Gorenstein-projective modules and any Gorenstein-projective module is a direct sum of finitely generated modules.*

If R is of finite CM-type, then any indecomposable Gorenstein-projective module is finitely generated.

Proof. (i) $\Rightarrow$ (ii) By the results of [35], R is Gorenstein. Let T be a finitely generated Gorenstein-projective module such that $\mathsf{Gproj}\,R = \mathsf{add}\,T$. Then $\underline{\mathsf{Gproj}}\,R = \mathsf{add}\,\underline{T}$ and let $\Lambda = \underline{\mathsf{End}}_R(T)^{\mathsf{op}}$

be the stable endomorphism ring of T. Consider the triangulated category $\underline{\mathsf{GProj}}\,R$ and the representation functor $\mathsf{H}\colon \underline{\mathsf{GProj}}\,R \longrightarrow \mathsf{Mod}\text{-}\underline{\mathsf{Gproj}}\,R = \mathsf{Mod}\text{-}\Lambda$, given by $\mathsf{H}(\underline{A}) = \underline{\mathsf{Hom}}_R(T, A)$. Since R is of finite CM-type, by [35, Lemma 4.1], the R-module $\mathsf{Ext}^1_R(M, N)$ is of finite length as an R-module, $\forall M, N \in \mathsf{Gproj}\,R$. In particular $\Lambda = \underline{\mathsf{Hom}}_R(T, T) \cong \mathsf{Ext}^1_R(\Omega^{-1}T, T)$ has finite length as an R-module. It follows that the R-algebra Λ is Noetherian and therefore, by [32], Λ is self-injective Artinian, so the Frobenius category $\mathsf{Mod}\text{-}\Lambda$ is a locally finite. By results of Krause [51], the stable category $\overline{\mathsf{GInj}}\,R$ is compactly generated and its compact objects are given, up to direct factors, by the Verdier quotient $\mathbf{D}^b(\mathsf{mod}\text{-}R)/\mathsf{perf}R$, where $\mathsf{perf}R$ is the full subcategory of $\mathbf{D}^b(\mathsf{mod}\text{-}R)$ formed by the perfect complexes. However, since R is Gorenstein, it is well known that $\mathbf{D}^b(\mathsf{mod}\text{-}R)/\mathsf{perf}R$ is triangle equivalent to $\underline{\mathsf{Gproj}}\,R$, see [32]. Also, by [27, Theorem VI.3.2], Gorensteinness implies the existence of a triangle equivalence $\underline{\mathsf{GProj}}\,R \approx \overline{\mathsf{GInj}}\,R$. We infer that $\underline{\mathsf{GProj}}\,R$ is compactly generated and $(\underline{\mathsf{GProj}}\,R)^{\mathsf{cpt}}$ is triangle equivalent to $\underline{\mathsf{Gproj}}\,R$. Since $\mathsf{Mod}\text{-}\underline{\mathsf{Gproj}}\,R$ is locally finite, by [21, Theorem 9.3], any object $\underline{A} \in \underline{\mathsf{GProj}}\,R$ admits a direct sum decomposition $\underline{A} = \bigoplus_{i\in I} \underline{X}_i$, where $\{X_i\}_{i\in I}$ is a small subset of $\mathsf{Gproj}\,R$. Then in $\mathsf{Mod}\text{-}R$ we have an isomorphism $A \oplus P \cong \bigoplus_{i\in I} X_i \oplus Q$, where $P, Q \in \mathsf{Proj}\,R$, and therefore A is pure-projective as a direct summand of a direct sum of finitely presented modules. Since R is complete, by a classical result of Warfield [66, Corollary 4] it follows that A is a direct sum of finitely generated modules. We conclude that any Gorenstein-projective module is a direct sum of indecomposable finitely generated (Gorenstein-projective) modules.

(ii) $\Rightarrow$ (i) We have an equality $\mathsf{GProj}\,R = \mathsf{Add}\,\mathsf{Gproj}\,R$, and therefore an equality $\underline{\mathsf{GProj}}\,R = \mathsf{Add}\,\underline{\mathsf{Gproj}}\,R$. This implies that the triangulated category $\underline{\mathsf{GProj}}\,R$ is phantomless and compactly generated with $(\underline{\mathsf{GProj}}\,R)^{\mathsf{cpt}} = \underline{\mathsf{Gproj}}\,R$. By [21, Theorem 9.3], the module category $\mathsf{Mod}\text{-}\underline{\mathsf{Gproj}}\,R$ is Frobenius and then by [23, Lemma 10.1], the category $\underline{\mathsf{Gproj}}\,R$ has right Auslander-triangles. Since the functor $\mathsf{Hom}_R(-, R)$ gives a self-duality of $\underline{\mathsf{Gproj}}\,R$, it follows that $\underline{\mathsf{Gproj}}\,R$ has Auslander–Reiten triangles and therefore by [23, Theorems 10.2, 10.3], the Frobenius category $\mathsf{Mod}\text{-}\underline{\mathsf{Gproj}}\,R$ is locally finite and $|\mathsf{Supp}\,\underline{\mathsf{Hom}}_R(-, X)| < \infty$, for any indecomposable $X \in \mathsf{Gproj}\,R$. In particular $|\mathsf{Supp}\,\underline{\mathsf{Hom}}_R(-, X_{R/\mathfrak{m}})| < \infty$, where $X_{R/\mathfrak{m}}$ is the (minimal) $\mathsf{Gproj}\,R$-approximation of $R/\mathfrak{m}$, which exists since R is Gorenstein. Let X be a non-zero indecomposable module in $\underline{\mathsf{Gproj}}\,R$ and assume that $X \notin \mathsf{Supp}\,\underline{\mathsf{Hom}}_R(-, X_{R/\mathfrak{m}})$, i.e. $\underline{\mathsf{Hom}}_R(X, X_{R/\mathfrak{m}}) = 0$. Since R is Gorenstein, there exists an exact sequence $0 \longrightarrow Y_{R/\mathfrak{m}} \longrightarrow X_{R/\mathfrak{m}} \longrightarrow R/\mathfrak{m} \longrightarrow 0$, where $Y_{R/\mathfrak{m}}$ has finite projective dimension. Applying $\mathsf{Hom}_R(X, -)$ we see easily that $\underline{\mathsf{Hom}}_R(X, R/\mathfrak{m}) = 0$, i.e. any map $X \longrightarrow R/\mathfrak{m}$ factors through a finite free R-module. This easily implies that X admits R as a direct summand and this is impossible since X is non-free indecomposable. We infer that $\mathsf{Ind}\,\underline{\mathsf{Gproj}}\,R = \mathsf{Supp}\,\underline{\mathsf{Hom}}_R(-, X_{R/\mathfrak{m}})$. Since $\mathsf{Mod}\text{-}\underline{\mathsf{Gproj}}\,R$ is locally finite, the functor $\underline{\mathsf{Hom}}_R(-, X_{R/\mathfrak{m}})$ has finite length. Then as in the proof of Theorem 3.1 we infer that $\mathsf{Supp}\,\underline{\mathsf{Hom}}_R(-, X_{R/\mathfrak{m}}) = \bigcup_{i=0}^{n} \mathsf{Supp}\,\mathbb{S}_i$, where the $\mathbb{S}_i$ are the simple functors $(\underline{\mathsf{Gproj}}\,R)^{\mathsf{op}} \longrightarrow \mathfrak{Ab}$ appearing in the composition series of $\underline{\mathsf{Hom}}_R(-, X_{R/\mathfrak{m}})$. Since $|\mathsf{Supp}\,\mathbb{S}_i| < \infty$, $\forall i$, we infer that $|\mathsf{Ind}\,\underline{\mathsf{Gproj}}\,R| < \infty$. This implies that $\mathsf{Gproj}\,R$ is of finite representation type, i.e. R is of finite CM-type.

(i) $\Leftrightarrow$ (iii) The proof follows as in Proposition 4.18 using Corollary 3.5.

(iv) $\Leftrightarrow$ (ii) By recent results of Jørgensen and Holm, see [40, Theorem 2.7], the first part of condition (iv) implies that R is Gorenstein, so (iv) $\Rightarrow$ (ii). If (ii) holds, then by [37, Theorem 10.3.8] any Gorenstein-flat module is a filtered colimit of finitely generated Gorenstein-projective modules, so (iv) holds. $\square$

Corollary 4.21. *Let R be a commutative Noetherian complete local Gorenstein ring.*

(i) *If $\mathsf{Gproj}\, R = \mathsf{proj}\, R$, then R is non-singular, i.e. regular, of finite CM-type.*
(ii) *If $\mathsf{Gproj}\, R \neq \mathsf{proj}\, R$ and any Gorenstein-projective module is a direct sum of finitely generated modules, then R is a simple (hypersurface) singularity of finite CM-type.*

5. Stable and torsion-free modules

Let Λ be a Noetherian ring. Recall that a finitely generated right Λ-module X is called **stable** if $\mathsf{Ext}^n_\Lambda(X, \Lambda) = 0$, $\forall n \geqslant 1$, i.e. $X \in {}^\perp\Lambda$. And X is called **torsion-free** if the transpose $\mathsf{Tr}\, X$ of X is stable: $\mathsf{Ext}^n_{\Lambda^{\mathsf{op}}}(\mathsf{Tr}\, X, \Lambda^{\mathsf{op}}) = 0$, $\forall n \geqslant 1$, i.e. $\mathsf{Tr}\, X \in {}^\perp\Lambda^{\mathsf{op}}$. Here $\mathsf{Tr} : \underline{\mathsf{mod}}\text{-}\Lambda \longrightarrow \underline{\mathsf{mod}}\text{-}\Lambda^{\mathsf{op}}$ is the Auslander–Bridger transpose duality functor given by $\mathsf{Tr}(M) = \mathsf{Coker}(\mathsf{Hom}_\Lambda(P^0, \Lambda) \longrightarrow \mathsf{Hom}_\Lambda(P^1, \Lambda))$, where $P^1 \longrightarrow P^0 \longrightarrow M \longrightarrow 0$ is a finite projective presentation of M, see [11]. In this terminology a finitely generated module X is Gorenstein-projective if and only if X is stable and torsion-free.

It was an open question if stable implies torsion-free, i.e. if stableness is sufficient for a finitely generated module to be Gorenstein-projective. Recently Jorgensen and Sega answered this question in the negative. In fact they constructed a commutative Noetherian local (graded Koszul) ring and a finitely generated stable module which is not torsion-free, see [44] for details. On the other hand Yoshino proved that the answer to the above question is positive for Henselian local Noetherian rings $(R, \mathfrak{m}, k)$ such that the full subcategory of stable modules is of finite representation type, see [68]. Recently R. Takahashi proved a far reaching generalization: if $\mathcal{X}$ is a contravariantly finite resolving subcategory of $\mathsf{mod}\text{-}R$ and if any module X in $\mathcal{X}$ satisfies $\mathsf{Ext}^n_R(X, R) = 0$, for $n \gg 0$, then $\mathcal{X} = \mathsf{proj}\, R$ or R is Gorenstein and either $\mathcal{X} = \mathsf{Gproj}\, R$, or $\mathcal{X} = \mathsf{mod}\text{-}R$, see [65]. In the non-commutative setting, where in the Artin algebra case Takahashi's result is false (e.g. consider $\mathcal{X} = \mathsf{proj}^{<\infty}\, \Lambda$ for a Gorenstein algebra Λ of infinite global dimension), it is known that all stable modules are Gorenstein-projective if the Artin algebra Λ is Gorenstein or more generally if $\mathsf{id}\, {}_\Lambda\Lambda < \infty$, see [24, Proposition 4.4]. Also any stable module with finite Gorenstein-projective dimension is Gorenstein-projective, see [24, Proposition 3.9], and all stable right modules are Gorenstein-projective if Λ is left CoGorenstein in the sense that arbitrary syzygy left modules are stable, see [24, Proposition 4.4].

5.1. Finitely generated stable modules

One reason of why the condition ${}^\perp\Lambda = \mathsf{Gproj}\, \Lambda$ is important is that it allows the computation of the *Gorenstein dimension* $\mathsf{G\text{-}dim}\, M := \mathsf{res.dim}_{\mathsf{Gproj}\, \Lambda}\, M$ of a module $M \in \mathsf{mod}\text{-}\Lambda$, without using resolutions. Recall that the *Gorenstein dimension* of $\mathsf{mod}\text{-}\Lambda$ is defined by

$$\mathsf{G\text{-}dim}\, \Lambda = \mathsf{res.dim}_{\mathsf{Gproj}\, \Lambda}\, \mathsf{mod}\text{-}\Lambda = \max\{\mathsf{id}\, {}_\Lambda\Lambda, \mathsf{id}\, \Lambda_\Lambda\}$$

Thus Λ is Gorenstein if and only if $\mathsf{G\text{-}dim}\, \Lambda < \infty$.

Lemma 5.1. *Let Λ be a Noetherian ring. Then the following are equivalent.*

(i) ${}^\perp\Lambda = \mathsf{Gproj}\, \Lambda$.
(ii) $\forall M \in \mathsf{mod}\text{-}\Lambda$*:* $\mathsf{G\text{-}dim}\, M = \sup\{t \in \mathbb{Z} \mid \mathsf{Ext}^t_\Lambda(M, \Lambda) \neq 0\}$.

Proof. Assume that ${}^{\perp}\Lambda = \mathsf{Gproj}\,\Lambda$ and let M be in $\mathsf{mod}\text{-}\Lambda$. If $\mathsf{G}\text{-}\mathsf{dim}\,M < \infty$, then the assertion (ii) is easy, see for instance [24, Proposition 3.9]. Assume that $\mathsf{G}\text{-}\mathsf{dim}\,M = \infty$. If $\sup\{t \in \mathbb{Z} \mid \mathsf{Ext}^t_{\Lambda}(M, \Lambda) \neq 0\} = d < \infty$, then $\mathsf{Ext}^n_{\Lambda}(M, \Lambda) = 0$, $\forall n > d$. Hence $\mathsf{Ext}^k_{\Lambda}(\Omega^d M, \Lambda) = 0$, $\forall k \geqslant 1$, so $\Omega^d M \in {}^{\perp}\Lambda = \mathsf{Gproj}\,\Lambda$. However this implies that $\mathsf{G}\text{-}\mathsf{dim}\,M \leqslant d$ and this is not true. Hence $d = \infty$. Conversely if $\mathsf{G}\text{-}\mathsf{dim}\,M = \sup\{t \in \mathbb{Z} \mid \mathsf{Ext}^t_{\Lambda}(M, \Lambda) \neq 0\}$, then $\mathsf{G}\text{-}\mathsf{dim}\,M = 0$, for any module $M \in {}^{\perp}\Lambda$. Hence ${}^{\perp}\Lambda = \mathsf{Gproj}\,\Lambda$. $\square$

Our aim in this section is to prove the following result which, in the context of virtually Gorenstein Artin algebras, gives a complete answer to the question of when a stable module is torsion-free, hence Gorenstein-projective. In particular we extend properly Yoshino's result in the non-commutative setting to the case where the category of stable modules is not necessarily of finite representation type.

Theorem 5.2. *Let Λ be an Artin algebra. If $\mathsf{Gproj}\,\Lambda$ is contravariantly finite, for instance if Λ is virtually Gorenstein or of finite CM-type, then the following are equivalent.*

(i) ${}^{\perp}\Lambda = \mathsf{Gproj}\,\Lambda$.
(ii) ${}^{\perp}\Lambda \cap (\mathsf{Gproj}\,\Lambda)^{\perp} = \mathsf{proj}\,\Lambda$.
(iii) ${}^{\perp}\Lambda \cap (\mathsf{Gproj}\,\Lambda)^{\perp}$ *is of finite representation type.*

Before we proceed to the proof of Theorem 5.2 we need some preparations. Let $\mathscr{A}$ be an abelian category with enough projectives and let $\mathcal{X}$ be a resolving subcategory of $\mathscr{A}$. We view the stable category $\underline{\mathscr{A}}$ modulo projectives as a left triangulated category; then clearly the stable category $\underline{\mathcal{X}}$ of $\mathcal{X}$ is a left triangulated subcategory of $\underline{\mathscr{A}}$, see [26], so $\underline{\mathcal{X}}$ has weak kernels and therefore the category of coherent functors $\mathsf{mod}\text{-}\underline{\mathcal{X}}$ over $\underline{\mathcal{X}}$ is abelian with enough projectives. Moreover the Yoneda embedding $\underline{\mathcal{X}} \longrightarrow \mathsf{mod}\text{-}\underline{\mathcal{X}}$, $\underline{X} \longmapsto \underline{\mathsf{Hom}}(-, X)$ has image in $\mathsf{Proj}\,\mathsf{mod}\text{-}\underline{\mathcal{X}}$ and if idempotents split in $\underline{\mathcal{X}}$, it induces an equivalence $\underline{\mathcal{X}} \approx \mathsf{Proj}\,\mathsf{mod}\text{-}\underline{\mathcal{X}}$. In the sequel we identify $\mathsf{mod}\text{-}\underline{\mathcal{X}}$ with the full subcategory of $\mathsf{mod}\text{-}\mathcal{X}$ consisting of all coherent functors $F : \mathcal{X}^{\mathsf{op}} \longrightarrow \mathfrak{Ab}$ which admit a presentation $\mathcal{X}(-, X_1) \longrightarrow \mathcal{X}(-, X_0) \longrightarrow F \longrightarrow 0$ such that the map $X_1 \longrightarrow X_0$ is an epimorphism in $\mathscr{A}$. This clearly coincides with the full subcategory of $\mathsf{mod}\text{-}\mathcal{X}$ consisting of all functors vanishing on the projectives.

