Relative homology, higher cluster-tilting theory and categorified Auslander–Iyama correspondence

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ABSTRACT

We study homological properties of contravariantly finite rigid subcategories of an arbitrary triangulated or abelian category, concentrating at the class of higher cluster tilting subcategories. In both the triangulated and abelian context we present several new characterizations of such subcategories and we give conditions ensuring that the associated cluster tilted category is Gorenstein and/or stably Calabi–Yau. Thus we generalize and improve on several recent results of the literature. In the context of abelian categories, we prove a categorified version of higher Auslander correspondence, as developed by Iyama in the case of module categories. In this connection we investigate in detail homological properties of (stable) Auslander categories associated to higher cluster tilting subcategories in an abelian category, and we present applications to (possibly infinitely generated) cluster tilting modules over an arbitrary ring. Finally we give bounds on the global and representation dimension of certain categories of coherent functors associated to Cohen–Macaulay objects.

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Introduction

Cluster tilting theory, depending on the context, emerged from two rather different sources. On one hand, in the setting of finitely generated modules over an Artin algebra, Iyama [40,41] introduced cluster tilting modules in order to build a higher Auslander correspondence, generalizing the classical correspondence between algebras of finite representation type and Auslander algebras as developed by Auslander in the early seventies [3]. On the other hand and approximately at the same time, Buan, Marsh, Reineke, Reiten and Todorov, see [21–23], [24], [55,56], inspired by classical tilting theory, introduced cluster tilting objects in the context of cluster categories associated to a hereditary algebra, in order to categorify certain phenomena occurring in the recently developed theory of cluster algebras by Fomin and Zelevinsky [32,33]. There is a nice interplay between the two approaches, which converge in some important special cases, by providing links between representation theory of hereditary Artin algebras and the combinatorics of cluster algebras, and by helping to understand several hidden features in classical tilting theory, see [59]. Originally cluster tilting objects in the triangulated setting, were defined and investigated for the cluster category associated to a hereditary Artin algebra, and in the abelian setting for the category of finitely generated modules over an Artin algebra and slight generalizations of those. Since then cluster tilting theory was rapidly developed in both settings and at present it plays an increasingly important role in categorifying, and providing new perspectives for, several phenomena of geometric, combinatorial, or homological nature in various different areas of research. It should be noted that in the course of time many new ideas and techniques have emerged and new connections to other areas have been established.

The principal aims of the present paper are the following:
(A) Our first aim is to develop a categorified version of Iyama’s higher Auslander correspondence in the context of arbitrary abelian categories, to study in detail the structure of the associated (stable) Auslander categories, and to give applications to cluster tilting theory in categories of, possibly infinitely generated, modules over associative rings, not necessarily Artin algebras.

(T) Our second aim is to develop a relative homological theory in a triangulated category based on rigid subcategories, concentrating at the class of higher cluster tilting subcategories, and in this context to extend several results of homological nature about them which are known to be true in lower dimensions or under strong finiteness conditions. In particular we are interested in the Gorenstein and the Calabi–Yau property of the cluster tilted category associated to a higher-dimensional cluster tilting subcategory.

In order to explain further our motivation, we present below, as Theorems A and B, two of the main results of the paper in the abelian (A) and triangulated (T) setting, which generalize and extend related results of Keller and Reiten [46,47], Iyama [40], and others, referring to the text for definitions of all unexplained terms:

(A) Auslander in the early seventies proved the following remarkable result which played a fundamental role for much of the later developments in the representation theory of Artin algebras. Firstly, we recall that the dominant dimension \( \text{dom} \dim \mathcal{F} \) of an abelian category \( \mathcal{F} \) is the largest \( n \in \mathbb{N} \cup \{ \infty \} \) such that any projective object of \( \mathcal{F} \) has an injective resolution whose first \( n \) terms are projective; the dominant dimension \( \text{dom} \dim \Gamma \) of an Artin algebra \( \Gamma \) is the dominant dimension of the abelian category \( \text{mod} \Gamma \) of finitely generated \( \Gamma \)-modules.

**Theorem 1 (Auslander).** (See [3].) There is a bijective correspondence between Morita equivalence classes of Artin algebras \( \Lambda \) of finite representation type and Artin algebras \( \Gamma \) such that \( \text{gl.dim} \Gamma \leq 2 \leq \text{dom} \dim \Gamma \).

A categorified version of Auslander’s correspondence was proved in [13]. First recall that the free abelian category of an additive category \( \mathcal{C} \) is an abelian category \( \mathfrak{F}(\mathcal{C}) \) together with an additive functor \( \mathcal{C} \rightarrow \mathfrak{F}(\mathcal{C}) \) which is universal for additive functors out of \( \mathcal{C} \) to abelian categories in the sense that any additive functor \( \mathcal{C} \rightarrow \mathcal{A} \) to an abelian category \( \mathcal{A} \) factorizes uniquely through \( \mathfrak{F}(\mathcal{C}) \) via an exact functor. The free abelian category \( \mathfrak{F}(\mathcal{C}) \) of any additive category \( \mathcal{C} \) always exists, it is unique up to equivalence, and it admits the following descriptions: \( \text{mod}-(\mathcal{C}-\text{mod})^{\text{op}} \approx \mathfrak{F}(\mathcal{C}) \approx ((\text{mod}-(\mathcal{C}-\text{mod}))^{\text{op}}, \) where \( \text{mod}-\mathcal{C} \) denotes the category of (contravariant) coherent functors \( \mathcal{C}^{\text{op}} \rightarrow \mathfrak{A} \) from \( \mathcal{C} \) to the category \( \mathfrak{A} \) of abelian groups, and \( \mathcal{C}-\text{mod} = \text{mod}-\mathcal{C}^{\text{op}} \). On the other hand we call an abelian category \( \mathcal{F} \) with enough projectives and injectives, an Auslander category if \( \text{gl.dim} \mathcal{F} \leq 2 \leq \text{dom} \dim \mathcal{F} \). A main example is the category of finitely generated modules over the Auslander algebra \( \Gamma \) of a representation finite Artin algebra \( \Lambda \).
Theorem 2. (See [13, Theorems 6.1 and 6.6].) An abelian category \( \mathcal{F} \) is free if and only if \( \mathcal{F} \) is an Auslander category. Moreover the maps

\[
\mathcal{C} \mapsto \mathfrak{S}(\mathcal{C}) \quad \text{and} \quad \mathcal{F} \mapsto \text{Proj}\, \mathcal{F} \cap \text{Inj}\, \mathcal{F}
\]

give, up to equivalence, mutually inverse bijections between:

1. Additive categories \( \mathcal{C} \) (with split idempotents).
2. Auslander categories \( \mathcal{F} \).

Recently Iyama proved, as one of the main results of [40], the following far reaching generalization of Auslander’s correspondence which in addition gives interesting connections with unstable cluster tilting theory. Recall that a finitely generated module \( M \) over a finite-dimensional algebra \( \Lambda \) is called an \( n \)-cluster tilting module if \( \{ X \in \text{mod-}\Lambda \mid \text{Ext}^k_{\Lambda}(M, X) = 0, 1 \leq k \leq n \} = \text{add} \, M = \{ X \in \text{mod-}\Lambda \mid \text{Ext}^k_{\Lambda}(X, M) = 0, 1 \leq k \leq n \} \), see [40,41], where the terminology \((n + 1)\)-cluster tilting module or maximal \( n \)-orthogonal module is used.

Theorem 3 (Iyama). (See [40].) For any \( n \geq 0 \), the map \( M \mapsto \text{End}_{\Lambda}(M) \) gives a bijection between equivalence classes of \( n \)-cluster tilting \( \Lambda \)-modules \( M \) for finite-dimensional algebras \( \Lambda \) and Morita-equivalence classes of finite-dimensional algebras \( \Gamma \) such that \( \text{gl.dim} \, \Gamma \leq n + 2 \leq \text{dom.dim} \, \Gamma \).

Our first aim in this paper is to refine and generalize Theorems 1, 2 and 3 in the context of abelian categories. In fact we shall prove the following categorification of Theorem 3 which shows that Theorem 2 is the zero case of a natural correspondence at a higher level. First we need some definitions. A full subcategory \( \mathcal{M} \) of an abelian category \( \mathcal{A} \) is called an \( n \)-cluster tilting subcategory, if (a) \( \mathcal{M} \) is contravariantly finite in \( \mathcal{A} \) and any right \( M \)-approximation is an epimorphism, and (b) \( \mathcal{M} \) coincides with the full subcategory \( \mathcal{M}^+_n := \{ A \in \mathcal{A} \mid \text{Ext}^k_{A}(M, A) = 0, 1 \leq k \leq n \} \). An abelian category \( \mathcal{B} \) is called an \( n \)-Auslander category if \( \mathcal{B} \) has enough projectives and satisfies \( \text{gl.dim} \, \mathcal{B} \leq n + 2 \leq \text{dom.dim} \, \mathcal{B} \). A ring \( \Lambda \) is called a (right) \( n \)-Auslander ring if \( \text{Mod-}\Lambda \) is an \( n \)-Auslander category. Finally an object \( X \) in an additive category \( \mathcal{A} \) with infinite coproducts is called self-compact if the functor \( \mathcal{A}(X, -) \colon \text{Add} \, X \to \mathfrak{Ab} \) preserves coproducts, where \( \text{Add} \, X \) denotes the full subcategory of \( \mathcal{A} \) consisting of all direct factors of arbitrary coproducts of copies of \( X \).

Theorem 3 provides motivation for the following result.
Theorem A (Generalized Auslander correspondence [Theorems 8.23 and 9.9]). For any integer \( n \geq 0 \), the map

\[
\mathcal{M} \mapsto \text{mod-}\mathcal{M}
\]

gives, up to equivalence, a bijective correspondence between:

(I) \( n \)-cluster tilting subcategories \( \mathcal{M} \) in abelian categories \( \mathcal{A} \) with enough injectives.

(II) \( n \)-Auslander categories \( \mathcal{B} \).

Moreover the map

\[
\{ \mathcal{A}, M \} \mapsto \Lambda = \text{End}_{\mathcal{A}}(M)
\]

gives a bijective correspondence between (cluster/Morita) equivalence classes of

(III) Pairs \( \{ \mathcal{A}, M \} \) where \( \mathcal{A} \) is a Grothendieck category and \( M \in \mathcal{A} \) is a self-compact \( n \)-cluster tilting object.

(IV) Left coherent and right perfect \( n \)-Auslander rings \( \Lambda \).

Note that “Grothendieck” can be replaced by “cocomplete abelian” in Theorem A; the latter when specialized to \( \mathcal{M} = \text{add } M \) for an \( n \)-cluster tilting \( \Lambda \)-module \( M \) over an Artin algebra \( \Lambda \), gives Iyama’s Theorem 3.

Note that the abelian category \( \mathcal{A} \) in the above correspondence is uniquely determined, up to equivalence, by the \( n \)-cluster tilting subcategory \( \mathcal{M} \) it contains: in fact \( \mathcal{A} \) is equivalent to the dual of the category of coherent covariant functors over the full subcategory of projective–injective objects of \( \text{mod-}\mathcal{M} \). We also define \( n \)-cluster cotilting subcategories and \( n \)-coAuslander categories and we prove a dual version of the above result. In fact we show that in most cases a subcategory is \( n \)-cluster tilting iff it is \( n \)-cluster cotilting. Concerning functoriality of the construction, we define the (large) category \( \text{ClustTilt}(n) \) consisting of pairs \( (\mathcal{A}, M) \) consisting of an abelian category \( \mathcal{A} \) with enough injectives and an \( n \)-cluster tilting subcategory \( \mathcal{M} \) of \( \mathcal{A} \), and the (large) category \( \text{AuslCat}(n) \) with objects \( n \)-Auslander categories, and we prove that there is an equivalence of categories

\[
\text{ClustTilt}(n) \xrightarrow{\cong} \text{AuslCat}(n), \quad (\mathcal{A}, M) \mapsto \text{mod-}\mathcal{M}
\]

(\( T \)) Let \( \mathcal{T} \) be a triangulated category. Recall that a full subcategory \( \mathcal{X} \) of \( \mathcal{T} \) is called \((n + 1)\)-cluster tilting subcategory if: (\( \alpha \)) \( \mathcal{X} \) is contravariantly finite in \( \mathcal{T} \), and (\( \beta \)) \( \mathcal{X} \) coincides with the full subcategory \( \mathcal{X}^+_n := \{ A \in \mathcal{T} \mid \mathcal{T}(\mathcal{X}, A[k]) = 0, \ 1 \leq k \leq n \} \). The associated \( n \)-cluster tilted category is defined to be the (abelian) category \( \text{mod-}\mathcal{X} \) of coherent functors over \( \mathcal{X} \). In this context we give several characterization of when a subcategory of \( \mathcal{T} \) is \( n \)-cluster tilting. It should be noted that this definition of cluster
tilting subcategory is more general than the definition used in the current literature, e.g. see [44], but we prove that both definitions are equivalent.

Usually one is interested in homological properties of the associated cluster tilted category \( \text{mod-}X \) which reflect combinatorial or geometric properties of \( X \) and \( T \). In this connection, Keller and Reiten [46] proved the following remarkable result, see also [16] for a more general version.

**Theorem 4** (Keller–Reiten). (See [46].) Let \( T \) be a 2-Calabi–Yau triangulated category and \( X \) a 2-cluster tilting subcategory of \( T \). Then the cluster tilted category \( \text{mod-}X \) is 1-Gorenstein and stably 3-Calabi–Yau.

Recall that an abelian category \( \mathcal{A} \) with enough projectives and injectives, is called \( n \)-Gorenstein if the projective, resp. injective, dimension of any injective, resp. projective, object of \( \mathcal{A} \) is bounded by \( n \). Such a category is called stably \( k \)-Calabi–Yau if the stable triangulated category \( \text{CM}(\mathcal{A}) \) of Cohen–Macaulay objects of \( \mathcal{A} \) is \( k \)-Calabi–Yau. It is known that the conclusions of Theorem 4 are false for \( n \)-cluster tilting subcategories \( X \), if \( n > 2 \). This provides motivation for the next result which generalizes, and gives a higher dimensional analog of, Theorem 4. First we say that a full subcategory \( X \) of \( T \) is \( k \)-corigid, \( k \geq 1 \), if \( T(X, X[-t]) = 0 \), \( 1 \leq t \leq k \).

**Theorem B** (Theorems 6.4 and 7.6). Let \( T \) be a triangulated category and \( X \) an \( (n + 1) \)-cluster tilting subcategory of \( T \) which is \((n - k)\)-corigid, for some non-negative integer \( k \) with: \( 0 \leq k \leq \frac{n+1}{2} \). Then:

1. The cluster tilted category \( \text{mod-}X \) is \( k \)-Gorenstein.
2. If \( T \) is \((n + 1)\)-Calabi–Yau and any object of \( \text{mod-}X \) of the form \( T(\cdot, A)|_X \), where \( A \) lies in the extension category \( X[-n + 1] \star \cdots \star X[-n + k] \), has finite projective dimension, then the cluster tilted category \( \text{mod-}X \) is stably \((n + 2)\)-Calabi–Yau, i.e. the triangulated category \( \text{CM}(\text{mod-}X) \) is \((n + 2)\)-Calabi–Yau.

In particular for any \((n - 1)\)-corigid \((n + 1)\)-cluster tilting subcategory \( X \) of \( T \), i.e. the case \( k = 1 \) in Theorem B, the cluster tilted category \( \text{mod-}X \) is 1-Gorenstein and \( \text{mod-}X \) is stably \((n + 2)\)-Calabi–Yau. This special case was proved independently by Iyama and Oppermann [43].

Now we briefly describe the organization and the contents of the paper, which is divided into three parts.

1. In the first part, which consists of the first four sections, we develop a homological theory in a general triangulated category \( T \) based on a contravariantly or covariantly finite subcategory \( X \) of \( T \), concentrating at \( n \)-rigid subcategories, in the sense that \( T(X, X[k]) = 0 \), for \( 1 \leq k \leq n \). We analyze the structure of \( T \) in connection with homological properties of the associated abelian category \( \text{mod-}X \) of coherent functors over \( X \).
The main tools for this study is the ideal \( \mathcal{G}_{\mathcal{X}}(\mathcal{T}) \) of \( \mathcal{X} \)-ghost maps in \( \mathcal{T} \) with respect to \( \mathcal{X} \), i.e. maps that are invisible by the homological functor \( H: \mathcal{T} \to \text{mod-}\mathcal{X} \), \( H(A) = \mathcal{T}(-, A)|_{\mathcal{X}} \), and the category \( \mathcal{X} \star \mathcal{X}[1] \star \cdots \star \mathcal{X}[n] \) of extensions, in the sense of [10], of (positively) shifted copies of \( \mathcal{X} \). In this connection we characterize when the ideal of \( \mathcal{X} \)-ghost maps is nilpotent in an appropriate sense or when the category of shifted extensions of \( \mathcal{X} \) covers \( \mathcal{T} \), in terms of finiteness of the global dimension \( \text{gl.dim}_{\mathcal{X}} \mathcal{T} \) of \( \mathcal{T} \) with respect to \( \mathcal{X} \), see Theorem 3.4. Also we construct suitable resolutions of objects of \( \mathcal{T} \) by objects of (shifted copies of) \( \mathcal{X} \), see Theorem 4.2, and we give necessary conditions in order to realize the category of objects of finite projective or injective dimension of \( \text{mod-}\mathcal{X} \) as a full subcategory of \( \mathcal{T} \), see Theorems 4.6 and 4.7.

The results of this part are used in the second part for deriving the main properties of a cluster tilting subcategory and its associated cluster tilted category. It should be noted that a similar relative homological theory was developed in [12], see also [29], in case \( \mathcal{X} \) is stable in the sense that \( \mathcal{X} = \mathcal{X}[1] \). However the non-stable case behaves differently and, besides some formal analogies, the obtained results are of a different nature.

2. The second part of the paper, consisting of Sections 5, 6 and 7 is devoted to the analysis of the structure of a cluster tilting subcategory \( \mathcal{X} \) in a triangulated category \( \mathcal{T} \), and also to deriving the main homological properties of the cluster tilted category \( \text{mod-}\mathcal{X} \) associated to \( \mathcal{X} \).

First in Section 5 we present a host of characterizations of when a subcategory \( \mathcal{X} \) of \( \mathcal{T} \) is \((n+1)\)-cluster tilting, see Theorem 5.3. These characterizations are new in this generality. In particular we show that half of the conditions in the definition of cluster tilting used in the current literature can be removed. In Section 6 we show in Theorem 6.4 half of Theorem B concerning the Gorenstein property of the cluster tilted category, we prove its analogue for maximal rigid subcategories, and we analyze some direct consequences. More precisely we give applications to the Morita Theorem for cluster categories due to Keller and Reiten [47], to representation dimension of the cluster tilted category, and also we present conditions ensuring that the latter has finite global dimension.

In Section 7, we prove in Theorem 7.6 the second half of Theorem B concerning the stable Calabi–Yau property of the cluster tilted category and we present some of its immediate consequences.

3. The third part of the paper, which consists of Sections 8, 9, 10, and 11, is devoted to the categorification of the Auslander–Iyama correspondence in the context of abelian categories and to the analysis of the structure of (stable) Auslander categories \( \mathcal{B} \) associated to a cluster tiling subcategory \( \mathcal{M} \) of an abelian category \( \mathcal{A} \).

More precisely in Section 8 we prove the first part of Theorem A and we discuss its variations: for instance we prove in Theorem 8.32 that suitably defined big categories of \( n \)-cluster tilting subcategories in abelian categories with injectives and \( n \)-Auslander categories are equivalent, so the categorified Auslander–Iyama correspondence is functorial. We also discuss universality of the correspondence and we give some applications to K-theory of Auslander categories.
In Section 9, we first show in Theorem 9.6 that cluster tilting subcategories coincide with cluster cotilting ones, so half of the conditions of the definition of cluster tilting subcategories in abelian categories can be removed. Then we characterize in Theorem 9.9, left coherent and right perfect \( n \)-Auslander rings as the endomorphism rings of self-compact \( n \)-cluster tilting objects in Grothendieck categories, or more generally in cocomplete abelian categories with enough injectives, thus completing the proof of Theorem A.

Section 10 is devoted to analyzing the structure of the Auslander category \( \text{mod-} \mathcal{M} \) and its stable analogue \( \text{mod-} \mathcal{M}_\mathcal{A} \), associated to an \( n \)-cluster tilting subcategory \( \mathcal{M} \) in an abelian category \( \mathcal{A} \). After observing that the \( n \)-Auslander category \( \text{mod-} \mathcal{M} \) appears in the middle of a natural recollement situation with end terms the stable \( n \)-Auslander category \( \text{mod-} \mathcal{M}_\mathcal{A} \) and the abelian category \( \mathcal{A} \), we use duality between \( \text{mod-} \mathcal{M} \) and \( \mathcal{M}_\mathcal{A} \)-\text{mod} to describe the projective/injective objects of \( \text{mod-} \mathcal{M} \) and we present the latter as a suitable subquotient category of \( \mathcal{A} \), see Theorem 10.7. Next in Theorems 10.13, 10.17 and 10.14, we give necessary conditions for a stable Auslander category \( \text{mod-} \mathcal{M} \) to be of finite global dimension or Gorenstein, and when \( \text{mod-} \mathcal{M} \) is itself an Auslander category. Finally in Theorem 10.22 we give applications to \( n \)-Auslander rings.

In the final Section 11 of the paper we investigate homological properties of subcategories \( \mathcal{M} \subseteq \mathcal{A} \) arising from \( n \)-cluster tilting subcategories \( \mathcal{M} \subseteq \text{CM}(\mathcal{A}) \) of the triangulated category \( \text{CM}(\mathcal{A}) \) of Cohen–Macaulay objects in a Gorenstein abelian category \( \mathcal{A} \). In the main result in this section, see Theorem 11.3, we describe homological properties of \( \text{mod-} \mathcal{M} \) and we compute its global dimension as the maximum of \( n + 2 \) and the Gorenstein dimension of \( \mathcal{A} \). In addition we characterize when \( \text{mod-} \mathcal{M} \) is an Auslander category and we apply our results to the category of finitely generated modules over an Artin algebra, see Theorem 11.10.

**Conventions.** A general convention used in the paper is that the composition of morphisms in a given category, but not the composition of functors, is meant in the diagrammatic order. Our additive categories contain finite direct sums and their subcategories are assumed to be closed under isomorphisms and direct summands.

**Part 1. Relative homology of rigid subcategories**

In [12] a relative homological theory in a triangulated category \( \mathcal{T} \) was developed based on a contra(co)variantly finite subcategory \( \mathcal{X} \) which is *stable*, that is, closed under the suspension functor: \( \mathcal{X} = \mathcal{X}[1] \). In the first part of the paper we develop an analogous, but in many aspects radically different, homological theory on \( \mathcal{T} \) based on a contra(co)variantly finite subcategory \( \mathcal{X} \) which is not stable but usually is *rigid*, that is, \( \mathcal{T}(\mathcal{X}, \mathcal{X}[1]) = 0 \). In this context the, appropriately defined, ideal of ghost maps and categories of extensions with respect to \( \mathcal{X} \) play an important role in connection with the behavior of certain (co)resolutions and (co)homological dimensions.
1. Relative homology: Adams resolutions and cellular towers

Let $\mathcal{C}$ be an additive category and $X$ a full additive subcategory of $\mathcal{C}$. Recall that $X$ is called contravariantly finite in $\mathcal{C}$ if for any object $A$ in $\mathcal{C}$ there exists a map $f_A: X_A \to A$, where $X_A$ lies in $X$, such that the induced map $\mathcal{C}(X, f_A): \mathcal{C}(X, X_A) \to \mathcal{C}(X, A)$ is surjective. In this case the map $f_A$ is called a right $X$-approximation of $A$. Dually we have the notions of covariant finiteness and left approximations. Then $X$ is called functorially finite if it is both contravariantly and covariantly finite.

For a class of objects $\mathcal{V} \subseteq \mathcal{C}$ we denote by $\text{add}\, \mathcal{V}$ the full subcategory of $\mathcal{C}$ consisting of the direct factors of finite direct sums of copies of objects from $\mathcal{V}$. If $\mathcal{C}$ contains all small coproducts, resp. products, then $\text{Add}\, \mathcal{V}$, resp. $\text{Prod}\, \mathcal{V}$, denotes the full subcategory of $\mathcal{C}$ consisting of the direct factors of coproducts, resp. products, of copies of objects from $\mathcal{V}$. If $\mathcal{C}$ is an abelian category, then we denote by $\text{Proj}\, \mathcal{C}$, resp. $\text{Inj}\, \mathcal{C}$, the full subcategory of projective, resp. injective, objects of $\mathcal{C}$. We denote by $\text{Ab}$ the category of abelian groups.

The stable category of $\mathcal{C}$ with respect to a full subcategory $X$ is denoted by $\mathcal{C}/X$. Recall that the objects of $\mathcal{C}/X$ are the objects of $\mathcal{C}$, and the morphism space $\text{Hom}_{\mathcal{C}/X}(A, B)$ is the quotient $\mathcal{C}(A, B)/\mathcal{C}_X(A, B)$, where $\mathcal{C}_X(A, B)$ is the subgroup of $\mathcal{C}(A, B)$ consisting of all maps $A \to B$ factoring through an object from $X$. We then have a full dense additive functor $\pi: \mathcal{C} \to \mathcal{C}/X$, $\pi(A) = A$ and $\pi(f) = f$.

An additive functor $F: \mathcal{C}^{op} \to \text{Ab}$ is called coherent, see [2], if there exists an exact sequence $\mathcal{C}(-, A^1) \to \mathcal{C}(-, A^0) \to F \to 0$, where the $A^i$ lie in $\mathcal{C}$. We denote by $\text{mod}\, \mathcal{C}$ the category of coherent functors over $\mathcal{C}$. Recall that a weak kernel of a morphism $B \to C$ in $\mathcal{C}$ is a map $A \to B$ such that the sequence of functors $\mathcal{C}(-, A) \to \mathcal{C}(-, B) \to \mathcal{C}(-, C)$ is exact. The additive category $\mathcal{C}$ is called right coherent if $\mathcal{C}$ has weak kernels, i.e. any morphism in $\mathcal{C}$ has a weak kernel. It is easy to see that a contravariantly finite subcategory of a right coherent category is right coherent. Note that the additive category $\mathcal{C}$ is right coherent if and only if the category $\text{mod}\, \mathcal{C}$ is abelian; in this case $\text{mod}\, \mathcal{C}$ has enough projectives and if idempotents split in $\mathcal{C}$, then the Yoneda full embedding $\mathcal{C} \to \text{mod}\, \mathcal{C}$, $A \mapsto \mathcal{C}(-, A)$, induces an equivalence $\mathcal{C} \approx \text{Proj\, mod}\, \mathcal{C}$. Weak cokernels and left coherent categories are defined dually and they have dual properties. Finally if $\mathcal{C}$ is skeletally small, we denote by $\text{Mod}\, \mathcal{C}$ the category of contravariant functors from $\mathcal{C}$ to the category $\text{Ab}$ of abelian groups.

From now on and for the rest of the paper we denote by $\mathcal{T}$ a triangulated category with split idempotents; its suspension functor is denoted by $A \mapsto A[1]$. We also fix a full additive contravariantly finite subcategory $X$ of $\mathcal{T}$ which is closed under isomorphisms and direct summands.

For any object $A$ in $\mathcal{T}$ we construct inductively triangles, $\forall t \geq 0$:

$$\Omega_{X}^{t+1}(A) \xrightarrow{g_A^t} X_A^t \xrightarrow{f_A^t} \Omega_X^t(A) \xrightarrow{h_A^t} \Omega_X^{t+1}(A)[1]$$

(T\textsuperscript{\text{I}}_{\textsuperscript{\text{A}}})$

where $\Omega_X^0(A) := A$, and the middle map $X_A^t \to \Omega_X^t(A)$ is a right $X$-approximation of $\Omega_X^t(A)$. Applying the homological functor $H: \mathcal{T} \to \text{Mod}\, \mathcal{X}$, defined by $H(A) =$
\[ \mathcal{T}(-, A)|_{\mathcal{X}}, \] to the above triangles we have exact sequences \( H(\Omega_{\mathcal{X}}^{t+1} A) \to H(X^t_A) \to H(\Omega_{\mathcal{X}}^t A) \to 0, \forall t \geq 0. \) In particular we have an exact sequence

\[
H(X^1_A) \to H(X^0_A) \to H(A) \to 0
\]

which is a projective presentation of \( H(A). \) Since \( \mathcal{T}, \) as a triangulated category, is right coherent and \( \mathcal{X} \) is contravariantly finite in \( \mathcal{T}, \) it follows that \( \mathcal{X} \) is right coherent and therefore \( \text{mod-}\mathcal{X} \) is abelian with enough projectives. Since \( \mathcal{T} \) has split idempotents and \( \mathcal{X} \) is closed under direct summands, the functor \( H \) induces an equivalence \( \mathcal{X} \approx \text{Proj mod-}\mathcal{X}. \)

It follows that the functor \( H(A) \) is coherent and therefore the restricted Yoneda functor \( H \) induces a homological functor

\[
H : \mathcal{T} \to \text{mod-}\mathcal{X}, \quad H(A) = \mathcal{T}(-, A)|_{\mathcal{X}}
\]

Clearly we have: \( \text{Ker} H = \mathcal{X}^\perp := \{ A \in \mathcal{T} \mid \mathcal{T}(\mathcal{X}, A) = 0 \}. \)

**Remark 1.1.** Dually if \( \mathcal{X} \) is covariantly finite in \( \mathcal{T}, \) then \( \mathcal{X} \) is left coherent and therefore the category \( \mathcal{X}-\text{mod} := \text{mod-}\mathcal{X}^{\text{op}} \) of covariant coherent functors is abelian. For any object \( A \) in \( \mathcal{T} \) we have triangles, \( \forall t \geq 0: \)

\[
\Sigma_{\mathcal{X}}^{t+1}(A)[-1] \xrightarrow{h^A_t} \Sigma_{\mathcal{X}}^t(A) \xrightarrow{f^A_t} X^t_A \xrightarrow{g^A_t} \Sigma_{\mathcal{X}}^{t+1}(A) \quad (T^A_t)
\]

where \( \Sigma^t_{\mathcal{X}}(A) := A, \) and the middle map \( \Sigma^t_{\mathcal{X}}(A) \to X^t_A \) is a left \( \mathcal{X} \)-approximation of \( \Sigma^t_{\mathcal{X}}(A). \) Further we have a contravariant cohomological functor

\[
H^{\text{op}} : \mathcal{T}^{\text{op}} \to \mathcal{X}-\text{mod}, \quad H^{\text{op}}(A) = \mathcal{T}(A, -)|_{\mathcal{X}}
\]

which induces a duality \( \mathcal{X}^{\text{op}} \approx \text{Proj} (\mathcal{X}-\text{mod}) \) and clearly: \( \text{Ker} H^{\text{op}} = \mathcal{X}^{\perp} := \{ A \in \mathcal{T} \mid \mathcal{T}(A, \mathcal{X}) = 0 \}. \)

We denote by \( \mathcal{T}/\mathcal{X} \) the stable category of \( \mathcal{T} \) with respect to \( \mathcal{X}; \) recall that the objects of \( \mathcal{T}/\mathcal{X} \) are the objects of \( \mathcal{T}, \) and the morphism space \( \text{Hom}_{\mathcal{T}/\mathcal{X}}(A, B) \) is the quotient \( \mathcal{T}(A, B)/\mathcal{T}_\mathcal{X}(A, B), \) where \( \mathcal{T}_\mathcal{X}(A, B) \) is the subgroup of \( \mathcal{T}(A, B) \) consisting of all maps \( A \to B \) factorizing through an object from \( \mathcal{X}. \) We then have an additive functor \( \pi : \mathcal{T} \to \mathcal{T}/\mathcal{X}, \pi(A) = A \) and \( \pi(f) = f. \)

**Remark 1.2.** Although the object \( \Omega_{\mathcal{X}}^1(A) \) appearing in the triangle \( (T^A_0) \) depends on the choice of the right \( \mathcal{X} \)-approximation of \( A, \) it is not difficult to see that it is uniquely determined in the stable category \( \mathcal{T}/\mathcal{X}. \) Moreover the assignment \( A \mapsto \Omega_{\mathcal{X}}^1(A) \) induces a well-defined additive functor \( \Omega_{\mathcal{X}}^1 : \mathcal{T}/\mathcal{X} \to \mathcal{T}/\mathcal{X}. \) Dually if \( \mathcal{X} \) is covariantly finite, then the object \( \Sigma_{\mathcal{X}}^1(A) \) defined by the triangle \( (T^A_0) \) is uniquely determined in \( \mathcal{T}/\mathcal{X} \) and the assignment \( A \mapsto \Sigma_{\mathcal{X}}^1(A) \) induces a well-defined additive functor \( \Sigma_{\mathcal{X}}^1 : \mathcal{T}/\mathcal{X} \to \mathcal{T}/\mathcal{X}. \) If \( \mathcal{X} \) is functorially finite then we have an adjoint pair of endofunctors \( (\Sigma_{\mathcal{X}}^1, \Omega^1_{\mathcal{X}}) : \mathcal{T}/\mathcal{X} \to \mathcal{T}/\mathcal{X}; \) we refer to [16] for more details.
1.1. Ghost and cellular towers

Fix an object \( A \) in \( \mathcal{T} \) and consider as before the triangles, \( \forall t \geq 0: \)

\[
\Omega_{X}^{t+1}(A) \xrightarrow{\beta_{A}^{t}} X_{A}^{t} \xrightarrow{f_{A}^{t}} \Omega_{X}^{t}(A) \xrightarrow{h_{A}^{t}} \Omega_{X}^{t+1}(A)[1] \quad (T_{A}^{t})
\]

Then we have as in [12, Section 5], a tower of triangles in \( \mathcal{T} \), henceforth denoted by \((C_{A}^{\bullet})\):

\[
\begin{array}{ccccccc}
\Omega_{X}^{1}(A) & \xrightarrow{\beta_{A}^{1}} & \text{Cell}_{0}(A) & \xrightarrow{\gamma_{A}^{1}} & A & \xrightarrow{\omega_{A}^{0}} & \Omega_{X}^{0}(A)[1] & (C_{A}^{0}) \\
& h_{A}^{1} & \downarrow & & \alpha_{A}^{1} & \downarrow & & \h_{A}^{1}[1] & \\
\Omega_{X}^{2}(A)[1] & \xrightarrow{\beta_{A}^{2}} & \text{Cell}_{1}(A) & \xrightarrow{\gamma_{A}^{2}} & A & \xrightarrow{\omega_{A}^{1}} & \Omega_{X}^{1}(A)[2] & (C_{A}^{1}) \\
& h_{A}^{2}[1] & \downarrow & & \alpha_{A}^{2} & \downarrow & & \h_{A}^{2}[2] & \\
\Omega_{X}^{3}(A)[2] & \xrightarrow{\beta_{A}^{3}} & \text{Cell}_{2}(A) & \xrightarrow{\gamma_{A}^{3}} & A & \xrightarrow{\omega_{A}^{2}} & \Omega_{X}^{2}(A)[3] & (C_{A}^{2}) \\
& & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & (1.2) \\
\Omega_{X}^{n}(A)[n-1] & \xrightarrow{\beta_{A}^{n-1}} & \text{Cell}_{n-1}(A) & \xrightarrow{\gamma_{A}^{n-1}} & A & \xrightarrow{\omega_{A}^{n-1}} & \Omega_{X}^{n-1}(A)[n] & (C_{A}^{n-1}) \\
h_{A}^{n}[n-1] & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \cdots
\end{array}
\]

where: \( \text{Cell}_{0}(A) := X_{A}^{0}, \omega_{A}^{0} := h_{A}^{0}, \gamma_{A}^{0} := f_{A}^{0}, \) and \( \beta_{A}^{0} := g_{A}^{0}. \) The tower \((C_{A}^{\bullet})\) is constructed inductively by forming cobase change, in the sense of [12, Proposition 2.1], or taking homotopy push-outs in the sense of [51], of the triangles \((C_{A}^{n-1})\) along the maps \( h_{A}^{n}[n-1]: \Omega_{X}^{n}(A)[n-1] \longrightarrow \Omega_{X}^{n+1}(A)[n], \forall n \geq 1. \)

**Definition 1.3.** For any object \( A \in \mathcal{T} \), the map \( \gamma_{A}^{n}: \text{Cell}_{n}(A) \longrightarrow A \) constructed above is called the \textbf{\( n \)-th-cellular approximation} of \( A \) and the induced tower \((\text{Cell}_{A}^{\bullet})\):

\[
X_{A}^{0} = \text{Cell}_{0}(A) \xrightarrow{\alpha_{A}^{1}} \text{Cell}_{1}(A) \xrightarrow{\alpha_{A}^{2}} \text{Cell}_{2}(A) \longrightarrow \cdots \longrightarrow \text{Cell}_{n-1}(A) \xrightarrow{\alpha_{A}^{n-1}} \text{Cell}_{n}(A) \longrightarrow \cdots
\]
is called the **cellular tower** of \( A \) with respect to \( \mathcal{X} \). The tower of objects and \( \mathcal{X} \)-ghost maps \( (\text{Gh}_A^*) \):

\[
A = \Omega^n_\mathcal{X}(A) \xrightarrow{h^0_A} \Omega^1_\mathcal{X}(A)[1] \xrightarrow{h^1_A[1]} \Omega^2_\mathcal{X}(A)[2] \rightarrow \cdots
\rightarrow \Omega^n_\mathcal{X}(A)[n] \xrightarrow{h^n_A[n]} \Omega^{n+1}_\mathcal{X}(A)[n+1] \rightarrow \cdots
\]

is called the **Ghost tower**, or an **Adams resolution**, of \( A \) with respect to \( \mathcal{X} \). It is convenient to represent an Adams resolution by the following tower of triangles

\[
A \xrightarrow{h^0_A} \Omega^1_\mathcal{X}(A)[1] \xrightarrow{h^1_A[1]} \Omega^2_\mathcal{X}(A)[2] \xrightarrow{h^2_A[2]} \Omega^3_\mathcal{X}(A)[3] \xrightarrow{h^3_A[3]} \Omega^4_\mathcal{X}(A)[4] \rightarrow \cdots
\]

where \( \Omega^0_\mathcal{X}(A)[n] \twoheadrightarrow X^{n-1}_A[n-1] \) denotes a map of degree +1, i.e. a map \( \Omega^0_\mathcal{X}(A)[n] \rightarrow X^{n-1}_A[n], \forall n \geq 1 \).

Recall from [10] that if \( \mathcal{U} \) and \( \mathcal{V} \) are subcategories of \( \mathcal{T} \), then the **extension category** \( \mathcal{U} \star \mathcal{V} \) consists of all direct summands of objects \( A \) for which there exists a triangle \( U \longrightarrow A \longrightarrow V \longrightarrow U[1] \), where \( U \in \mathcal{U} \) and \( V \in \mathcal{V} \). Note that if \( \mathcal{T}(\mathcal{U}, \mathcal{V}) = 0 \), then \( \mathcal{U} \star \mathcal{V} \) is closed under direct summands, see [44]. The extension category \( \mathcal{U}_1 \star \mathcal{U}_2 \star \cdots \star \mathcal{U}_n \) is defined inductively for subcategories \( \mathcal{U}_i \) of \( \mathcal{T} \), \( 1 \leq i \leq n \).

**Remark 1.4.**

(i) By the construction of the tower \( (C^*_A) \) we have triangles, \( \forall t \geq 1 \):

\[
X^t_A[t-1] \longrightarrow \text{Cell}_{t-1}(A) \xrightarrow{\alpha^t_A} \text{Cell}_t(A) \longrightarrow X^t_A[t]
\]

and

\[
X^t_A[t] \longrightarrow \Omega^t_\mathcal{X}(A)[t] \xrightarrow{h^t_A[t]} \Omega^{t+1}_\mathcal{X}(A)[t+1] \longrightarrow X^t_A[t+1]
\]

Since \( \text{Cell}_0(A) = X^0_A \in \mathcal{X} \), it follows that we have a triangle \( X^0_A \longrightarrow \text{Cell}_1(A) \longrightarrow X^1_A[1] \longrightarrow X^0_A[1] \) and therefore \( \text{Cell}_1(A) \in \mathcal{X} \star \mathcal{X}[1] \). By induction we infer that:

\[
\text{Cell}_t(A) \in \mathcal{X} \star \mathcal{X}[1] \star \cdots \star \mathcal{X}[t], \quad \forall t \geq 0
\]

(ii) If \( \Omega^n_\mathcal{X}(A) \) lies in \( \mathcal{X} \), then \( \Omega^{n-t}_\mathcal{X}(A) \) lies in \( \mathcal{X} \star \mathcal{X}[1] \star \cdots \star \mathcal{X}[t] \) and in particular \( A \in \mathcal{X} \star \mathcal{X}[1] \star \cdots \star \mathcal{X}[n] \). In this case we may choose \( X^n_A = \Omega^n_\mathcal{X}(A) \) and \( \Omega^{n+t}_\mathcal{X}(A) = 0 \), \( \forall t \geq 1 \).
(iii) Splicing together the triangles \((T^n_A)\) and setting \(e^n_A = f^n_A \circ g^{n-1}_A, \forall n \geq 1\), we obtain a complex

\[
\cdots \rightarrow X^n_A \xrightarrow{\varepsilon^n_A} X^{n-1}_A \rightarrow \cdots \rightarrow X^1_A \xrightarrow{\varepsilon^1_A} X^0_A \xrightarrow{f^0_A} A \rightarrow 0 \quad (X^A_\bullet)
\]

over \(A\) by objects of \(\mathcal{X}\). Note that although the sequence \(H(X^1_A) \rightarrow H(X^0_A) \rightarrow H(A) \rightarrow 0\) is exact, in general the induced complex \(H(X^A_\bullet)\) in \(\text{mod}-\mathcal{X}\) is not necessarily exact.

1.2. Cocellular and Coghost towers

Dually if \(\mathcal{X}\) is covariantly finite in \(\mathcal{T}\), then as in Remark 1.1, for any object \(A \in \mathcal{T}\) we may construct inductively triangles, \(t \geq 0\):

\[
\Sigma^{t+1}_\mathcal{X}(A)[-1] \xrightarrow{h^t_A} \Sigma^t_\mathcal{X}(A) \xrightarrow{f^t_A} X^t_A \xrightarrow{g^t_A} \Sigma^{t+1}_\mathcal{X}(A) \quad (T^A_t)
\]

where the middle map is a left \(\mathcal{X}\)-approximation of \(\Sigma^t(A)\), and \(\Sigma^0_\mathcal{X}(A) := A\). As in Subsection 1.1, taking inductively homotopy pull-backs of the triangle \((T^n_A)\) along the maps \(h^n_A[-n+1], \forall n \geq 0\), we obtain the following inverse tower of triangles, henceforth denoted by \((C^A_\bullet)\):

\[
\begin{align*}
&\vdots & & \vdots & & \vdots & & \vdots \\
&\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\Sigma^n_\mathcal{X}(A)[-n] & \xrightarrow{\omega^n_n} A & \xrightarrow{\beta^n_n} \text{Cell}^{n-1}(A) & \xrightarrow{\gamma^n_n} \Sigma^n_\mathcal{X}(A)[-n+1] & & & & \\
&h^{n-1}_n[-n+1] & & & & & & & h^{n-1}_n[-n+2] \\
\Sigma^{n-1}_\mathcal{X}(A)[-n+1] & \xrightarrow{\omega^{n-1}_{n-1}} A & \xrightarrow{\beta^{n-1}_{n-1}} \text{Cell}^{n-2}(A) & \xrightarrow{\gamma^{n-1}_{n-1}} \Sigma^{n-1}_\mathcal{X}(A)[-n+2] & & & & \\
& \downarrow & & \downarrow & & \downarrow & & \downarrow \\
& \vdots & & \vdots & & \vdots & & \vdots \\
& \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\Sigma^3_\mathcal{X}(A)[-3] & \xrightarrow{\omega^3_3} A & \xrightarrow{\beta^3_3} \text{Cell}^2(A) & \xrightarrow{\gamma^3_2} \Sigma^3_\mathcal{X}(A)[-2] & & & & \\
&h^3_3[-2] & & & & & & & h^3_3[-1] \\
\Sigma^2_\mathcal{X}(A)[-2] & \xrightarrow{\omega^2_2} A & \xrightarrow{\beta^2_2} \text{Cell}^1(A) & \xrightarrow{\gamma^2_1} \Sigma^2_\mathcal{X}(A)[-1] & & & & \\
&h^2_2[-1] & & & & & & & h^2_2 \\
\Sigma^1_\mathcal{X}(A)[-1] & \xrightarrow{\omega^1_1} A & \xrightarrow{\beta^1_1} \text{Cell}^0(A) & \xrightarrow{\gamma^1_0} \Sigma^1_\mathcal{X}(A) & & & & \\
&h^1_1 & & & & & & & h^1_1
\end{align*}
\]
where we set: $\text{Cell}^0(A) := X_0^A$, $\omega_0^A := h_0^A$, $\beta_0^A := f_0^A$, and $\gamma_0^A := g_0^A$. The inverse tower of objects $(\text{Cell}^A_*)$:

$$\cdots \to \text{Cell}^{n+1}(A) \xrightarrow{\alpha_{n+1}^A} \text{Cell}_n(A) \to \cdots$$

$$\to \text{Cell}^2(A) \xrightarrow{\alpha_2^A} \text{Cell}^1(A) \xrightarrow{\alpha_1^A} \text{Cell}^0(A) = X_0^A$$

is called the **cocellular tower** of $A$ with respect to $\mathcal{X}$, and the inverse tower of objects $(\text{CoGh}^A_*)$:

$$\cdots \to \Sigma^{n+1}_n(A)[-n-1] \xrightarrow{h_n^A[-n]} \Sigma^n_n(A)[-n] \to \cdots$$

$$\to \Sigma^2_n(A)[-2] \xrightarrow{h^2_1[-1]} \Sigma^1_n(A)[-1] \xrightarrow{h^0_n} \Sigma^0_n(A) = A$$

is called the **coghost tower**, or an **Adams coresolution**, of $A$ with respect to $\mathcal{X}$. We leave to the reader to state the dual version of **Remark 1.4**.

1.3. The functor $H : \mathcal{T} \to \text{mod-}\mathcal{X}$

We close this section by analyzing the main properties of the homological functor $H : \mathcal{T} \to \text{mod-}\mathcal{X}$, $H(A) = \mathcal{T}(-, A)|_{\mathcal{X}}$, which will play a basic role in the rest of the paper.

We call an additive functor $F : \mathcal{A} \to \mathcal{B}$ between additive categories $\mathcal{A}$ and $\mathcal{B}$ **almost full** if for any map $\alpha : F(A) \to F(B)$ in $\mathcal{B}$, there are objects $A^*, B^*$ in $\mathcal{A}$ and maps $\alpha^* : A^* \to B^*$, $\omega_A : A^* \to A$ and $\omega_B : B^* \to B$, such that the maps $F(\omega_A) : F(A^*) \to F(A)$ and $F(\omega_B) : F(B^*) \to F(B)$ are invertible and the following square commutes:

$$\begin{array}{ccc}
F(A^*) & \xrightarrow{F(\alpha^*)} & F(B^*) \\
\downarrow F(\omega_A) & \cong & \downarrow F(\omega_B) \\
F(A) & \xrightarrow{\alpha} & F(B)
\end{array}$$

Clearly $F$ is full if and only if $F$ is almost full and $F$ is full on isomorphisms, i.e. any isomorphism $g : F(X) \to F(Y)$ in $\mathcal{B}$ is of the form $F(f)$ for some (not necessarily invertible) map $f : X \to Y$ in $\mathcal{A}$.

**Lemma 1.5.** Assume that $\mathcal{T}(\mathcal{X}, \mathcal{X}[1]) = 0$.

(i) $\forall A \in \mathcal{T} : H(\Omega^k_{\mathcal{X}}(A)[1]) = 0$, $\forall k \geq 1$.

(ii) For any object $A \in \mathcal{T}$, the map $H(\gamma^1_A) : H(\text{Cell}_1(A)) \to H(A)$ is invertible.

(iii) The functor $H : \mathcal{T} \to \text{mod-}\mathcal{X}$ is almost full and essentially surjective.
(iv) (See \cite{46}.) The functor $H : \mathcal{T} \rightarrow \mod\mathcal{X}$ induces an equivalence

$$H : (\mathcal{X} \times \mathcal{X}[1]) / \mathcal{X}[1] \xrightarrow{\approx} \mod\mathcal{X}$$

**Proof.** (i) Since $H(h_A^k) = 0$, $\forall k \geq 0$, applying the homological functor $H$ to the triangle $\Omega_X^k(A) \rightarrow X_A^{k-1} \rightarrow \Omega_X^{k-1}A H_k \rightarrow \Omega_X^k(A)[1]$ and using that $\mathcal{T}(\mathcal{X}, \mathcal{X}[1]) = 0$, we see that $H(\Omega_X^k(A)[1]) = 0$, $\forall k \geq 1$.

(ii) Since $H(h_A^0) = 0$, applying $H$ to the triangle $(C_A^1 : \Omega_X^2(A)[1] \rightarrow \Cell_1(A) H_1 \rightarrow A \xrightarrow{\omega_1} \Omega_X^2(A)[2]$ and using that $H(\Omega_X^2(A)[1]) = 0$ and $H(\omega_1^1) = H(h_A^0) \circ H(h_A^1[1]) = 0$, we infer that the map $H(\gamma_A^1)$ is invertible.

(iii) Let $H(X^1) \rightarrow H(X^0) \rightarrow F \rightarrow 0$ be a projective presentation of $F \in \mod\mathcal{X}$. If $X^1 \rightarrow X^0 \rightarrow A \rightarrow X^1[1]$ is a triangle in $\mathcal{T}$, then applying $H$ and using that $\mathcal{T}(\mathcal{X}, \mathcal{X}[1]) = 0$, we have $H(A) \cong F$, so $H$ is essentially surjective. Let $\alpha : H(A) \rightarrow H(B)$ be a map in $\mod\mathcal{X}$ and let $H(X_A^1) \rightarrow H(X_A^0) \rightarrow H(A) \rightarrow 0$ and $H(X_B^1) \rightarrow H(X_B^0) \rightarrow H(B) \rightarrow 0$ be projective presentations of $H(A)$ and $H(B)$ as in (1.1). Since $H|_{\mathcal{X}}$ is full and $H(\mathcal{X}) = \Proj \mod\mathcal{X}$, we get a map of projective presentations in $\mod\mathcal{X}$ as in diagram (*) below

$$
\begin{array}{c}
H(X_A^1) \xrightarrow{H(\epsilon_A^1)} H(X_A^0) \xrightarrow{H(f_A^0)} H(A) \rightarrow 0 \\
\downarrow \quad \downarrow \quad \downarrow \alpha \\
H(X_B^1) \xrightarrow{H(\epsilon_B^1)} H(X_B^0) \xrightarrow{H(f_B^0)} H(B) \rightarrow 0
\end{array}
(*)
$$

which, since $H|_{\mathcal{X}}$ is faithful, lifts to a morphism of triangles as in diagram (**) above. By the construction of the towers $(C_A^\bullet)$ and $(C_B^\bullet)$ we have: $\alpha_A^1 \circ \gamma_A^1 = f_A^0$ and $\alpha_B^1 \circ \gamma_B^1 = f_B^0$, and by part (i) the maps $H(\gamma_A^1)$ and $H(\gamma_B^1)$ are invertible. Consider the diagram:

$$
\begin{array}{c}
H(X_A^0) \xrightarrow{H(\alpha_A^1)} H(\Cell_1(A)) \xrightarrow{H(\gamma_A^1)} H(A) \\
\downarrow \quad \downarrow \quad \downarrow \alpha \\
H(X_B^0) \xrightarrow{H(\alpha_B^1)} H(\Cell_1(B)) \xrightarrow{H(\gamma_B^1)} H(B)
\end{array}
$$

(\ast\ast)
where the left square is commutative. Then $H(\alpha^1_A) \circ H(\gamma^1_A) \circ \alpha = H(f^0_A) \circ \alpha = H(\beta) \circ H(f^0_B) = H(\beta) \circ H(\alpha^B) \circ H(\gamma^1_B) = H(\alpha^A) \circ H(\gamma^1_B)$. Since the cone of $\alpha^A$ lies in $X[1]$, $H(\alpha^1_A)$ is an epimorphism. Then $H(\gamma^1_A) \circ \alpha = H(\alpha^*) \circ H(\gamma^1_B)$ and the right square is commutative. This shows that $H$ is almost full.