Lemma 5.3. *Let $\mathscr{A}$ be an abelian category and $(\mathcal{X}, \mathcal{Y})$ a cotorsion pair in $\mathscr{A}$ with heart $\omega = \mathcal{X} \cap \mathcal{Y}$.*

(i) *The category $\mathsf{mod}\text{-}\mathcal{X}$ is abelian with enough projectives and the restricted Yoneda functor $\mathsf{H} : \mathscr{A} \longrightarrow \mathsf{Mod}\text{-}\mathcal{X}$, $\mathsf{H}(A) = \mathscr{A}(-, A)|_{\mathcal{X}}$ has image in $\mathsf{mod}\text{-}\mathcal{X}$.*
(ii) *The category $\mathcal{X}/\omega$ is right triangulated and the category $(\mathcal{X}/\omega\text{-}\mathsf{mod})^{\mathsf{op}}$ is abelian with enough injectives. If idempotents split in $\mathcal{X}/\omega$, then we have an equivalence $\mathcal{X}/\omega \approx \mathsf{Inj}(\mathcal{X}/\omega\text{-}\mathsf{mod})^{\mathsf{op}}$.*
(iii) *If $\mathscr{A}$ has enough projectives, then $\mathsf{mod}\text{-}\underline{\mathcal{X}}$ has enough projectives and enough injectives. Moreover if idempotents split in $\mathcal{X}/\omega$, then the assignment $X \longmapsto \mathsf{Ext}^1_{\mathscr{A}}(-, X)|_{\mathcal{X}}$ gives rise to an equivalence*

$$\mathcal{X}/\omega \approx \mathsf{Inj}\,\mathsf{mod}\text{-}\underline{\mathcal{X}}$$

Proof. (i), (ii) Since $\mathcal{X}$ is contravariantly finite in $\mathscr{A}$ it follows that $\mathcal{X}$ has weak kernels and therefore $\mathsf{mod}\text{-}\mathcal{X}$ is abelian. For any object $A \in \mathscr{A}$, let $X_A^0 \longrightarrow A$ be a right $\mathcal{X}$-approximation of A, and let $X_A^1 \longrightarrow K$ be a right $\mathcal{X}$-approximation of the kernel K of $X_A^0 \longrightarrow A$. Applying the functor H we have clearly an exact sequence $\mathsf{H}(X_A^1) \longrightarrow \mathsf{H}(X_A^0) \longrightarrow \mathsf{H}(A) \longrightarrow 0$ which shows that $\mathsf{H}(A)$ is coherent. Hence $\operatorname{Im}\mathsf{H} \subseteq \mathsf{mod}\text{-}\mathcal{X}$. By [26, Lemma VI.1.1] the stable category $\mathcal{X}/\omega$ is right triangulated, in particular $\mathcal{X}/\omega$ has weak cokernels. Therefore $\mathcal{X}/\omega\text{-}\mathsf{mod}$ is abelian with enough projectives and part (ii) follows.

(iii) Consider the Yoneda embedding $\mathsf{H}: \mathcal{X} \longrightarrow \mathsf{mod}\text{-}\mathcal{X}$, $\mathsf{H}(X) = \mathcal{X}(-, X)$. Since ω is an Ext-injective cogenerator of $\mathcal{X}$, we may choose, $\forall X \in \mathcal{X}$, an *$\omega$-injective copresentation* of X, i.e. an exact sequence $0 \longrightarrow X \longrightarrow T \longrightarrow X' \longrightarrow 0$, where $T \in \omega$ and $X' \in \mathcal{X}$. Then we have a projective resolution in $\mathsf{mod}\text{-}\mathcal{X}$:

$$0 \longrightarrow \mathsf{H}(X) \longrightarrow \mathsf{H}(T) \longrightarrow \mathsf{H}\big(X'\big) \longrightarrow \mathsf{Ext}^1_{\mathscr{A}}(-, X)|_{\mathcal{X}} \longrightarrow 0$$

This shows that $\mathsf{Ext}^1_{\mathscr{A}}(-, X)|_{\mathcal{X}}$ is coherent and lies in $\mathsf{mod}\text{-}\underline{\mathcal{X}}$, $\forall X \in \mathcal{X}$. We show that $\mathsf{Ext}^1_{\mathscr{A}}(-, X)|_{\mathcal{X}}$ is injective in $\mathsf{mod}\text{-}\underline{\mathcal{X}}$. Let F be in $\mathsf{mod}\text{-}\underline{\mathcal{X}}$ with a presentation $\mathcal{X}(-, X_1) \longrightarrow \mathcal{X}(-, X_0) \longrightarrow F \longrightarrow 0$ in $\mathsf{mod}\text{-}\mathcal{X}$ such that the map $X_1 \longrightarrow X_0$ is an epimorphism. Since $\mathcal{X}$ is resolving, we have an exact sequence $0 \longrightarrow X_2 \longrightarrow X_1 \longrightarrow X_0 \longrightarrow 0$ in $\mathcal{X}$ which induces a projective resolution of F in $\mathsf{mod}\text{-}\mathcal{X}$:

$$0 \longrightarrow \mathsf{H}(X_2) \longrightarrow \mathsf{H}(X_1) \longrightarrow \mathsf{H}(X_0) \longrightarrow F \longrightarrow 0 \tag{5.1}$$

and a triangle $\Omega \underline{X}_0 \longrightarrow \underline{X}_2 \longrightarrow \underline{X}_1 \longrightarrow \underline{X}_0$ in $\underline{\mathcal{X}}$ which induces a projective resolution of F in $\mathsf{mod}\text{-}\underline{\mathcal{X}}$:

$$\cdots \longrightarrow \underline{\mathcal{X}}(-, \Omega \underline{X}_0) \longrightarrow \underline{\mathcal{X}}(-, \underline{X}_2) \longrightarrow \underline{\mathcal{X}}(-, \underline{X}_1) \longrightarrow \underline{\mathcal{X}}(-, \underline{X}_0) \longrightarrow F \longrightarrow 0$$

Then $\mathsf{Ext}^1_{\mathsf{mod}\text{-}\underline{\mathcal{X}}}(F, \mathsf{Ext}^1_{\mathscr{A}}(-, X)|_{\mathcal{X}})$ is the homology of the complex $\mathsf{Ext}^1_{\mathscr{A}}(X_0, X) \longrightarrow \mathsf{Ext}^1_{\mathscr{A}}(X_1, X) \longrightarrow \mathsf{Ext}^1_{\mathscr{A}}(X_2, X)$ which is exact. Hence $\mathsf{Ext}^1_{\mathsf{mod}\text{-}\underline{\mathcal{X}}}(F, \mathsf{Ext}^1_{\mathscr{A}}(-, X)|_{\mathcal{X}}) = 0$ and this implies that $\mathsf{Ext}^1_{\mathscr{A}}(-, X)|_{\mathcal{X}}$ is injective in $\mathsf{mod}\text{-}\underline{\mathcal{X}}$. Let $\alpha: X_1 \longrightarrow X_2$ be a map in $\mathcal{X}$ and $0 \longrightarrow X_i \longrightarrow T_i \longrightarrow X_i' \longrightarrow 0$ be ω-injective copresentations of the X_i. Since $\mathsf{Ext}^1_{\mathscr{A}}(X_1', T_2) = 0$, there exists an exact commutative diagram in $\mathcal{X}$:

$$\begin{tikzcd}
0 \arrow[r] & X_1 \arrow[r] \arrow[d, "\alpha"'] & T_1 \arrow[r] \arrow[d, "\beta"'] & X_1' \arrow[r] \arrow[d, "\gamma"'] & 0 \\
0 \arrow[r] & X_2 \arrow[r] & T_2 \arrow[r] & X_2' \arrow[r] & 0
\end{tikzcd}$$

which induces an exact commutative diagram in $\mathsf{mod}\text{-}\mathcal{X}$:

$$\begin{tikzcd}
0 \arrow[r] & \mathsf{H}(X_1) \arrow[r] \arrow[d, "\mathsf{H}(\alpha)"'] & \mathsf{H}(T_1) \arrow[r] \arrow[d, "\mathsf{H}(\beta)"'] & \mathsf{H}(X_1') \arrow[r] \arrow[d, "\mathsf{H}(\gamma)"'] & \mathsf{Ext}^1_{\mathscr{A}}(-, X_1)|_{\mathcal{X}} \arrow[r] \arrow[d, "\alpha^*"'] & 0 \\
0 \arrow[r] & \mathsf{H}(X_2) \arrow[r] & \mathsf{H}(T_2) \arrow[r] & \mathsf{H}(X_2') \arrow[r] & \mathsf{Ext}^1_{\mathscr{A}}(-, X_2)|_{\mathcal{X}} \arrow[r] & 0
\end{tikzcd}$$

By diagram chasing it is easy to see that the objects $\mathsf{Ext}^1_{\mathscr{A}}(-, X_i)|_{\mathcal{X}}$ are independent of the chosen ω-injective copresentations of the X_i, and the map α^* is independent of the liftings β and γ of α. In this way we obtain a functor $\mathsf{H}^* : \mathcal{X} \longrightarrow \mathsf{mod}\text{-}\underline{\mathcal{X}}$, $\mathsf{H}^*(X) = \mathsf{Ext}^1_{\mathscr{A}}(-, X)|_{\mathcal{X}}$, and $\mathsf{H}^*(\alpha) = \alpha^*$. Since $\mathsf{H}^*(\omega) = 0$, H^* induces a functor $\mathsf{H}^\vee : \mathcal{X}/\omega \longrightarrow \mathsf{mod}\text{-}\underline{\mathcal{X}}$, and by the above analysis we have $\operatorname{Im} \mathsf{H}^\vee \subseteq \mathsf{Inj}\,\mathsf{mod}\text{-}\underline{\mathcal{X}}$. If $\alpha^* = 0$, then chasing the above diagram we see that α factors through $X_1 \longrightarrow T_1$ and therefore α is zero in $\mathcal{X}/\omega$, i.e. $\mathsf{H}^\vee$ is faithful. In the same way, using that H is fully faithful and its image consists of projective objects of $\mathsf{mod}\text{-}\mathcal{X}$, we see that $\mathsf{H}^\vee$ is full. Let F be an injective object in $\mathsf{mod}\text{-}\underline{\mathcal{X}}$ with projective resolution in $\mathsf{mod}\text{-}\mathcal{X}$ as in (5.1). Then the induced long exact sequence

$$0 \longrightarrow \mathsf{H}(X_2) \longrightarrow \mathsf{H}(X_1) \longrightarrow \mathsf{H}(X_0) \longrightarrow \mathsf{Ext}^1_{\mathscr{A}}(-, X_2)|_{\mathcal{X}} \longrightarrow \cdots$$

shows that we have a monomorphism $F \rightarrowtail \mathsf{Ext}^1_{\mathscr{A}}(-, X_2)|_{\mathcal{X}}$ in $\mathsf{mod}\text{-}\underline{\mathcal{X}}$ which splits since F is injective. This produces an idempotent endomorphism f of $\mathsf{Ext}^1_{\mathscr{A}}(-, X_2)|_{\mathcal{X}}$ which clearly comes from an idempotent endomorphism $\underline{e} : \underline{X}_2 \longrightarrow \underline{X}_2$ in $\mathcal{X}/\omega$. By our hypothesis on $\mathcal{X}/\omega$, the idempotent $\underline{e}$ splits: there is an object $X_3 \in \mathcal{X}$ and morphisms $\underline{\kappa} : \underline{X}_2 \longrightarrow \underline{X}_3$ and $\underline{\lambda} : \underline{X}_3 \longrightarrow \underline{X}_2$ such that $\underline{e} = \underline{\kappa} \circ \underline{\lambda}$ and $1_{\underline{X}_3} = \underline{\lambda} \circ \underline{\kappa}$. This clearly implies that F is isomorphic to $\mathsf{H}^\vee(\underline{X}_3)$ and therefore $\mathsf{H}^\vee : \mathcal{X}/\omega \longrightarrow \mathsf{Inj}\,\mathsf{mod}\text{-}\underline{\mathcal{X}}$ is an equivalence. □

We have the following consequence, see also [32, Proposition 4.6] for a related result.

Corollary 5.4. *Let $\mathscr{A}$ be an abelian category and $(\mathcal{X}, \mathcal{Y})$ a cotorsion pair in $\mathscr{A}$ with heart $\omega = \mathcal{X} \cap \mathcal{Y}$.*

(i) *If $\mathscr{A}$ has enough projectives, then there is an equivalence:*

$$\mathsf{mod}\text{-}\underline{\mathcal{X}} \xrightarrow{\approx} (\mathcal{X}/\omega\text{-}\mathsf{mod})^{\mathsf{op}}$$

(ii) *If $\mathscr{A}$ has enough injectives, then there is an equivalence:*

$$\overline{\mathcal{Y}}\text{-}\mathsf{mod} \xrightarrow{\approx} (\mathsf{mod}\text{-}\mathcal{Y}/\omega)^{\mathsf{op}}$$

Proof. By Lemma 5.3, the abelian category $\mathsf{mod}\text{-}\underline{\mathcal{X}}$ has enough injectives and we have a full embedding $\mathsf{H}^* : \mathcal{X}/\omega \longrightarrow \mathsf{Inj}\,\mathsf{mod}\text{-}\underline{\mathcal{X}}$ and $\mathsf{Inj}\,\mathsf{mod}\text{-}\underline{\mathcal{X}} = \mathsf{add}\,\mathsf{Im}\,\mathsf{H}^*$. Taking coherent functors we have $\mathsf{mod}\text{-}\underline{\mathcal{X}} \approx ((\mathsf{Inj}\,\mathsf{mod}\text{-}\underline{\mathcal{X}})\text{-}\mathsf{mod})^{\mathsf{op}} \approx ((\mathsf{add}\,\mathsf{Im}\,\mathsf{H}^\vee)\text{-}\mathsf{mod})^{\mathsf{op}} \approx (\mathcal{X}/\omega\text{-}\mathsf{mod})^{\mathsf{op}}$. The second part is proved in a dual way. □

Example 5.5. (α) If $\mathscr{A}$ has enough projectives and enough injectives, then considering the cotorsion pairs $(\mathscr{A}, \mathsf{Inj}\,\mathscr{A})$ and $(\mathsf{Proj}\,\mathscr{A}, \mathscr{A})$ in $\mathscr{A}$, we deduce an equivalence $\mathsf{mod}\text{-}\underline{\mathscr{A}} \approx (\overline{\mathscr{A}}\text{-}\mathsf{mod})^{\mathsf{op}}$.

(β) Let T, resp. S, be a cotilting, resp. tilting, module over an Artin algebra Λ. Since we have cotorsion pairs $({}^\perp T, \mathcal{Y})$ and $(\mathcal{X}, S^\perp)$ in $\mathsf{mod}\text{-}\Lambda$, we deduce equivalences: $\mathsf{mod}\text{-}\underline{{}^\perp T} \approx \{({}^\perp T/\,\mathsf{add}\,T)\text{-}\mathsf{mod}\}^{\mathsf{op}}$ and $\overline{S^\perp}\text{-}\mathsf{mod} \approx \{\mathsf{mod}\text{-}(S^\perp/\,\mathsf{add}\,S)\}^{\mathsf{op}}$. If $\mathsf{id}\,T \leqslant 1$, resp. $\mathsf{pd}\,S \leqslant 1$, then it is easy to see that $\mathcal{Y}/\,\mathsf{add}\,T \approx \mathcal{Z} := \{A \in \mathsf{mod}\text{-}\Lambda \mid \mathsf{Hom}_\Lambda(A, T) = 0\}$, resp. $\mathcal{X}/\,\mathsf{add}\,S \approx \mathcal{W} := \{A \in \mathsf{mod}\text{-}\Lambda \mid \mathsf{Hom}_\Lambda(S, A) = 0\}$, see [27, Proposition V.5.2]. It follows that in this case we have equivalences: $\overline{\mathcal{Y}}\text{-}\mathsf{mod} \approx (\mathsf{mod}\text{-}\mathcal{Z})^{\mathsf{op}}$ and $\mathsf{mod}\text{-}\underline{\mathcal{X}} \approx (\mathcal{W}\text{-}\mathsf{mod})^{\mathsf{op}}$.

For the proof of the following we refer to [11, 2.43], [4, Theorem 3.3] and [22, Corollary 4.9].

Lemma 5.6. *Let $\mathcal{X}$ be a resolving subcategory of an abelian category $\mathscr{A}$ with enough projectives. If $\mathcal{X} \subseteq {}^{\perp}\{\mathsf{Proj}\,\mathscr{A}\}$, then the loop functor $\Omega : \underline{\mathcal{X}} \longrightarrow \underline{\mathcal{X}}$ is fully faithful and any projective object in* $\mathsf{mod}\text{-}\underline{\mathcal{X}}$ *is injective.*

Theorem 5.7. *Let $\mathscr{A}$ be an abelian category and $(\mathcal{X}, \mathcal{Y})$ a cotorsion pair in $\mathscr{A}$ with heart $\omega = \mathcal{X} \cap \mathcal{Y}$.*

(i) *Assume that $\mathscr{A}$ has enough projectives and $\mathcal{X} \subseteq {}^{\perp}\,\mathsf{Proj}\,\mathscr{A}$. Then $\mathcal{X} \subseteq \mathsf{GProj}\,\mathscr{A}$ provided that one of the following conditions holds:*
   (a) $\mathsf{res.dim}_{\mathcal{X}}\,\mathscr{A} < \infty$.
   (b) *$\mathcal{X}$ is of finite representation type and idempotents split in $\mathcal{X}/\omega$.*

(ii) *Assume that $\mathscr{A}$ has enough injectives and $\mathcal{Y} \subseteq (\mathsf{Inj}\,\mathscr{A})^{\perp}$. Then $\mathcal{Y} \subseteq \mathsf{GInj}\,\mathscr{A}$ provided that one of the following conditions holds:*
   (a) $\mathsf{cores.dim}_{\mathcal{Y}}\,\mathscr{A} < \infty$.
   (b) *$\mathcal{Y}$ is of finite representation type and idempotents split in $\mathcal{Y}/\omega$.*

Proof. We only prove (i) since part (ii) follows by duality.

Since $\mathcal{X} \subseteq {}^{\perp}\,\mathsf{Proj}\,\mathscr{A}$, it follows that $\mathsf{Proj}\,\mathscr{A} \subseteq \mathcal{Y}$ and therefore $\mathsf{Proj}\,\mathscr{A} \subseteq \omega$. We first show that in both cases the conclusion $\mathcal{X} \subseteq \mathsf{GProj}\,\mathscr{A}$ follows from the equality $\omega = \mathsf{Proj}\,\mathscr{A}$. Indeed if this equality holds, then since ω, as the heart of the cotorsion pair $(\mathcal{X}, \mathcal{Y})$, is an injective cogenerator of $\mathcal{X}$, it follows that for any object $M \in \mathcal{X}$ there exists a short exact sequence $0 \longrightarrow M \longrightarrow P \longrightarrow M' \longrightarrow 0$, where P is projective and $M' \in \mathcal{X}$. Since $\mathcal{X} \subseteq {}^{\perp}\,\mathsf{Proj}\,\mathscr{A}$, this clearly implies that M is Gorenstein-projective. Hence $\mathcal{X} \subseteq \mathsf{GProj}\,\mathscr{A}$.

(a) If $\mathsf{res.dim}_{\mathcal{X}}\,\mathscr{A} = d < \infty$, then by [27, Proposition VII.1.1] there exists a cotorsion pair $(\mathcal{X}, \mathsf{Proj}^{<\infty}\,\mathscr{A})$ with heart $\mathsf{Proj}\,\mathscr{A}$. It follows that $\mathcal{Y} = \mathsf{Proj}^{<\infty}\,\mathscr{A}$ and then clearly $\omega = \mathsf{Proj}\,\mathscr{A}$.

(b) Since $\mathcal{X}$ is a Krull–Schmidt category of finite representation type, it follows that so are $\mathsf{Proj}\,\mathscr{A}$ and ω. Assume that $\omega \setminus \mathsf{Proj}\,\mathscr{A} \neq \emptyset$. Let $\{T_1, T_2, \ldots, T_l\}$, where $l \geqslant 1$, be the set of indecomposable non-projective objects of ω and let $\{X_1, X_2, \ldots, X_k\}$ be the set of indecomposable objects of $\mathcal{X}$ which are not in ω. Note that if $\mathcal{X} = \omega$, then it is easy to see that $\mathcal{X} = \mathsf{Proj}\,\mathscr{A} \subseteq \mathsf{GProj}\,\mathscr{A}$, so we may exclude this trivial case. Set $T = \bigoplus_{i=1}^{l} T_i$, $S = \bigoplus_{i=1}^{k} X_i$, and $X = T \oplus S$. Then $\mathcal{X} = \mathsf{add}(P \oplus X)$, where P is the direct sum of the set of non-isomorphic indecomposable projective objects of $\mathscr{A}$. Consider the stable categories $\underline{\mathcal{X}}$ and $\mathcal{X}/\omega$. Then clearly $\underline{\mathcal{X}} = \mathsf{add}\,\underline{X}$ and $\mathcal{X}/\omega = \mathsf{add}\,\underline{S}$. Since idempotents split in $\mathcal{X}/\omega$, by Lemma 5.3 there is an equivalence $\mathsf{Inj}\,\mathsf{mod}\text{-}\underline{\mathcal{X}} \approx \mathcal{X}/\omega$. On the other hand, since $\mathcal{X} \subseteq {}^{\perp}\,\mathsf{Proj}\,\mathscr{A}$, by Lemma 5.6 it follows that any projective object of $\mathsf{mod}\text{-}\underline{\mathcal{X}}$ is injective, i.e. $\mathsf{Proj}\,\mathsf{mod}\text{-}\underline{\mathcal{X}} \subseteq \mathsf{Inj}\,\mathsf{mod}\text{-}\underline{\mathcal{X}}$. Putting things together we infer the existence of a full embedding $\underline{\mathcal{X}} \longrightarrow \mathcal{X}/\omega$ which is the composite $\underline{\mathcal{X}} \xrightarrow{\approx} \mathsf{Proj}\,\mathsf{mod}\text{-}\underline{\mathcal{X}} \longrightarrow \mathsf{Inj}\,\mathsf{mod}\text{-}\underline{\mathcal{X}} \xrightarrow{\approx} \mathcal{X}/\omega$. Therefore $k + l = |\underline{\mathcal{X}}| \leqslant |\mathcal{X}/\omega| = k$. This contradiction shows that $l = 0$, i.e. any indecomposable object of the heart ω is projective. We infer that $\omega = \mathsf{Proj}\,\mathscr{A}$. $\square$

Remark 5.8. Let $\mathscr{A}$ be an abelian Krull–Schmidt category with enough projectives and enough injectives. If $\mathcal{X}$ is a contravariantly finite resolving subcategory of $\mathscr{A}$, then it is easy to see that there exists a cotorsion pair $(\mathcal{X}, \mathcal{Y})$ in $\mathscr{A}$, where $\mathcal{Y} = \mathcal{X}^{\perp}$. Hence Theorem 5.7 applies to resolving subcategories of finite representation type in $\mathscr{A}$ such that idempotents split in $\mathcal{X}/\mathcal{X} \cap \mathcal{X}^{\perp}$ and $\mathcal{X} \subseteq {}^{\perp}\,\mathsf{Proj}\,\mathscr{A}$.

A full subcategory $\mathcal{X}$ of an abelian category $\mathscr{A}$ is called **projectively thick** if $\mathcal{X}$ is resolving and closed under direct summands and cokernels of $\mathsf{Proj}\,\mathscr{A}$-monics. By [24], the map $\mathcal{X} \longmapsto \underline{\mathcal{X}}$ gives a bijection between projectively thick subcategories of $\mathsf{GProj}\,\mathscr{A}$, and thick subcategories of $\underline{\mathsf{GProj}}\,\mathscr{A}$. The following gives a non-commutative generalization of the above mentioned result of Yoshino, see [68, Theorem 5.5].

Proposition 5.9. *Let Λ be an Artin algebra and $\mathcal{X}$ a resolving subcategory of $\mathsf{mod}\text{-}\Lambda$. Then the following statements are equivalent.*

(i) *$\mathcal{X}$ is of finite representation type and $\mathcal{X} \subseteq {}^{\perp}\Lambda$.*
(ii) (a) *$\mathcal{X}$ is contravariantly finite in $\mathsf{mod}\text{-}\Lambda$,*
  (b) *$\mathcal{X}$ is a projectively thick subcategory of $\mathsf{Gproj}\,\Lambda$, and*
  (c) *the functor category $\mathsf{Mod}\text{-}\underline{\mathcal{X}}$ is Frobenius.*
(iii) (a) *$\mathcal{X}$ is contravariantly finite in $\mathsf{mod}\text{-}\Lambda$,*
  (b) *$\underline{\mathcal{X}}$ is a triangulated subcategory of $\underline{\mathsf{mod}}\text{-}\Lambda$, and*
  (c) *the stable triangulated category $\varinjlim \mathcal{X}/\varinjlim \mathsf{proj}\,\Lambda$ is phantomless.*

In particular if $\mathcal{X}$ is a representation-finite resolving subcategory of ${}^{\perp}\Lambda$, then $\mathcal{X} \subseteq \mathsf{Gproj}\,\Lambda$, $\mathsf{mod}\text{-}\underline{\mathcal{X}}$ and $\mathsf{Mod}\text{-}\underline{\mathcal{X}}$ are Frobenius, and there is an equivalence $\varinjlim\{\mathcal{X}/\mathsf{proj}\,\Lambda\} \approx \varinjlim \mathcal{X}/\varinjlim \mathsf{proj}\,\Lambda$.

Proof. (i) $\Rightarrow$ (ii) Clearly $\mathcal{X}$ is contravariantly finite in $\mathsf{mod}\text{-}\Lambda$. Since $\mathsf{mod}\text{-}\Lambda$ is Krull–Schmidt with enough injectives, by Remark 5.8, there exists a cotorsion pair $(\mathcal{X}, \mathcal{Y})$ in $\mathsf{mod}\text{-}\Lambda$, where $\mathcal{Y} = \mathcal{X}^{\perp}$, and idempotents split in the stable category $\mathcal{X}/\mathcal{X} \cap \mathcal{Y}$. Then $\mathcal{X} \subseteq \mathsf{Gproj}\,\Lambda$ by Theorem 5.7. Since $\mathsf{Mod}\text{-}\underline{\mathcal{X}} \approx \mathsf{Mod}\text{-}\Gamma$ where Γ is the stable endomorphism algebra of a representation generator of $\mathcal{X}$, it follows that $\mathsf{Mod}\text{-}\underline{\mathcal{X}}$ is Frobenius. Finally as in [24, Proof of Proposition 3.8(iii)] we see that $\mathcal{X}$ is projectively thick.

(ii) $\Rightarrow$ (i) Clearly $\mathcal{X} \subseteq {}^{\perp}\Lambda$ and the stable category $\underline{\mathcal{X}}$ is a thick triangulated subcategory of $\underline{\mathsf{Gproj}}\,\Lambda$. By [24, Theorem 9.4] the stable category $\varinjlim \mathcal{X}/\varinjlim \mathsf{proj}\,\Lambda$ is triangulated and compactly generated, and we have an equivalence $\underline{\mathcal{X}} \approx \{\varinjlim \mathcal{X}/\varinjlim \mathsf{proj}\,\Lambda\}^{\mathsf{cpt}}$. Hence the representation category of $\varinjlim \mathcal{X}/\varinjlim \mathsf{proj}\,\Lambda$ is equivalent to the module category $\mathsf{Mod}\text{-}\underline{\mathcal{X}}$ which by hypothesis is Frobenius; this clearly implies that $\mathsf{Mod}\text{-}\underline{\mathcal{X}}$ is perfect. On the other hand since $\mathcal{X}$ is contravariantly finite, as in the proof of Theorem 3.1, it follows that $\mathcal{X}$ is a dualizing variety. Then by [16] so is $\underline{\mathcal{X}}$ and consequently $\underline{\mathcal{X}}$ has left almost split maps. This implies by [23, Theorem 10.2] that $\mathsf{Mod}\text{-}\underline{\mathcal{X}}$ is locally finite. Then a similar argument as in the proof of Theorem 3.1 shows that $\underline{\mathcal{X}}$, or equivalently $\mathcal{X}$, is of finite representation type.

(ii) $\Leftrightarrow$ (iii) This follows from the following facts: (α) $\underline{\mathcal{X}}$ is a triangulated subcategory of $\underline{\mathsf{mod}}\text{-}\Lambda$ if and only if $\mathcal{X}$ is projectively thick and consists of Gorenstein-projective modules [24], and (β) $\varinjlim \mathcal{X}/\varinjlim \mathsf{proj}\,\Lambda$ is compactly generated and phantomless if and only if the module category $\mathsf{Mod}\text{-}\underline{\mathcal{X}}$ is Frobenius [21]. $\square$

If $\mathscr{A}$ is an abelian category, we let $\Omega^{\infty}(\mathscr{A})$ be the full subcategory of $\mathscr{A}$ consisting of arbitrary syzygy objects, i.e. objects A admitting an exact sequence $0 \longrightarrow A \longrightarrow P^0 \longrightarrow P^1 \longrightarrow \cdots$, with $P^n \in \mathsf{Proj}\,\mathscr{A}$, $\forall n \geqslant 0$.

Corollary 5.10. *Let Λ be an Artin algebra and let $\mathcal{X}$ be a resolving subcategory of $\mathsf{mod}\text{-}\Lambda$. If $\mathcal{X} \subseteq {}^{\perp}\Lambda$ and $\Omega^{\infty}(\mathsf{mod}\text{-}\Lambda^{\mathsf{op}})$ is of finite representation type, then $\mathcal{X} \subseteq \mathsf{Gproj}\,\Lambda$.*

Proof. Since $\mathcal{X} \subseteq {}^{\perp}\Lambda$, it follows directly that $\mathsf{Tr}(\mathcal{X}) \subseteq \Omega^{\infty}(\mathsf{mod}\text{-}\Lambda^{\mathsf{op}})$. Since $\mathsf{Tr}: \underline{\mathsf{mod}}\text{-}\Lambda \longrightarrow \underline{\mathsf{mod}}\text{-}\Lambda^{\mathsf{op}}$ is a duality, we have a fully faithful functor $\mathsf{Tr}: \underline{\mathcal{X}} \longrightarrow \Omega^{\infty}(\underline{\mathsf{mod}}\text{-}\Lambda^{\mathsf{op}})$. It follows that $\underline{\mathcal{X}}$ is of finite representation type and therefore $\mathcal{X} \subseteq \mathsf{Gproj}\,\Lambda$ by Proposition 5.9. □

Now we can give the proof of Theorem 5.2.

Proof of Theorem 5.2. (i) ⇒ (ii) ⇒ (iii) If ${}^{\perp}\Lambda = \mathsf{Gproj}\,\Lambda$, then ${}^{\perp}\Lambda \cap (\mathsf{Gproj}\,\Lambda)^{\perp} = \mathsf{Gproj}\,\Lambda \cap (\mathsf{Gproj}\,\Lambda)^{\perp}$. Clearly any finitely generated projective module lies in $\mathsf{Gproj}\,\Lambda \cap (\mathsf{Gproj}\,\Lambda)^{\perp}$. If $X \in \mathsf{Gproj}\,\Lambda \cap (\mathsf{Gproj}\,\Lambda)^{\perp}$, then since X is Gorenstein-projective, there is a short exact sequence $0 \longrightarrow X \longrightarrow P \longrightarrow X' \longrightarrow 0$, where $P \in \mathsf{proj}\,\Lambda$ and $X' \in \mathsf{Gproj}\,\Lambda$. This extension splits since $X \in (\mathsf{Gproj}\,\Lambda)^{\perp}$, so X is projective as a direct summand of P. Hence $\mathsf{Gproj}\,\Lambda \cap (\mathsf{Gproj}\,\Lambda)^{\perp} = \mathsf{proj}\,\Lambda$ which is always of finite representation type.