(iv) If $A$ lies in $X \star X[1]$ then clearly $\Omega^1_X(A) := X^1_A \in X$ and $A = \text{Cell}_1(A)$. Then (ii) shows that the map $\mathcal{T}(A, B) \rightarrow \text{Hom}(H(A), H(B))$ is surjective, $\forall B \in \mathcal{T}$, i.e. $H_{|X \star X[1]}$ is full, and essentially surjective by (i). If $\alpha: A \rightarrow B$ is a map such that $H(\alpha) = 0$, then the composition $X^0_A \rightarrow A \rightarrow B$ is zero and therefore it factorizes through the cone $X^1_A[1]$ of $X^0_A \rightarrow A$. It follows that $\text{Ker} H_{|X \star X[1]} = X[1]$ and the assertion follows. □

We refer to [16], see also Corollary 2.6 below, for necessary and sufficient conditions ensuring that $H$ is full.

2. Ghosts and extensions

Let as before $X$ be a contravariantly finite subcategory of $\mathcal{T}$. Our aim here is to analyze the structure of maps in $\mathcal{T}$ which are invisible by the functor $H: \mathcal{T} \rightarrow \text{mod-}X$, in the sense of the following definition.

**Definition 2.1.** A map $f: A \rightarrow B$ in $\mathcal{T}$ is called $X$-ghost if $\mathcal{T}(X, f) = 0$; equivalently $H(f) = 0$. Dually the map $f$ is called $X$-coghost if $\mathcal{T}(f, X) = 0$, or equivalently $H^0(f) = 0$.

We let $\text{Gh}_X(A, B)$ be the subset of $\mathcal{T}(A, B)$ consisting of all $X$-ghost maps, and let $\text{CoGh}_X(A, B)$ be the subset of $\mathcal{T}(A, B)$ consisting of all $X$-coghost maps. Note that $\text{Gh}_X(A, -) = 0$ if and only if $A \in X$ if and only if $\text{CoGh}_X(-, A) = 0$. Clearly $\text{Gh}_X(A, B)$ is a subgroup of $\mathcal{T}(A, B)$ and it is easy to see that in this way we obtain an ideal $\text{Gh}_X(\mathcal{T})$ of $\mathcal{T}$. For $n \geq 1$, we denote by $\text{Gh}_X^{[n+1]}(A, B)$ the subset of $\mathcal{T}(A, B)$ consisting of all maps $f: A \rightarrow B$ which can be written as a composition $f = f_0 \circ f_1 \cdots \circ f_n$, where each $f_i$ is $X[i]$-ghost, $0 \leq i \leq n$. Then for $n \geq 0$, we have an ideal $\text{Gh}_X^{[n+1]}(\mathcal{T})$, where $\text{Gh}_X^{[1]}(\mathcal{T}) := \text{Gh}_X(\mathcal{T})$. In other words, $\text{Gh}_X^{[n+1]}(\mathcal{T})$ is the product of the ideals $\text{Gh}_X[i](\mathcal{T})$, $0 \leq i \leq n$:

$$\text{Gh}_X^{[n+1]}(\mathcal{T}) = \text{Gh}_X(\mathcal{T}) \circ \text{Gh}_X[1](\mathcal{T}) \circ \text{Gh}_X[2](\mathcal{T}) \circ \cdots \circ \text{Gh}_X[n](\mathcal{T})$$

Note that if $X$ is stable, that is, $X = X[1]$, then $\text{Gh}_X^{[n]}(\mathcal{T})$ is the genuine $n$th power $\text{Gh}_X^n(\mathcal{T})$ of the ideal $\text{Gh}_X(\mathcal{T})$. We call the elements of $\text{Gh}_X^{[n]}(\mathcal{T})$, $X$-ghost maps of depth $n$. For instance, for any object $A \in \mathcal{T}$ the tower of triangles $(C^1_A)$ shows that, for any $n \geq 1$,
the map

\[ (-1)^{n+1}h_A^0 \circ h_A^1 \circ \cdots \circ h_A^{n-1}[n-1] := \omega_A^{n-1} : A \rightarrow \Omega^0_X(A)[n] \]

is an \( X \)-ghost map of depth \( n \), i.e. lies in \( \text{Gh}^{[n]}_X(A, \Omega^0_X(A)[n]) \). Sometimes \( \omega_A^{n-1} \) is called the \( n \)th Atiyah class of \( A \) with respect to \( X \) and is denoted by \( \text{at}^n_X(A) \), see [27] for the terminology in the stable case.

2.1. The Ghost Lemma

The following version of the Ghost Lemma (see for instance [12, Proposition 5.1 and Corollary 5.5] in the stable case) shows that

\[ \text{Gh}^{[n]}_X(A, B) = \omega_A^{n-1} \circ T(\Omega^0_X(A)[n], B) \]

In other words the left ideal \( \text{Gh}^{[n]}_X(A, -) \) of \( X \)-ghost maps out of \( A \) of depth \( n \) is principal generated by the \( n \)th Atiyah class \( \text{at}^n_X(A) = \omega_A^{n-1} \). As another consequence of the Ghost Lemma it follows that the complexity of the ideal of \( X \)-ghost maps measures the possibility for building \( T \) from \( X \) using positive shifts and extensions.

Proposition 2.2 (The Ghost Lemma).

(i) For a map \( f : A \rightarrow B \) in \( T \), the following are equivalent:
   (a) \( f \in \text{Gh}^{[n]}_X(A, B) \).
   (b) \( f \) factors through \( \omega_A^{n-1} \), that is, there exists a map \( g : \Omega^0_X(A)[n] \rightarrow B \) such that \( f = \omega_A^{n-1} \circ g \).

(ii) Let \( A \) be an object in \( T \), and consider the following statements:
   (a) \( \Omega^0_X(A) \in X \).
   (b) \( A \in X \ast X[1] \ast \cdots \ast X[n] \).
   (c) \( \text{Gh}^{[n+1]}_X(A, -) = 0 \).

Then (a) \( \Rightarrow \) (b) \( \iff \) (c). In particular: \( T = X \ast X[1] \ast \cdots \ast X[n] \) if and only if \( \text{Gh}^{[n+1]}_X(T) = 0 \).

Proof. (i) If \( f \) factors through \( \omega_A^{n-1} \), then clearly \( f \) lies in \( \text{Gh}^{[n]}_X(A, B) \) since by construction \( h_A^i[i] \) is \( \mathcal{X}[i] \)-ghost, \( 0 \leq i \leq n-1 \), cf. (1.4). To show the converse, let \( n = 1 \) and \( f : A \rightarrow B \) be \( X \)-ghost. Then the composition \( X_A^0 \rightarrow A \rightarrow B \) is zero and therefore it factorizes through \( \omega_A^0 = h_A^0 : A \rightarrow \Omega^1_X(A)[1] \). If \( n = 2 \) and \( f : A \rightarrow B \) lies in \( \text{Gh}^{[2]}_X(A, B) \), then \( f \) admits a factorization \( f = f_0 \circ f_1 \), where \( f_0 : A \rightarrow B_0 \) is \( X \)-ghost and \( f_1 : B_0 \rightarrow B \) is \( X[1] \)-ghost. Then there exists a map \( g_0 : \Omega^1_X(A)[1] \rightarrow B_0 \) such that \( f_0 = h_A^0 \circ g_0 \). Since \( f_1 \) is \( X[1] \)-ghost, so is \( g_0 \circ f_1 \) and therefore its composition with the map \( X_A^1[1] \rightarrow \Omega^1_X(A)[1] \) is zero. Then \( g_0 \circ f_1 \) factorizes through the map \( -h_A^1[1] : \Omega^1_X(A)[1] \rightarrow \Omega^2_X(A)[2] \), so there exists a map \( g : \Omega^2_X(A)[2] \rightarrow B \) such that
(-h^1_A(1)) \circ g = g_0 \circ f_1. \quad \text{Then} \quad f = f_0 \circ f_1 = h^0_A \circ g_0 \circ f_1 = h^0_A \circ (-h^1_A(1)) \circ g. \quad \text{Hence} \quad f = -(h^0_A \circ h^1_A(1)) \circ g = \omega_A^1 \circ g, \quad \text{and this completes the proof for the case} \ n = 2. \quad \text{The assertion for} \ n \geq 3 \ \text{follows easily by induction.}

(ii) \ (a) \Rightarrow (b) \text{ If} \ \Omega^2_X(A) \text{ lies in} \ X, \text{then since} \ \text{Cell}_{n-1}(A) \text{ lies in} \ X \times X[1] \ast \cdots \ast X[n-1], \text{from the triangle} \ (C^0_A)^{-1} \text{ in the tower of triangles} \ (C^*_A) \text{ it follows that} \ A \text{ lies in} \ X \times X[1] \ast \cdots \ast X[n]. \quad \text{(b)} \Leftrightarrow (c) \text{ If} \ n = 0, \text{ i.e.} \ A \in X, \text{then clearly} \ \text{Gr}^1_X(A, -) = \text{Gr}_X(A, -) = 0. \text{ Let} \ n = 1, \text{ so that} \ A \in X \ast X[1]. \text{ Let} \ X_0 \xrightarrow{\alpha} A \xrightarrow{\beta} X_1[1] \xrightarrow{\gamma} X_0[1] \text{ be a triangle, where} \ X_i \in X, \text{ and} \ f \in \text{Gr}^2_X(A, B), \text{ i.e.} \ f \text{ admits a factorization} \ f = f_0 \circ f_1, \text{ where} \ f_0 : A \to B_0 \text{ is} \ X\text{-ghost and} \ f_1 : B_0 \to B \text{ is} \ X[1]\text{-ghost. Then} \ f_0 = \beta \circ g \text{ for some map} \ g : X_1[1] \to B_0. \text{ Since the map} \ f_1 \text{ is} \ X[1]\text{-ghost, the composition} \ g \circ f_1 : X_1[1] \to B \text{ is zero. Then} \ f = f_0 \circ f_1 = \beta \circ g \circ f_1 = 0, \text{ hence} \ \text{Gr}^2_X(A, B) = 0. \text{ For} \ n \geq 3 \text{ the assertion follows by induction. Now assume} \ \text{Gr}^{n+1}_X(A, -) = 0. \text{ Then} \ \omega^n_A = 0 \text{ since by definition} \ \omega^n_A \text{ is in} \ \text{Gr}^{n+1}_X(A, \Omega^{n+1}_X(A)[n+1]). \text{ This implies that} \ A \text{ lies in} \ X \ast X[1] \ast \cdots \ast X[n] \text{ as a direct summand of Cell}_n(A). \quad \Box

As a consequence of the Ghost Lemma, we have the following result which gives an alternative description of the “powers” of the ideal of \ X\text{-ghost maps. Let} \ A \text{ be an object of} \ T \text{ and consider the triangle}

\begin{align*}
\Omega^{t+1}_X(A)[t] \xrightarrow{\beta^t_A} \text{Cell}_t(A) \xrightarrow{\gamma^t_A} A \xrightarrow{\omega^t_A} \Omega^{t+1}_X(A)[t+1] = (C^+_A)^{t+1}
\end{align*}

Corollary 2.3.

(i) \text{ For any} \ t \geq 0, \text{ the map} \ \gamma^t_A : \text{Cell}_t(A) \to A \text{ is a right} \ (X \ast X[1] \ast \cdots \ast X[t])\text{-approximation of} \ A. \text{ In particular} \ X \ast X[1] \ast \cdots \ast X[t] \text{ is contravariantly finite in} \ T.

(ii) \text{ For any objects} \ A, B \in T \text{ and any} \ t \geq 1, \text{ we have an equality:}

\begin{align*}
\text{Gr}^{t+1}_X(A, B) = \text{Gr}_{X \ast X[1] \ast \cdots \ast X[t]}(A, B)
\end{align*}

Proof. (i) \text{ Let} \ g : C \to A \text{ be a map, where} \ C \text{ lies in} \ X \ast X[1] \ast \cdots \ast X[t]. \text{ Since the map} \ \omega^t_A \text{ lies in} \ \text{Gr}^{t+1}_X(A, \Omega^{t+1}_X(A)[t+1]), \text{ it follows that} \ g \circ \omega^t_A \text{ lies in} \ \text{Gr}^{t+1}_X(C, \Omega^{t+1}_X(A)[t+1]). \text{ By the Ghost Lemma, we have} \ g \circ \omega^t_A = 0 \text{ and therefore} \ g \text{ factorizes through} \ \text{Cell}_t(A), \text{ i.e.} \ \gamma^t_A \text{ is a right} \ (X \ast X[1] \ast \cdots \ast X[t])\text{-approximation of} \ A. \text{ We infer that} \ X \ast X[1] \ast \cdots \ast X[t] \text{ is contravariantly finite in} \ T.

(ii) \text{ If} \ f : A \to B \text{ lies in} \ \text{Gr}_{X \ast X[1] \ast \cdots \ast X[t]}(A, B), \text{ then the composition} \ \gamma^t_A \circ f \text{ is zero since} \ \text{Cell}_t(A) \in X \ast X[1] \ast \cdots \ast X[t]. \text{ Hence} \ f \text{ factorizes through} \ \omega^t_A : A \to \Omega^{t+1}_X(A)[t+1], \text{ say as} \ f = \omega^t_A \circ g. \text{ Then} \ f \text{ lies in} \ \text{Gr}^{t+1}_X(A, B) \text{ since} \ \omega^t_A \text{ lies in} \ \text{Gr}^{t+1}_X(A, \Omega^{t+1}_X(A)[t+1]). \text{ Hence} \ \text{Gr}_{X \ast X[1] \ast \cdots \ast X[t]}(A, B) \subseteq \text{Gr}^{t+1}_X(A, B). \text{ Conversely if} \ f \text{ lies in} \ \text{Gr}^{t+1}_X(A, B), \text{ then by the Ghost Lemma we have} \ f = \omega^t_A \circ g \text{ for some} \ g : \Omega^{t+1}_X(A)[t+1] \to B. \text{ This implies that} \ \gamma^t_A \circ f = 0. \text{ Since, by part (i),} \ \gamma^t_A \text{ is a right} \ (X \ast X[1] \ast \cdots \ast X[t])\text{-approximation of} \ A, \text{ we have} \ T(C, f) = 0, \forall C \in X \ast X[1] \ast \cdots \ast X[t], \text{ so} \ f \text{ lies in} \ \text{Gr}_{X \ast X[1] \ast \cdots \ast X[t]}(A, B). \quad \Box
Corollary 2.3 shows that for any contravariantly finite subcategory $\mathcal{X}$ of $\mathcal{T}$ and any $n \geq 0$, the pair

$$(\mathcal{X} \star \mathcal{X}[1] \star \cdots \star \mathcal{X}[n], \, \text{Gh}_{\mathcal{X}}^{[n+1]}(\mathcal{T}))$$

forms a projective class in $\mathcal{T}$ in the sense of [29] and [31], see also [12] in the stable case.

Remark 2.4.

(i) For any integer $n \geq 1$, we define $\text{Gh}_{\mathcal{X}}^{[-n]}(A, B)$ to be the subgroup of $\mathcal{T}(A, B)$ consisting of all maps $f : A \to B$ which can be written as a composition $f = f_{-n+1} \circ f_{-n+2} \circ \cdots \circ f_{-1} \circ f_0$, where $f_{-i} : B_{-i-1} \to B_{-i}$ is $\mathcal{X}[-i]$-ghost, $A = B_{-n}$ and $B_0 = B$. Clearly then the map

$$\text{Gh}_{\mathcal{X}}^{[-n]}(A, B) \to \text{Gh}_{\mathcal{X}}^{[n]}(A[n-1], B[n-1]), \quad f \mapsto f[n-1]$$

is an isomorphism. By Corollary 2.3 we then have $\text{Gh}_{\mathcal{X}[-n], \cdots, \mathcal{X}[-1], \mathcal{X}}(A, B) \cong \text{Gh}_{\mathcal{X}, \mathcal{X}[1], \cdots, \mathcal{X}[n]}(A[n], B[n])$.

(ii) The above results as well as the results that follow concerning $\mathcal{X}$-ghost maps have dual versions for $\mathcal{X}$-coghost maps. We leave their formulation to the reader.

Clearly if $\mathcal{X} \subseteq \mathcal{Y}$, then $\text{Gh}_{\mathcal{Y}}(A, B) \subseteq \text{Gh}_{\mathcal{X}}(A, B)$. Hence the increasing filtration by subcategories of $\mathcal{T}$

$$\mathcal{X} \subseteq \mathcal{X} \star \mathcal{X}[1] \subseteq \mathcal{X} \star \mathcal{X}[1] \star \mathcal{X}[2] \subseteq \cdots \subseteq \mathcal{X} \star \mathcal{X}[1] \star \cdots \star \mathcal{X}[t] \subseteq \cdots \subseteq \mathcal{T} \quad (2.1)$$

induces, by Corollary 2.3, a decreasing filtration of the $\text{Hom}$-functor $\mathcal{T}(-, -)$ of $\mathcal{T}$ by $\mathcal{X}$-ghost ideals:

$$\cdots \subseteq \text{Gh}_{\mathcal{X}}^{[t+1]}(\mathcal{T}) \subseteq \cdots \subseteq \text{Gh}_{\mathcal{X}}^{[2]}(\mathcal{T}) \subseteq \text{Gh}_{\mathcal{X}}(\mathcal{T}) \subseteq \mathcal{T}(-, -) \quad (2.2)$$

By the Ghost Lemma the above filtrations have the same length. For any objects $A, B \in \mathcal{T}$, we denote by $H_{A, B}$ the canonical map

$$H_{A, B} : \mathcal{T}(A, B) \to \text{Hom}(H(A), H(B)), \quad f \mapsto H(f)$$

A version of the next useful result was proved in [17] in case $\mathcal{X}$ is stable, i.e. $\mathcal{X} = \mathcal{X}[1]$.

**Proposition 2.5.** For any objects $A, B \in \mathcal{T}$, there exists a natural exact sequence

$$0 \to \text{Gh}_{\mathcal{X}}(A, B) \to \mathcal{T}(A, B) \to \text{Hom}(H(A), H(B))$$

$$\to \text{Gh}_{\mathcal{X}}(\Omega_{\mathcal{X}}^1(A), B) \to \text{Gh}_{\mathcal{X}}^{[2]}(A, B[1]) \to 0$$
Proof. We construct maps
\[ \vartheta_{A,B} : \text{Hom}(H(A), H(B)) \rightarrow \text{Gh}_X(\Omega^1_X(A), B) \quad \& \quad \zeta_{A,B} : \text{Gh}_X(\Omega^1_X(A), B) \rightarrow \text{Gh}^{[2]}_X(A, B[1]) \]
as follows. Let \( \alpha : H(A) \rightarrow H(B) \) be a map and consider the triangle \( \Omega^1_X(A) \rightarrow X^0_A \rightarrow A \rightarrow \Omega^0_X(A)[1] \). Then we have an exact sequence \( \text{H}(\Omega^1_X(A)) \rightarrow \text{H}(X^0_A) \rightarrow \text{H}(A) \rightarrow 0 \) in mod-\( X \) and clearly the composition \( \text{H}(f^0_A) \circ \alpha : \text{H}(X^0_A) \rightarrow \text{H}(A) \rightarrow \text{H}(B) \) is of the form \( \text{H}(\alpha^*) \) for any map \( \alpha^* : X^0_A \rightarrow B \). Define \( \vartheta_{A,B}(\alpha) = g^0_A \circ \alpha^* : \Omega^1_X(A) \rightarrow B \). Clearly \( g^0_A \circ \alpha^* \) is \( X \)-ghost since \( \text{H}(g^0_A \circ \alpha^*) = \text{H}(g^0_A \circ \delta^*) = \text{H}(g^0_A \circ \text{H}(f^0_A) \circ \alpha) = 0 \). Now if \( \alpha \in \text{Hom}(H(A), H(B)) \) lies in \( \text{Ker} \vartheta_{A,B} \), then \( \vartheta_{A,B}(\alpha) = g^0_A \circ \alpha^* = 0 \). Hence \( \alpha^* = f^0_A \circ \beta \) for some map \( \beta : A \rightarrow B \) and then \( \text{H}(f^0_A) \circ \alpha = \text{H}(\alpha^*) = \text{H}(f^0_A) \circ \beta \) and therefore \( \alpha = \text{H}(\beta) \). Conversely if \( \alpha = \text{H}(\beta) \), for some map \( \beta : A \rightarrow B \), then \( \alpha^* = f^0_A \circ \beta \) and then \( \vartheta_{A,B}(\alpha) = g^0_A \circ \alpha^* = g^0_A \circ f^0_A \circ \beta = 0 \). Hence \( \text{Ker} \vartheta_{A,B} = \text{Im} \text{H}_{A,B} \). On the other hand, if \( \alpha : \Omega^1_X(A) \rightarrow B \) is an \( X \)-ghost map, then since \( h^0_A[-1] \) is clearly \( X[-1] \)-ghost and \( \alpha \) is \( X \)-ghost, it follows by Corollary 2.3 that \( h^0_A[-1] \circ \alpha \in \text{Gh}_X[-1] \circ \text{A}(A[-1], B) \), hence by Remark 2.4, we have a map \( \zeta_{A,B} : \text{Gh}_X(\Omega^1_X(A), B) \rightarrow \text{Gh}^{[2]}_X(A, B[1]) \), \( \alpha \rightarrow \zeta_{A,B}(\alpha) = h^0_A \circ \alpha \). We show that \( \zeta_{A,B} \) is surjective. Let \( \alpha : A \rightarrow B[1] \) be a map in \( \text{Gh}^{[2]}_X(A, B[1]) \), so \( \alpha = \beta \circ \gamma \), where \( \beta \in \text{Gh}_X(A, C) \) and \( \gamma \in \text{Gh}_{X[1]}(C, B[1]) \). By the Ghost Lemma, \( \beta = h^0_A \circ \rho \), for some map \( \rho : \Omega^1_X(A)[1] \rightarrow C \). Then the map \( (\rho \circ \gamma)[-1] : \Omega^1_X(A) \rightarrow B \) is \( X \)-ghost and \( \zeta_{A,B}((\rho \circ \gamma)[-1]) = \alpha \), so \( \zeta_{A,B} \) is surjective. Finally we show that \( \text{Ker} \zeta_{A,B} = \text{Im} \vartheta_{A,B} \). Let \( \beta : \Omega^1_X(A) \rightarrow B \) be \( X \)-ghost such that \( \zeta_{A,B}(\beta) = h^0_A \circ \beta[1] = 0 \). Then \( \beta = g^0_A \circ \gamma \), for some map \( \gamma : X^0_A \rightarrow B \). Since \( 0 = \text{H}(\beta) = \text{H}(g^0_A \circ \gamma) \), there exists a unique map \( \alpha : H(A) \rightarrow H(B) \) such that \( \text{H}(f^0_A) \circ \alpha = \text{H}(\gamma) \). It follows that \( \vartheta_{A,B}(\alpha) = g^0_A \circ \alpha \circ \gamma = \beta \). i.e. \( \text{Ker} \zeta_{A,B} \subseteq \text{Im} \vartheta_{A,B} \). Finally if \( \delta : \Omega^1_X(A) \rightarrow B \) is in the image of \( \vartheta_{A,B} \), then \( \delta = g^0_A \circ \delta^* \) for some map \( \delta^* : X^0_A \rightarrow B \). Plainly \( \zeta_{A,B}(\delta) = h^0_A \circ g^0_A[1] \circ \delta^*[1] = 0 \), i.e. \( \delta \in \text{Ker} \zeta_{A,B} \). Hence \( \text{Ker} \zeta_{A,B} = \text{Im} \vartheta_{A,B} \) and the sequence is exact. \( \square \)

The obstruction group \( \mathcal{O}_H(A, B) \) associated to the objects \( A, B \in \mathcal{T} \) is defined to be the cokernel of the natural map \( \mathcal{O}_{A,B} : \mathcal{T}(A, B) \rightarrow \text{Hom}(H(A), H(B)) \). In this way we obtain the obstruction bifunctor \( \mathcal{O}_H : \mathcal{T}^{op} \times \mathcal{T} \rightarrow \mathfrak{A}b \), and clearly \( \mathcal{O}_H(A, -) = 0 \) iff any map \( H(A) \rightarrow - \) is in the image of \( H \), and \( \mathcal{O}_H = 0 \), i.e. \( \mathcal{O}_H(A, B) = 0 \), \( \forall A, B \in \mathcal{T} \), iff \( H \) is full. The next result describes the largest subcategory \( \mathcal{U} \) of \( \mathcal{T} \) such that the restriction \( H|_{\mathcal{U}} \) is full. First we need the following notation: if \( \mathcal{A}, \mathcal{B} \) are classes of objects of \( \mathcal{T} \), then \( \mathcal{A} \oplus \mathcal{B} \) denotes the full subcategory of \( \mathcal{T} \) consisting of the direct factors of coproducts \( A \oplus B \), where \( A \in \mathcal{A} \) and \( B \in \mathcal{B} \).

Corollary 2.6. If \( \mathcal{T}(X, X[1]) = 0 \), then:

\[ X \ast X[1] = \{ A \in \mathcal{T} \mid \text{Gh}^{[2]}(A, -) = 0 \} \subseteq \{ A \in \mathcal{T} \mid \mathcal{O}_H(A, -) = 0 \} = (X \ast X[1]) \oplus X^\top \]
In particular: $\mathsf{Gh}_X^{[2]}(T) = 0 \iff T = X \star X[1]$ and $\mathcal{O}_H = 0 \iff H$ is full $\iff T = (X \star X[1]) \oplus X^\top$.

Proof. The first equality follows directly by using that $\mathsf{Gh}_X^{[2]}(A, -) \cong \mathsf{Gh}_{X \star X[1]}(A, -)$, see Corollary 2.3. If $A \in X \star X[1]$, then there is a triangle $X^1 \rightarrow X^0 \rightarrow A \rightarrow X^1[1]$, where the $X^i$ lie in $X$. Since $T(X, X[1]) = 0$, the map $X^0 \rightarrow A$ is a right $X$-approximation, hence $X^1 = \Omega^1_X(A)$. It follows that $\mathsf{Gh}_X(\Omega^1_X(A), -) = 0$ and then $\mathcal{O}_H(A, -) = 0$ by Proposition 2.5. For the proof of the second equality we refer to [16]. □

2.2. Extensions of coherent functors

The following results extend the exact sequence of Proposition 2.5 and involves extensions of coherent functors.

Proposition 2.7. For any objects $A, B \in \mathcal{T}$ such that $\mathcal{T}(X, A[-1]) = 0$, there exists a 7-term exact sequence:

$$
0 \rightarrow \mathsf{Gh}_X(A, B) \rightarrow \mathcal{T}(A, B) \xrightarrow{H_{A,B}} \mathsf{Hom}(H(A), H(B)) \xrightarrow{\vartheta_{A,B}} \mathsf{Gh}_X(\Omega^1_X(A), B) \xrightarrow{\eta_{A,B}} \mathsf{Gh}_X(A, B[1]) \xrightarrow{\xi_{A,B}} \mathsf{Ext}^1(H(A), H(B)) \rightarrow \mathcal{O}_H(\Omega^1_X(A), B) \rightarrow 0 \quad (2.3)
$$

Moreover: $\text{Im}(\vartheta_{A,B}) = \mathcal{O}_H(A, B)$ and $\text{Im}(\eta_{A,B}) \cong \mathsf{Gh}_X^{[2]}(A, B[1])$.

Proof. Applying $H$ to the triangle $\Omega^1_X(A) \rightarrow X^0_A \rightarrow A \rightarrow \Omega^1_X(A)[1]$ and using that $H(A[-1]) = 0$, we have an exact sequence $0 \rightarrow H(\Omega^1_X(A)) \rightarrow H(X^0_A) \rightarrow H(A) \rightarrow 0$ and an exact commutative diagram

$$
\cdots \rightarrow \mathcal{T}((A, B)) \xrightarrow{\mathcal{T}(s^0_{A,B})} \mathcal{T}(\Omega^1_X(A), B) \xrightarrow{\mathcal{T}(h^0_{A,B})} \mathcal{T}(h^0_A[-1], B) \rightarrow 0
$$

where $\mathsf{Ker} H_{A,B} = \mathsf{Gh}_X(A, B)$, $\mathsf{Coker} H_{A,B} = \mathcal{O}_H(A, B)$, and $\mathsf{Ker} H_{\Omega^1_X(A), B} = \mathsf{Gh}_X(\Omega^1_X(A), B)$, and $\mathsf{Coker} H_{\Omega^1_X(A), B} = \mathcal{O}_H(\Omega^1_X(A), B)$. By chasing the above commutative diagram, it is not difficult to see that we have an exact sequence

$$
0 \rightarrow \mathcal{O}_H(A, B) \rightarrow \mathsf{Gh}_X(\Omega^1_X(A), B) \rightarrow \text{Im} \mathcal{T}(h^0_A[-1], B) \xrightarrow{\phi} \mathsf{Ext}^1(H(A), H(B)) \rightarrow \mathsf{Coker} \phi \rightarrow 0
$$

and identifications: $\mathsf{Coker} \phi \cong \mathsf{Coker} H_{\Omega^1_X(A), B} = \mathcal{O}_H(\Omega^1_X(A), B)$, $\mathsf{Ker} \phi = \mathsf{Gh}_X(-1 \star X)(A[-1], B)$, and $\text{Im} \mathcal{T}(h^0_A[-1], B) = \mathsf{Gh}_X[-1](A[-1], B)$. Since, by Remark 2.4, we have isomorphisms: $\mathsf{Gh}_X[-1](A[-1], B) \cong \mathcal{Gh}_A(B[1])$ and $\mathsf{Gh}_X[-1 \star X](A[-1], B) \cong \mathsf{Gh}_X^{[2]}(A, B[-1])$, the exact sequence (2.3) follows by splicing together the above exact sequence with the exact sequence of Proposition 2.5. □
**Corollary 2.8.** Let \( t \geq 2 \) and assume that \( \mathcal{T}(\mathcal{X}, \mathcal{X}[i]) = 0, 1 \leq i \leq t - 1 \). Then for any \( A \in \mathcal{T} \) such that \( \mathcal{T}(\mathcal{X}, A[i]) = 0, 1 \leq i \leq t \), and any object \( B \in \mathcal{T} \), there exists an exact sequence, for any \( 1 \leq i \leq t - 1 \):

\[
0 \rightarrow \mathcal{Gh}_{\mathcal{X}}(\Omega^i_{\mathcal{X}}(A), B) \rightarrow \mathcal{T}(\Omega^i_{\mathcal{X}}(A), B) \rightarrow \text{Hom}(\Omega^i_{\mathcal{X}}(A), H(B))
\]

\[
\rightarrow \mathcal{Gh}_{\mathcal{X}}(\Omega^{i+1}_{\mathcal{X}}(A), B) \rightarrow \mathcal{Gh}_{\mathcal{X}}(\Omega^i_{\mathcal{X}}(A), B[1]) \rightarrow \text{Ext}^{i+1}(H(A), H(B))
\]

\[
\rightarrow \mathcal{O}_H(\Omega^{i+1}_{\mathcal{X}}(A), B) \rightarrow 0
\]

\[(2.4)\]

**Proof.** Applying the functor \( H \) to the triangles \( \Omega^i_{\mathcal{X}}(A) \rightarrow X^i_{A} \rightarrow \Omega^{i-1}_{\mathcal{X}}(A) \rightarrow \Omega^i_{\mathcal{X}}[1], i \geq 1 \), and using the vanishing conditions of the statement, we see easily that \( \mathcal{T}(\mathcal{X}, \Omega^i_{\mathcal{X}}(A)[-t+k]) = 0, 1 \leq k \leq t - 1 \). As a consequence we have short exact sequences \( 0 \rightarrow H(\Omega^i_{\mathcal{X}}(A)) \rightarrow H(X^i_{A}) \rightarrow H(\Omega^{i-1}_{\mathcal{X}}(A)) \rightarrow 0, 1 \leq k \leq t \), so \( \Omega^k H(A) \cong H(\Omega^k_{\mathcal{X}}(A)), 1 \leq k \leq t \). Since \( \mathcal{T}(\mathcal{X}, \Omega^k_{\mathcal{X}}(A)[-1]) = 0, 1 \leq k \leq t - 1 \), the existence of \((2.4)\) follows from the exact sequence \((2.3)\) of Proposition 2.7, by replacing \( A \) with \( \Omega^k_{\mathcal{X}}(A) \). \( \square \)

**Corollary 2.9.** Let \( A \in \mathcal{T} \) be such that \( \mathcal{T}(\mathcal{X}, A[-1]) = 0 \).

(i) If \( \Omega^2_{\mathcal{X}}(A) \in \mathcal{X} \), then, \( \forall B \in \mathcal{T} \), there is an isomorphism

\[
\text{Ext}^1(H(A), H(B)) \cong \frac{\mathcal{Gh}_{\mathcal{X}}(A, B[1])}{\mathcal{Gh}^2_{\mathcal{X}}(A, B[1])}
\]

(ii) If \( A \in \mathcal{X} \ast \mathcal{X}[1] \) and \( \mathcal{T}(\mathcal{X}, \mathcal{X}[1]) = 0 \), then, \( \forall B \in \mathcal{T} \), there is an isomorphism:

\[
\text{Ext}^1(H(A), H(B)) \cong \mathcal{Gh}_{\mathcal{X}}(A, B[1])
\]

(iii) If \( A, B \in \mathcal{X} \ast \mathcal{X}[1] \) and \( \mathcal{T}(\mathcal{X}, \mathcal{X}[1]) = 0 = \mathcal{T}(\mathcal{X}, \mathcal{X}[2]) \), then there is an isomorphism:

\[
\text{Ext}^1(H(A), H(B)) \cong \mathcal{T}(A, B[1])
\]

**Proof.** (i) By Proposition 2.7 we have an exact sequence

\[
\mathcal{Gh}_{\mathcal{X}}(\Omega^1_{\mathcal{X}}(A), B) \xrightarrow{\eta_{A,B}} \mathcal{Gh}_{\mathcal{X}}(A, B[1]) \xrightarrow{\kappa_{A,B}} \text{Ext}^1(H(A), H(B))
\]

\[
\rightarrow \mathcal{O}_H(\Omega^1_{\mathcal{X}}(A), B) \rightarrow 0
\]

\[(2.5)\]

and an isomorphism \( \text{Im}(\eta_{A,B}) \cong \mathcal{Gh}^2_{\mathcal{X}}(A, B[1]) \). Hence we have an exact sequence

\[
0 \rightarrow \frac{\mathcal{Gh}_{\mathcal{X}}(A, B[1])}{\mathcal{Gh}^2_{\mathcal{X}}(A, B[1])} \rightarrow \text{Ext}^1(H(A), H(B)) \rightarrow \mathcal{O}_H(\Omega^1_{\mathcal{X}}(A), B) \rightarrow 0
\]

and it suffices to show that \( \mathcal{O}_H(\Omega^1_{\mathcal{X}}(A), B) = 0 \). Since \( \Omega^2_{\mathcal{X}}(A) \in \mathcal{X} \), it follows that there is a triangle \( X^1 \rightarrow X^0 \rightarrow \Omega^1_{\mathcal{X}}(A) \rightarrow X^1[1] \), where the \( X^i \) lie in \( \mathcal{X} \), such that the sequence
H(X^1) \to H(X^0) \to H(\Omega_X^1(A)) \to 0 \text{ is exact. Applying the functor } \text{Hom}(\cdot, H(B)) \text{ to this exact sequence we have an exact sequence}

\[ 0 \to \text{Hom}(H(\Omega_X^1(A)), H(B)) \to \text{Hom}(H(X^0), H(B)) \to \text{Hom}(H(X^1), H(B)) \]

Since the objects \(X^0, X^1\) lie in \(X\), the canonical map \(H_{X^i,B}: T(X^i, B) \to \text{Hom}(H(X^i), H(B)), \ i = 1, 2\), is invertible and we have an exact commutative diagram

\[
\begin{array}{ccc}
\cdots & \to & T(\Omega_X^1(A), B) \\
& & \downarrow H_{\Omega_X^1(A), B} & \to & T(X^0, B) \\
& & \downarrow H_{X^0, B} & \cong & T(X^1, B) \\
0 & \to & \text{Hom}(H(\Omega_X^1(A)), H(B)) \\
\end{array}
\]

Chasing the above exact commutative diagram, it follows directly that the map \(H_{\Omega_X^1(A), B}\) is an epimorphism and then by definition we have \(\mathcal{O}_H(\Omega_X^1(A), B) = 0\) as desired.

(ii) If \(A \in \mathcal{X} \times [1]\) and \(T(\mathcal{X}, \mathcal{X}[1]) = 0\), then \(\Omega_X^1(A) \in \mathcal{X}\). Hence \(\text{Gh}_{\mathcal{X}}(\Omega_X^1(A), B) = 0 = \mathcal{O}_H(\Omega_X^1(A), B)\) and the map \(\text{Gh}_{\mathcal{X}}(A, B[1]) \to \text{Ext}^1(\text{H}(A), H(B))\) in (2.5) is invertible.

(iii) Under the imposed assumptions, there exists a triangle \(X^1 \to X^0 \to B \to X^1[1]\) where \(X^j \in \mathcal{X}\). Applying \(H\) and using that \(T(\mathcal{X}, \mathcal{X}[i]) = 0, 1 \leq i \leq 2\), it follows that \(H(B[1]) = 0\) in particular any map \(A \to B[1]\) is \(\mathcal{X}\)-ghost. Hence \(\text{Gh}_{\mathcal{X}}(A, B[1]) \cong T(A, B[1])\) and the assertion follows from (ii).

To get an analogous result for the higher extension functors we need the following.

Lemma 2.10. For any objects \(A, B \in \mathcal{T}\) and any \(k \geq 0\), there is an epimorphism:

\[
\varphi_{A,B}^k : \text{Gh}_{\mathcal{X}}(\Omega_X^k(A), B[1]) \to \text{Gh}_{\mathcal{X}}^{k+1}(A, B[k + 1]) \to 0
\]

which is an isomorphism if \(T(\mathcal{X}, B[i]) = 0, 1 \leq i \leq k\).

Proof. Define \(\varphi_{A,B}^k(\alpha) = \omega_A^{k-1} \circ \alpha[k]\). Since \(\omega_A^{k-1}\) lies in \(\text{Gh}_{\mathcal{X}}^k(A, \Omega_X^k(A)[k])\) and since \(\alpha[k]\) is \(\mathcal{X}[k]\)-ghost, it follows that \(\varphi_{A,B}^k(\alpha)\) lies in \(\text{Gh}_{\mathcal{X}}^{k+1}(A, B[k + 1])\). If \(\beta : A \to B[k + 1]\) lies in \(\text{Gh}_{\mathcal{X}}^{k+1}(A, B[k + 1])\), by the Ghost Lemma, there exists a map \(\gamma : \Omega_X^{k+1}(A) \to B[k + 1]\) such that \(\beta = \omega_A^k \circ \gamma = \omega_A^{k-1} \circ h_A^{k}[k] \circ \gamma\). Clearly the map \(h_A^{k} \circ \gamma[-k] : \Omega_X^{k}(A) \to B[1]\) is \(\mathcal{X}\)-ghost and \(\varphi_{A,B}^k(h_A^{k} \circ \gamma[-k]) = \omega_A^{k-1} \circ h_A^{k}[k] \circ \gamma = \omega_A^k \circ \gamma = \beta\), so \(\varphi_{A,B}^k\) is surjective. Assume now that \(T(\mathcal{X}, B[i]) = 0, 1 \leq i \leq k\) and let \(\alpha \in \text{Ker} \varphi_{A,B}^k\). Then \(\omega_A^{k-1} \circ \alpha[k] = 0\) and therefore \(\alpha[k]\) factorizes through the cone \(\text{Cell}_{k-1}(A)[1]\) of \(\omega_A^{k-1}\). It is easy to see that the condition \(T(\mathcal{X}, B[i]) = 0, 1 \leq i \leq k\), forces any map from an object in \(\mathcal{X}[1] \star \cdots \star \mathcal{X}[k]\) to \(B[k + 1]\) to be zero. Since \(\text{Cell}_{k-1}(A)[1]\) lies in \(\mathcal{X}[1] \star \cdots \star \mathcal{X}[k]\), we therefore have that \(\alpha[k]\) or equivalently \(\alpha\) is zero. We infer that \(\varphi_{A,B}^k\) is injective.

Corollary 2.11. Let \(t \geq 2\) and assume that \(T(\mathcal{X}, \mathcal{X}[i]) = 0, -t + 1 \leq i \leq t, i \neq 0\). If \(A \in \mathcal{X} \star \mathcal{X}[1] \star \cdots \star \mathcal{X}[t]\) and \(T(\mathcal{X}, A[-i]) = 0, 1 \leq i \leq t\), then \(\text{Ext}^{t+1}(\text{H}(A), H(-)) = 0\), so \(\text{pd} \text{H}(A) \leq t\). Moreover:
\[
\text{Ext}^t(H(A), H(B)) \xrightarrow{\cong} \text{Gh}_X(\Omega_X^{t-1}(A), B[1]) \quad \& \\
\text{Ext}^{t-1}(H(A), H(B)) \xrightarrow{\cong} \frac{\text{Gh}_X(\Omega_X^{t-2}(A), B[1])}{\text{Gh}_X^2(\Omega_X^{t-2}(A), B[1])}
\]

If in addition \(\mathcal{T}(X, B[i]) = 0\), for \(1 \leq i \leq t - 1\), then: \(\text{Ext}^t(H(A), H(B)) \xrightarrow{\cong} \text{Gh}_X^{i}(A, B[t]).\)

**Proof.** The assumptions on \(A\) imply easily that \(\Omega_X^t(A) \in \mathcal{X}\). Then \(\Omega_X^{t-1}(A) \in \mathcal{X} \ast \mathcal{X}[1]\), and therefore we have \(\mathcal{O}_H(\Omega_X^k(A), -) = 0\), for \(k = t, t - 1\), and \(\text{Gh}_X(\Omega_X^t(A), -) = 0\). It follows from (2.4), for \(i = t\), that \(\text{Ext}^{t+1}(H(A), H(B)) = 0\), \(\forall B \in \mathcal{T}\). Since any object of \(\text{mod-} \mathcal{X}\) lies in the image of \(H\) it follows that \(\text{pd} H(A) \leq t\). Next setting \(i = t - 1\) in (2.4), we have the first isomorphism, and setting \(i = t - 2\) in (2.4) we have the second isomorphism. Finally if \(\mathcal{T}(X, B[i]) = 0\), \(1 \leq i \leq t - 1\), Lemma 2.10 shows that \(\text{Gh}_X(\Omega_X^{t-1}(A), B[1]) \cong \text{Gh}_X^{i}(A, B[t])\) and therefore the last isomorphism follows. \(\square\)

**Remark 2.12.** Assume that \(\mathcal{T}(X, X[1]) = 0\). If \(A \in \mathcal{X} \ast \mathcal{X}[1]\) satisfies \(\mathcal{T}(X, A[-1]) = 0\), then combining Proposition 2.7 and Corollary 2.9(ii), we have for any \(B \in \mathcal{T}\) a “universal coefficient” exact sequence:

\[
0 \to \text{Ext}^1(H(A), H(B[-1])) \to \mathcal{T}(A, B) \to \text{Hom}(H(A), H(B)) \to 0
\]

For instance if \(\mathcal{X}[2] \subseteq \mathcal{X} \ast \mathcal{X}[1]\), then the above exact sequence gives \(\text{Ext}^1(H(X[2]), H(X)) \cong \mathcal{T}(X, X[-1])\). Hence if \(X\) is 1-rigid and \(X[2] \subseteq X \ast X[1]\), then \(\mathcal{T}(X, X[-1]) = 0\) if and only if \(\text{Ext}^1(H(X[2]), H(X)) = 0\).

### 2.3. Algebraic ghosts

We close this section by analyzing ghost maps in algebraic triangulated categories. In this context ghost maps of any finite depth admit a concrete realization as elements of relative extension groups.

Let \(\mathcal{T}\) be an algebraic triangulated category, i.e. \(\mathcal{T} = \mathcal{C}\) is the stable category, modulo the full subcategory \(\mathcal{P}\) of projective–injective objects, of some exact Frobenius category \((\mathcal{C}, \mathcal{E})\), where \(\mathcal{E}\) denotes its exact structure. We denote by \(\pi: \mathcal{C} \to \mathcal{T} = \mathcal{C}\), \(\pi(A) = A\), the projection functor, by \(\Omega(A), \text{resp. } \Omega^{-1}(A)\), the first syzygy, resp. cosyzygy, of an object \(A\) in \(\mathcal{C}\), and by \([1] = \Omega^{-1}\) the suspension functor in \(\mathcal{T}\).

Let \(X\) be a contravariantly finite subcategory of \(\mathcal{T}\). Then setting

\[
\mathcal{M}_n := \pi^{-1}(X \ast X[1] \ast \cdots \ast X[n]), \quad n \geq 0
\]

we obtain a chain of full subcategories of \(\mathcal{C}\)

\[
\mathcal{P} \subseteq \mathcal{M} := \mathcal{M}_0 \subseteq \mathcal{M}_1 \subseteq \mathcal{M}_2 \subseteq \cdots \subseteq \mathcal{M}_n \subseteq \cdots \subseteq \mathcal{C}
\]
Since by Corollary 2.3 each subcategory $\mathcal{X} \star \mathcal{X}[1] \star \cdots \star \mathcal{X}[n]$ is contravariantly finite in $\mathcal{T}$, it is easy to see that $\mathcal{M}_n$ is contravariantly finite in $\mathcal{C}$, $\forall n \geq 0$. As a consequence the subcategories $\mathcal{M}_n$ define new exact structures $\mathcal{E}_n(M)$ on $\mathcal{C}$ by declaring a diagram $A \rightarrow B \rightarrow C$ in $\mathcal{C}$ to be in $\mathcal{E}_n(M)$ if the induced sequence $0 \rightarrow \mathcal{C}(M_n, A) \rightarrow \mathcal{C}(M_n, B) \rightarrow \mathcal{C}(M_n, C) \rightarrow 0$ is exact in $\mathsf{Ab}$. Then the identity functor $\text{Id}_\mathcal{C}$ of $\mathcal{C}$ induces an exact functor $(\mathcal{C}, \mathcal{E}_n(M)) \rightarrow (\mathcal{C}, \mathcal{E})$, $\forall n \geq 0$. The exact category $(\mathcal{C}, \mathcal{E}_n(M))$ has enough projective objects and the projective objects are the objects of the subcategory $\mathcal{M}_n$. We denote by $\text{Ext}^*_\mathcal{C}(A, B)$ and $\text{Ext}^*_\mathcal{M}_n(A, B)$ the extension functors of the exact categories $(\mathcal{C}, \mathcal{E})$ and $(\mathcal{C}, \mathcal{E}_n(M))$ respectively, $\forall n \geq 0$. Note that since the exact category $(\mathcal{C}, \mathcal{E})$ is Frobenius, $\forall A, B \in \mathcal{T}$ we have isomorphisms $\text{Ext}^n_{\mathcal{C}}(A, B) \cong \mathcal{T}(A, B[n]), \forall n \geq 1$. In particular $\text{Ext}^k_{\mathcal{C}}(\mathcal{M}, \mathcal{M}) = 0$, for $1 \leq k \leq n$, if and only if $\mathcal{T}(\mathcal{X}, \mathcal{X}[k]) = 0$, for $1 \leq k \leq n$.

The $\mathcal{M}$-resolution dimension of $A \in \mathcal{C}$, denoted by $\text{res.dim}_\mathcal{M} A$, is defined to be the minimum $n \geq 0$ (if it exists) such that there exists an exact sequence $(\ast) : 0 \rightarrow M_n \rightarrow M_{n-1} \rightarrow \cdots \rightarrow M_0 \rightarrow A \rightarrow 0$ in $\mathcal{C}$, where the $M_i$ lie in $\mathcal{M}$. The $\mathcal{M}$-resolution dimension of $\mathcal{C}$, denoted by $\text{res.dim}_\mathcal{M} \mathcal{C}$, is defined as $\sup\{\text{res.dim}_\mathcal{M} A \mid A \in \mathcal{C}\}$. We also denote by $\text{pd}_\mathcal{M} A$ the projective dimension of $A$ as an object of the exact category $(\mathcal{C}, \mathcal{E}_0(M))$ and by $\text{gl.dim}_\mathcal{M} \mathcal{C} = \sup\{\text{pd}_\mathcal{M} A \mid A \in \mathcal{C}\}$ its global dimension. Clearly $\text{pd}_\mathcal{M} A$ is the minimum $n \geq 0$ (or infinity) such that there exists an exact sequence $(\ast)$ where the $M_i$ lie in $\mathcal{M}$, which remains exact after applying $\mathcal{C}(\mathcal{M}, \ast)$.

The next result collects basic properties of ghost maps in the context of algebraic triangulated categories.

**Proposition 2.13.** With the above notations we have the following.

(i) There are isomorphisms: $\text{Gh}_\mathcal{X}^{[n+1]}(A, B) \cong \text{Ext}^1_{\mathcal{M}_n}(A, \Omega B), \forall n \geq 0$. In particular:

\[ \text{Gh}_\mathcal{X}(A, B) \cong \text{Ext}^1_{\mathcal{M}}(A, \Omega B) \]

(ii) $\mathcal{M}_n = \{ A \in \mathcal{C} \mid \text{res.dim}_\mathcal{M} A \leq n \}$.

(iii) $\text{Gh}_\mathcal{X}^{[n+1]}(\mathcal{T}) = 0$ if and only if $\text{res.dim}_\mathcal{M} \mathcal{C} \leq n$.

(iv) If $\mathcal{T}(\mathcal{X}, \mathcal{X}[k]) = 0$, $1 \leq k \leq n$, then: $\mathcal{M}_k = \{ A \in \mathcal{C} \mid \text{pd}_\mathcal{M} A \leq k \}$, for $0 \leq k \leq n$.

(v) If $\mathcal{T}(\mathcal{X}, \mathcal{X}[k]) = 0$, $1 \leq k \leq n$, then: $\text{Gh}_\mathcal{X}^{[n+1]}(\mathcal{T}) = 0$ if and only if $\text{gl.dim}_\mathcal{M} \mathcal{C} \leq n$.

(vi) For any objects $A, B$ in $\mathcal{C}$, there exists an exact sequence:

\[ 0 \rightarrow \text{Ext}^1_{\mathcal{M}}(A, \Omega B) \rightarrow \mathcal{C}(A, B) \rightarrow \text{Hom}(\mathcal{C}(-, A)|_\mathcal{M}, \mathcal{C}(-, B)|_\mathcal{M}) \rightarrow \text{Ext}^2_{\mathcal{M}}(A, \Omega B) \rightarrow \text{Ext}^1_{\mathcal{M}_1}(A, B) \rightarrow 0 \]

(vii) Assume that $\mathcal{T}(\mathcal{X}, \mathcal{X}[k]) = 0$, $1 \leq k \leq n$, and $\text{Gh}_\mathcal{X}^{[n+1]}(\mathcal{T}) = 0$. If any map between objects of $\mathcal{M}$ admits a kernel in $\mathcal{C}$, then $\mathcal{mod} \cdot \mathcal{M}$ is an abelian category of finite global dimension, in fact:

\[ \text{gl.dim} \mathcal{mod} \cdot \mathcal{M} \leq n + 2 \]
Proof. (i) Let \( h : A \rightarrow B \) be an \( \mathcal{X} \)-ghost map in \( \mathcal{F} = \mathcal{C} \) and let \( \Omega B \rightarrow K \rightarrow A \rightarrow 0 \) be a triangle in \( \mathcal{F} \). By the construction of the triangulation of \( \mathcal{F} \), this triangle is induced by a short exact sequence \( (E_h) : 0 \rightarrow \Omega B \rightarrow K \rightarrow A \rightarrow 0 \) in \( \mathcal{C} \), i.e. an element of \( \mathcal{E} \). This sequence is \( \mathcal{M} \)-exact, i.e. is an element of \( \mathcal{E}_0(\mathcal{M}) \); indeed since \( \mathcal{X} = \pi(\mathcal{M}) \) and \( h \) is \( \mathcal{X} \)-ghost, it follows easily that any map \( M \rightarrow A \), where \( M \in \mathcal{M} \), factorizes through \( K \). Therefore the extension \( (E_h) \) is an element of \( \text{Ext}^1_\mathcal{M}(A, \Omega B) \). We leave to the reader to show that the assignment \( \text{Gh}_\mathcal{X}(A, B) \rightarrow \text{Ext}^1_\mathcal{M}(A, \Omega B) \), \( h \mapsto (E_h) \) is an isomorphism.

The isomorphisms \( \text{Gh}^{[n+1]}_\mathcal{X}(A, B) \cong \text{Ext}^1_\mathcal{M}(A, \Omega B) \), \( \forall n \geq 1 \), are proved similarly using Corollary 2.3.