(iii) ⇒ (i) Since by [24], the subcategory $(\mathsf{Gproj}\,\Lambda)^{\perp}$ is resolving, it follows that so is ${}^{\perp}\Lambda \cap (\mathsf{Gproj}\,\Lambda)^{\perp}$. Hence ${}^{\perp}\Lambda \cap (\mathsf{Gproj}\,\Lambda)^{\perp}$ is a resolving subcategory of $\mathsf{mod}\text{-}\Lambda$ of finite representation type. Then by Proposition 5.9 we have ${}^{\perp}\Lambda \cap (\mathsf{Gproj}\,\Lambda)^{\perp} \subseteq \mathsf{Gproj}\,\Lambda$. Let T be in $\mathsf{mod}\text{-}\Lambda$ such that $\mathsf{add}\,T = {}^{\perp}\Lambda \cap (\mathsf{Gproj}\,\Lambda)^{\perp}$. Then T is Gorenstein-projective and since T lies in $(\mathsf{Gproj}\,\Lambda)^{\perp}$ we infer that T is projective. On the other hand since any projective module lies in both ${}^{\perp}\Lambda$ and $(\mathsf{Gproj}\,\Lambda)^{\perp}$, we infer that ${}^{\perp}\Lambda \cap (\mathsf{Gproj}\,\Lambda)^{\perp} = \mathsf{proj}\,\Lambda$. Since $\mathsf{Gproj}\,\Lambda$ is contravariantly finite, there is a cotorsion pair $(\mathsf{Gproj}\,\Lambda, (\mathsf{Gproj}\,\Lambda)^{\perp})$ in $\mathsf{mod}\text{-}\Lambda$. Hence $\forall A \in {}^{\perp}\Lambda$ there exists a short exact sequence $0 \longrightarrow A \longrightarrow Y^A \longrightarrow X^A \longrightarrow 0$, where $Y^A \in (\mathsf{Gproj}\,\Lambda)^{\perp}$ and $X^A \in \mathsf{Gproj}\,\Lambda$. Clearly then Y^A lies in ${}^{\perp}\Lambda \cap (\mathsf{Gproj}\,\Lambda)^{\perp}$, hence Y^A is projective and therefore $A \cong \Omega X^A$. Since X^A is Gorenstein-projective, this implies that so is A. We infer that ${}^{\perp}\Lambda = \mathsf{Gproj}\,\Lambda$. □

Corollary 5.11. *For an Artin algebra Λ, the following are equivalent.*

(i) *${}^{\perp}\Lambda$ is of finite representation type.*
(ii) *Λ is of finite CM-type and ${}^{\perp}\Lambda = \mathsf{Gproj}\,\Lambda$.*

5.2. Infinitely generated stable modules

Now we turn our attention to the question of when infinitely generated stable modules, i.e. modules in ${}^{\perp}\Lambda = \{A \in \mathsf{Mod}\text{-}\Lambda \mid \mathsf{Ext}^n_{\Lambda}(A, \Lambda) = 0,\ \forall n \geqslant 1\}$, are Gorenstein-projective.

Proposition 5.12. *If Λ is an Artin algebra then the following are equivalent.*

(i) *Λ is virtually Gorenstein and $\mathsf{GProj}\,\Lambda = {}^{\perp}\Lambda$.*
(ii) *Any module in ${}^{\perp}\Lambda \cap (\mathsf{Gproj}\,\Lambda)^{\perp}$ is a direct sum of finitely generated modules.*
(iii) (a) *${}^{\perp}\Lambda \cap (\mathsf{Gproj}\,\Lambda)^{\perp}$ is of finite representation type.*
  (b) *Any finitely generated module admits a left $\{{}^{\perp}\Lambda \cap (\mathsf{Gproj}\,\Lambda)^{\perp}\}$-approximation which is finitely generated.*
(iv) *${}^{\perp}\Lambda \cap (\mathsf{Gproj}\,\Lambda)^{\perp}$ is a pure-semisimple locally finitely presented category.*

If (i) *holds, then* ${}^{\perp}\Lambda = \mathsf{Gproj}\,\Lambda$.

Proof. Set $\mathcal{H} = {}^{\perp}\Lambda \cap (\mathsf{Gproj}\,\Lambda)^{\perp}$ and $\omega = \mathcal{H} \cap \mathsf{mod}\text{-}\Lambda$. Then clearly $\omega = {}^{\perp}\Lambda \cap (\mathsf{Gproj}\,\Lambda)^{\perp}$.

(i) ⇒ (ii) follows from Proposition 4.7 since in this case we have $(\mathsf{Gproj}\,\Lambda)^{\perp} = (\mathsf{GProj}\,\Lambda)^{\perp} = \varinjlim(\mathsf{Gproj}\,\Lambda)^{\perp}$.

(ii) ⇒ (iii) We have $\mathcal{H} = \varinjlim \omega = \mathsf{Add}\,\omega$. Clearly then any finitely generated module admits a finitely generated left $\mathcal{H}$-approximation. Since ω is resolving, by Theorem 3.1, ω is of finite representation type.

(iii) ⇒ (iv) Since the subcategories ${}^{\perp}\Lambda$ and $(\mathsf{Gproj}\,\Lambda)^{\perp}$ are definable and resolving, it follows that so is $\mathcal{H}$. Then by [50] condition (b) implies that $\mathcal{H} = \varinjlim \omega$, so $\mathcal{H}$ is locally finitely presented. On the other hand condition (a) together with Theorem 3.1 implies that $\mathcal{H}$ is pure-semisimple.

(iv) ⇒ (i) Since ω is resolving, by Theorem 3.1 it follows that $\omega = \mathsf{fp}\,\mathcal{H}$ is of finite representation type. Then as in the proof of Theorem 5.2 we infer that $\omega = \mathsf{proj}\,\Lambda$ and therefore $\mathcal{H} = \varinjlim \mathsf{proj}\,\Lambda = \mathsf{Proj}\,\Lambda$. Consider the cotorsion pair $(\mathscr{A}, (\mathsf{Gproj}\,\Lambda)^{\perp})$ in $\mathsf{Mod}\text{-}\Lambda$ cogenerated by $\mathsf{Gproj}\,\Lambda$. Clearly then we have inclusions $\mathscr{A} \subseteq \varinjlim \mathsf{Gproj}\,\Lambda \subseteq \mathsf{GProj}\,\Lambda \subseteq {}^{\perp}\Lambda$. If M lies in ${}^{\perp}\Lambda$, let $0 \longrightarrow M \longrightarrow Y^M \longrightarrow A^M \longrightarrow 0$ be a short exact sequence, where $Y^M \in (\mathsf{Gproj}\,\Lambda)^{\perp}$ and $A^M \in \mathscr{A}$. Plainly Y^M lies in ${}^{\perp}\Lambda \cap (\mathsf{Gproj}\,\Lambda)^{\perp} = \mathcal{H}$, hence Y^M is projective. Then M lies in $\mathscr{A}$ since the latter is resolving. We infer that $\mathscr{A} = {}^{\perp}\Lambda$ and therefore $\mathscr{A} = \varinjlim \mathsf{Gproj}\,\Lambda = \mathsf{GProj}\,\Lambda = {}^{\perp}\Lambda$. Then ${}^{\perp}\Lambda = \mathsf{Gproj}\,\Lambda$ and, by 4.6.1, the equality $\mathsf{GProj}\,\Lambda = \varinjlim \mathsf{Gproj}\,\Lambda$ shows that Λ is virtually Gorenstein. □

Recall that the Gorenstein Symmetry Conjecture, (GSC) for short, see [18,27,24], asserts that an Artin algebra with finite one-sided self-injective dimension is Gorenstein. This conjecture is still open. However virtually Gorenstein algebras, in particular algebras which are derived or stably equivalent to algebras of finite representation type satisfy (GSC), see [24]. The next consequence shows that also algebras for which stable modules are Gorenstein-projective satisfy the conjecture.

Corollary 5.13. *Let Λ be an Artin algebra. If $\mathsf{id}\,\Lambda_{\Lambda} < \infty$, then the following are equivalent.*

(i) *Λ is Gorenstein.*
(ii) *$\mathsf{Gproj}\,\Lambda$ is contravariantly finite and ${}^{\perp}\Lambda \cap (\mathsf{Gproj}\,\Lambda)^{\perp}$ is of finite representation type.*
(iii) *Any module in ${}^{\perp}\Lambda \cap (\mathsf{Gproj}\,\Lambda)^{\perp}$ is a direct sum of finitely generated modules.*

Proof. If (i) holds then Λ is virtually Gorenstein and ${}^{\perp}\Lambda = \mathsf{GProj}\,\Lambda$. Then assertion (iii) holds by Proposition 5.12. If (iii) holds, then, by Proposition 5.12, Λ is virtually Gorenstein and then (i) holds, by Example 4.5.1. If Λ is Gorenstein, then by 4.6.5, $\mathsf{Gproj}\,\Lambda$ is contravariantly finite and clearly we have ${}^{\perp}\Lambda \cap (\mathsf{Gproj}\,\Lambda)^{\perp} = \mathsf{proj}\,\Lambda$ which is of finite representation type, so (i) implies (ii). If (ii) holds, then by Theorem 5.2 we have ${}^{\perp}\Lambda = \mathsf{Gproj}\,\Lambda$. Since $\mathsf{id}\,\Lambda_{\Lambda} = d < \infty$, it follows that for any module $A \in \mathsf{mod}\text{-}\Lambda$, we have $\mathsf{Ext}_{\Lambda}^{d+k}(A, \Lambda) \cong \mathsf{Ext}_{\Lambda}^{k}(\Omega^d A, \Lambda) = 0$, $\forall k \geqslant 1$. Hence $\Omega^d A \in {}^{\perp}\Lambda = \mathsf{Gproj}\,\Lambda$, $\forall A \in \mathsf{mod}\text{-}\Lambda$. This implies that $\mathsf{res.dim}_{\mathsf{Gproj}\,\Lambda}\,\mathsf{mod}\text{-}\Lambda = \mathsf{G\text{-}dim}\,\Lambda \leqslant d$. Therefore Λ is Gorenstein. □

5.3. Local rings

Let throughout R be a commutative Noetherian local ring. The following result shows that if R is Gorenstein but not Artinian, then the condition ${}^{\perp}R = \mathsf{GProj}\,R$ does not necessarily holds.

Proposition 5.14. *If R is Gorenstein, then the following statements are equivalent.*

(i) *R is complete and ${}^{\perp}R = \mathsf{GProj}\,R$.*
(ii) *R is Artinian.*

Proof. Since R is Gorenstein, there exists a cotorsion pair $(\mathsf{GProj}\,R, \mathsf{Proj}^{<\infty} R)$ in $\mathsf{Mod}\text{-}R$, where $\mathsf{Proj}^{<\infty} R$ denotes the full subcategory of all R-modules with finite projective dimension, see [20,37]. If A lies in ${}^{\perp}\,\mathsf{Proj}\,R$, let $(\dagger): 0 \longrightarrow Y_A \longrightarrow X_A \longrightarrow A \longrightarrow 0$ be an exact sequence, where X_A is Gorenstein-projective and Y_A has finite projective dimension. Then clearly Y_A lies in ${}^{\perp}\,\mathsf{Proj}\,R$, and since Y_A has finite projective dimension, we infer that Y_A is projective. Then $(\dagger)$ splits and therefore A is Gorenstein-projective, i.e. $\mathsf{GProj}\,R = {}^{\perp}\,\mathsf{Proj}\,R$.

(i) $\Rightarrow$ (ii) Since R is complete, it follows that R is pure-injective as an R-module, see [43]. On the other hand it is well known that if E is a pure-injective module over a ring S, then for any filtered system $\{A_i \mid i \in I\}$ of S-modules, we have an isomorphism $\varprojlim \mathsf{Ext}^n_S(A_i, E) \cong \mathsf{Ext}^n_S(\varinjlim A_i, E)$, $\forall n \geqslant 0$, see [10, Chap. 1, Proposition 10.1]. We infer that ${}^{\perp}R$, and therefore $\mathsf{GProj}\,R$, is closed under filtered colimits. Since $\mathsf{Gproj}\,R \subseteq \mathsf{GProj}\,R$, we have $\varinjlim \mathsf{Gproj}\,R \subseteq \mathsf{GProj}\,R$. Since R is Gorenstein, by [37] we also have $\mathsf{GFlat}\,R \subseteq \varinjlim \mathsf{Gproj}\,R$. Hence any Gorenstein-flat R-module is Gorenstein-projective. In particular any flat R-module is Gorenstein-projective. However since R is local, it follows that the big finitistic dimension, which is equal to the Krull dimension of R, is finite. By a result of Jensen [42] this implies that any flat R-module has finite projective dimension. Since clearly any Gorenstein-projective module with finite projective dimension is projective, we infer that any flat R-module is projective and therefore R is perfect. This is equivalent to R being Artinian.

(ii) $\Rightarrow$ (i) Since R is Artinian, it follows that R is complete and moreover we have ${}^{\perp}R = {}^{\perp}\,\mathsf{Proj}\,R$, since in this case any projective R-module is a direct summand of a product of copies of R, see [53]. Hence $\mathsf{GProj}\,R = {}^{\perp}R$. $\square$

Corollary 5.15. *The following statements are equivalent.*

(i) *R is complete and $\Omega^{\infty}(\mathsf{Mod}\text{-}R) \subseteq {}^{\perp}R \subseteq \mathsf{GProj}\,R$.*
(ii) *R is Artinian Gorenstein.*

If (ii) *holds, then the inclusions in* (i) *are equalities. If* $\mathsf{Gproj}\,R \neq \mathsf{proj}\,R$*, then the above are also equivalent to:*

(iii) *R is complete,* $\mathsf{Gproj}\,R$ *is contravariantly finite and* $\mathsf{GProj}\,R = {}^{\perp}R$.

Proof. (i) $\Rightarrow$ (ii) Since $\mathsf{GProj}\,R \subseteq {}^{\perp}\,\mathsf{Proj}\,R$, the hypothesis implies that $\Omega^{\infty}(\mathsf{Mod}\text{-}R) \subseteq {}^{\perp}\,\mathsf{Proj}\,R$. Let $P^{\bullet}$ be an acyclic complex of projective R-modules. Then $\forall n \in \mathbb{Z}$, the R-module $A^n := \mathsf{Ker}(P^n \longrightarrow P^{n+1})$ lies in $\Omega^{\infty}(\mathsf{Mod}\text{-}R)$. Since the latter is contained in ${}^{\perp}\,\mathsf{Proj}\,R$, it follows that the complex $\mathsf{Hom}_R(P^{\bullet}, P)$ is acyclic, $\forall P \in \mathsf{Proj}\,R$. Hence any acyclic complex of projective R-modules is totally acyclic. Since R is complete, it is well known that R admits a dualizing complex. Then by [52] it follows that R is Gorenstein. Then R is Artinian by Proposition 5.14.

(ii) $\Rightarrow$ (i) Clearly R is complete, and ${}^{\perp}R = \mathsf{GProj}\,R$ by Proposition 5.14. If $d = \mathsf{id}\,R$, then any R-module admits a finite resolution of length at most d by Gorenstein-projective modules, equivalently $\Omega^d(A)$ is Gorenstein-projective, $\forall A \in \mathsf{Mod}\text{-}R$. Let A be in $\Omega^{\infty}(\mathsf{Mod}\text{-}R)$ and let $0 \longrightarrow A \longrightarrow P^0 \longrightarrow P^1 \longrightarrow \cdots$ be an exact sequence, where the P^ns are projective. Setting $A^0 = A$ and $A^n := \mathsf{Ker}(P^n \longrightarrow P^{n+1})$, $\forall n \geqslant 1$, it follows that $A = \Omega^d A^d$ and therefore A is Gorenstein-projective. Hence $\Omega^{\infty}(\mathsf{Mod}\text{-}R) \subseteq \mathsf{GProj}\,R$. $\square$

Corollary 5.16. *If R is of finite CM-type and $\mathsf{Gproj}\,R \neq \mathsf{proj}\,R$, then the following are equivalent.*

(i) *R is Artinian (Gorenstein).*
(ii) *R is complete and any module in ${}^{\perp}R \cap (\mathsf{GProj}\,R)^{\perp}$ is a direct sum of finitely generated modules.*

Proof. By Theorem 4.20, R is Gorenstein and therefore $(\mathsf{GProj}\,R)^{\perp} = \mathsf{Proj}^{<\infty}\,R$.

(i) $\Rightarrow$ (ii) Clearly R is complete and by Proposition 5.14 we have ${}^{\perp}R = \mathsf{GProj}\,R$. Hence ${}^{\perp}R \cap (\mathsf{GProj}\,R)^{\perp} = \mathsf{GProj}\,R \cap \mathsf{Proj}^{<\infty}\,R = \mathsf{Proj}\,R$ and (ii) follows.