(ii) We show by induction that \( M_n = \{ A \in \mathcal{C} \mid \text{res.dim}_\mathcal{M} A \leq n \} \), the case \( n = 0 \) being clear. Let \( n = 1 \) and let \( A \in M_1 \), so \( A \) lies in \( \mathcal{X} \times \mathcal{X}[1] \). Then there exists a triangle \( M_1 \rightarrow A \rightarrow \Omega^{-1}M_2 \rightarrow \Omega^{-1}M_1 \) in \( \mathcal{F} \), where the \( M_i \) lie in \( \mathcal{M} \). The last triangle is induced by a short exact sequence \( 0 \rightarrow M_1 \rightarrow A \rightarrow \Omega^{-1}M_2 \rightarrow 0 \) in \( \mathcal{C} \) and we have a short exact sequence \( 0 \rightarrow M_2 \rightarrow P \rightarrow \Omega^{-1}M_2 \rightarrow 0 \), where \( P \) is projective–injective in \( \mathcal{C} \). It is easy to see then that there is induced an exact sequence \( 0 \rightarrow M_2 \rightarrow M_1 \otimes P \rightarrow A \rightarrow 0 \) in \( \mathcal{C} \) and this shows that \( \text{res.dim}_\mathcal{M} A \leq 1 \), since \( P \subseteq \mathcal{M} \). Conversely if \( \text{res.dim}_\mathcal{M} A \leq 1 \) and \( 0 \rightarrow M_2 \rightarrow M_1 \rightarrow A \rightarrow 0 \) is an exact sequence in \( \mathcal{C} \), where the \( M_i \) lie in \( \mathcal{M} \), then the induced triangle \( M_1 \rightarrow A \rightarrow \Omega^{-1}M_2 \rightarrow \Omega^{-1}M_1 \) in \( \mathcal{F} \) shows that \( A \in \mathcal{X} \times \mathcal{X}[1] \) and this means that \( A \) lies in \( \mathcal{M}_1 = \pi^{-1}(\mathcal{X} \times \mathcal{X}[1]) \). We infer that \( \mathcal{M}_1 = \{ A \in \mathcal{C} \mid \text{res.dim}_\mathcal{M} A \leq 1 \} \). The general case follows easily by induction.

(iii) By Proposition 2.2 and part (ii) we have \( \text{Gh}^{[n+1]}_\mathcal{X}(\mathcal{F}) = 0 \) if and only if \( \mathcal{F} = \mathcal{X} \times \mathcal{X}[1] \times \cdots \times \mathcal{X}[n] \) if and only if \( \mathcal{F} = M_n \) if and only if \( \text{res.dim}_\mathcal{M} \mathcal{C} \leq n \).

(iv) Using (ii), we have clearly an inclusion \( \{ A \in \mathcal{C} \mid \text{pd}_\mathcal{M} A \leq k \} \subseteq \mathcal{M}_k \), \( \forall k \geq 0 \). Assume that \( \mathcal{F}(\mathcal{X}, \mathcal{X}[1]) = 0 \) and let \( A \in M_1 \), so there exists an exact sequence \( (E) : 0 \rightarrow M_2 \rightarrow M_1 \rightarrow A \rightarrow 0 \) in \( \mathcal{C} \), where the \( M_i \) lie in \( \mathcal{M} \). Then we have a triangle \( M_2 \rightarrow M_1 \rightarrow A \rightarrow \Omega^{-1}M_2 \) in \( \mathcal{F} \), where the \( M_i \) lie in \( \mathcal{M} \). Since \( \mathcal{F}(\mathcal{X}, \mathcal{X}[1]) = 0 \), any map \( M \rightarrow A \), where \( M \in \mathcal{M} \), factorizes through the map \( M_1 \rightarrow A \). This easily implies that any map \( M \rightarrow A \), where \( M \in \mathcal{M} \), factorizes through the map \( M_1 \rightarrow A \) and therefore the extension \( (E) \) is \( \mathcal{M} \)-exact. This means that \( (E) \) is a projective resolution of \( A \) of length \( \leq 1 \), so \( \text{pd}_\mathcal{M} A \leq 1 \). We infer that \( \{ A \in \mathcal{C} \mid \text{pd}_\mathcal{M} A \leq 1 \} = \mathcal{M}_1 \). The inclusion \( \mathcal{M}_k \subseteq \{ A \in \mathcal{C} \mid \text{pd}_\mathcal{M} A \leq k \} \), for \( 0 \leq k \leq n \), provided that \( \mathcal{F}(\mathcal{X}, \mathcal{X}[k]) = 0 \), \( 1 \leq k \leq n \), follows easily by induction.

(v, vi) Part (v) follows directly from (iv) and Proposition 2.2, and part (vi) follows from Proposition 2.5.

(vii) Since any map between objects of \( \mathcal{M} \) has a kernel in \( \mathcal{C} \) and \( \mathcal{M} \) is contravariantly finite in \( \mathcal{C} \), it follows that \( \mathcal{M} \) has weak kernels: in fact a weak kernel of the map \( f : M_2 \rightarrow M_1 \) in \( \mathcal{M} \) is the composition \( f_A \circ g : M_A \rightarrow M_1 \) of the kernel \( g : A \rightarrow M_1 \) in \( \mathcal{C} \) of \( f \) followed by a right \( \mathcal{M} \)-approximation \( f_A : M_A \rightarrow A \) of \( A \). We infer that \( \text{mod-}M \) is abelian. Let \( A \) be in \( \mathcal{C} \). Then by (iii) and (iv) there exists an exact sequence \( 0 \rightarrow M_n \rightarrow M_{n-1} \rightarrow \cdots \rightarrow M_0 \rightarrow A \rightarrow 0 \) in \( \mathcal{C} \), where the \( M_i \) lie in \( \mathcal{M} \) which remains exact after applying \( \mathcal{C}(\mathcal{M}, -) \). It follows that the sequence \( 0 \rightarrow \mathcal{C}(\mathcal{M}, -) \rightarrow \mathcal{C}(\mathcal{M}, -) \rightarrow \cdots \rightarrow \mathcal{C}(\mathcal{M}, -) \rightarrow 0 \) is a projective resolution.
of $\mathcal{C}(-, A)|_{\mathcal{M}}$ in $\text{mod-}\mathcal{M}$ and therefore $\text{pd} \mathcal{C}(-, A)|_{\mathcal{M}} \leq n$. Finally let $F$ be in $\text{mod-}\mathcal{M}$ and let $\mathcal{C}(-, M^1)|_{\mathcal{M}} \rightarrow \mathcal{C}(-, M^0)|_{\mathcal{M}} \rightarrow F \rightarrow 0$ be a projective presentation of $F$. If $A$ is the kernel of the map $M^1 \rightarrow M^0$ in $\mathcal{C}$, then $\mathcal{C}(-, A)|_{\mathcal{M}} = \Omega^2 F$ and therefore $\text{pd} F \leq n + 2$. Hence $\text{gl.dim mod-}\mathcal{M} \leq n + 2$. □

We leave as an exercise to the reader the interpretation of the results of this and subsequent sections in the setting of algebraic triangulated categories.

3. Rigid and corigid subcategories

Throughout this section we fix as before a triangulated category $\mathcal{T}$ with split idempotents and a contravariantly finite subcategory $\mathcal{X}$ of $\mathcal{T}$ which is closed under direct summands and isomorphisms.

3.1. Rigid subcategories

For $n \geq 1$, we consider the following subcategories naturally associated to $\mathcal{X}$:

$$\mathcal{X}_n^\triangledown := \{ A \in \mathcal{T} \mid \mathcal{T}(A, A[i]) = 0, 1 \leq i \leq n \} \quad \text{and}$$

$$\mathcal{T} \ni \mathcal{X} := \{ A \in \mathcal{T} \mid \mathcal{T}(A, \mathcal{X}[i]) = 0, 1 \leq i \leq n \}$$

We also set: $\mathcal{X}_0^\triangledown := \mathcal{X}$ and $\mathcal{X}_{n^{-}} := \mathcal{T} \mathcal{X}$. Note that we have filtrations:

$$\mathcal{X} \subseteq \mathcal{X} \ast \mathcal{X}[1] \subseteq \cdots \subseteq \mathcal{X} \ast \mathcal{X}[1] \ast \cdots \ast \mathcal{X}[n] \subseteq \cdots \subseteq \mathcal{T}$$

(3.1)

(3.2)

**Definition 3.1.** A full subcategory $\mathcal{X} \subseteq \mathcal{T}$ is called $n$-rigid, $n \geq 1$, if: $\mathcal{T}(\mathcal{X}, \mathcal{X}[i]) = 0$, $1 \leq i \leq n$.

It follows that $\mathcal{X}$ is $n$-rigid if and only if $\mathcal{X} \subseteq \mathcal{X}_n^\triangledown$ or equivalently $\mathcal{X} \subseteq \mathcal{X}_{n^{-}}$. Recall from [44] that a pair $(\mathcal{X}, \mathcal{Y})$ of full subcategories of $\mathcal{T}$ is called a torsion pair in $\mathcal{T}$, if $\mathcal{T}(\mathcal{X}, \mathcal{Y}) = 0$ and for any object $A$ in $\mathcal{T}$ there is a triangle $X_A \rightarrow A \rightarrow Y^A \rightarrow X_A[1]$, where $X_A \in \mathcal{X}$ and $Y^A \in \mathcal{Y}$. It follows in particular that $\mathcal{T} = \mathcal{X} \ast \mathcal{Y}$ for any torsion pair $(\mathcal{X}, \mathcal{Y})$ in $\mathcal{T}$. We call a triple $(\mathcal{X}, \mathcal{Y}, \mathcal{Z})$ of subcategories of $\mathcal{T}$ a torsion triple if $(\mathcal{X}, \mathcal{Y})$ and $(\mathcal{Y}, \mathcal{Z})$ are torsion pairs. If $(\mathcal{X}, \mathcal{Y})$ is a torsion pair in $\mathcal{T}$, then clearly the map $X_A \rightarrow A$ is a right $\mathcal{X}$-approximation of $A$, and the map $A \rightarrow Y^A$ is a left $\mathcal{Y}$-approximation of $A$. Hence $\mathcal{X}$ is contravariantly finite and $\mathcal{Y}$ is covariantly finite in $\mathcal{T}$. Moreover it is easy to see that $\mathcal{X}^\triangledown = \mathcal{Y}$ and $\mathcal{T} \mathcal{Y} = \mathcal{X}$.

**Throughout this subsection:** $\mathcal{X}$ denotes a contravariantly finite $n$-rigid subcategory of $\mathcal{T}$, $n \geq 1$. We begin with the following result which will be useful in the rest of the paper.
Proposition 3.2. For any object $A$ in $\mathcal{T}$, we have the following:

(i) $\Omega_X^T(A) \in X^T_t$, $t \geq 1$, and $\Omega_X^T(A) \in X^T_1$, $1 \leq t \leq n$. Moreover:

$$H(X[1] \star X[2] \star \cdots \star X[k]) = 0 = H^\op(X[-1] \star X[-2] \star \cdots \star X[-k]), \quad 1 \leq k \leq n$$

(ii) There is a torsion pair

$$(X \star X[1] \star X[2] \star \cdots \star X[t-1], X^T_t), \quad 1 \leq t \leq n \quad (\ast)$$

The map $\gamma_A^{t-1} : \text{Cell}_{t-1}(A) \to A$ is a right $(X \star X[1] \star \cdots \star X[t-1])$-approximation of $A$, the map $H(\gamma_A^{t-1})$ is invertible, and the map $\omega_A^{t-1} : A \to \Omega_X^T(A)[t]$ is a left $X^T_t$-approximation of $A$.

(iii) $\forall t = 1, 2, \cdots, n$: $\Omega_X^T(A) \in X$ if and only if $A \in X \star X[1] \star \cdots \star X[t]$.

Proof. (i) Applying the homological functor $H : \mathcal{T} \to \text{mod-}X$ to the triangles

$$\Omega_X^T(A) \to X^{t-1}_A \to \Omega_X^{t-1}(A) \to \Omega_X^T(A)[1] \quad (T_A^t)$$

and using that $X$ is $n$-rigid, it follows that $H(\Omega_X^T(A)[1]) = 0$, $\forall t \geq 1$, and $H(\Omega_X^T(A)[i]) \cong H(\Omega_X^{t-1}(A)[i+1])$, for $1 \leq i, t \leq n - 1$. Hence for any $t = 1, 2, \cdots, n$, we have:

$H(\Omega_X^T(A)[t]) \cong H(\Omega_X^{t-1}(A)[t-1]) \cong \cdots \cong H(\Omega_X^2(A)[2]) \cong H(\Omega_X^1(A)[1]) = 0.$

It follows that $\Omega_X^T(A) \in X^T_t$, $1 \leq t \leq n$, and $\Omega_X^T(A) \in X^T_1$, $\forall t \geq 1$.

We show by induction that $H(X[1] \star X[2] \star \cdots \star X[k]) = 0$, or equivalently $\mathcal{T}(X, X[1] \star X[2] \star \cdots \star X[k]) = 0$, for $1 \leq k \leq n$. Since $X$ is $1$-rigid we have $\mathcal{T}(X, X[1]) = 0$. Assume that $\mathcal{T}(X, X[1] \star X[2] \star \cdots \star X[k-1]) = 0$ and let $A \in X[1] \star X[2] \star \cdots \star X[k]$. Then there exists a triangle $B \to A \to X'[k] \to B[1]$, where $B \in X[1] \star X[2] \star \cdots \star X[k-1]$ and $X' \in X$. Since $X$ is $n$-rigid and $k \leq n$, we have $\mathcal{T}(X, X[k]) = 0$, hence for any object $X \in X$ and any map $f : X \to A$, the composition $X \to A \to X'[k]$ is zero and therefore it factorizes through $B$, say via a map $g : X \to B$. Then by hypothesis $g = 0$ and therefore $f = 0$. We infer that any map from an object of $X$ to an object in $X[1] \star X[2] \cdots \star X[k]$ is zero, so $H(X[1] \star X[2] \star \cdots \star X[k]) = 0$. The proof that $H^\op(X[-1] \star X[-2] \star \cdots \star X[-k]) = 0$ is similar.

(ii) Since for any object $A \in \mathcal{T}$ we have a triangle

$$\text{Cell}_{t-1}(A) \xrightarrow{\gamma_A^{t-1}} A \xrightarrow{\omega_A^{t-1}} \Omega_X^T(A)[t] \xrightarrow{-\beta_A^{-1}} \text{Cell}_{t-1}(A)[1] \quad (C_A^t)$$

where $\text{Cell}_{t-1}(A) \in X \star X[1] \star \cdots \star X[t-1]$ and $\Omega_X^T(A)[t] \in X^T_t$, it suffices to show that $\mathcal{T}(A, B) = 0$, for any object $A$ in $X \star X[1] \star \cdots \star X[t-1]$ and any object $B \in X^T_t$. Equivalently we show by induction that any map $f : A \to B$ is zero provided that $A$ lies in $X[1] \star X[2] \star \cdots \star X[t]$ and $B$ lies in $X^T_t[t+1]$, i.e. $\mathcal{T}(X, B[-i]) = 0$, $1 \leq i \leq t$. If $t = 1$, then $A = X[1]$ for some $X \in X$ and $B$ satisfies $\mathcal{T}(X, B[-1]) = 0$. Hence $\mathcal{T}(A, B) = 0$. Assume now that $A \in X[1] \star X[2] \star \cdots \star X[t]$ and $B \in X^T_t[t+1]$. Then there is a triangle
\( A' \xrightarrow{g} A \xrightarrow{h} A'' \longrightarrow A'[1] \), where \( A' \in \mathcal{X}[1] \star \mathcal{X}[2] \star \cdots \star \mathcal{X}[t-1] \) and \( A'' \in \mathcal{X}[t] \), i.e. \( A'' = \mathcal{X}[t] \) for some \( X \in \mathcal{X} \). Since clearly \( B \in \mathcal{X}_{t-1}[t] \), by the induction hypothesis, the composition \( g \circ f = 0 \) and therefore \( f = h \circ \phi \) for some map \( \phi : A'' \longrightarrow B \). Since \( A'' = \mathcal{X}[t] \) and \( \mathcal{J}(\mathcal{X}, B[-t]) = 0 \), we have \( \mathcal{J}(A'', B) = 0 \). Hence \( \phi = 0 \) and therefore \( f = 0 \). We infer that \((*)\) is a torsion pair in \( \mathcal{J} \). Finally using (i) and the construction of the tower of triangles \((C^n_A)\) associated to \( A \), we see directly that \( H(\gamma^n_A) \) is invertible, \( 1 \leq t \leq n-1 \).

(iii) By Remark 1.4(ii), it follows that \( A \in \mathcal{X} \star \mathcal{X}[1] \star \cdots \star \mathcal{X}[t] \). Conversely assume that \( A \in \mathcal{X} \star \mathcal{X}[1] \star \cdots \star \mathcal{X}[t] \). Then the right \((\mathcal{X} \star \mathcal{X}[1] \star \cdots \star \mathcal{X}[t])\)-approximation \( \gamma^t_A : \text{Cell}_t(A) \longrightarrow A \) splits and therefore from the tower of triangles \((C^n_A)\) we have \( \omega^n_A = \omega^{t-1}_A \circ h^n_A[t] = 0 \). Hence the map \( h^n_A[t] : \Omega^t_X(A)[t] \longrightarrow \Omega^{t+1}_X(A)[t+1] \) factorizes through the cone \( \text{Cell}_{t-1}(A)[1] \) of \( \omega^{t-1}_A \), say as \( h^n_A[t] = \beta^{t-1}_A \circ f \), for some map \( f : \text{Cell}_{t-1}(A) \longrightarrow \Omega^{t+1}_X(A)[t+1] \). We show that \( f = 0 \). Observe first that \( \Omega^{t+1}_X(A) \in \mathcal{X}^t \). Indeed this follows form (i) if \( t \leq n-1 \) and by applying \( H \) to the triangle \( \Omega^{n+1}_X(A) \longrightarrow X^n_A \longrightarrow \Omega^n_X(A) \longrightarrow \Omega^{n+1}_X(A)[1] \) and using that \( \Omega^n_X(A) \in X^n \), if \( t = n \). Since \( \text{Cell}_{t-1}(A) \) lies in \( X[1] \star \cdots \star X[t] \) and \( \Omega^{n+1}_X(A) \in \mathcal{X}^t \), by (ii) it follows that \( f \) is zero and therefore \( h^n_A[t] = 0 \). This implies that \( \Omega^n_X(A)[t] \) is a direct summand of the left cone \( X^n_A[t] \) of \( h^n_A[t] \), see Remark 1.4(i). We infer that \( \Omega^n_X(A) \) lies in \( X \) as a direct summand of \( X^n_A \).

Combining Remark 1.4, the Ghost Lemma (Proposition 2.2) and Proposition 3.2 we have the following.

**Corollary 3.3.** For any object \( A \in \mathcal{J} \) and \( 1 \leq t \leq n \), the following are equivalent:

(i) \( \Omega^n_X(A) \in \mathcal{X} \).

(ii) \( A \in \mathcal{X} \star \mathcal{X}[1] \star \cdots \star \mathcal{X}[t] \).

(iii) \( \text{Gh}^t_X(A, -) = 0 \).

Moreover the maps \( \text{Cell}_{t-1}(A) \xrightarrow{\alpha^t_A} \text{Cell}_t(A) \xrightarrow{\gamma^n_A} A \) in the cellular tower of \( A \) induce isomorphisms:

\[
H(A) \xrightarrow{\cong} H(\text{Cell}_1(A)) \xrightarrow{\cong} H(\text{Cell}_2(A)) \xrightarrow{\cong} \cdots \xrightarrow{\cong} H(\text{Cell}_n(A))
\]

and exact sequences:

\[
\begin{cases}
H(X^n_A) \longrightarrow H(\text{Cell}_0(A)) \longrightarrow H(\text{Cell}_1(A)) \longrightarrow 0,
0 \longrightarrow H(\text{Cell}_n(A)) \longrightarrow H(\text{Cell}_{n+1}(A)) \longrightarrow H(X^{n+1}_A[n+1]).
\end{cases}
\]

The following main result of this section characterizes when the filtrations (2.2), (3.1) and (3.2) stabilize:
Theorem 3.4. If $\mathcal{X}$ is a contravariantly finite n-rigid subcategory of $\mathcal{T}$, then the following are equivalent:

(i) $\mathcal{X}^\top_n = \mathcal{X}$.
(ii) $\Omega^0_n(A) \in \mathcal{X}$, $\forall A \in \mathcal{T}$.
(iii) $\text{Gh}^{(n+1)}_\mathcal{X}(\mathcal{T}) = 0$.
(iv) $\mathcal{T} = \mathcal{X} \star \mathcal{X}[1] \star \cdots \star \mathcal{X}[n]$.
(v) $\mathcal{X}^\top = \mathcal{X}[1] \star \mathcal{X}[2] \star \cdots \star \mathcal{X}[n]$.
(vi) For any $1 \leq t \leq n$: $\mathcal{X}^\top_t[t] = \mathcal{X}[t] \star \mathcal{X}[t+1] \star \cdots \star \mathcal{X}[n]$. 

If (i) holds, then $\mathcal{X}$ and $\mathcal{X}^\top$ are functorially finite and $\mathcal{X}^\top_{n+k} = 0$, $\forall k \geq 1$.

Proof. Since by Proposition 3.2, $\Omega^0_n(A)[n]$ lies in $\mathcal{X}^\top_n[n]$, condition (i) implies that $\Omega^0_n(A) \in \mathcal{X}$, $\forall A \in \mathcal{T}$. Hence (i) $\Rightarrow$ (ii), and by Corollary 3.3 we have (ii) $\Leftrightarrow$ (iii) $\Leftrightarrow$ (iv). We show that (iv) $\Rightarrow$ (i). By Proposition 3.2 we know that, for any object $A \in \mathcal{T}$, the map $\omega^{n-1}_A : A \rightarrow \Omega^0_n(A)[n]$ is a left $\mathcal{X}^\top[n]$-approximation of $A$ and $\Omega^0_n(A)[n]$ lies in $\mathcal{X}[n]$ since $\Omega^0_n(A) \in \mathcal{X}$ by Corollary 3.3. Hence if $A \in \mathcal{X}^\top_n[n]$, the map $\omega^{n-1}_A$ is split monic and therefore $A$ lies in $\mathcal{X}[n]$ as a direct summand of $\Omega^0_n(A)[n]$. We infer that $\mathcal{X}^\top_n[n] \subseteq \mathcal{X}[n]$, or equivalently $\mathcal{X}^\top_n \subseteq \mathcal{X}$. Since $\mathcal{X}$ is n-rigid, we have $\mathcal{X} \subseteq \mathcal{X}^\top$, hence $\mathcal{X}^\top = \mathcal{X}$, i.e. (iv) $\Rightarrow$ (i).

By Proposition 3.2, for $t = 0$, we have a torsion pair $(\mathcal{X}, \mathcal{X}^\top)$, hence $\mathcal{T} = \mathcal{X} \star \mathcal{X}^\top$. Clearly then (v) implies (iv). Conversely if (iv) holds, then by (ii) we have $\Omega^0_n(A) \in \mathcal{X}$, $\forall A \in \mathcal{T}$. This implies that $\Omega^0_{n-1}(A) \in \mathcal{X} \star \mathcal{X}[1]$ and inductively $\Omega^0_1(A) \in \mathcal{X} \star \mathcal{X}[1] \star \cdots \star \mathcal{X}[n-1]$. Hence $\Omega^0_{n}(A)[1] \in \mathcal{X}[1] \star \mathcal{X}[2] \star \cdots \star \mathcal{X}[n]$. If $A$ lies in $\mathcal{X}^\top$, then the map $\mathcal{X}^\top_0 \rightarrow A$ is zero and therefore $A$ lies in $\mathcal{X}[1] \star \cdots \star \mathcal{X}[n]$ as a direct summand of $\Omega^0_1(A)[1]$. Hence $\mathcal{X}^\top \subseteq \mathcal{X}[1] \star \cdots \star \mathcal{X}[n]$. Since by Proposition 3.2(ii) we have $\mathcal{X}[1] \star \cdots \star \mathcal{X}[n] \subseteq \mathcal{X}^\top$, we infer that $\mathcal{X}^\top = \mathcal{X}[1] \star \cdots \star \mathcal{X}[n]$. Taking $t = n$ in (vi) we get (i). Conversely assume that (i) holds and let $1 \leq t \leq n$. If $A$ lies in $\mathcal{X}[X][1] \star \cdots \star \mathcal{X}[n-t]$, then using n-rigidity of $\mathcal{X}$ it follows easily that $A \in \mathcal{X}^\top_t$. This implies that $\mathcal{X} \star \mathcal{X}[1] \star \cdots \star \mathcal{X}[n-t] \subseteq \mathcal{X}^\top_t$ and therefore $\mathcal{X}[t] \star \mathcal{X}[t+1] \star \cdots \star \mathcal{X}[n] \subseteq \mathcal{X}^\top_t$. If $A$ lies in $\mathcal{X}^\top_t[t]$, then the left $\mathcal{X}^\top_t[t]$-approximation $\omega^{t-1}_A : A \rightarrow \Omega^0_t(A)[t]$ splits. Since, by (ii), $\Omega^0_t(A)$ lies in $\mathcal{X}$, by Remark 1.4(iii), $\Omega^0_t(A)[t]$ lies in $\mathcal{X}[t] \star \mathcal{X}[t+1] \star \cdots \star \mathcal{X}[n]$. Hence $A$ lies in $\mathcal{X}[t] \star \mathcal{X}[t+1] \star \cdots \star \mathcal{X}[n]$ as a direct summand of $\Omega^0_t(A)[t]$. We infer that $\mathcal{X}^\top_t[t] \subseteq \mathcal{X}[t] \star \mathcal{X}[t+1] \star \cdots \star \mathcal{X}[n]$. This concludes the proof of the equivalence of conditions (i)–(vi).

If (i) holds, and $A \in \mathcal{X}^\top_{n+1}$, then clearly $A \in \mathcal{X}^\top_n = \mathcal{X}$, so $A[1] \in \mathcal{X}[1]$, and $A[1] \in \mathcal{X}^\top_n = \mathcal{X}$. It follows that $\mathcal{A}[1] \in \mathcal{X} \cap \mathcal{X}[1] = 0$, hence $A = 0$. Therefore $\mathcal{X}^\top_{n+1} = 0$ and then $\mathcal{X}^\top_{n+k} = 0$, $\forall k \geq 1$. From Proposition 3.2(ii), for $t = 0$, it follows that the subcategory $\mathcal{X}^\top$ is covariantly finite, and, for $t = n$, the subcategory $\mathcal{X} \star \mathcal{X}[1] \star \cdots \star \mathcal{X}[n-1]$ is contravariantly finite. This clearly implies that $(\mathcal{X} \star \mathcal{X}[1] \star \cdots \star \mathcal{X}[n-1])[1] = \mathcal{X}[1] \star \mathcal{X}[2] \star \cdots \star \mathcal{X}[n] = \mathcal{X}^\top$ is contravariantly finite. Since, by Proposition 3.2, $\mathcal{X}^\top_n[n]$ is covariantly finite, condition (i) implies that $\mathcal{X}[n]$ or equivalently $\mathcal{X}$, is covariantly finite. We infer that $\mathcal{X}$ is functorially finite.
Remark 3.5. By duality, if $\mathcal{X}$ is a covariantly finite subcategory of $\mathcal{T}$, $n \geq 1$, such that $\mathcal{T} \subset \mathcal{X}$, then $\mathcal{T} = \mathcal{X}[-n] \star \mathcal{X}[-n+1] \star \cdots \star \mathcal{X}[-1] = \text{Ker} H^0$ is functorially finite. We shall show later that: $\mathcal{X}$ is contravariantly finite and $\mathcal{X}^\perp_n = \mathcal{X}$ if and only if $\mathcal{X}$ is covariantly finite and $\mathcal{T} \subset \mathcal{X}$, see Proposition 5.1.

We close this subsection with the following vanishing results which will be useful later.

Corollary 3.6. Let $\mathcal{X}$ be a contravariantly finite $n$-rigid subcategory of $\mathcal{T}$, and $A \in \mathcal{X} \star \mathcal{X}[1] \star \cdots \star \mathcal{X}[k]$.

(i) If $k = n$, then $\omega_A^t = 0$, $\forall t \geq n$, there is a decomposition: $\text{Cell}_t(A) \cong A \oplus \Omega^{t+1}_\mathcal{X}(A)[t]$, $\forall t \geq n$, and:

$$\mathcal{T}(\Omega^{t}_\mathcal{X}(A), \mathcal{X}[n - t + 1]) = 0, \quad 1 \leq t \leq n$$

(ii) If $0 \leq k \leq n$, then for any object $B \in \mathcal{T}$ such that $\mathcal{T}(\mathcal{X}, B[-i]) = 0$, $1 \leq i \leq k - 1$, we have:

$$\text{Gh}_\mathcal{X}(\Omega^1_\mathcal{X}(A), B) = 0$$

and the map $H_{A,B} : \mathcal{T}(A, B) \rightarrow \text{Hom}(H(A), H(B))$, $f \mapsto H(f)$, is surjective.

Proof. (i) Since $A \in \mathcal{X} \star \mathcal{X}[1] \star \cdots \star \mathcal{X}[n]$, we have $\Omega^n_\mathcal{X}(A) \in \mathcal{X}$ by Corollary 3.3, and then $\Omega^t_\mathcal{X}(A)$ lies in $\mathcal{X} \star \mathcal{X}[1] \star \cdots \star \mathcal{X}[n - t]$. On the other hand, since $\mathcal{X} \subset \mathcal{X}^\perp_{n-t+1}$, we have $\mathcal{X}[n - t + 1] \subset \mathcal{X}^\perp_{n-t+1}[n - t + 1]$. Then the torsion pair $(\mathcal{X} \star \mathcal{X}[1] \star \cdots \star \mathcal{X}[n - t], \mathcal{X}^\perp_{n-t+1}[n - t + 1])$ in $\mathcal{T}$, see Proposition 3.2(ii), shows that $\mathcal{T}(\Omega^t_\mathcal{X}(A), \mathcal{X}[n - t + 1]) = 0$, $1 \leq t \leq n$. Finally note that the right $(\mathcal{X} \star \mathcal{X}[1] \star \cdots \star \mathcal{X}[n])$-approximation $\gamma^n_A$ of $A$ splits and therefore $\omega_A^t = 0$. Then from the tower of triangles $(C^n_A)$, we have $\omega_A^t = 0$, $\forall t \geq n$.

(ii) Since $A \in \mathcal{X} \star \mathcal{X}[1] \star \cdots \star \mathcal{X}[k]$, $k \leq n$, it follows that $\Omega^k_\mathcal{X}(A) \in \mathcal{X}$ and then by induction $\Omega^k_\mathcal{X}(A) \in \mathcal{X} \star \mathcal{X}[1] \star \cdots \star \mathcal{X}[k-1]$. Hence there is a triangle $X \rightarrow \Omega^k_\mathcal{X}(A) \rightarrow C \rightarrow \mathcal{X}[1]$, where $X \in \mathcal{X}$ and $C \in \mathcal{X}[1] \star \cdots \star \mathcal{X}[k-1]$. If $\alpha : \Omega^k_\mathcal{X}(A) \rightarrow B$ is $\mathcal{X}$-ghost, then the composition $X \rightarrow \Omega^k_\mathcal{X}(A) \rightarrow B$ is zero, hence $\Omega^k_\mathcal{X}(A) \rightarrow B$ factorizes through $\Omega^k_\mathcal{X}(A) \rightarrow C$. Since any map from an object from $\mathcal{X}[1] \star \cdots \star \mathcal{X}[k-1]$ to an object $B$ satisfying $\mathcal{T}(\mathcal{X}, B[-i]) = 0$, $1 \leq i \leq k - 1$, is clearly zero, it follows that $\mathcal{T}(C, B) = 0$. This implies that the map $\Omega^k_\mathcal{X}(A) \rightarrow B$ is zero and consequently $\text{Gh}_\mathcal{X}(\Omega^k_\mathcal{X}(A), B) = 0$. \hfill \Box

3.2. Homological dimensions

Theorem 3.4 naturally suggests to introduce the following homological invariants of objects of $\mathcal{T}$ with respect to a contravariantly finite subcategory $\mathcal{X}$ of $\mathcal{T}$. First recall that by Remark 1.2 we have additive functors $\Omega^*_\mathcal{X} : \mathcal{T}/\mathcal{X} \rightarrow \mathcal{T}/\mathcal{X}$, $\forall n \geq 0$, showing in particular that the homological dimensions $\text{pd}_\mathcal{X} A$ and $\text{gl.dim}_\mathcal{X} \mathcal{T}$ introduced below are well-defined.
Definition 3.7. The $\mathcal{X}$-projective dimension $\text{pd}_\mathcal{X} A$ of $A \in \mathcal{T}$ is the smallest $n \geq 0$ such that $\Omega^n_{\mathcal{X}}(A) = 0$ in $\mathcal{T}/\mathcal{X}$ or equivalently $\Omega^n_{\mathcal{X}}(A) \in \mathcal{X}$. If $\Omega^n_{\mathcal{X}}(A) \neq 0$, i.e. $\Omega^n_{\mathcal{X}}(A) \notin \mathcal{X}$, $\forall n \geq 0$, then we set $\text{pd}_\mathcal{X} A = \infty$.

The $\mathcal{X}$-global dimension of $\mathcal{T}$ is defined by: $\text{gl.dim}_\mathcal{X} \mathcal{T} = \sup\{\text{pd}_\mathcal{X} A \mid A \in \mathcal{T}\}$.

Dually if $\mathcal{X}$ is covariantly finite, the $\mathcal{X}$-injective dimension $\text{id}_\mathcal{X} A$ is defined by using the functors $\Sigma^n_{\mathcal{X}} : \mathcal{T}/\mathcal{X} \rightarrow \mathcal{T}/\mathcal{X}$, $\forall n \geq 0$. If $\mathcal{X}$ is functorially finite in $\mathcal{T}$, then the adjoint pair $(\Sigma^n_{\mathcal{X}}, \Omega^n_{\mathcal{X}})$ in $\mathcal{T}/\mathcal{X}$, $\forall n \geq 0$, shows that

$$\sup\{\text{pd}_\mathcal{X} A \mid A \in \mathcal{T}\} = \text{gl.dim}_\mathcal{X} \mathcal{T} = \sup\{\text{id}_\mathcal{X} A \mid A \in \mathcal{T}\}$$

Recall from [16] that the triangulated category $\mathcal{T}$ is called $\mathcal{X}$-Frobenius, for a functorially finite subcategory $\mathcal{X}$ of $\mathcal{T}$, if $\text{Gh}_\mathcal{X}(-, \mathcal{X}[1]) = \text{CoGh}_\mathcal{X}(\mathcal{X}[-1], -)$. It is shown in [16, Proposition 3.1] that $\mathcal{T}$ is $\mathcal{X}$-Frobenius if and only if the stable category $\mathcal{T}/\mathcal{X}$ carries a natural triangulated structure with suspension functor $\Sigma^n_{\mathcal{X}}$.

Lemma 3.8. Let $A$ be an object of $\mathcal{T}$.

(i) If $\text{pd}_\mathcal{X} A = m < \infty$, then $A \in \mathcal{X} \ast \mathcal{X}[1] \ast \cdots \ast \mathcal{X}[m]$.

(ii) If $\mathcal{T}$ is $\mathcal{X}$-Frobenius, then for any object $A \in \mathcal{T}$: $\text{pd}_\mathcal{X} A = 0$ or $\infty$, and $\text{id}_\mathcal{X} A = 0$ or $\infty$. In particular if $\mathcal{X} \neq \mathcal{T}$, then $\text{gl.dim}_\mathcal{X} \mathcal{T} = \infty$.

(iii) If $\mathcal{X}$ is $n$-rigid, then for any $m \leq n$ we have: $\text{pd}_\mathcal{X} A \leq m$ if and only if $A \in \mathcal{X} \ast \mathcal{X}[1] \ast \cdots \ast \mathcal{X}[m]$.

(iv) If $\mathcal{X}$ is $n$-rigid, and $\text{pd}_\mathcal{X} A = m < n$, then $A \in \mathcal{X}^\ast_{n-m}$, i.e. $\mathcal{T}(\mathcal{X}, A[i]) = 0$, $1 \leq i \leq n - m$.

Proof. (i), (ii), (iii) Part (i) follows from Proposition 2.2 and part (iii) follows from Proposition 3.2. Since $\mathcal{T}$ is $\mathcal{X}$-Frobenius, part (ii) follows directly from the fact that $\Omega^n_{\mathcal{X}}$ is an equivalence with quasi-inverse $\Sigma^n_{\mathcal{X}}$, see [16, Proposition 3.1]. For part (iv), let $\text{pd}_\mathcal{X} A = m$. Then $A \in \mathcal{X} \ast \mathcal{X}[1] \ast \cdots \ast \mathcal{X}[m]$, hence there exists a triangle $X_0 \rightarrow A \rightarrow B \rightarrow X_0[1]$, where $X_0 \in \mathcal{X}$ and $B \in \mathcal{X}[1] \ast \mathcal{X}[2] \ast \cdots \ast \mathcal{X}[m]$. It follows that $B[i] \in \mathcal{X}[i+1] \ast \cdots \ast \mathcal{X}[i+m]$ and this implies that for $1 \leq i \leq n - m$ we have $\mathcal{T}(\mathcal{X}, B[i]) = 0$ since $\mathcal{X}$ is $n$-rigid and $m < n$. Then applying $\mathcal{T}(\mathcal{X}, -)$ to the above triangle we see directly that $\mathcal{T}(\mathcal{X}, A[i]) = 0$, $1 \leq i \leq n - m$. $\square$

If $\mathcal{X}$ is $n$-rigid, then by Lemma 3.8 we have: $\text{gl.dim}_\mathcal{X} \mathcal{T} \leq m$ if and only if $\Omega^n_{\mathcal{X}}(A) \in \mathcal{X}$, $\forall A \in \mathcal{T}$, if and only if $\text{Gh}_{\mathcal{X}}(m+1)(A, -) = 0$, $\forall A \in \mathcal{T}$, if and only if $\mathcal{T} = \mathcal{X} \ast \mathcal{X}[1] \ast \cdots \ast \mathcal{X}[m]$. The following result shows that $n$ is a lower bound for the global dimension of $\mathcal{T}$ with respect to contravariantly finite $n$-rigid subcategories.
Corollary 3.9. Let $\mathcal{I}$ be a non-trivial triangulated category and $\mathcal{X}$ a contravariantly finite $n$-rigid subcategory of $\mathcal{I}$, where $n \geq 1$. Then $\text{gl.dim}_{\mathcal{X}} \mathcal{I} \geq n$. Moreover:

$$\text{gl.dim}_{\mathcal{X}} \mathcal{I} = n \iff \mathcal{I} = \mathcal{X} \star \mathcal{X}[1] \star \cdots \star \mathcal{X}[n]$$

**Proof.** Assume $\text{gl.dim}_{\mathcal{X}} \mathcal{I} = m < \infty$. If $m < n$, then $\mathcal{I} = \mathcal{X} \star \mathcal{X}[1] \star \cdots \star \mathcal{X}[m]$ by Lemma 3.8. Since $\mathcal{X}$ is $m$-rigid, by Theorem 3.4 we have $\mathcal{X}_{m+1}^\top = 0$. Since $\text{pd}_{\mathcal{X}} A[t] \leq m$, $\forall t \in \mathbb{Z}$, $\forall A \in \mathcal{I}$, it follows by Lemma 3.8 that $A[t] \in \mathcal{X}_{n-m}^\top$, in particular $\mathcal{I}(\mathcal{X}, A[t+n-m]) = 0$, $\forall t \in \mathbb{Z}$. Choosing successively $t = 2m-n+1, 2m-n, \cdots, m-n+1$, we have $\mathcal{I}(\mathcal{X}, A[1]) = \cdots = \mathcal{I}(\mathcal{X}, A[m+1]) = 0$. Then $A \in \mathcal{X}_{m+1}^\top = 0$, hence $A = 0$. Since $\mathcal{I} \neq 0$, we infer that $\text{gl.dim}_{\mathcal{X}} \mathcal{I} \geq n$. Finally by Lemma 3.8 we have $\text{gl.dim}_{\mathcal{X}} \mathcal{I} = n$ if and only if $\mathcal{I} = \mathcal{X} \star \mathcal{X}[1] \star \cdots \star \mathcal{X}[n]$. \(\square\)

3.3. Corigid subcategories

Let $\mathcal{I}$ be a triangulated category and $\mathcal{X}$ a full subcategory of $\mathcal{I}$. It is easy to see that for any $t \geq 1$:

$$\mathcal{X}_t^\top[t+1] = \{ A \in \mathcal{I} \mid \mathcal{I}(\mathcal{X}, A[-k]) = 0, \ 1 \leq k \leq t \}$$

and we have the following chain of full subcategories of $\mathcal{I}$:

$$\cdots \subseteq \mathcal{X}_n^\top[n+1] \subseteq \mathcal{X}_{n-1}^\top[n] \subseteq \cdots \cdots \subseteq \mathcal{X}_2^\top[3] \subseteq \mathcal{X}_1^\top[2] \subseteq \mathcal{X}_0^\top[1] \quad (\ast)$$

In the sequel we shall need the following notion of corigidity.

**Definition 3.10.** A full subcategory $\mathcal{X}$ of $\mathcal{I}$ is called $t$-corigid, where $t \geq 1$, if:

$$\mathcal{X} \subseteq \mathcal{X}_t^\top[t+1], \quad \text{i.e.} \quad \mathcal{I}(\mathcal{X}, \mathcal{X}[-1]) = \cdots = \mathcal{I}(\mathcal{X}, \mathcal{X}[-t]) = 0$$

A $t$-corigid subcategory $\mathcal{X}$ is called strictly $t$-corigid, if $\mathcal{X}$ is not $(t+1)$-corigid.

Clearly if $\mathcal{X}$ is $s$-corigid, then $\mathcal{X}$ is $t$-corigid for any $t \leq s$. The next two results give convenient characterizations of when rigid subcategories are corigid.

**Proposition 3.11.** Let $\mathcal{X}$ be a contravariantly finite $n$-rigid subcategory of $\mathcal{I}$, where $n \geq 2$. Then for $1 \leq k \leq n-1$, the following statements are equivalent:

(i) $\mathcal{X}$ is $(n-k)$-corigid.

(ii) $\mathcal{O}_H(\mathcal{X}[-i], -) = 0, \ 1 \leq i \leq n-k$.

In particular if the functor $H : \mathcal{I} \to \text{mod-}\mathcal{X}$ is full, then $\mathcal{X}$ is $(n-k)$-corigid, for any $k$ with $1 \leq k \leq n-1$. 

Proof. If $\mathcal{X}$ is $(n-k)$-corigid, then we have $H(\mathcal{X}[-i]) = \mathcal{T}(\mathcal{X}, \mathcal{X}[-i]) = 0$ and this clearly implies that $\mathcal{O}_H(\mathcal{X}[-i], \mathcal{X}) = 0$, for $1 \leq i \leq n-k$. Conversely if this holds, then by Corollary 2.6, we have $\mathcal{X}[-i] \subseteq (\mathcal{X} \oplus \mathbb{T})$. Assume that there exists $X \in \mathcal{X}$ such that $X[-i]$ lies in $\mathcal{X} \oplus \mathbb{T}$ for some $1 \leq i \leq n-k$. Then there exists a triangle $X^0 \to X[-i] \to X^1[i] \to X^0[i]$, and therefore a triangle $X^0[i] \to X \to X^1[i+1] \to X^0[i+1]$ in $\mathcal{T}$. Since $\mathcal{X}$ is $n$-rigid and $1 \leq i \leq n-1$, applying the functor $H$ to the last triangle, we see directly that $H(X) = 0$, i.e. $X = 0$. We infer that $\mathcal{X}[-i] \subseteq \mathbb{T}$ for $1 \leq i \leq n-k$ and this means that $\mathcal{X}$ is $(n-k)$-corigid.

The last assertion is clear since by Corollary 2.6, the functor $H$ is full if and only if $\mathcal{O}_H = 0$. $\square$

Proposition 3.12. Let $\mathcal{T}$ be a triangulated category and $\mathcal{X}$ a contravariantly finite $n$-rigid subcategory of $\mathcal{T}$. If $\mathcal{T} = \mathcal{X} \oplus \mathcal{X}[1] \oplus \cdots \oplus \mathcal{X}[n]$, then, for $1 \leq t \leq n$, the following are equivalent:

(i) $\mathcal{X}$ is $t$-corigid.
(ii) $\mathcal{X}[n+1] \subseteq \mathcal{X} \oplus \mathcal{X}[1] \oplus \cdots \oplus \mathcal{X}[n-t]$.
(iii) $\mathcal{X} \subseteq \mathcal{X}[t+1] \oplus \cdots \oplus \mathcal{X}[n] \oplus \mathcal{X}[n+1]$.
(iv) $\mathcal{G}_n^a = (\mathcal{X}[n+1], -) = 0$.

Proof. (i) $\Rightarrow$ (ii) Since $\mathcal{X}[n+1] \subseteq \mathcal{T} = \mathcal{X} \oplus \mathcal{X}[1] \oplus \cdots \oplus \mathcal{X}[n]$, for any object $X \in \mathcal{X}$, there exists a triangle $A \to X[n+1] \to B \to A[1]$, where $A \in \mathcal{X} \oplus \mathcal{X}[1] \oplus \cdots \oplus \mathcal{X}[n-t]$ and $B \in \mathcal{X}[n-t+1] \oplus \cdots \oplus \mathcal{X}[n]$. Since $\mathcal{X}$ is $t$-corigid, any map from an object from $\mathcal{X}$ to an object from $\mathcal{X}[n-t] \oplus \cdots \oplus \mathcal{X}[1]$ is zero and this implies that any map from an object from $\mathcal{X}[n+1]$ to an object from $\mathcal{X}[n-t+1] \oplus \cdots \oplus \mathcal{X}[n]$ is zero. Hence the map $X[n+1] \to B$ is zero and then $X[n+1]$ lies in $\mathcal{X} \oplus \mathcal{X}[1] \oplus \cdots \oplus \mathcal{X}[n-t]$ as a direct summand of $A$.

(ii) $\Rightarrow$ (i) The hypothesis implies that $\mathcal{X} \subseteq \mathcal{X}[-n-1] \oplus \mathcal{X}[-n] \oplus \cdots \oplus \mathcal{X}[-t-1]$. Hence for any object $X \in \mathcal{X}$, there exists a triangle $A \to X \to B \to A[1]$, where $A \in \mathcal{X}[-n-1]$ and $B \in \mathcal{X}[-n] \oplus \cdots \oplus \mathcal{X}[-t-1]$. Let $1 \leq k \leq t$ and consider any map $X \to X'[k]$, where $X' \in \mathcal{X}$. The composition $A \to X \to X'[k]$ is zero since it lies in $\mathcal{T}(\mathcal{X}, \mathcal{X}[n-k])$ which is zero since $\mathcal{X}$ is $n$-rigid. Hence the map $X \to X'[k]$ factorizes through the map $X \to B$. However using that $1 \leq k \leq t$ and $\mathcal{X}$ is $n$-rigid, it follows easily that any map from an object from $\mathcal{X}[-n] \oplus \cdots \oplus \mathcal{X}[-t-1]$ to an object from $\mathcal{X}[k]$ is zero. Hence the map $X \to X'[k]$ is zero, i.e. $\mathcal{T}(\mathcal{X}, \mathcal{X}[k]) = 0$, $1 \leq k \leq t$. We infer that $\mathcal{X}$ is $t$-corigid.

Finally, since $\mathcal{X}$ is $t$-corigid if $\mathcal{X} \subseteq \mathcal{X}[t+1]$, the equivalence (i) $\Leftrightarrow$ (iii) follows from Theorem 3.4(vi), and the equivalence (ii) $\Leftrightarrow$ (iv) is a consequence of Corollary 3.3. $\square$

The following gives another characterization of corigid subcategories.

Corollary 3.13. Let $\mathcal{X}$ be a contravariantly $n$-rigid subcategory of $\mathcal{T}$. If $\mathcal{T} = \mathcal{X} \oplus \mathcal{X}[1] \oplus \cdots \oplus \mathcal{X}[n]$, then, for $0 \leq k \leq n-1$, the following conditions are equivalent.
(i) $\mathcal{X}$ is $(n-k)$-corigid.
(ii) $\mathcal{X} = \mathcal{X}_k^+ \cap (\mathcal{X}[n-k+1] \ast \mathcal{X}[-k+1] \ast \cdots \ast \mathcal{X}[-1] \ast \mathcal{X})$.

**Proof.** (ii) $\Rightarrow$ (i) By Proposition 3.12 it suffices to show that $\mathcal{X}[n+1] \subseteq \mathcal{X} \ast \mathcal{X}[1] \ast \cdots \ast \mathcal{X}[k]$. By Proposition 3.2(i) we have $\Omega_X^k(\mathcal{X}[n+1]) \subseteq \mathcal{X}_k^+$. On the other hand we clearly have $\Omega_X^k(\mathcal{X}[n+1]) \subseteq \mathcal{X}[n] \ast \mathcal{X}$. This implies that $\Omega_X^k(\mathcal{X}[n+1]) \subseteq \mathcal{X}[n-1] \ast \mathcal{X}[-1] \ast \mathcal{X}$. Then by induction we have $\Omega_X^k(\mathcal{X}[n+1]) \subseteq \mathcal{X}[n-k+1] \ast \mathcal{X}[-k+1] \ast \cdots \ast \mathcal{X}[-1] \ast \mathcal{X}$. Hence the condition in (ii) implies that $\Omega_X^k(\mathcal{X}[n+1]) \subseteq \mathcal{X}$. It follows from Corollary 3.3 that $\mathcal{X}[n+1] \subseteq \mathcal{X} \ast \mathcal{X}[1] \ast \cdots \ast \mathcal{X}[k]$ and therefore $\mathcal{X}$ is $(n-k)$-corigid.

(i) $\Rightarrow$ (ii) Clearly $\mathcal{X} \subseteq (\mathcal{X}[n-k+1] \ast \mathcal{X}[-k+1] \ast \cdots \ast \mathcal{X}[-1] \ast \mathcal{X}) \cap \mathcal{X}_k^+$. Let $A$ be in $(\mathcal{X}[n-k+1] \ast \mathcal{X}[-k+1] \ast \cdots \ast \mathcal{X}[-1] \ast \mathcal{X}) \cap \mathcal{X}_k^+$. Then there exists a triangle $M \rightarrow A \rightarrow X \rightarrow M[1]$, where $X \in \mathcal{X}$ and $M \in \mathcal{X}[n-k+1] \ast \mathcal{X}[-k+1] \ast \cdots \ast \mathcal{X}[-1]$. Since $A \in \mathcal{X}_k^+$, by Theorem 3.4(vi) it follows that $A$ lies in $\mathcal{X} \ast \mathcal{X}[1] \ast \cdots \ast \mathcal{X}[n-k]$. Using that $\mathcal{X}$ is $(n-k)$-corigid, it is easy to see that any map from an object of $\mathcal{X}[n-k+1] \ast \mathcal{X}[-k+1] \ast \cdots \ast \mathcal{X}[-1]$ to an object of $\mathcal{X} \ast \mathcal{X}[1] \ast \cdots \ast \mathcal{X}[n-k]$ is zero. Hence the map $M \rightarrow A$ is zero and then $A$ lies in $\mathcal{X}$ as a direct summand of $\mathcal{X}$. Therefore $\mathcal{X}$ is $(n-k)$-corigid.

The following consequence, which will be useful in the next section, characterizes the $(n-k)$-corigid subcategories satisfying the conditions of Theorem 3.4.

**Corollary 3.14.** Let $\mathcal{X}$ be contravariantly finite in $\mathcal{T}$, and $0 \leq k \leq n-1$. Then the following are equivalent.

(i) $\mathcal{X}$ is $n$-rigid, $(n-k)$-corigid, and $\mathcal{T} = \mathcal{X} \ast \mathcal{X}[1] \ast \cdots \ast \mathcal{X}[n]$.
(ii) $\mathcal{X} = \mathcal{X}_n^+ \subseteq \mathcal{X}[n-k+1] \ast \mathcal{X}[n-k+2] \ast \cdots \ast \mathcal{X}[n] \ast \mathcal{X}[n+1]$.
(iii) $\mathcal{X} = \mathcal{X}_n^+ = \mathcal{X}_k^+ \cap (\mathcal{X}[n-k+1] \ast \mathcal{X}[-k+1] \ast \cdots \ast \mathcal{X}[-1] \ast \mathcal{X})$.

4. Resolutions and homological dimensions

In this section we construct (co)resolutions of objects of a triangulated category by objects of certain subcategories and we study the associated homological invariants in connection with the homological dimensions defined in Section 3.

4.1. (Co)resolutions

Let $\mathcal{T}$ be a triangulated category with split idempotents and $\mathcal{X}$ a contravariantly finite subcategory of $\mathcal{T}$. Let $\mathcal{H} : \mathcal{T} \rightarrow \text{mod-}\mathcal{X}$ be the associated homological functor.