(ii) $\Rightarrow$ (i) Since R is of finite CM-type, we have $\mathsf{GProj}\,R = \mathsf{Add}\,\mathsf{Gproj}\,R$. This clearly implies that ${}^{\perp}R = \mathsf{Gproj}\,R$ and $(\mathsf{GProj}\,R)^{\perp} = (\mathsf{Gproj}\,R)^{\perp}$. Since R is complete, this in turn implies that the resolving subcategory ${}^{\perp}R \cap (\mathsf{GProj}\,R)^{\perp}$ is definable, and in particular it is closed under products. Since any module in ${}^{\perp}R \cap (\mathsf{GProj}\,R)^{\perp}$ is a direct sum of modules in ${}^{\perp}R \cap (\mathsf{Gproj}\,R)^{\perp} = \mathsf{proj}\,R$, it follows that ${}^{\perp}R \cap (\mathsf{GProj}\,R)^{\perp} = \mathsf{Proj}\,R$ and therefore $\mathsf{Proj}\,R$ is closed under products. Then by Chase's Theorem, see [43], R is perfect and consequently Artinian. $\square$

6. Relative Auslander algebras

Let Λ be an Artin algebra. If Λ is of finite representation type and Γ is the endomorphism ring of a representation generator of $\mathsf{mod}\text{-}\Lambda$, then by a classical result of Auslander [7], Γ is an *Auslander algebra*, i.e. $\mathsf{gl.dim}\,\Gamma \leqslant 2 \leqslant \mathsf{dom.dim}\,\Gamma$, where $\mathsf{dom.dim}\,\Gamma$ denotes the dominant dimension of Γ, that is, the smallest integer $d \geqslant 0$ such that the first d terms of a minimal injective resolution of Γ are projective. In this section we are concerned with the global dimension of the endomorphism ring of a representation generator of a resolving subcategory $\mathcal{X}$ of $\mathsf{mod}\text{-}\Lambda$. Then we apply our results to the case $\mathcal{X} = \mathsf{Gproj}\,\Lambda$ for an Artin algebra Λ of finite CM-type.

6.1. *Auslander algebras of resolving subcategories of finite representation type*

To proceed further we need the following general observation.

Lemma 6.1. *Let $\mathscr{A}$ be an abelian category with enough projectives and $(\mathcal{X}, \mathcal{Y})$ a cotorsion pair in $\mathscr{A}$ with heart $\omega = \mathcal{X} \cap \mathcal{Y}$. If $\mathsf{H}: \mathscr{A} \longrightarrow \mathsf{mod}\text{-}\mathcal{X}$, $\mathsf{H}(A) = \mathscr{A}(-, A)|_{\mathcal{X}}$ is the restricted Yoneda functor, then:*

(i) *$\mathsf{H}: \mathscr{A} \longrightarrow \mathsf{mod}\text{-}\mathcal{X}$ is fully faithful and admits an exact left adjoint $\mathsf{T}: \mathsf{mod}\text{-}\mathcal{X} \longrightarrow \mathscr{A}$.*
(ii) *$\forall A \in \mathscr{A}$: $\mathsf{pd}\,\mathsf{H}(A) = \mathsf{res.dim}_{\mathcal{X}}\,A$.*

Proof. (i) Define a functor $\mathsf{T}: \mathsf{mod}\text{-}\mathcal{X} \longrightarrow \mathscr{A}$ as follows. If $\mathsf{H}(X_1) \longrightarrow \mathsf{H}(X_0) \longrightarrow F \longrightarrow 0$ is a finite presentation of $F \in \mathsf{mod}\text{-}\mathcal{X}$, then set $\mathsf{T}(F) = \mathsf{Coker}(X_1 \longrightarrow X_0)$. It is easy to see that this defines a functor $\mathsf{T}: \mathsf{mod}\text{-}\mathcal{X} \longrightarrow \mathscr{A}$ which is left adjoint to H. For any object $A \in \mathscr{A}$ let $0 \longrightarrow Y_A^0 \longrightarrow X_A^0 \longrightarrow A \longrightarrow 0$ and $0 \longrightarrow Y_A^1 \longrightarrow X_A^1 \longrightarrow Y_A^0 \longrightarrow 0$ be right $\mathcal{X}$-approximation sequences of A and Y_A^0 respectively, and observe that $X_A^1 := \omega_A^1$ lies in $\mathcal{X} \cap \mathcal{Y} = \omega$ since $\mathcal{Y}$ is closed under extensions. Continuing in this way we have an exact resolution

$$\cdots \longrightarrow \omega_A^n \longrightarrow \omega_A^{n-1} \longrightarrow \cdots \longrightarrow \omega_A^1 \longrightarrow X_A^0 \longrightarrow A \longrightarrow 0 \tag{6.1}$$

where the ω^i_A lie in ω and the $Y^i_A = \mathsf{Im}(\omega^i_A \longrightarrow \omega^{i-1}_A)$ lie in $\mathcal{Y}$. By construction this sequence remains exact after applying H and therefore

$$\cdots \longrightarrow \mathsf{H}\big(\omega^n_A\big) \longrightarrow \mathsf{H}\big(\omega^{n-1}_A\big) \longrightarrow \cdots \longrightarrow \mathsf{H}\big(\omega^1_A\big) \longrightarrow \mathsf{H}\big(X^0_A\big) \longrightarrow \mathsf{H}(A) \longrightarrow 0 \tag{6.2}$$

is a projective resolution of $\mathsf{H}(A)$. Applying T and using that $\mathsf{TH}|_{\mathcal{X}} \cong \mathsf{Id}_{\mathcal{X}}$, we infer that the counit $\mathsf{TH} \longrightarrow \mathsf{Id}_{\mathscr{A}}$ is invertible and the resulted complex is acyclic. Hence H is fully faithful and it is easy to see that T is exact.

(ii) Set $d := \mathsf{res.dim}_{\mathcal{X}} A$. If $d = 0$, then $A \in \mathcal{X}$ and therefore $\mathsf{pd}\,\mathsf{H}(A) = 0$, since $\mathsf{H}(A)$ is projective. Assume that $d \geqslant 1$. Since $\mathcal{X}$ is resolving and $\mathsf{res.dim}_{\mathcal{X}} A = d$, the object $Y^d_A \in \mathcal{Y}$ in (6.1) lies in $\mathcal{X}$, see [11, Lemma 3.12]. Hence $Y^d_A := \omega^d_A$ lies in ω and therefore (6.2) gives a projective resolution

$$0 \longrightarrow \mathsf{H}\big(\omega^d_A\big) \longrightarrow \mathsf{H}\big(\omega^{d-1}_A\big) \longrightarrow \cdots \longrightarrow \mathsf{H}\big(\omega^1_A\big) \longrightarrow \mathsf{H}\big(X^0_A\big) \longrightarrow \mathsf{H}(A) \longrightarrow 0$$

We infer that $\mathsf{pd}\,\mathsf{H}(A) \leqslant d$. If $\mathsf{pd}\,\mathsf{H}(A) = t < d$, then $\mathsf{H}(Y^t_A)$ will be projective, or equivalently $Y^t_A \in \mathcal{X} \cap \mathcal{Y} = \omega$. Then $\mathsf{res.dim}_{\mathcal{X}} A \leqslant t$ and this is not true. We infer that $\mathsf{pd}\,\mathsf{H}(A) = d = \mathsf{res.dim}_{\mathcal{X}} A$. This argument also shows that $\mathsf{res.dim}_{\mathcal{X}} A = \infty$ if and only if $\mathsf{pd}\,\mathsf{H}(A) = \infty$. □

Now let $\mathcal{X}$ be a contravariantly finite subcategory of $\mathsf{mod}\text{-}\Lambda$. As before we denote by $\mathcal{L}(\mathcal{X}) = \mathsf{Mod}\text{-}\mathcal{X}$ the representation category of $\varinjlim \mathcal{X}$. In this subsection we denote by H the functor $\mathsf{Mod}\text{-}\Lambda \longrightarrow \mathcal{L}(\mathcal{X})$, which is given by $\mathsf{H}(A) = \mathsf{Hom}_\Lambda(-, A)|_{\mathcal{X}}$. Contravariant finiteness of $\mathcal{X}$ implies that the category $\mathsf{mod}\text{-}\mathcal{X}$ of coherent functors $\mathcal{X}^{\mathsf{op}} \longrightarrow \mathfrak{Ab}$ is abelian and then H induces a functor $\mathsf{H} : \mathsf{mod}\text{-}\Lambda \longrightarrow \mathsf{mod}\text{-}\mathcal{X}$.

Corollary 6.2. *Let $\mathcal{X}$ be a contravariantly finite resolving subcategory of* $\mathsf{mod}\text{-}\Lambda$*. Then:*

(i) $\forall A \in \mathsf{mod}\text{-}\Lambda$*:* $\mathsf{pd}\,\mathsf{H}(A) = \mathsf{res.dim}_{\mathcal{X}} A$.
(ii) $\forall A \in \mathsf{Mod}\text{-}\Lambda$*:* $\mathsf{fd}\,\mathsf{H}(A) = \mathsf{res.dim}_{\varinjlim \mathcal{X}} A$.
(iii) $\forall A \in \mathsf{Mod}\text{-}\Lambda$*:* $\mathsf{pd}\,\mathsf{H}(A) = \mathsf{res.dim}_{\varinjlim \mathcal{X}} A$ *iff $\mathcal{X}$ is of finite representation type.*

Proof. By a result of Auslander–Reiten, see [14], there exists a cotorsion pair $(\mathcal{X}, \mathcal{Y})$ in $\mathsf{mod}\text{-}\Lambda$. Then (i) follows from Lemma 6.1. By a result of Krause and Solberg, see [54], $(\mathcal{X}, \mathcal{Y})$ extends to a cotorsion pair $(\varinjlim \mathcal{X}, \varinjlim \mathcal{Y})$ in $\mathsf{Mod}\text{-}\Lambda$. Since $\mathcal{X}$ consists of finitely presented modules, the functor $\mathsf{H} : \mathsf{Mod}\text{-}\Lambda \longrightarrow \mathsf{Mod}\text{-}\mathcal{X}$ commutes with filtered colimits. This implies that H induces an equivalence between $\varinjlim \mathcal{X}$ and $\mathsf{Flat}(\mathsf{Mod}\text{-}\mathcal{X})$ and an equivalence between $\mathsf{Add}\,\mathcal{X}$ and $\mathsf{Proj}(\mathsf{Mod}\text{-}\mathcal{X})$. Then working as in Lemma 6.1, part (ii) follows. Finally (iii) holds iff any flat $\mathcal{X}$-module is projective iff $\mathsf{Mod}\text{-}\mathcal{X}$ is perfect. As in the proof of Theorem 3.1 this is equivalent to saying that $\mathcal{X}$ is of finite representation type. □

Definition 6.3. Let $\mathcal{X}$ be a resolving subcategory of $\mathsf{mod}\text{-}\Lambda$ which is of finite representation type. If T is a representation generator of $\mathcal{X}$, i.e. $\mathcal{X} = \mathsf{add}(T)$, then the Artin algebra

$$\mathsf{A}(\mathcal{X}) := \mathsf{End}_\Lambda(T)^{\mathsf{op}}$$

is called the **Auslander algebra** of $\mathcal{X}$, or the $\mathcal{X}$-Auslander algebra of Λ.

Clearly the Auslander algebra of $\mathcal{X}$ is unique up to Morita equivalence, that is $\mathsf{A}(\mathcal{X})$ is up to Morita equivalence independent of the choice of the representation generator of $\mathcal{X}$, and in case $\mathcal{X} = \mathsf{mod}\text{-}\Lambda$, i.e. Λ is of finite representation type, $\mathsf{A}(\mathsf{mod}\text{-}\Lambda)$ is the Auslander algebra of Λ. Auslander proved that $\mathsf{gl.dim}\,\mathsf{A}(\mathsf{mod}\text{-}\Lambda) = 2$ provided that Λ is not semisimple. The following result treats the general case.

Proposition 6.4. *Let $\mathcal{X}$ be a representation-finite resolving subcategory of* $\mathsf{mod}\text{-}\Lambda$.

(i) $\mathsf{gl.dim}\,\mathsf{A}(\mathcal{X}) = 0$ *if and only if Λ is semisimple (and then $\mathcal{X} = \mathsf{proj}\,\Lambda$).*
(ii) $\mathsf{gl.dim}\,\mathsf{A}(\mathcal{X}) = 1$ *if and only if* $\mathsf{gl.dim}\,\Lambda = 1$ *and* $\mathcal{X} = \mathsf{proj}\,\Lambda$.
(iii) $\mathsf{gl.dim}\,\mathsf{A}(\mathcal{X}) = 2$ *if and only if either*
  (a) $\mathcal{X} = \mathsf{proj}\,\Lambda$ *and* $\mathsf{gl.dim}\,\Lambda = 2$, *or*
  (b) $\mathcal{X} \neq \mathsf{proj}\,\Lambda$ *and* $\mathsf{res.dim}_{\mathcal{X}}\,\mathsf{mod}\text{-}\Lambda \leqslant 2$.
(iv) *If* $\mathsf{res.dim}_{\mathcal{X}}\,\mathsf{mod}\text{-}\Lambda \geqslant 3$, *then:* $\mathsf{gl.dim}\,\mathsf{A}(\mathcal{X}) = \mathsf{res.dim}_{\mathcal{X}}\,\mathsf{mod}\text{-}\Lambda$.

In any case we have the following bounds:

$$\mathsf{gl.dim}\,\mathsf{A}(\mathcal{X}) = \begin{cases} \mathsf{gl.dim}\,\Lambda & \text{if } \mathcal{X} = \mathsf{proj}\,\Lambda \\ \max\{2, \mathsf{res.dim}_{\mathcal{X}}\,\mathsf{mod}\text{-}\Lambda\} & \text{if } \mathcal{X} \neq \mathsf{proj}\,\Lambda \end{cases}$$

Proof. (i) If Λ is semisimple, then $\mathcal{X} = \mathsf{proj}\,\Lambda = \mathsf{mod}\text{-}\Lambda$ and clearly $\mathsf{gl.dim}\,\mathsf{A}(\mathcal{X}) = 0$. Conversely if this holds, then let $A \in \mathsf{mod}\text{-}\Lambda$ and let $f : P \longrightarrow A$ be the projective cover of A. Then the projection $\mathsf{H}(P) \longrightarrow \mathsf{Im}\,\mathsf{H}(f)$ splits and therefore $\mathsf{Im}\,\mathsf{H}(f)$ is projective, hence $\mathsf{Im}\,\mathsf{H}(f) \cong \mathsf{H}(X)$ for some direct summand X of P. Applying T we have $A \cong X$. Hence A is projective and therefore Λ is semisimple.

(ii) If $\mathcal{X} = \mathsf{proj}\,\Lambda$ and $\mathsf{gl.dim}\,\Lambda = 1$, then $\mathsf{gl.dim}\,\mathsf{A}(\mathcal{X}) = 1$, since $\mathsf{A}(\mathcal{X})$ is Morita equivalent to Λ. Conversely assume that $\mathsf{gl.dim}\,\mathsf{A}(\mathcal{X}) = 1$ and there is a non-projective module $X \in \mathcal{X}$. Let $0 \longrightarrow \Omega X \longrightarrow P \longrightarrow X \longrightarrow 0$ be exact where P is projective. Then we have an exact sequence $0 \longrightarrow \mathsf{H}(\Omega X) \longrightarrow \mathsf{H}(P) \longrightarrow \mathsf{H}(X) \longrightarrow M \longrightarrow 0$ in $\mathsf{mod}\text{-}\mathsf{A}(\mathcal{X})$ and M is not zero since otherwise X will be projective as a direct summand of P. Let $N := \mathsf{Im}(\mathsf{H}(P) \longrightarrow \mathsf{H}(X))$. Then N is projective, hence $N = \mathsf{H}(X_1)$ for some module X_1 in $\mathcal{X}$ which necessarily is a direct summand of P. Applying T we then have $\mathsf{T}(M) = 0$ and $X \cong X_1$. This means that X is projective and this is not the case. We conclude that $\mathcal{X} = \mathsf{proj}\,\Lambda$. Hence $\mathsf{A}(\mathcal{X})$ is Morita equivalent to Λ and then $\mathsf{gl.dim}\,\mathsf{A}(\mathcal{X}) = \mathsf{gl.dim}\,\Lambda = 1$.