**Definition 4.1.** Let $\mathcal{U}$ and $\mathcal{V}$ be full subcategories of $\mathcal{T}$ and let $A$ be an object of $\mathcal{T}$.

(i) A $\mathcal{U}$-coreolution of $A$ (with respect to $\mathcal{H}$) is a complex

$$0 \rightarrow A \rightarrow U^0 \rightarrow U^1 \rightarrow \cdots \rightarrow U^k \rightarrow \cdots$$

(j)
in \( \mathcal{T} \), where \( U^i \in \mathcal{U}, \forall i \geq 0 \), such that the induced complex in \( \text{mod-} \mathcal{X} \)
\[
0 \rightarrow H(A) \rightarrow H(U^0) \rightarrow H(U^1) \rightarrow \cdots \rightarrow H(U^k) \rightarrow \cdots
\]
is exact. The minimum \( k \in \mathbb{N} \cup \{ \infty \} \) such that there exists a \( \mathcal{U} \)-coresolution (\( \dagger \)) of \( A \) with \( U^i = 0, \forall i > k \), is called the \( \mathcal{U} \)-coresolution dimension of \( A \) and is denoted by \( \text{cores.dim}_\mathcal{U} A \).

(ii) A \( \mathcal{V} \)-resolution of \( A \) (with respect to \( \mathcal{H} \)) is a complex
\[
\cdots \rightarrow V^k \rightarrow \cdots \rightarrow V^1 \rightarrow V^0 \rightarrow A \rightarrow 0 \quad (\dagger\dagger)
\]
in \( \mathcal{T} \), where \( V^i \in \mathcal{V}, \forall i \geq 0 \), such that the induced complex in \( \text{mod-} \mathcal{X} \)
\[
\cdots \rightarrow H(V^k) \rightarrow \cdots \rightarrow H(V^1) \rightarrow H(V^0) \rightarrow H(A) \rightarrow 0
\]
is exact. The minimum \( k \in \mathbb{N} \cup \{ \infty \} \) such that there exists a \( \mathcal{V} \)-resolution (\( \dagger\dagger \)) of \( A \) with \( V^i = 0, \forall i > k \), is called the \( \mathcal{V} \)-resolution dimension of \( A \) and is denoted by \( \text{res.dim}_\mathcal{V} A \).

(iii) If \( W \subseteq \mathcal{T} \), then we set:
\[
\text{res.dim}_\mathcal{U} W = \sup \{ \text{res.dim}_\mathcal{U} W \mid W \in W \} \quad \text{and} \quad \text{cores.dim}_\mathcal{U} W = \sup \{ \text{cores.dim}_\mathcal{V} W \mid W \in W \}
\]
Observe that \( \text{res.dim}_\mathcal{X} A = \text{pd} H(A) \). We are mostly interested in the following cases:
(\( \alpha \)) \( \mathcal{U} = \mathcal{X} \) and \( W = \mathcal{X}_n^\top [n+1] \), and (\( \beta \)) \( \mathcal{V} = \mathcal{X}_n^\top [n+1] \) and \( W = \mathcal{X} \). In this connection we have the following main result of this section which shows existence of \( \mathcal{X} \)-resolutions and \( \mathcal{X}_n^\top [n+1] \)-coresolutions for certain objects of \( \mathcal{T} \).

**Theorem 4.2.** Let \( \mathcal{X} \) be a contravariantly finite \( n \)-rigid subcategory of \( \mathcal{T} \), \( n \geq 1 \). If \( n \geq 2 \), we assume further that \( \mathcal{X} \) is \((n-k)\)-corigid, for some \( k \) with \( 0 \leq k \leq \frac{n+1}{2} \).

(i) \( \forall A \in (\mathcal{X} \ast \mathcal{X}[1] \ast \cdots \ast \mathcal{X}[k]) \bigcap \mathcal{X}_k^\top [k+1] : \text{res.dim}_\mathcal{X} A \leq k \)

(ii) If \( \mathcal{X}_n^\top \subseteq \mathcal{X}[-n-1] \ast \mathcal{X}[-n] \ast \cdots \ast \mathcal{X}[-n+k-1] \), then:
\[
\forall A \in \mathcal{X}_{n-k}^\top [n-k+1] \bigcap \mathcal{X}_k^\top : \text{cores.dim}_{\mathcal{X}_n^\top [n+1]} A \leq k
\]

Note that if \( n = 1 \), then the only values for the non-negative integer \( k \) in the condition “\( \mathcal{X} \) is \((n-k)\)-corigid, for some \( k \) with \( 0 \leq k \leq \frac{n+1}{2} \)” appearing in Theorem 4.2, are \( k = 0 \) and \( k = 1 \). Clearly the above condition is vacuous in case \( n = 1 = k \). Note also that if \( n \geq 2 \) and \( 0 \leq k \leq \frac{n+1}{2} \), then \( 0 \leq k \leq n-1 \).

We split the proof of Theorem 4.2 into several steps. The first one is a direct consequence of Corollary 2.11.
Proposition 4.3. Let $\mathcal{X}$ be a contravariantly finite subcategory of $\mathcal{T}$ and $A \in \mathcal{T}$.

(i) If $\mathcal{X}$ is 1-rigid and $A \in (\mathcal{X} \ast \mathcal{X}[1]) \cap \mathcal{X}^\top_1[2]$, then: $\text{pd } H(A) \leq 1$.

(ii) Let $t \geq 2$ and assume that $\mathcal{X}$ is $t$-rigid and $(t - 1)$-corigid. Then:

$$A \in (\mathcal{X} \ast \mathcal{X}[1] \ast \cdots \ast \mathcal{X}[t]) \cap \mathcal{X}^\top_t[t + 1] \implies \text{pd } H(A) \leq t.$$  

Proof. Part (ii) follows from Corollary 2.11. For part (i), let $\mathcal{X}$ be 1-rigid and $A \in (\mathcal{X} \ast \mathcal{X}[1]) \cap \mathcal{X}^\top_1[2]$. Then $\mathcal{T}(\mathcal{X}, A[-1]) = 0$ and there exists a triangle $X_0 \to A \to X_1[1] \to X_0[1]$, where the $X_i$ lie in $\mathcal{X}$. Applying $H$ we have an exact sequence $0 \to H(X_1) \to H(X_0) \to H(A) \to 0$. Hence $\text{pd } H(A) \leq 1$. $\square$

Proposition 4.4. Let $\mathcal{X}$ be a contravariantly finite $n$-rigid subcategory of $\mathcal{T}$, $n \geq 1$. If $n \geq 2$, we assume further that $\mathcal{X}$ is $(n - k)$-corigid, for some $k$ with $0 \leq k \leq n - 1$. Let $A$ be an object in $\mathcal{X}^\top_{n - k}[n - k + 1]$.

(i) There exist triangles:

$$C^i \to E^i \to C^{i+1} \to C^i[1], \quad 0 \leq i \leq k - 1, \quad C^0 = A, \quad C^k = E^k$$

where each $E^i$ lies in $\mathcal{X}^\top_{n + 1}[n + 1]$, $0 \leq i \leq k$, which induce exact sequences in $\text{mod-} \mathcal{X}$

$$0 \to H(C^i) \to H(E^i) \to H(C^{i+1})$$

(ii) If $A \in \mathcal{X}^\top_k$, and if $\mathcal{X}^\top_n \subseteq \mathcal{X}[-n - 1] \ast \mathcal{X}[-n] \ast \cdots \ast \mathcal{X}[-n + k - 1]$ in case $k \geq 2$, then we have an exact sequence:

$$0 \to H(A) \to H(E^0) \to H(E^1) \to \cdots \to H(E^k) \to 0$$

provided that $k \leq \frac{n + 1}{2}$.

Proof. We split the proof in two parts, dividing the second part into several steps.

1. Case $n = 1$: We use Proposition 3.2: since $\mathcal{X}$ is 1-rigid, we have a triangle

$$\Omega^1_x(A[-1])[1] \to \text{Cell}_0(A[-1])[1] \to A \to \Omega^1_x(A[-1])[2]$$

where $\text{Cell}_0(A[-1]) = X^0_{A[-1]} \in \mathcal{X} \subseteq \mathcal{X}^\top_1$. Setting $E^0 = \Omega^1_x(A[-1])[2]$ and $E^1 = \text{Cell}_0(A[-1])[2]$ it follows directly that the $E^i$ lie in $\mathcal{X}^\top_1[2]$, that is $H(E^i[-1]) = 0$, for $0 \leq i \leq 1$, so we have the desired triangle $A \to E^0 \to E^1 \to A[1]$, and an exact sequence

$$0 \to H(A) \to H(E^0) \to H(E^1)$$
Observe that if \( A \in \mathcal{X}_1^\top \), i.e. \( H(A[1]) = 0 \), then we have an exact sequence
\[
0 \rightarrow H(A) \rightarrow H(E^0) \rightarrow H(E^1) \rightarrow 0
\]

2. **Case** \( n \geq 2 \): We assume that \( \mathcal{X} \) is \((n-k)\)-corigid, for some \( k \) with \( 0 \leq k \leq \frac{n+1}{2} \). For the convenience of the reader, we first treat the cases \( 0 \leq k \leq 3 \). Let \( A \in \mathcal{X}_{n-k}^\top[n-k+1] \).

(a) Let \( k = 0 \). Then \( A \) lies in \( \mathcal{X}_{n}^\top[n+1] \), so we may choose \( E^0 = A \), and \( C^i = 0 = E^i \) for \( i \geq 1 \).

(b) Let \( k = 1 \). Then \( A \in \mathcal{X}_{n-1}^\top[n] \), i.e. \( \mathcal{T}(\mathcal{X}, A[-i]) = 0 \), \( 1 \leq i \leq n-1 \). Consider the triangle arising from the tower of triangles \( (C^*_A[-1]) \) associated to the object \( A[-1] \):
\[
\Omega^n_X(A[-1])[n-1] \rightarrow \text{Cell}_{n-1}(A[-1]) \rightarrow A[-1] \rightarrow \Omega^n_X(A[-1])[n]
\]

By **Proposition 3.2**, \( \Omega^n_X(A[-1]) \in \mathcal{X}_n^\top \) and \( \text{Cell}_{n-1}(A[-1]) \in \mathcal{X} \times \mathcal{X}[1] \cdots \times \mathcal{X}[n-1] \). Then we have a triangle
\[
\Omega^n_X(A[-1])[n] \rightarrow \text{Cell}_{n-1}(A[-1])[1] \rightarrow A \rightarrow \Omega^n_X(A[-1])[n+1] \quad (4.1)
\]

where \( \text{Cell}_{n-1}(A[-1])[1] \) lies in \( \mathcal{X}[1] \times \cdots \times \mathcal{X}[n] \), so \( H(\text{Cell}_{n-1}(A[-1])[1]) = 0 \) and \( \Omega^n_X(A[-1])[n+1] \) lies in \( \mathcal{X}_n^\top[n+1] \). We set \( E^0 := \Omega^n_X(A[-1])[n+1] \) and \( E^1 := \text{Cell}_{n-1}(A[-1])[2] \). Applying the homological functor \( H \) to the triangle \( A \rightarrow E^0 \rightarrow E^1 \rightarrow A[1] \) and using that \( A \) lies in \( \mathcal{X}_{n-1}^\top[n] \), and \( E^0 \) lies in \( \mathcal{X}_n^\top[n+1] \), we see directly from the induced long exact sequence that \( E^1 \) lies in \( \mathcal{X}_n^\top[n+1] \) and we have an exact sequence in \( \text{mod-} \mathcal{X} \):
\[
0 \rightarrow H(A) \rightarrow H(E^0) \rightarrow H(E^1)
\]

Observe that if \( A \in \mathcal{X}_1^\top \), i.e. we have \( H(A[1]) = 0 \), then we get, as required, an exact sequence:
\[
0 \rightarrow H(A) \rightarrow H(E^0) \rightarrow H(E^1) \rightarrow 0
\]

where the \( E^i \) lie in \( \mathcal{X}_n^\top[n+1] \).

(c) Let \( k = 2 \). Then \( n \geq 3 \) and \( \mathcal{X} \) is \((n-2)\)-corigid, and \( A \in \mathcal{X}_{n-2}^\top[n-1] \). Equivalently \( H(\mathcal{X}[-i]) = 0 \), \( 1 \leq i \leq n-2 \). Consider the triangle \( (4.1) \) and set \( E^0 := \Omega^n_X(A[-1])[n+1] \) and \( C^1 := \text{Cell}_{n-1}(A[-1])[2] \). Then as before \( E^0 \) lies in \( \mathcal{X}_n^\top[n+1] \). Applying \( H \) to the triangle
\[
A \rightarrow E^0 \rightarrow C^1 \rightarrow A[1] \quad (4.2)
\]

and using that \( A \) lies in \( \mathcal{X}_{n-2}^\top[n-1] \) and \( H(C^1[-1]) = 0 \), we have an exact sequence
\[
0 \rightarrow H(A) \rightarrow H(E^0) \rightarrow H(C^1)
\]
and $H(C^1[-i]) = 0$ for $1 \leq i \leq n - 1$. It follows that $C^1$ lies in $\mathcal{X}^T_{n-1}[n]$. Setting $E^1 := \Omega^n_X(C^1[-1])[n + 1]$ and $E^2 := \text{Cell}_{n-1}(C^1[-1])[2]$, this implies, as in the case $k = 1$, the existence of a triangle

$$C^1 \longrightarrow E^1 \longrightarrow E^2 \longrightarrow C^1[1] \quad (4.3)$$

where the $E^i$ lie in $\mathcal{X}^T_n[n+1]$. Since $E^2[-1] = \text{Cell}_{n-1}(C^1[-1])[1] \in \mathcal{X}[1] \cdots \mathcal{X}[n]$, by Proposition 3.2 we have $H(E^2[-1]) = 0$ and therefore we have an exact sequence

$$0 \longrightarrow H(C^1) \longrightarrow H(E^1) \longrightarrow H(E^2)$$

Observe that if $A \in \mathcal{X}^T_2$, so $H(A[i]) = 0$ for $1 \leq i \leq 2$, then applying $H$ to the triangle $(4.2)$ we get an exact sequence

$$0 \longrightarrow H(A) \longrightarrow H(E^0) \longrightarrow H(C^1) \longrightarrow 0$$

and an isomorphism $H(E^0[1]) \cong H(C^1[1])$. We infer that we have an exact sequence

$$0 \longrightarrow H(A) \longrightarrow H(E^0) \longrightarrow H(E^1) \longrightarrow H(E^2)$$

If moreover $\mathcal{X}^T_n \subseteq \mathcal{X}[-n-1] \cdots \mathcal{X}[-n] \cdots \mathcal{X}[-n+1]$, then $\mathcal{X}^T_n[n+1] \subseteq \mathcal{X} \cdots \mathcal{X}[1] \cdots \mathcal{X}[2]$, so $\mathcal{X}^T_n[n+2] \subseteq \mathcal{X}[1] \cdots \mathcal{X}[2] \cdots \mathcal{X}[3]$. Since $n \geq 3$, it follows that $\mathcal{X}[1] \cdots \mathcal{X}[2] \cdots \mathcal{X}[3] \subseteq \mathcal{X}^T$ and therefore $H(\mathcal{X}^T_n[n+2]) = 0$. Since $E^0[1]$ lies in $\mathcal{X}^T_n[n+2]$, we infer that $H(C^1[1]) \cong H(E^0[1]) = 0$. It follows that the map $H(E^1) \longrightarrow H(E^2)$ is an epimorphism and therefore we have the desired exact sequence

$$0 \longrightarrow H(A) \longrightarrow H(E^0) \longrightarrow H(E^1) \longrightarrow H(E^2) \longrightarrow 0$$

(d) Now we discuss the case $k = 3$. Then $n \geq 2 \cdot 3 - 1 = 5$ and $\mathcal{X}$ is $(n-3)$-corigid, i.e. $H(\mathcal{X}[-1]) = 0$, $1 \leq i \leq n - 3$. Let $A \in \mathcal{X}^T_{n-3}[n-2]$. Then as above, we have triangles:

$$A \longrightarrow E^0 \longrightarrow C^1 \longrightarrow A[1] \quad \& \quad C^1 \longrightarrow E^1 \longrightarrow E^2 \longrightarrow C^1[1] \quad (4.4)$$

where as before: $E^0 := \Omega^n_X(A[-1])[n + 1]$, $C^1 := \text{Cell}_{n-1}(A[-1])[2]$, $E^1 := \Omega^n_X(C^1[-1])[n + 1]$, and $C^2 := \text{Cell}_{n-1}(C^1[-1])[2]$. Clearly $E^i$ lie in $\mathcal{X}^T_n[n + 1]$ for $0 \leq i \leq 1$. Applying the homological functor $H$ to the triangles $(4.4)$, we see directly that $C^1 \in \mathcal{X}^T_{n-2}[n - 1]$ and $C^2 \in \mathcal{X}^T_{n-1}[n]$. Now setting $E^2 := \Omega^n_X(C^2[-1])[n + 1]$ and $E^3 := \text{Cell}_{n-1}(C^2[-1])[2]$, we have as before a triangle

$$C^2 \longrightarrow E^2 \longrightarrow E^3 \longrightarrow C^2[1] \quad (4.5)$$
where the objects \( E^i \) lie in \( X_n^\top [n+1] \), for \( 2 \leq i \leq 3 \), and exact sequences:

\[
\begin{align*}
0 \rightarrow H(A) & \rightarrow H(E^0) \rightarrow H(C^1) \quad \& \\
0 \rightarrow H(C^1) & \rightarrow H(E^1) \rightarrow H(C^2) \\
0 \rightarrow H(C^2) & \rightarrow H(E^2) \rightarrow H(E^3)
\end{align*}
\] (4.6)

Now assume that \( A \in X_3^\top \), i.e. \( H(A[i]) = 0 \), for \( 1 \leq i \leq 3 \). Then applying \( H \) to the first triangle in (4.4) we have an exact sequence (*) : \( 0 \rightarrow H(A) \rightarrow H(E^0) \rightarrow H(C^1) \rightarrow 0 \) and isomorphisms \( H(E^0[i]) \cong H(C^1[i]) \), for \( 1 \leq i \leq 2 \). Assume moreover that \( X_n^\top \subseteq X[-n-1] \ast X[-n] \ast X[-n+1] \ast X[-n+2] \). Then \( X_n^\top [n+2] \subseteq X[1] \ast X[2] \ast X[3] \ast X[4] \) and \( X_{n+k} \subseteq X[2] \ast X[3] \ast X[4] \ast X[5] \).

Since \( n \geq 5 \), it follows that \( X[1] \ast X[2] \ast X[3] \ast X[4] \subseteq X^\top \supseteq X[2] \ast X[3] \ast X[4] \ast X[5] \) and therefore \( H(X_n^\top [n+2]) = 0 = H(X_{n+k}[n+3]) \). Since the \( E^i[1] \) lie in \( X_n^\top [n+2] \), we have \( H(E^i[1]) = 0 \) and since the \( E^i[2] \) lie in \( X_{n+k}[n+3] \), we have \( H(E^i[2]) = 0 \). It follows easily then that \( H(C^1[1]) \cong H(E^0[1]) = 0 \) and \( H(C^1[2]) \cong H(E^0[2]) = 0 \). Therefore applying \( H \) to the second triangle in (4.4) we obtain a short exact sequence (**) : \( 0 \rightarrow H(C^1) \rightarrow H(E^1) \rightarrow H(C^2) \rightarrow 0 \). On the other hand applying \( H \) to the second triangle in (4.4) we have a long exact sequence \( \cdots \rightarrow H(E^1[1]) \rightarrow H(C^2[1]) \rightarrow H(C^2[2]) \rightarrow H(E^1[2]) \rightarrow \cdots \) .

Since \( H(E^1[1]) = 0 = H(E^1[2]) \), the map \( H(C^2[1]) \rightarrow H(C^2[2]) \) is invertible and therefore \( H(C^2[1]) = 0 \). Hence applying \( H \) to the triangle (4.5) we obtain that the map \( H(E^2) \rightarrow H(E^3) \) in (4.7) is an epimorphism and we have a short exact sequence (***) : \( 0 \rightarrow H(C^2) \rightarrow H(E^2) \rightarrow H(E^3) \rightarrow 0 \). Splicing together the short exact sequences (*), (**), and (***) , we obtain, as required, the following exact sequence, where the objects \( E^i \) lie in \( X_n^\top [n+1] \), for \( 0 \leq i \leq 3 \):

\[
0 \rightarrow H(A) \rightarrow H(E^0) \rightarrow H(E^1) \rightarrow H(E^2) \rightarrow H(E^3) \rightarrow 0
\]

(e) Continuing in this way for \( n-1 \geq k \geq 4 \), under the hypotheses of part (ii) and working as above, we construct inductively triangles

\[
C^i \rightarrow E^i \rightarrow C^{i+1} \rightarrow C^i[1], \quad 0 \leq i \leq k-1, \quad C^0 = A, \quad C^k = E^k
\]

\[
E^i = \Omega^\alpha_{X^\top} C^i[\ast-1][n+1] \quad \text{and} \quad C^j = \text{Cell}_{n-1}(C^j[\ast-1][n+1])[2],
\]

for \( 1 \leq j \leq k-1, \quad 0 \leq i \leq k-1 \), such that

\[
E^i \in X_n^\top [n+1], \quad 0 \leq i \leq k-1 \quad \text{and} \quad C^i \in X_{n-k-i}^\top [n-k+i+1]
\]

for \( 1 \leq i \leq k \), and the induced sequences

\[
0 \rightarrow H(C^i) \rightarrow H(E^i) \rightarrow H(C^{i+1}), \quad 0 \leq i \leq k-1
\]
are exact in mod-$\mathcal{X}$. Moreover if $A \in \mathcal{X}_k^\top$ and $\mathcal{X}_n^\top \subseteq \mathcal{X}[−n − 1] \ast \mathcal{X}[−n] \ast \cdots \ast \mathcal{X}[−n + k − 1]$, then as above the maps $H(E^i) \to H(C^i)$ are epimorphisms. Therefore we have an exact sequence

$$0 \to H(A) \to H(E^0) \to H(E^1) \to \cdots \to H(E^{k−1}) \to H(E^k) \to 0$$

We infer that the induced complex $0 \to A \to E^0 \to E^1 \to \cdots \to E^k \to 0$ is an $\mathcal{X}_n^\top[n+1]$-coresolution of $A$ and therefore $\text{cores.dim}_{\mathcal{X}_n^\top[n+1]} A \leq k$. □

Now we can give the proof of Theorem 4.2.

**Proof of Theorem 4.2.** Part (ii) follows from Proposition 4.4. For part (i) first note that the case $n = 1$ follows from Proposition 4.3(i). Assume now that $n \geq 2$, and $\mathcal{X}$ is $(n−k)$-corigid, where $0 \leq k \leq \frac{n+1}{2}$. Observe that since $n \geq 2 \cdot k−1$, we have $n−k \geq k−1$. Since $\mathcal{X}$ is $(n−k)$-corigid, this easily implies that $\mathcal{X}$ is $(k−1)$-corigid. Since $k \leq n$, $\mathcal{X}$ is also $k$-rigid and therefore the assertion follows from Proposition 4.3(ii). □

As a consequence we have the following interesting symmetry between $\mathcal{X}$ and $\mathcal{X}_n^\top[n+1]$.

**Corollary 4.5.** Let $\mathcal{X}$ be a contravariantly finite $n$-rigid subcategory of $\mathcal{T}$, $n \geq 1$. If $n \geq 2$, assume that $\mathcal{X}$ is $(n−k)$-corigid, $0 \leq k \leq \frac{n+1}{2}$, and $\mathcal{X}_n^\top[n+1] \subseteq \mathcal{X} \ast \mathcal{X}[1] \ast \cdots \ast \mathcal{X}[k]$. Then:

$$\text{res.dim}_{\mathcal{X}} \mathcal{X}_n^\top[n+1] \leq k \quad \text{and} \quad \text{cores.dim}_{\mathcal{X}_n^\top[n+1]} \mathcal{X} \leq k.$$

**Proof.** Since $k \leq n$, it follows that $\mathcal{X}_n^\top[n+1] \subseteq \mathcal{X}_n^\top[k+1]$. Now the condition $\mathcal{X}_n^\top \subseteq \mathcal{X}[−n−1] \ast \mathcal{X}[−n] \ast \cdots \ast \mathcal{X}[−n+k−1]$ implies that $\mathcal{X}_n^\top[n+1] \subseteq \mathcal{X} \ast \mathcal{X}[1] \ast \cdots \ast \mathcal{X}[k]$, hence $\mathcal{X}_n^\top[n+1] \subseteq (\mathcal{X} \ast \mathcal{X}[1] \ast \cdots \ast \mathcal{X}[k]) \cap \mathcal{X}_k^\top[k+1]$. Then the bound $\text{res.dim}_{\mathcal{X}} \mathcal{X}_n^\top[n+1] \leq k$ follows from Theorem 4.2(ii). Since $\mathcal{X}$ is $(n−k)$-corigid and $k$-rigid, we have $\mathcal{X} \subseteq \mathcal{X}_n^\top[n−k+k] \cap \mathcal{X}_k^\top$ and the bound $\text{cores.dim}_{\mathcal{X}_n^\top[n+1]} \mathcal{X} \leq k$ follows from Theorem 4.2(i). □

Although the condition $\mathcal{X}_n^\top \subseteq \mathcal{X}[−n−1] \ast \mathcal{X}[−n] \ast \cdots \ast \mathcal{X}[−n+k−1]$ or equivalently $\mathcal{X}_n^\top[n+1] \subseteq \mathcal{X} \ast \mathcal{X}[1] \ast \cdots \ast \mathcal{X}[k]$, looks somewhat artificial, we shall see later, cf. Theorem 5.3, that it is directly connected with $\mathcal{X}$ being an $(n+1)$-cluster tilting subcategory.

4.2. Finite projective dimension

We give a description of the full subcategories $\text{Proj}^{\leq k} \mathcal{A}$, where $\mathcal{A} = \text{mod-}\mathcal{X}$ or $\mathcal{X}$-mod, consisting of all objects with projective dimension $\leq k$.

**Theorem 4.6.** Let $\mathcal{X}$ be a contravariantly finite $t$-rigid subcategory of $\mathcal{T}$, $t \geq 1$, and assume that $\mathcal{X}$ is $(t−1)$-corigid, if $t \geq 2$. Then the functor $H : \mathcal{T} \to \text{mod-}\mathcal{X}$ induces
a full embedding

\[ H : (X \star X[1] \star \cdots \star X[t]) \cap X_i^t \cap [t + 1] \hookrightarrow \text{Proj}^{\leq t} \text{mod-}X \]

which is an equivalence if \( X \) is \( t \)-corigid.