(iii) If (a) holds, then clearly $\mathsf{gl.dim}\,\mathsf{A}(\mathcal{X}) = 2$ since $\mathsf{A}(\mathcal{X})$ is Morita equivalent to Λ. If (b) holds, and $\mathsf{res.dim}_{\mathcal{X}}\,\mathsf{mod}\text{-}\Lambda = 0$, then $\mathcal{X} = \mathsf{mod}\text{-}\Lambda$ and therefore Λ is a non-semisimple algebra of finite representation type. Then by Auslander's Theorem [7] we have $\mathsf{gl.dim}\,\mathsf{A}(\mathcal{X}) = 2$. Now assume that $1 \leqslant \mathsf{res.dim}_{\mathcal{X}}\,\mathsf{mod}\text{-}\Lambda \leqslant 2$. Since there is a non-projective module $X \in \mathcal{X}$, we have a non-split exact sequence $0 \longrightarrow \Omega X \longrightarrow P \longrightarrow X \longrightarrow 0$ where P is projective. Then ΩX lies in $\mathcal{X}$ since $\mathcal{X}$ is resolving, and we have a projective resolution $0 \longrightarrow \mathsf{H}(\Omega X) \longrightarrow \mathsf{H}(P) \longrightarrow \mathsf{H}(X) \longrightarrow M \longrightarrow 0$ in $\mathsf{mod}\text{-}\mathsf{A}(\mathcal{X})$. As in the proof of (ii) above, M is non-zero and N is not projective. Hence $\mathsf{pd}\,M = 2$. On the other hand any finitely generated $\mathsf{A}(\mathcal{X})$-module F admits a presentation $0 \longrightarrow \mathsf{H}(A) \longrightarrow \mathsf{H}(X^1) \longrightarrow \mathsf{H}(X^0) \longrightarrow F \longrightarrow 0$ where the X^i lie in $\mathcal{X}$ and $A = \mathsf{Ker}(X^1 \longrightarrow X^0)$. If $B = \mathsf{Coker}(X^1 \longrightarrow X^0)$, then we have an exact sequence $0 \longrightarrow A \longrightarrow X^1 \longrightarrow X^0 \longrightarrow B \longrightarrow 0$ in $\mathsf{mod}\text{-}\Lambda$. If $B = 0$, then $A \in \mathcal{X}$ since $\mathcal{X}$ is resolving. If $B \neq 0$, then $A \in \mathcal{X}$ since $\mathsf{res.dim}_{\mathcal{X}}\,B \leqslant 2$. In any case the above exact sequence is a projective resolution of F and therefore $\mathsf{pd}\,F \leqslant 2$. Hence $\mathsf{gl.dim}\,\mathsf{A}(\mathcal{X}) = 2$. If $\mathsf{gl.dim}\,\mathsf{A}(\mathcal{X}) = 2$, then for any Λ-module A we have $\mathsf{res.dim}_{\mathcal{X}} A = \mathsf{pd}\,\mathsf{H}(A) \leqslant 2$, so $\mathsf{res.dim}_{\mathcal{X}}\,\mathsf{mod}\text{-}\Lambda \leqslant 2$. If $\mathcal{X} = \mathsf{proj}\,\Lambda$, then (i), (ii) imply that $\mathsf{gl.dim}\,\Lambda = 2$.

(iv) Let $d = \mathsf{res.dim}_{\mathcal{X}}\,\mathsf{mod}\text{-}\Lambda \geqslant 3$. Let M be a finitely generated $\mathsf{A}(\mathcal{X})$-module and let $\mathsf{H}(X^{d-1}) \longrightarrow \mathsf{H}(X^{d-2}) \longrightarrow \cdots \longrightarrow \mathsf{H}(X^0) \longrightarrow M \longrightarrow 0$ be part of the projective resolution of M. Then we have an exact complex $X^{d-1} \longrightarrow X^{d-2} \longrightarrow \cdots \longrightarrow X^0 \longrightarrow A \longrightarrow 0$ in $\mathsf{mod}\text{-}\Lambda$ where the X^i lie in $\mathcal{X}$ and $A = \mathsf{Coker}(X^1 \longrightarrow X^0)$. Since $\mathsf{res.dim}_{\mathcal{X}} A \leqslant d$ and $\mathcal{X}$ is resolving, it follows that $B := \mathsf{Ker}(X^{d-1} \longrightarrow X^{d-2})$ lies in $\mathcal{X}$, see [11]. Setting $X^d = B$, we have a finite projective resolution $0 \longrightarrow \mathsf{H}(X^d) \longrightarrow \mathsf{H}(X^{d-1}) \longrightarrow \mathsf{H}(X^{d-2}) \longrightarrow \cdots \longrightarrow \mathsf{H}(X^0) \longrightarrow M \longrightarrow 0$ of M and therefore $\mathsf{gl.dim}\,\mathsf{A}(\mathcal{X}) \leqslant d$. Then in fact $\mathsf{gl.dim}\,\mathsf{A}(\mathcal{X}) = d$ since otherwise $\mathsf{res.dim}_{\mathcal{X}}\,\mathsf{mod}\text{-}\Lambda < d$. Since clearly $\mathsf{res.dim}_{\mathcal{X}}\,\mathsf{mod}\text{-}\Lambda = \infty$ if and only if $\mathsf{gl.dim}\,\mathsf{A}(\mathcal{X}) = \infty$, we infer that $\mathsf{res.dim}_{\mathcal{X}}\,\mathsf{mod}\text{-}\Lambda = \mathsf{gl.dim}\,\mathsf{A}(\mathcal{X})$. □

Remark 6.5. Let Λ be an Artin algebra of finite representation type. Also let T be a finitely generated cotilting module of injective dimension $n \geqslant 0$, see [14]. Then we have a cotorsion pair $(\mathcal{X}, \mathcal{Y})$ in $\mathsf{mod}\text{-}\Lambda$, where $\mathcal{X} = {}^{\perp}T$ and clearly $\mathcal{X}$ is a representation-finite resolving subcategory of $\mathsf{mod}\text{-}\Lambda$. It is well known that $\mathsf{res.dim}_{\mathcal{X}}\,\mathsf{mod}\text{-}\Lambda = n$. Hence by Proposition 6.4 we have $\mathsf{gl.dim}\,\mathsf{A}(\mathcal{X}) = n$, if $n \geqslant 3$; if T is not projective and $n \leqslant 2$, then $\mathsf{gl.dim}\,\mathsf{A}(\mathcal{X}) = 2$; finally $\mathsf{gl.dim}\,\mathsf{A}(\mathcal{X}) = n$, if T is projective and $n \leqslant 1$. We infer that the global dimension of the Auslander algebra of a (non-trivial) resolving subcategory of finite representation type can take any value $n \geqslant 0$. On the other hand let $\mathcal{X}$ be a representation-finite thick subcategory of $\mathsf{mod}\text{-}\Lambda$ containing Λ, i.e. $\mathcal{X}$ is resolving and closed under cokernels of monomorphisms, e.g. $\mathcal{X} = (\mathsf{Gproj}\,\Lambda)^{\perp}$. Then $\mathsf{gl.dim}\,\mathsf{A}(\mathcal{X}) = \infty$, except if $\mathcal{X} = \mathsf{mod}\text{-}\Lambda$ in which case $\mathsf{gl.dim}\,\mathsf{A}(\mathcal{X}) \leqslant 2$.

Let $\mathcal{X}$ be a resolving contravariantly finite subcategory of $\mathsf{mod}\text{-}\Lambda$. We set $\mathsf{inj}\,\mathcal{X} := \mathcal{X} \cap \mathsf{inj}\,\Lambda$. We say that $\mathcal{X}$ has $\mathcal{X}$-dominant dimension at least two: $\mathsf{dom.dim}_{\mathsf{inj}\,\mathcal{X}}\,\mathcal{X} \geqslant 2$, if for any $X \in \mathcal{X}$, there exists an exact sequence $0 \longrightarrow X \longrightarrow I^0 \longrightarrow I^1$ where the I^i lie in $\mathsf{inj}\,\mathcal{X}$. If $\mathcal{X} = \mathsf{proj}\,\Lambda$, then $\mathsf{dom.dim}_{\mathsf{proj}\,\Lambda \cap \mathsf{inj}\,\Lambda}\,\mathsf{proj}\,\Lambda := \mathsf{dom.dim}\,\Lambda$ is the dominant dimension of Λ.

The following result describes when the $\mathcal{X}$-Auslander algebra of Λ is an Auslander algebra.

Proposition 6.6.

(1) *Let Λ be an Artin algebra and $\mathcal{X}$ a representation-finite resolving subcategory of* $\mathsf{mod}\text{-}\Lambda$. *Then the following are equivalent.*
   (i) *The $\mathcal{X}$-Auslander algebra of Λ is an Auslander algebra.*
   (ii) $\mathsf{res.dim}_{\mathcal{X}}\,\mathsf{mod}\text{-}\Lambda \leqslant 2 \leqslant \mathsf{dom.dim}_{\mathsf{inj}\,\mathcal{X}}\,\mathcal{X}$.
   If (i) *holds, then the Artin algebra $\Gamma = \mathsf{End}_{\Lambda}(J)^{\mathsf{op}}$, where* $\mathsf{add}\,J = \mathsf{inj}\,\mathcal{X}$, *is of finite representation type and there are equivalences $\mathcal{X} \approx \mathsf{mod}\text{-}\Gamma$ and $\mathsf{mod}\text{-}\mathsf{A}(\mathcal{X}) \approx \mathsf{A}(\mathsf{mod}\text{-}\Gamma)$. In particular $\mathcal{X}$ is abelian.*

(2) *Let Γ be an Artin algebra of finite representation type. Then $\mathcal{X} = \mathsf{proj}\,\mathsf{A}(\mathsf{mod}\text{-}\Gamma)$ is a resolving representation-finite subcategory of* $\mathsf{mod}\text{-}\mathsf{A}(\mathsf{mod}\text{-}\Gamma)$ *with* $\mathsf{res.dim}_{\mathcal{X}}\,\mathsf{mod}\text{-}\mathsf{A}(\mathsf{mod}\text{-}\Gamma) \leqslant 2 \leqslant \mathsf{dom.dim}_{\mathsf{inj}\,\mathcal{X}}\,\mathcal{X}$.

Proof. (1) If (ii) holds, then by Proposition 6.4, the $\mathcal{X}$-Auslander algebra $\mathsf{A}(\mathcal{X})$ of Λ has $\mathsf{gl.dim}\,\mathsf{A}(\mathcal{X}) \leqslant 2$. Since $\mathsf{dom.dim}_{\mathsf{Inj}\,\mathcal{X}}\,\mathcal{X} \geqslant 2$, $\forall X \in \mathcal{X}$, there exists an exact sequence $0 \longrightarrow X \longrightarrow I^0 \longrightarrow I^1$, where the I^i lie in $\mathsf{inj}\,\mathcal{X}$. Since H preserves injectives as a right adjoint of the exact functor T, we infer that $0 \longrightarrow \mathsf{H}(X) \longrightarrow \mathsf{H}(I^0) \longrightarrow \mathsf{H}(I^1)$ is exact in $\mathsf{mod}\text{-}\mathsf{A}(\mathcal{X})$, where the $\mathsf{H}(I^i)$ are projective–injective. Hence $\mathsf{dom.dim}\,\mathsf{A}(\mathcal{X}) \geqslant 2$ and therefore $\mathsf{A}(\mathcal{X})$ is an

Auslander algebra. It is not difficult to see that the functor $\mathsf{H}: \text{mod-}\Lambda \longrightarrow \text{mod-}\mathsf{A}(\mathcal{X})$ induces an equivalence $\mathsf{H}: \operatorname{inj}\mathcal{X} \approx \operatorname{proj}\mathsf{A}(\mathcal{X}) \cap \operatorname{inj}\mathsf{A}(\mathcal{X})$. If J is Λ-module such that $\operatorname{add} J = \operatorname{inj}\mathcal{X}$, then $\operatorname{add}\mathsf{H}(J) = \mathsf{H}(\operatorname{inj}\mathcal{X})$. By Auslander's results [7], the ring $\Gamma := \operatorname{End}_\Lambda(J)^{\mathrm{op}} \cong \operatorname{End}_{\mathsf{A}(\mathcal{X})}\mathsf{H}(J)^{\mathrm{op}}$ is of finite representation type and $\text{mod-}\Gamma$ is equivalent to $\mathcal{X}$. Clearly then $\text{mod-}\mathsf{A}(\mathcal{X})$ is equivalent to $\mathsf{A}(\text{mod-}\Gamma)$. Conversely if (i) holds, then $\operatorname{gl.dim}\mathsf{A}(\mathcal{X}) \leqslant 2$. By Proposition 6.4 we have $\operatorname{res.dim}_{\mathcal{X}} \text{mod-}\Lambda \leqslant 2$. Since $\operatorname{dom.dim}\mathsf{A}(\mathcal{X}) \geqslant 2$, and since H is fully faithful and induces an equivalence between $\operatorname{inj}\mathcal{X}$ and $\operatorname{proj}\mathsf{A}(\mathcal{X}) \cap \operatorname{inj}\mathsf{A}(\mathcal{X})$, it follows that $\operatorname{dom.dim}_{\operatorname{inj}\mathcal{X}}\mathcal{X} \geqslant 2$.

(2) Let Γ be an algebra of finite representation type and let $\Lambda = \mathsf{A}(\text{mod-}\Gamma)$ be its Auslander algebra. Then $\mathcal{X} = \text{mod-}\Gamma \approx \operatorname{proj}\Lambda$ is a resolving subcategory of finite representation type of $\text{mod-}\Lambda$, $\operatorname{inj}\mathcal{X}$ is the full subcategory of projective–injective modules of $\text{mod-}\Lambda$ and $\operatorname{res.dim}_{\mathcal{X}} \text{mod-}\Lambda = \operatorname{gl.dim}\Lambda \leqslant 2$ and $\operatorname{dom.dim}_{\operatorname{Inj}\mathcal{X}}(\mathcal{X}) = \operatorname{dom.dim}\Lambda \geqslant 2$, since Λ is an Auslander algebra. $\square$

As a consequence we have the following bound on the dimension $\dim \mathbf{D}^b(\text{mod-}\Lambda)$ of the bounded derived category of $\text{mod-}\Lambda$ in the sense of Rouquier, see [64,29]. Note that if $\mathcal{X} = \operatorname{proj}\Lambda$, then $\dim \mathbf{D}^b(\text{mod-}\Lambda) \leqslant \operatorname{gl.dim}\Lambda = \operatorname{res.dim}_{\mathcal{X}} \text{mod-}\Lambda$, see [64].

Corollary 6.7. *Let $\mathcal{X}$ representation-finite resolving subcategory of* $\text{mod-}\Lambda$. *Then:*

$$\dim \mathbf{D}^b(\text{mod-}\Lambda) \leqslant \max\{2, \operatorname{res.dim}_{\mathcal{X}} \text{mod-}\Lambda\}$$

Proof. The exact left adjoint functor $\mathsf{T}: \text{mod-}\mathsf{A}(\mathcal{X}) \longrightarrow \text{mod-}\Lambda$ of H induces a triangulated functor $\mathbf{D}^b\mathsf{T}: \mathbf{D}^b(\text{mod-}\mathsf{A}(\mathcal{X})) \longrightarrow \mathbf{D}^b(\text{mod-}\Lambda)$ which admits as a right adjoint the fully faithful right derived functor of H. Hence $\mathbf{D}^b(\text{mod-}\Lambda)$ is a Verdier quotient of $\mathbf{D}^b(\text{mod-}\mathsf{A}(\mathcal{X}))$ and therefore by [64] we have $\dim \mathbf{D}^b(\text{mod-}\Lambda) \leqslant \dim \mathbf{D}^b(\text{mod-}\mathsf{A}(\mathcal{X}))$. Since $\dim \mathbf{D}^b(\text{mod-}\Gamma) \leqslant \operatorname{gl.dim}\Gamma$, for any Artin algebra Γ, we have $\dim \mathbf{D}^b(\text{mod-}\mathsf{A}(\mathcal{X})) \leqslant \operatorname{gl.dim}\mathsf{A}(\mathcal{X})$ and the assertion follows from Proposition 6.4. $\square$

6.2. Cohen–Macaulay Auslander algebras

Let Λ be an Artin algebra. If Λ is of finite CM-type, then we call $\mathsf{A}(\operatorname{Gproj}\Lambda)$, resp. $\mathsf{A}(\underline{\operatorname{Gproj}}\,\Lambda)$, the resp. *stable*, *Cohen–Macaulay Auslander algebra* of Λ.

Now as a consequence of Propositions 6.4 and 6.6 we have the following.

Corollary 6.8. *Let Λ be an Artin algebra of finite CM-type. Then we have the following:*

(i) $\operatorname{gl.dim}\mathsf{A}(\operatorname{Gproj}\Lambda) = 0$ *if and only if Λ is semisimple.*
(ii) $\operatorname{gl.dim}\mathsf{A}(\operatorname{Gproj}\Lambda) = 1$ *if and only if* $\operatorname{gl.dim}\Lambda = 1$.
(iii) $\operatorname{gl.dim}\mathsf{A}(\operatorname{Gproj}\Lambda) = 2$ *if and only if either*
   (a) $\operatorname{Gproj}\Lambda = \operatorname{proj}\Lambda$ *and* $\operatorname{gl.dim}\Lambda = 2$, *or*
   (b) $\operatorname{Gproj}\Lambda \neq \operatorname{proj}\Lambda$ *and Λ is a Gorenstein algebra with* $\operatorname{id}\Lambda \leqslant 2$.
(iv) *If* $\text{G-dim}\,\text{mod-}\Lambda \geqslant 3$, *then:*

$$\operatorname{gl.dim}\mathsf{A}(\operatorname{Gproj}\Lambda) = \text{G-dim}\,\Lambda = \max\{\operatorname{id}{}_\Lambda\Lambda, \operatorname{id}\Lambda_\Lambda\}$$

(v) *Λ is Gorenstein iff its Cohen–Macaulay Auslander algebra* $\mathsf{A}(\operatorname{Gproj}\Lambda)$ *has finite global dimension.*
(vi) $\mathsf{A}(\operatorname{Gproj}\Lambda)$ *is an Auslander algebra if and only if* $\text{G-dim}\,\Lambda \leqslant 2 \leqslant \operatorname{dom.dim}\Lambda$.

Note that the Artin algebras Λ satisfying $\mathsf{G}\text{-}\mathsf{dim}\,\Lambda = 2 = \mathsf{dom.dim}\,\Lambda$ have been described in [17].

The following consequence gives an upper bound for the dimension of the bounded derived category of an Artin algebra of finite CM-type.

Corollary 6.9. *Let Λ be a virtually Gorenstein Artin algebra of finite CM-type. Then*

$$\mathsf{dim}\,\mathbf{D}^b(\mathsf{mod}\text{-}\Lambda) \leqslant \mathsf{max}\{2, \mathsf{id}\,\Lambda\}$$

Proof. If $\mathsf{id}\,\Lambda = \infty$, there is nothing to prove. Assume that $\mathsf{id}\,\Lambda_\Lambda = d < \infty$. Since Λ is virtually Gorenstein, it follows that Λ is Gorenstein and $\mathsf{id}\,{}_\Lambda\Lambda = \mathsf{id}_\Lambda\,\Lambda = d$. If $\mathsf{Gproj}\,\Lambda = \mathsf{proj}\,\Lambda$, i.e. Λ is Morita equivalent to $\mathsf{A}(\mathsf{Gproj}\,\Lambda)$, then $\mathsf{gl.dim}\,\Lambda = d$. Then the assertion follows from Corollaries 6.7 and 6.8. □

In the commutative case we have the following consequence. Note that if R is Cohen–Macaulay, then the second inequality below was first shown by Leuschke [55].

Corollary 6.10. *Let R be a commutative Noetherian complete local ring of finite CM-type. Then:*

$$\mathsf{dim}\,\mathbf{D}^b(\mathsf{mod}\text{-}R) \leqslant \mathsf{gl.dim}\,\mathsf{A}(\mathsf{Gproj}\,R) \leqslant \mathsf{max}\{2, \mathsf{dim}\,R\} < \infty$$

and the second inequality is an equality if $\mathsf{dim}\,R \geqslant 2$.

Proof. By Theorem 4.20, R is Gorenstein. Since the injective dimension $\mathsf{id}\,R$ of R coincides with the Krull dimension $\mathsf{dim}\,R$, the assertions follow as in Corollaries 6.8 and 6.9. □

Example 6.11. Let Λ be a non-semisimple Artin algebra. If the resolving subcategory $\mathsf{Sub}(\mathsf{mod}\text{-}\Lambda)$ of $T_2(\Lambda)$ is of finite representation type, it follows by Proposition 6.4 that $\mathsf{gl.dim}\,\mathsf{A}(\mathsf{Sub}(\mathsf{mod}\text{-}\Lambda)) = 2$, since $\mathsf{Sub}(\mathsf{mod}\text{-}\Lambda)$ is closed under submodules in $\mathsf{mod}\text{-}T_2(\Lambda)$. If in addition Λ is self-injective, e.g. $\Lambda_n = k[t]/(t^n)$ with $n \leqslant 5$, then, by Example 4.17, we have $\mathsf{G}\text{-}\mathsf{dim}\,T_2(\Lambda_n) = 1$ and therefore $\mathsf{gl.dim}\,\mathsf{A}(\mathsf{Gproj}\,T_2(\Lambda_n)) = 2$.

The following gives a convenient characterization of the full subcategory of Gorenstein-projectives.

Lemma 6.12. *Let $\mathscr{A}$ be an abelian category with enough projectives. If $\mathcal{X}$ is a full subcategory of $\mathsf{GProj}\,\mathscr{A}$, then the following are equivalent:*

(i) $\mathcal{X} = \mathsf{GProj}\,\mathscr{A}$.
(ii) $\mathcal{X}$ *is resolving and* $\mathsf{res.dim}_{\mathcal{X}}\,\mathsf{GProj}\,\mathscr{A} < \infty$.

Proof. If $\mathcal{X} = \mathsf{GProj}\,\mathscr{A}$, then clearly $\mathcal{X}$ is resolving and $\mathsf{res.dim}_{\mathcal{X}}\,\mathsf{GProj}\,\mathscr{A} = 0$. Conversely let $\mathsf{res.dim}_{\mathcal{X}}\,\mathsf{GProj}\,\mathscr{A} = d < \infty$ and let $A \in \mathsf{GProj}\,\mathscr{A}$. Then $A \cong \mathsf{Im}(P^{-1} \longrightarrow P^0)$ for some (totally) acyclic complex $\cdots \longrightarrow P^{-1} \longrightarrow P^0 \longrightarrow P^1 \longrightarrow \cdots$ of projectives. Clearly then $B = \mathsf{Im}(P^{d-1} \longrightarrow P^d)$ is Gorenstein-projective and $\Omega^d B = A$. Since $\mathsf{res.dim}_{\mathcal{X}}\,B \leqslant d$, by [11, 3.12] it follows that $A \in \mathcal{X}$. We infer that $\mathsf{GProj}\,\mathscr{A} = \mathcal{X}$. □

Let $\mathscr{A}$ be an abelian category with enough projectives. Following [27, VII.1] we say that $\mathscr{A}$ is *Gorenstein* if $\mathsf{res.dim}_{\mathsf{GProj}\,\mathscr{A}}\,\mathscr{A} < \infty$. For instance if Λ is a Noetherian ring, then Λ is Gorenstein if and only if the category $\mathsf{Mod}\text{-}\Lambda$ is Gorenstein in the above sense.

Part (ii) of the following consequence gives a trivial proof to (a generalization of) the main result of [56].

Corollary 6.13. *Let $\mathscr{A}$ be an abelian category with enough projectives.*

(i) *$\mathscr{A}$ is Gorenstein, resp. Gorenstein of finite CM-type, if and only if $\mathsf{GProj}\,\mathscr{A}$ admits a resolving subcategory $\mathcal{X}$, resp. of finite representation type, such that $\mathsf{res.dim}_{\mathcal{X}}\,\mathscr{A} < \infty$.*
(ii) *If $\mathscr{A}$ is R-linear over a commutative ring R and the R-module $\mathscr{A}(A, B)$ is finitely generated, $\forall A, B \in \mathscr{A}$, then the following are equivalent:*
 (a) *$\mathscr{A}$ is Gorenstein of finite CM-type.*
 (b) *$\mathsf{GProj}\,\mathscr{A}$ admits a resolving subcategory $\mathcal{X}$ of finite representation type such that $\mathsf{gl.dim}\,\mathsf{A}(\mathcal{X}) < \infty$.*

Proof. Part (i) follows from Lemma 6.12. For part (ii) note that for any object T of $\mathscr{A}$, the subcategory $\mathsf{add}\,T$ is contravariantly finite in $\mathscr{A}$. Therefore if $\mathcal{X}$ is of finite representation type, then the category $\mathcal{X}$ has weak kernels or equivalently the category $\mathsf{A}(\mathcal{X}) = \mathsf{mod}\text{-}\mathcal{X} \approx \mathsf{mod}\text{-}\,\mathsf{End}_{\mathscr{A}}(T)^{\mathsf{op}}$ of coherent functors over $\mathcal{X} = \mathsf{add}\,T$ is abelian. The rest follows from Lemma 6.12 using that $\mathsf{res.dim}_{\mathcal{X}}\,\mathscr{A} < \infty$ if and only if $\mathsf{gl.dim}\,\mathsf{A}(\mathcal{X}) < \infty$, see Proposition 6.4. □

We close this section with an application of Corollary 6.8 and results of Buchweitz [32] to algebras of finite CM-type having Cohen–Macaulay Auslander algebras of global dimension 2.

Let Λ be a finite-dimensional k-algebra of finite CM-type over a field k and assume that Λ has infinite global dimension. Let T be a representation generator of $\mathsf{Gproj}\,\Lambda$, so $\mathsf{A}(\mathsf{Gproj}\,\Lambda)$ is Morita equivalent to $\Gamma := \mathsf{End}_\Lambda(T)^{\mathsf{op}}$. Then $\underline{T}$ is a representation generator of $\underline{\mathsf{Gproj}}\,\Lambda$ and the stable Cohen–Macaulay Auslander algebra $\mathsf{A}(\underline{\mathsf{Gproj}\,\Lambda})$ of Λ is self-injective and Morita equivalent to $\Delta := \underline{\mathsf{End}}_\Lambda(T)^{\mathsf{op}}$. By [32], the canonical ring epimorphism $\Gamma \longrightarrow \Delta$, is pseudo-flat in the sense that $\mathsf{Tor}_1^\Gamma(\Delta, \Delta) = 0$. For the notion of a quasi-periodic resolution we refer to [32].

Corollary 6.14. *Let Λ be Gorenstein of dimension 2 and consider the Δ-bimodule $L = \mathsf{Tor}_2^\Gamma(\Delta, \Delta)$.*

(i) *$L \cong \underline{\mathsf{Hom}}_\Lambda(T, \Omega T)$ is an invertible Δ-bimodule with inverse $L^{-1} = \mathsf{Ext}^2_\Gamma(\Delta, \Gamma) \cong \underline{\mathsf{Hom}}_\Lambda(\Omega T, T)$.*
(ii) *Δ admits a quasi-periodic projective resolution over the k-enveloping algebra $\Delta \otimes_k \Delta^{\mathsf{op}}$ of Δ of period 3 with periodicity factor $\underline{\mathsf{Hom}}_\Lambda(T, \Omega T)$.*
(iii) *If $\underline{\mathsf{Hom}}_\Lambda(T, \Omega T)$ is of finite order n in the Picard group $\mathsf{Pic}\,\Delta$, then the Hochschild (co)-homology of Δ is periodic of period dividing $3n$.*

Proof. Since Λ is Gorenstein of dimension 2, by Corollary 6.8 we have that Γ is of global dimension 2, so its Hochschild dimension is 2. Then the assertions follow by [32, Theorem 1.5]. □

7. Rigid Gorenstein-projective modules

Our aim in this section is to study rigid or cluster tilting finitely generated Gorenstein-projective modules T over an Artin algebra Λ in connection with Cohen–Macaulay finiteness of Λ and representation-theoretic properties of the stable endomorphism algebra of T. In particular using recent results of Keller–Reiten and Amiot we give in some cases convenient descriptions of the stable triangulated category of Gorenstein-projectives in terms of the cluster category $\mathscr{C}_Q$ associated to the quiver Q of the stable endomorphism algebra of T. Moreover we show that, in this setting, derived equivalent virtually Gorenstein algebras share the same cluster category.

Recall that if $\mathscr{C}$ is an abelian, resp. triangulated, category, then an object $X \in \mathscr{C}$ is called **rigid** if $\mathsf{Ext}^1_{\mathscr{C}}(X, X) = 0$, resp. $\mathscr{C}(X, X[1]) = 0$. If $\mathscr{C}$ is triangulated, then X is called **cluster tilting**, see [46,41], if $\mathsf{add}\, X$ is functorially finite and $\{C \in \mathscr{C} \mid \mathscr{C}(C, X[1]) = 0\} = \mathsf{add}\, X = \{C \in \mathscr{C} \mid \mathscr{C}(X, C[1]) = 0\}$.

Let Λ be an Artin algebra. If T is a finitely generated Gorenstein-projective module, then we denote by $(\mathsf{Gproj}\,\Lambda)^{\leqslant 1}_T$ the full subcategory of $\mathsf{Gproj}\,\Lambda$ consisting of all modules A admitting an exact sequence $0 \longrightarrow T_1 \longrightarrow T_0 \longrightarrow A \longrightarrow 0$, where the T_i lie in $\mathsf{add}\, T$, i.e. $\mathsf{res.dim}_{\mathsf{add}\, T}\, A \leqslant 1$.

Recall that the **representation dimension** $\mathsf{rep.dim}\,\Lambda$ of Λ in the sense of Auslander [7] is defined by

$$\mathsf{rep.dim}\,\Lambda = \inf\bigl\{\mathsf{gl.dim}\,\mathsf{End}_\Lambda\bigl(\Lambda \oplus \mathsf{D}(\Lambda) \oplus X\bigr) \bigm| X \in \mathsf{mod}\text{-}\Lambda\bigr\}$$

Note that $\mathsf{rep.dim}\,\Lambda \leqslant 2$ iff Λ is of finite representation type, see [7]. The following result gives a connection between Artin algebras of finite CM-type and representation-theoretic properties of stable endomorphism algebras of rigid Gorenstein-projective modules.

Theorem 7.1. *Let T be a finitely generated Gorenstein-projective module, and assume that $\Lambda \in \mathsf{add}\, T$ and the module $\Omega^n T$ is rigid, for some $n \geqslant 0$.*

(i) *There is an equivalence*

$$(\underline{\mathsf{Gproj}}\,\Lambda)^{\leqslant 1}_T / \mathsf{add}\, \underline{T} \xrightarrow{\approx} \mathsf{mod}\text{-}\underline{\mathsf{End}}_\Lambda(T)^{\mathsf{op}}$$

and the following statements are equivalent.
   (a) $(\mathsf{Gproj}\,\Lambda)^{\leqslant 1}_T$ *is of finite representation type.*
   (b) *The stable endomorphism algebra $\underline{\mathsf{End}}_\Lambda(T)^{\mathsf{op}}$ is of finite representation type.*
   In particular if Λ is of finite CM-type, then $\underline{\mathsf{End}}_\Lambda(T)^{\mathsf{op}}$ is of finite representation type.

(ii) *The inclusion $(\mathsf{Gproj}\,\Lambda)^{\leqslant 1}_T \subseteq \mathsf{Gproj}\,\Lambda$ is an equality if and only $\underline{T}$ is a cluster tilting object in $\underline{\mathsf{Gproj}}\,\Lambda$. If this is the case, then:*
   (a) *Λ is of finite CM-type if and only if $\underline{\mathsf{End}}_\Lambda(T)^{\mathsf{op}}$ is of finite representation type.*
   (b) *The following statements are equivalent:*
      1. *$\mathcal{U} = \{X \in \mathsf{Gproj}\,\Lambda \mid$ any map $\Omega T \longrightarrow X$ factorizes through a module in $\mathsf{add}\,\Omega^{-1}T\}$ is of finite representation type.*
      2. *$\mathcal{V} = \{X \in \mathsf{Gproj}\,\Lambda \mid$ any map $X \longrightarrow \Omega^{-3}T$ factorizes through a module in $\mathsf{add}\,\Omega^{-1}T\}$ is of finite representation type.*
      3. *The stable endomorphism algebra $\underline{\mathsf{End}}_\Lambda(T)^{\mathsf{op}}$ is of finite CM-type.*

      If 3 holds, then $\mathsf{rep.dim}\,\underline{\mathsf{End}}_\Lambda(T)^{\mathsf{op}} \leqslant 3$.

Proof. (i) Since T is Gorenstein-projective, we have isomorphisms:

$$\begin{aligned}\mathsf{Ext}^1_\Lambda\big(\Omega^n T, \Omega^n T\big) &\cong \underline{\mathsf{Hom}}_\Lambda\big(\Omega^{n+1} T, \Omega^n T\big) \cong \underline{\mathsf{Hom}}_\Lambda(\Omega T, T)\\ &\cong \underline{\mathsf{Hom}}_\Lambda\big(T, \Omega^{-1} T\big) \cong \mathsf{Ext}^1_\Lambda(T, T)\end{aligned}$$

Hence T is rigid if and only if $\Omega^n T$ is rigid, $\forall n \geqslant 0$, if and only if $\underline{T}$ is rigid in $\underline{\mathsf{Gproj}}\,\Lambda$. So we may assume that T is rigid. Consider the stable category $(\underline{\mathsf{Gproj}}\,\Lambda)^{\leqslant 1}_T$ of $(\mathsf{Gproj}\,\Lambda)^{\leqslant 1}_T$ modulo projectives, and the functor

$$\mathsf{H} : (\underline{\mathsf{Gproj}}\,\Lambda)^{\leqslant 1}_T \longrightarrow \text{mod-}\underline{\mathsf{End}}_\Lambda(T)^{\mathsf{op}}, \quad \mathsf{H}(\underline{A}) = \underline{\mathsf{Hom}}_\Lambda(T, A)$$

which induces an equivalence between $\mathsf{add}\,\underline{T}$ and $\mathsf{proj}\,\underline{\mathsf{End}}_\Lambda(T)^{\mathsf{op}}$. Since $\Lambda \in \mathsf{add}\,T$, it follows that $\mathsf{proj}\,\Lambda \subseteq \mathsf{add}\,T$. It is easy to see that $(\underline{\mathsf{Gproj}}\,\Lambda)^{\leqslant 1}_T$ coincides with the extension category $\mathsf{add}\,\underline{T} \star \mathsf{add}\,\Omega^{-1}\underline{T}$ which by definition consists of all objects $\underline{A}$ for which there exists a triangle $\underline{T}_0 \longrightarrow \underline{A} \longrightarrow \Omega^{-1}\underline{T}_1 \longrightarrow \Omega^{-1}\underline{T}_0$ in $\underline{\mathsf{Gproj}}\,\Lambda$, where the T_i lie in $\mathsf{add}\,T$. Since $\underline{T}$ is rigid, by a result of Keller and Reiten [46], the functor H induces an equivalence between the stable category $(\mathsf{add}\,\underline{T} \star \mathsf{add}\,\Omega^{-1}\underline{T})/\,\mathsf{add}\,\underline{T}$ and $\text{mod-}\,\underline{\mathsf{End}}_\Lambda(T)^{\mathsf{op}}$, hence we have an equivalence $(\underline{\mathsf{Gproj}}\,\Lambda)^{\leqslant 1}_T/\,\mathsf{add}\,\underline{T} \approx \text{mod-}\,\underline{\mathsf{End}}_\Lambda(T)^{\mathsf{op}}$. It follows directly from this that $(\mathsf{Gproj}\,\Lambda)^{\leqslant 1}_T$ is of finite representation type if and only if $\underline{\mathsf{End}}_\Lambda(T)^{\mathsf{op}}$ is of finite representation type.

(ii) If $\underline{T}$ is a cluster tilting object in $\underline{\mathsf{Gproj}}\,\Lambda$, then we have $\underline{\mathsf{Gproj}}\,\Lambda = \mathsf{add}\,\underline{T} \star \mathsf{add}\,\Omega^{-1}\underline{T}$, see [46], and therefore $\underline{\mathsf{Gproj}}\,\Lambda = (\underline{\mathsf{Gproj}}\,\Lambda)^{\leqslant 1}_T$. It follows that $\mathsf{Gproj}\,\Lambda = (\mathsf{Gproj}\,\Lambda)^{\leqslant 1}_T$. Conversely assume that $\mathsf{Gproj}\,\Lambda = (\mathsf{Gproj}\,\Lambda)^{\leqslant 1}_T$, or equivalently $\underline{\mathsf{Gproj}}\,\Lambda = (\underline{\mathsf{Gproj}}\,\Lambda)^{\leqslant 1}_T$. Let $\underline{A}$ be in $\underline{\mathsf{Gproj}}\,\Lambda$ such that $\underline{\mathsf{Hom}}(T, \Omega^{-1}A) = 0$. Since $\underline{\mathsf{Gproj}}\,\Lambda = \mathsf{add}\,\underline{T} \star \mathsf{add}\,\Omega^{-1}\underline{T}$, there exists a triangle $\underline{T}_0 \longrightarrow \Omega^{-1}\underline{A} \longrightarrow \Omega^{-1}\underline{T}_1 \longrightarrow \Omega^{-1}\underline{T}_0$ in $\underline{\mathsf{Gproj}}\,\Lambda$, where the T_i lie in $\mathsf{add}\,T$. Then by hypothesis the first map is zero and therefore $\Omega^{-1}\underline{A}$ is a direct summand of $\Omega^{-1}\underline{T}_1$, or equivalently $\underline{A}$ is a direct summand of $\underline{T}_1$, i.e. $\underline{A} \in \mathsf{add}\,\underline{T}$. Hence $\mathsf{add}\,\underline{T} = \{\underline{A} \in \underline{\mathsf{Gproj}}\,\Lambda \mid \underline{\mathsf{Hom}}(\underline{T}, \Omega^{-1}\underline{A}) = 0\}$. Since $\mathsf{add}\,\underline{T}$ is contravariantly finite, by [48] it follows that $\underline{T}$ is a cluster tilting object in $\underline{\mathsf{Gproj}}\,\Lambda$.

Assume now that $\underline{T}$ is a cluster tilting object in $\underline{\mathsf{Gproj}}\,\Lambda$. Then part (a) follows from (i). By a result of Keller and Reiten [46], the endomorphism algebra $\underline{\mathsf{End}}_\Lambda(T)^{\mathsf{op}}$ is Gorenstein of Gorenstein dimension $\leqslant 1$. Clearly in this case we have that $\mathsf{Gproj}\,\underline{\mathsf{End}}_\Lambda(T)^{\mathsf{op}}$ consists of the submodules of the finitely generated projective $\underline{\mathsf{End}}_\Lambda(T)^{\mathsf{op}}$-modules, and $\mathsf{Ginj}\,\underline{\mathsf{End}}_\Lambda(T)^{\mathsf{op}}$ consists of the factors of the finitely generated injective $\underline{\mathsf{End}}_\Lambda(T)^{\mathsf{op}}$-modules. Using these descriptions it is not difficult to see that the functor H induces an equivalence between $\mathcal{U}/\,\mathsf{add}\,\Omega^{-1}\underline{T}$ and $\mathsf{Gproj}\,\underline{\mathsf{End}}_\Lambda(T)^{\mathsf{op}}$ and between $\mathcal{V}/\,\mathsf{add}\,\Omega^{-1}\underline{T}$ and $\mathsf{Ginj}\,\underline{\mathsf{End}}_\Lambda(T)^{\mathsf{op}}$. Then part (b) is a direct consequence of these equivalences and the fact that the stable categories $\underline{\mathsf{Gproj}}\,\underline{\mathsf{End}}_\Lambda(T)^{\mathsf{op}}$ and $\overline{\mathsf{Ginj}}\,\underline{\mathsf{End}}_\Lambda(T)^{\mathsf{op}}$ are equivalent, see Remark 4.3. Finally by using a recent result of Ringel [60], it follows easily that the representation dimension of a Gorenstein algebra of finite CM-type and of Gorenstein dimension at most one, is at most 3. We infer that $\mathsf{rep.dim}\,\underline{\mathsf{End}}_\Lambda(T)^{\mathsf{op}} \leqslant 3$. $\square$

We have the following consequence.

Corollary 7.2. *Let Λ be an Artin algebra and assume that $\underline{\mathsf{Gproj}}\,\Lambda$ contains a cluster tilting object $\underline{T}$ such that the algebra $\underline{\mathsf{End}}_\Lambda(T)^{\mathsf{op}}$ is of finite CM-type. Then Λ is of infinite CM-type if and only if* $\mathsf{rep.dim}\,\underline{\mathsf{End}}_\Lambda(T)^{\mathsf{op}} = 3$.

Proof. Since $\mathsf{rep.dim}\,\underline{\mathsf{End}}_\Lambda(T)^{\mathsf{op}} \leqslant 2$ if and only if $\underline{\mathsf{End}}_\Lambda(T)^{\mathsf{op}}$ is of finite representation type if and only if Λ is of finite CM-type, the assertion follows from part (ii) of Theorem 7.1. □

We close this section by giving convenient descriptions of the triangulated category $\underline{\mathsf{Gproj}}\,\Lambda$, for a virtually Gorenstein k-algebra Λ of finite CM-type over an algebraically closed field k, under the presence of a cluster tilted object. These descriptions follow from recent results of Amiot [1] and Keller and Reiten [47]. First we recall that if Q is a quiver without oriented cycles, and $d \geqslant 2$ or $d = 1$ and Q is a Dynkin quiver, then the d-**cluster category** of Q over a field k, is the orbit category $\mathscr{C}_Q^{(d)} := \mathbf{D}^b(\mathsf{mod}\text{-}kQ)/(\mathbb{S}^{-1}[d])^{\mathbb{Z}}$ of the bounded derived category $\mathbf{D}^b(\mathsf{mod}\text{-}kQ)$ of finite-dimensional modules over the path algebra kQ under the action of the automorphism group generated by $X \longmapsto \mathbb{S}^{-1}(X[d])$, where $\mathbb{S} = - \bigotimes_{kQ}^{\mathbf{L}} \mathsf{D}(kQ)$. As shown by Keller [45], $\mathscr{C}_Q^{(d)}$ is a triangulated category and the projection $\mathbf{D}^b(\mathsf{mod}\text{-}kQ) \longrightarrow \mathscr{C}_Q^{(d)}$ is a triangle functor. If $d = 2$, we write $\mathscr{C}_Q^{(d)} = \mathscr{C}_Q$.

If $\mathcal{T}$ is a k-linear triangulated category over a field k with finite-dimensional Hom-spaces, then a **Serre functor** for $\mathcal{T}$, in the sense of Bondal and Kapranov [28], is a triangulated equivalence $\mathbb{S} : \mathcal{T} \longrightarrow \mathcal{T}$ such that:

$$\mathsf{D}\mathsf{Hom}_{\mathcal{T}}(A, B) \xrightarrow{\approx} \mathsf{Hom}_{\mathcal{T}}(B, \mathbb{S}A)$$

naturally $\forall A, B \in \mathcal{T}$. If $\mathcal{T}$ admits a Serre functor $\mathbb{S}$, then $\mathbb{S}$ is unique up to an isomorphism of triangulated functors, and then $\mathcal{T}$ is called (**weakly**) d-**Calabi–Yau** if there exists a natural isomorphism $\mathbb{S}(?) \xrightarrow{\approx} (?)[d]$ as (additive) triangulated functors. E.g. the d-cluster category $\mathscr{C}_Q^{(d)}$ is d-Calabi–Yau [45]. The following describes when for an Artin algebra Λ, the triangulated category $\underline{\mathsf{Gproj}}\,\Lambda$ admits a Serre functor.

Lemma 7.3.

(i) *The triangulated category $\underline{\mathsf{Gproj}}\,\Lambda$ admits a Serre functor if and only if for any $A \in \mathsf{Gproj}\,\Lambda$, the (minimal) right $\mathsf{GProj}\,\Lambda$-approximation $X_{\mathsf{DTr}(A)}$ of $\mathsf{DTr}A$ is finitely generated.*
(ii) *If Λ is virtually Gorenstein, then $\underline{\mathsf{Gproj}}\,\Lambda$ admits a Serre functor which is given by $\mathbb{S}(\underline{A}) = \Omega^{-1}X_{\mathsf{DTr}A}$.*
(iii) *If Λ is of finite CM-type and $\underline{\mathsf{Gproj}}\,\Lambda$ is d-Calabi–Yau, then the stable Cohen–Macaulay Auslander algebra $\mathsf{A}(\underline{\mathsf{Gproj}\,\Lambda})$ of $\mathsf{Gproj}\,\Lambda$ is self-injective and the stable category $\underline{\mathsf{mod}}\text{-}\mathsf{A}(\underline{\mathsf{Gproj}}\,\Lambda)$ is $(3d-1)$-Calabi–Yau.*

Proof. (i) If $\mathbb{S}$ is a Serre functor on $\underline{\mathsf{Gproj}}\,\Lambda$, then $\mathsf{D}\,\underline{\mathsf{Hom}}_\Lambda(A, B) \cong \underline{\mathsf{Hom}}_\Lambda(B, \mathbb{S}A)$ naturally $\forall A, B \in \mathsf{Gproj}\,\Lambda$. Since $\underline{\mathsf{GProj}}\,\Lambda$ is compactly generated, by Brown representability, we have natural isomorphisms $\mathsf{D}\,\underline{\mathsf{Hom}}_\Lambda(A, B) \cong \underline{\mathsf{Hom}}_\Lambda(B, \Omega^{-1}X_{\mathsf{DTr}A})$, see [24, Proposition 8.8]. Hence $\forall A \in \mathsf{Gproj}\,\Lambda$ we have a natural isomorphism $\mathbb{S}A \cong \Omega^{-1}X_{\mathsf{DTr}A}$. It follows that $\Omega^{-1}X_{\mathsf{DTr}A}$, hence $X_{\mathsf{DTr}A}$, is finitely generated, $\forall A \in \mathsf{Gproj}\,\Lambda$. Conversely if this holds, then clearly $A \longmapsto \Omega^{-1}X_{\mathsf{DTr}A}$ is a Serre functor on $\underline{\mathsf{Gproj}}\,\Lambda$.

(ii), (iii) Since Λ is virtually Gorenstein iff X_M is finitely generated for any $M \in \mathsf{mod}\text{-}\Lambda$, see [24, Corollary 8.3], (ii) follows from (i). Part (iii) follows from our previous results and a result of Keller, see [45, Lemma 8.5.2], which asserts that if $\mathcal{T}$ is a d-Calabi–Yau triangulated category, then the stable triangulated category $\underline{\mathsf{mod}}\text{-}\mathcal{T}$ of the Frobenius category $\mathsf{mod}\text{-}\mathcal{T}$ of coherent functors over $\mathcal{T}$ is $(3d-1)$-Calabi–Yau. □

We have the following consequence.

Corollary 7.4. *Let Λ be a virtually Gorenstein finite-dimensional k-algebra over a field k and let T be a cluster tilting object in the triangulated category $\underline{\mathsf{Gproj}}\,\Lambda$. Assume that $\underline{\mathsf{Gproj}}\,\Lambda$ is weakly 2-Calabi–Yau: $X_{\mathsf{DTr}A} \cong \Omega^{-1}A$, naturally in $\underline{\mathsf{Gproj}}\,\Lambda$, for any finitely generated Gorenstein-projective module A.*

(i) *If the quiver Q of $\underline{\mathsf{End}}_\Lambda(T)^{\mathsf{op}}$ has no oriented cycles, then there is a triangle equivalence*

$$\underline{\mathsf{Gproj}}\,\Lambda \xrightarrow{\approx} \mathscr{C}_Q$$

(ii) *If Λ is of finite CM-type and $\underline{\mathsf{Gproj}}\,\Lambda$ is standard, then there is a triangle equivalence*

$$\underline{\mathsf{Gproj}}\,\Lambda \xrightarrow{\approx} \mathscr{C}_Q/G$$

where $\mathscr{C}_Q/G$ is the orbit category of $\mathscr{C}_Q$ under the action of some cyclic group G of automorphisms.

(iii) *If $\underline{\mathsf{Gproj}}\,\Lambda$ is 2-Calabi–Yau and the algebra $\underline{\mathsf{End}}_\Lambda(T)^{\mathsf{op}}$ is of finite CM-type, then the stable Cohen–Macaulay Auslander algebra $\mathsf{A}(\underline{\mathsf{Gproj}}\,\underline{\mathsf{End}}_\Lambda(T)^{\mathsf{op}})$ of $\mathsf{Gproj}\,\underline{\mathsf{End}}_\Lambda(T)^{\mathsf{op}}$ is self-injective and the stable triangulated category $\underline{\mathsf{mod}}\text{-}\mathsf{A}(\underline{\mathsf{Gproj}}\,\underline{\mathsf{End}}_\Lambda(T)^{\mathsf{op}})$ is 8-Calabi–Yau.*

Proof. (i) Since $\Gamma := \underline{\mathsf{End}}_\Lambda(T)^{\mathsf{op}}$ is Gorenstein with $\mathsf{id}_\Gamma\,\Gamma \leqslant 1$, the assumption implies that Γ is hereditary. Then the assertion follows from [47]. Part (ii) follows easily from Amiot's results [1]. Finally part (iii) follows from Lemma 7.3 and the fact that $\underline{\mathsf{Gproj}}\,\Lambda$ being 2-Calabi–Yau implies that $\underline{\mathsf{Gproj}}\,\underline{\mathsf{End}}_\Lambda(T)^{\mathsf{op}}$ is 3-Calabi–Yau, see [46]. □

The following consequence shows that derived equivalent algebras satisfying the conditions of the above corollary share the same (orbit category of the) associated cluster category.

Corollary 7.5. *Let Λ and Γ be derived equivalent algebras. Assume that Λ is virtually Gorenstein and the stable triangulated category $\underline{\mathsf{Gproj}}\,\Lambda$ is (weakly) 2-Calabi–Yau and admits a cluster tilting object T.*

(i) *$\underline{\mathsf{Gproj}}\,\Gamma$ is (weakly) 2-Calabi–Yau and admits a cluster tilting object S.*
(ii) *If the quiver Q_Λ of $\underline{\mathsf{End}}_\Lambda(T)^{\mathsf{op}}$ has no oriented cycles, then so does the quiver Q_Γ of $\underline{\mathsf{End}}_\Gamma(S)^{\mathsf{op}}$ and the associated cluster categories are triangle equivalent: $\mathscr{C}_{Q_\Lambda} \approx \mathscr{C}_{Q_\Gamma}$.*
(iii) *If Λ is of finite CM-type and $\underline{\mathsf{Gproj}}\,\Lambda$ is standard, then Γ is of finite CM-type and the associated orbit categories of the cluster categories of Q_Λ and Q_Γ are triangle equivalent: $\mathscr{C}_{Q_\Lambda}/G \approx \mathscr{C}_{Q_\Gamma}/G'$.*

Proof. It is shown in [24] that if Λ and Γ are derived equivalent Artin algebras, then Λ is virtually Gorenstein if and only if so is Γ. Moreover in this case there is induced a triangle equivalence between $\underline{\mathsf{Gproj}}\,\Lambda$ and $\underline{\mathsf{Gproj}}\,\Gamma$. Then the assertions follow directly from Corollary 7.4. □

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