**Proof.** (a) Set \( U_t := (X \star X[1] \star \cdots \star X[t]) \cap X_i^t \cap [t + 1] \) and \( H_t := H|_{U_t} \). First note that by Proposition 4.3, we have \( pd H(A) \leq t \), for any object \( A \in U_t \), so \( H \) induces a functor \( H_t : U_t \to \text{Proj}^{\leq k} \text{mod-}X \).

Let \( \alpha : A \to B \) be a map in \( U_t \) such that \( H(\alpha) = 0 \). Then \( \alpha \) factorizes through the map \( h_A^0 : A \to \Omega^1_X(A)[1] \), say via a map \( \beta : \Omega^1_X(A)[1] \to B \). Since \( X \) is \( t \)-rigid and \( A \) lies in \( X \star X[1] \star \cdots \star X[t] \), it follows that \( \Omega^1_X(A) \) lies in \( X \) and then it is easy to see that \( \Omega^1_X(A) \) lies in \( X \star X[1] \star \cdots \star X[t − 1] \), hence \( \Omega^1_X(A)[1] \in X[1] \star X[2] \star \cdots \star X[t] \). Then \( B \in X_i^t \cap [t + 1] \), we have \( T(X, B[-i]) = 0 \), \( 1 \leq i \leq t \), and then \( \beta = 0 \) by Proposition 3.2(ii). Therefore \( \alpha = 0 \) and \( H_t \) is faithful. Next let \( \alpha : \Omega^1_X(A) \to B \) be an \( X \)-ghost map. Then \( \alpha \) factorizes through the map \( h_A : \Omega^1_X(A) \to \Omega^2_X(A)[1] \), say via a map \( \beta : \Omega^2_X(A)[1] \to B \). As above, \( \Omega^2_X(A) \) lies in \( X \star X[1] \star \cdots \star X[t − 2] \), hence \( \Omega^2_X(A)[1] \in X[1] \star X[2] \star \cdots \star X[t − 1] \). Since \( B \in X_i^t \cap [t + 1] \), again by Proposition 3.2(ii) it follows that \( \beta = 0 \), and therefore \( \alpha = 0 \). Consequently \( \text{Gr}_X(\Omega^1_X(A), B) = 0 \), and then, by Proposition 2.5, the map \( T(A, B) \to \text{Hom}(H(A), H(B)) \) is surjective. We infer that \( H_t \) is full.

(\( \beta \)) Let \( X \) be \( t \)-corigid and let \( F \) be in \( \text{mod-}X \) with \( pd F \leq t \). We have a projective resolution in \( \text{mod-}X \)

\[ 0 \to H(X^t) \to H(X^{t-1}) \to \cdots \to H(X^1) \to H(X^0) \to F \to 0 \]

where each \( X^i \) lies in \( X \). The last map of the resolution gives us a map \( X^t \to X^{t−1} \) which induces a triangle \( X^t \to X^{t-1} \to A^{t-1} \to X^t[1] \). Applying \( H \) to this triangle and using that \( X \) is \( t \)-rigid and \( t \)-corigid, we see easily that \( H(A^{t-1}[i]) = 0 \), \( 1 \leq i \leq t−1 \), and \( H(A^{t-1}[−i]) = 0 \), \( 1 \leq i \leq t \), so \( A^{t−1} \) lies in \( X_i^t \cap [t + 1] \). Moreover we have an exact sequence \( 0 \to H(X^t) \to H(X^{t-1}) \to H(A^{t-1}) \to 0 \) and \( \text{Im} (H(X^{t−1}) \to H(X^{t-2})) = H(A^{t-1}) \). Since \( X^t \in X \), we have \( A^{t−1} \in X \star X[1] \); hence by Corollary 2.6 the monomorphism \( H(A^{t-1}) \to H(X^{t-2}) \) is induced by a map \( A^{t-1} \to X^{t-2} \). Consider a triangle \( A^{t-1} \to X^{t-2} \to A^{t-2} \to A^{t-1}[1] \). As above, applying \( H \) to this triangle and using that \( X \) is \( t \)-rigid and \( t \)-corigid, we see easily that \( T(X, A^{t−2}[i]) = 0 \), \( 1 \leq i \leq t−1 \), and \( A^{t−2} \in X_i^t \cap [t + 1] \). Moreover we have a short exact sequence \( 0 \to H(A^{t-1}) \to H(X^{t-2}) \to H(A^{t-2}) \to 0 \) and \( \text{Im} (H(X^{t−2}) \to H(X^{t-3})) = H(A^{t-2}) \). Since \( A^{t-1} \in X \star X[1] \), we have \( A^{t−2} \in X \star X[1] \star X[2] \) and then by Corollary 3.6(ii) the obstruction group \( \text{Ob}_H(A^{t−2}, X^{t-3}) \) vanishes, since \( X \) is \( (t − 1) \)-corigid. Hence the natural monomorphism \( H(A^{t−2}) \to H(X^{t-3}) \) is induced by a map \( A^{t−2} \to X^{t-3} \). Considering a triangle \( A^{t−2} \to X^{t-3} \to A^{t-3} \to A^{t-2}[1] \) and continuing in this way by using Corollary 3.6(ii) and the fact that \( X \) is \( t \)-(co)rigid, we may construct inductively triangles \( A^j \to X^{j−1} \to A^{j−1} \to A^j[1] \), with \( X^j \in X \), and exact sequences \( 0 \to \)
$H(A^j) \to H(X^{j-1}) \to H(A^{j-1}) \to 0$, for $1 \leq j \leq t$, where $A^i = X^i$, and the objects $A^j$ satisfy $\mathcal{T}(X, A^j[i]) = 0$, $1 \leq i \leq j$, and $A^j \in (X \star X[1] \star \cdots \star X[t-j]) \cap X^*_t[t+1]$. In particular for $j = 1$, we have a triangle $A^1 \to X^0 \to A^0 \to A^1[1]$, where $A := A^0 \in X \star X[1] \star \cdots \star X[t] \cap X^*_t[t+1]$ which induces a short exact sequence $0 \to H(A^1) \to H(X^0) \to H(A) \to 0$. This implies that $H(A) \cong F$, where $A \in \mathcal{U}_t$. We infer that $H : \mathcal{U}_t \to \text{Proj}^{\leq t} \mathcal{X}$ is essentially surjective, hence an equivalence. $\square$

For latter use we include without proof the following dual version of Theorem 4.6.

**Theorem 4.7.** Let $\mathcal{X}$ be a covariantly finite $t$-rigid subcategory of $\mathcal{T}$ and assume that $\mathcal{X}$ is $(t-1)$-corigid, if $t \geq 2$. Then the functor $H^{\text{op}} : \mathcal{T}^{\text{op}} \to \mathcal{X}^{\text{mod}}$ induces a full contravariant embedding

$$H^{\text{op}} : (X[-t] \star \cdots \star X[-1] \star X) \cap \mathcal{T}_t \to \mathcal{X} \to \text{Proj}^{\leq t} \mathcal{X}^{\text{mod}}$$

which is a duality if $\mathcal{X}$ is $t$-corigid.

### 4.3. Serre duality and Calabi–Yau categories

Assume that $\mathcal{T}$ is a $k$-linear triangulated category over a field $k$. Recall that $\mathcal{T}$ is $\text{Hom}$-finite if $\mathcal{T}(A, B)$ is a finite-dimensional vector space over $k$, $\forall A, B \in \mathcal{T}$. In this case $\mathcal{T}$ admits Serre duality, if there exists a Serre functor $S : \mathcal{T} \to \mathcal{T}$ for $\mathcal{T}$, [20]. Thus $S$ is a triangulated auto-equivalence of $\mathcal{T}$, and for any objects $A, B$ in $\mathcal{T}$ there are natural isomorphisms

$$\text{D} \text{Hom}_\mathcal{T}(A, B) \cong \text{Hom}_\mathcal{T}(B, SA)$$

where $\text{D}$ denotes the $k$-dual functor. If $\mathcal{T}$ admits a Serre functor $S$, then $\mathcal{T}$ is called $d$-Calabi–Yau, for some $d \geq 1$, provided that $S(?) \cong (?)[d]$ as triangulated functors.

If $\mathcal{X}$ is functionally finite subcategory of $\mathcal{T}$ and $\mathcal{T}$ has a Serre functor, then it is easy to see that $\mathcal{X}$ is a dualizing variety in the sense of [5], compare [9, Theorem 2.3]. In particular the duality $\text{D} = \text{Hom}_k(\cdot, k)$ with respect to the base field induces a duality $\text{D} : (\text{mod-}\mathcal{X})^{\text{op}} \xrightarrow{\cong} \mathcal{X}^{\text{mod}}$.

For later use we point-out the following observation.

**Lemma 4.8.** Let $\mathcal{T}$ be a triangulated category with a Serre functor $S$ over a field $k$, and let $\mathcal{X}$ be a functorially finite 1-rigid subcategory of $\mathcal{T}$. Then we have the following:

(i) **The category mod-\mathcal{X} has enough projectives and enough injectives.**

(ii) **The homological functor $H : \mathcal{T} \to \text{mod-}\mathcal{X}$ induces equivalences:**

$$S(\mathcal{X}) \cong \text{Injmod-}\mathcal{X} \quad \text{and} \quad \mathcal{X} \cong \text{Projmod-}\mathcal{X}$$
Proof. Since $\mathcal{X}$ is a dualizing variety, the duality functor $D$ induces an equivalence between $(\text{Proj } \mathcal{X} \text{-mod})^\text{op}$ and $\text{Inj \text{-mod } \mathcal{X}}$ and therefore $\text{Inj \text{-mod } \mathcal{X}} = DH^\text{op}(\mathcal{X}) = D\mathcal{T}(\mathcal{X}, -)|_{\mathcal{X}} = \mathcal{T}(-, S(\mathcal{X}))|_{\mathcal{X}} = H(S(\mathcal{X}))$. Since $\mathcal{X}$ is covariantly finite, for any object $A$ in $\mathcal{T}$ there exists a triangle $\Sigma_1(S^{-1}(A))[-1] \rightarrow S^{-1}(A) \rightarrow X_0^{-S^{-1}(A)} \rightarrow \Sigma_1(S^{-1}(A))$, where the middle map is a left $\mathcal{X}$-approximation of $S^{-1}(A)$. Then we have a map $A \rightarrow S(X_0^{-S^{-1}(A)})$ which induces a map $H(A) \rightarrow H(S(X_0^{-S^{-1}(A)})$ and $H(S(X_0^{-S^{-1}(A)})$ is injective in $\text{mod } \mathcal{X}$. This map is a monomorphism since $H(\Sigma_1(S^{-1}(A))[-1]) = \mathcal{T}(-, S\Sigma_1(S^{-1}(A)))[-1]|_{\mathcal{X}} \cong D\mathcal{T}(\Sigma_1(S^{-1}(A))[-1], \mathcal{X})$ and $\Sigma_1(S^{-1}(A)) \in \mathcal{T} \mathcal{X}$ by the dual of Proposition 3.2(i). Since any object of $\text{mod } \mathcal{X}$ is of the form $H(A)$ for some $A \in \mathcal{T}$, this shows that $\text{mod } \mathcal{X}$ has enough injectives. Hence we have the functor $H : S(\mathcal{X}) \rightarrow \text{Inj \text{-mod } \mathcal{X}}$ which is surjective on objects. If $\alpha : S(X_1) \rightarrow S(X_2)$ is a map such that $H(\alpha) = 0$, then $\alpha$ factorizes through $\mathcal{X}^\top$. Since $\mathcal{T}(\mathcal{X}^\top, S(X_2)) \cong D\mathcal{T}(X_2, \mathcal{X}^\top) = 0$, it follows that $\alpha$ is zero and therefore $H|_{S(\mathcal{X})}$ is faithful. We show that $H|_{S(\mathcal{X})}$ is full. By Proposition 2.5 it suffices to show that any $\mathcal{X}$-ghost map $\alpha : \Omega_1^\mathcal{X}(S(X_1)) \rightarrow S(X_2)$, where the $X_i$ lie in $\mathcal{X}$, is zero. However $\alpha$ factorizes through the universal $\mathcal{X}$-ghost map $\Omega_1^\mathcal{X}(S(X_1)) \rightarrow \Omega_2^\mathcal{X}(S(X_1))[1]$ out of $\Omega_1^\mathcal{X}(S(X_1))$, say via a map $\beta : \Omega_2^\mathcal{X}(S(X_1))[1] \rightarrow S(X_2)$. By Serre duality, the last map corresponds to an element in $D\mathcal{T}(X_2, \Omega_2^\mathcal{X}(S(X_1))[1])$ which is zero by Proposition 3.2(i). It follows that $\beta$, hence $\alpha$, is zero. Hence $G_{X}(\Omega_1^\mathcal{X}(S(X_1)), S(X_2)) = 0, \forall X_1, X_2 \in \mathcal{X}$ and therefore the functor $H : S(\mathcal{X}) \rightarrow \text{Inj \text{-mod } \mathcal{X}}$ is full, hence an equivalence. The case of projectives was treated in Section 1. \[ \square \]

The following consequence of Theorems 4.6 and 4.7 extends Lemma 4.8 to objects of finite projective or injective dimension.

Corollary 4.9. Let $\mathcal{T}$ be a triangulated category with Serre duality over a field $k$, and let $\mathcal{X}$ be a functorially finite $n$-rigid subcategory of $\mathcal{T}$ which is $(n - 1)$-corigid, if $n \geq 2$. Let $S$ be the Serre functor of $\mathcal{T}$.

(i) The homological functor $H : \mathcal{T} \rightarrow \text{mod } \mathcal{X}$ induces a full embedding

$$H : (\mathcal{X} \star \mathcal{X}[1] \star \cdots \star \mathcal{X}[n]) \cap \mathcal{X}_n^\top[n + 1] \hookrightarrow \text{Proj}^{\leq n} \text{mod } \mathcal{X}$$

which is an equivalence if $\mathcal{X}$ is $n$-corigid.

(ii) The homological functor $D \circ H^\text{op} = H \circ S : \mathcal{T} \rightarrow \text{mod } \mathcal{X}$ induces a full embedding

$$H \circ S : (\mathcal{X}[-n] \star \cdots \star \mathcal{X}[-1] \star \mathcal{X}) \cap \mathcal{X} [-n - 1] \hookrightarrow \text{Inj}^{\leq n} \text{mod } \mathcal{X}$$

which is an equivalence if $\mathcal{X}$ is $n$-corigid.
Part 2. Higher cluster tilting subcategories

In this part we study higher cluster tilting and maximal rigid subcategories in an arbitrary triangulated category. After presenting several characterizations and properties of cluster tilting subcategories, we concentrate at the Gorenstein and the Calabi–Yau property of the associated cluster tilted category of coherent functors. In this context and under some natural assumptions, we give conditions ensuring finiteness of global dimension, and the Gorenstein and the stably Calabi–Yau property, and we give applications to representation dimension.

5. Cluster-tilting and maximal rigid subcategories

Let as before \( \mathcal{T} \) be a triangulated category with split idempotents and \( \mathcal{X} \) a full subcategory of \( \mathcal{T} \) which is closed under direct summands and isomorphisms. The results of Section 4 show that if \( \mathcal{X} \) is contravariantly finite and satisfies \( \mathcal{X} = \mathcal{X}^\top_n \), then \( \mathcal{X} \) enjoys special properties. In this section we analyze further the structure of such subcategories by giving several characterizations. We also study briefly \( n \)-rigid subcategories \( \mathcal{X} \) which are maximal among those which are contained in \( \mathcal{X}^\top_n \).

5.1. Cluster tilting subcategories

First we observe the following symmetry.

Proposition 5.1. Let \( \mathcal{X} \) be a full subcategory of \( \mathcal{T} \), and \( n \geq 1 \). Then the following are equivalent:

(i) \( \mathcal{X} \) is contravariantly finite and \( \mathcal{X} = \mathcal{X}^\top_n \).
(ii) \( \mathcal{X} \) is contravariantly finite and both \( \mathcal{X} \) and \( \mathcal{X}^\top_n \) are \( n \)-rigid.
(iii) \( \mathcal{X} \) is covariantly finite and \( \mathcal{X} = \mathcal{X}^\top_n \).
(iv) \( \mathcal{X} \) is covariantly finite and both \( \mathcal{X} \) and \( \mathcal{X}^\top_n \) are \( n \)-rigid.

Proof. (i) \( \Rightarrow \) (ii) and (iii) \( \Rightarrow \) (iv) The proof is trivial.

(ii) \( \Rightarrow \) (i) \( \Leftarrow \) (iv) Assume that (ii) holds. Since \( \mathcal{X} \) is \( n \)-rigid, we have \( \mathcal{X} \subseteq \mathcal{X}^\top_n \). By Proposition 3.2, for any object \( A \in \mathcal{T} \), the map \( \omega^{n-1}_A : A \longrightarrow \Omega^{n-1}_\mathcal{X}(A)[n] \) in the tower of triangles \( (C^*_A) \), is a left \( \mathcal{X}^\top_n \)-approximation of \( A \). If \( A \) lies in \( \mathcal{X}^\top_n \), then since the latter is \( n \)-rigid it follows that \( \omega^{n-1}_A = 0 \) and therefore \( \omega^{n-2}_A \) factorizes through the left cone \( \mathcal{X}^A_{A} \) of \( h^{n-1}_A \) [n] of \( h^{n-1}_A \) [n − 1]. Since both \( A \) and \( \mathcal{X}^\top_n \) lie in \( \mathcal{X}^\top_n \) and the latter is \( n \)-rigid, it follows that \( \mathcal{T}(A, \mathcal{X}^A_{A} [n−1]) = 0 \) and this implies that \( \omega^{n-2}_A = 0 \). Continuing in this way after \( n−1 \) steps we deduce that \( \omega^{1}_A = 0 \) and therefore the map \( \omega^{0}_A \) factorizes through the left cone \( \mathcal{X}^1_A [1] \) of \( h^{1}_A [1] \). Since both \( A \) and \( \mathcal{X}^1_A \) lie in \( \mathcal{X}^\top_n \) and the latter is \( n \)-rigid, we have \( \mathcal{T}(A, \mathcal{X}^1_A [1]) = 0 \) and this implies that \( h^{0}_A = \omega^{0}_A = 0 \). Then \( A \) lies in \( \mathcal{X} \) as a direct summand of \( \mathcal{X}^0_A \). Hence \( \mathcal{X}^\top_n \subseteq \mathcal{X} \) and therefore \( \mathcal{X} = \mathcal{X}^\top_n \). The proof that
(iv) \(\Rightarrow\) (i) is dual to the proof of the implication (ii) \(\Rightarrow\) (i) using cocellular towers, cf. Subsection 1.2, and is left to the reader.

(i) \(\Rightarrow\) (iii) By Theorem 3.4, \(\mathcal{X}\) is covariantly finite, and clearly \(\mathcal{X} \subseteq \frac{\mathcal{T}}{n}\mathcal{X}\) since \(\mathcal{X}\) is \(n\)-rigid. Let \(A\) be in \(\frac{\mathcal{T}}{n}\mathcal{X}\) and consider the map \(\omega_{A}^{n-1} : A \rightarrow \Omega_{\mathcal{X}}^{n}(A)[n]\). Then \(\Omega_{\mathcal{X}}^{n}(A)[n] \in \mathcal{X}[n]\) and therefore \(\omega_{A}^{n-1} = 0\) since \(A \in \frac{\mathcal{T}}{n}\mathcal{X}\). Hence \(\omega_{A}^{n-2} \circ h_{A}^{n-1}[n-1] = 0\) and therefore \(\omega_{A}^{n-1}\) factors through the left cone \(X_{A}^{n-1}[n-1]\) of \(h_{A}^{n-1}[n-1]\), say via a map \(A \rightarrow X_{A}^{n-1}[n-1]\). Since \(A \in \frac{\mathcal{T}}{n}\mathcal{X}\), the last map is zero and therefore \(\omega_{A}^{n-2} = 0\). Continuing in this way we deduce after \(n-1\) steps that \(\omega_{A}^{1} = 0\) and therefore \(\omega_{A}^{0} \circ h_{A}[1] = 0\). Then \(\omega_{A}^{0}\) factorizes through the left cone \(X_{A}^{1}[1]\) of \(h_{A}[1]\). However \(\mathcal{T}(A, X_{A}^{1}[1]) = 0\), since \(A \in \frac{\mathcal{T}}{n}\mathcal{X}\). This implies that \(h_{A}^{0} = \omega_{A}^{0} = 0\), so \(A\) lies in \(\mathcal{X}\) as a direct summand \(X_{A}^{0}\). Hence \(\frac{\mathcal{T}}{n}\mathcal{X} \subseteq \mathcal{X}\) and therefore \(\mathcal{X} = \frac{\mathcal{T}}{n}\mathcal{X}\). \(\square\)

The rest of this section will be devoted to the investigation of the structure and the main properties of the following important class of contravariantly finite rigid subcategories of \(\mathcal{T}\).

**Definition 5.2.** (See [39,46,44].) A full subcategory \(\mathcal{X}\) of \(\mathcal{T}\) is called a \((n+1)\)-cluster tilting, \(n \geq 1\), if:

(i) \(\mathcal{X}\) is contravariantly finite and \(\mathcal{X} = \{ A \in \mathcal{T} | \mathcal{T}(X, A[i]) = 0, \ 1 \leq i \leq n \}\), i.e. \(\mathcal{X} = \mathcal{X}_{n}^{\mathcal{T}}\).

(ii) \(\mathcal{X}\) is covariantly finite and \(\mathcal{X} = \{ A \in \mathcal{T} | \mathcal{T}(A, X[i]) = 0, \ 1 \leq i \leq n \}\), i.e. \(\mathcal{X} = \frac{\mathcal{T}}{n}\mathcal{X}\).

In this case the category of coherent functors \(\text{mod-}\mathcal{X}\) is called the **cluster tilted category** of \(\mathcal{X}\).

The following main result of this section gives, among other things, several convenient characterizations of \((n+1)\)-cluster subcategories and generalizes many results of the literature which are known only in case \(n = 1\) or \(n > 1\) and \(\mathcal{T}\) is \((n+1)\)-Calabi–Yau, see e.g. [46,48]. For instance if \(n = 1\), then the equivalence between (i) and (ii) and the last assertion concerning injective objects of \(\text{mod-}\mathcal{X}\) is due to Keller and Reiten [46] if \(\mathcal{T}\) is 2-Calabi–Yau (see [48] if \(\mathcal{T}\) is not necessarily Calabi–Yau). It should be noted that the characterizations (ii), (iii) below show that in the definition of \((n+1)\)-cluster subcategories we need only one of the two conditions in **Definition 5.2**.

**Theorem 5.3.** Let \(\mathcal{T}\) be a triangulated category and \(\mathcal{X}\) a full subcategory of \(\mathcal{T}\). For an integer \(n \geq 1\), the following statements are equivalent.

(i) \(\mathcal{X}\) is a \((n+1)\)-cluster tilting subcategory of \(\mathcal{T}\).

(ii) \(\mathcal{X}\) is contravariantly finite and \(\mathcal{X} = \mathcal{X}_{n}^{\mathcal{T}}\).

(iii) \(\mathcal{X}\) is covariantly finite and \(\mathcal{X} = \frac{\mathcal{T}}{n}\mathcal{X}\).

(iv) \(\mathcal{X}\) is contravariantly finite and both \(\mathcal{X}\) and \(\mathcal{X}_{n}^{\mathcal{T}}\) are \(n\)-rigid.
(v) \( X \) covariantly finite and both \( X \) and \( T X \) are n-rigid.
(vi) \( X \) is contravariantly finite n-rigid and: \( X^T = X[1] \ast X[2] \ast \cdots \ast X[n] \).
(vii) \( X \) is contravariantly finite n-rigid and for \( 1 \leq t \leq n \): \( X^{T}[t] = X[t] \ast X[t+1] \ast \cdots \ast X[n] \).
(viii) \( X \) is covariantly finite n-rigid and: \( T X = X[-n] \ast X[-n+1] \ast \cdots \ast X[-1] \).
(ix) \( X \) is contravariantly (or covariantly) finite n-rigid and: \( \text{gl.dim}_X T = n \).
(x) \( X \) is contravariantly (or covariantly) finite n-rigid and: \( \mathcal{T} = X \ast X[1] \ast \cdots \ast X[n] \).
(xi) \( X \) is contravariantly (or covariantly) finite n-rigid and, \( \forall A \in \mathcal{T} : \Omega X^n(A) \in X \).
(xii) \( X \) is contravariantly (or covariantly) finite n-rigid and, \( \forall A \in \mathcal{T} : \Sigma X^n(A) \in X \).
(xiv) \( X \) is covariantly finite n-rigid, any object of \( X^T_n[n+1] \) is injective in \( \text{mod} \cdot X \) and the functor \( H : X^T_n[n+1] \rightarrow \text{mod} \cdot X \) is full and reflects isomorphisms.

If \( X \) is a \( (n+1) \)-cluster subcategory of \( \mathcal{T} \), then the abelian category \( \text{mod} \cdot X \) has enough projectives and enough injectives, the functors \( X[n+1] \rightarrow \text{mod} \cdot X \leftarrow X \) are fully faithful and induce equivalences

\[
X[n+1] \xrightarrow{\cong} \text{Inj mod} \cdot X \quad \& \quad X \xrightarrow{\cong} \text{Proj mod} \cdot X
\]

By Proposition 5.1 and the results of Sections 3 and 4, the first thirteen conditions are equivalent. So to complete the proof of Theorem 5.3, it remains to show that (xiv) is equivalent to (i). This requires several steps.

**Lemma 5.4.** Let \( X \) be an \( (n+1) \)-cluster tilting subcategory of \( \mathcal{T} \). Let \( B \in \mathcal{T} \) be such that \( \mathcal{T}(X,B[-1]) = 0 \), \( 1 \leq i \leq n-1 \), i.e. \( B \in X_{n-1}^T[n] \). Then for any object \( A \in \mathcal{T} \), there exists a short exact sequence

\[
0 \rightarrow \text{Gh}_X^n(A,B) \rightarrow \mathcal{T}(A,B) \rightarrow \text{Hom}[H(A),H(B)] \rightarrow 0
\]

If, in addition, \( \mathcal{T}(X,B[-n]) = 0 \), i.e. if \( B \in X_n^T[n+1] \), then the map \( H_{A,B} \) is invertible.

**Proof.** By Corollary 3.6(ii), with \( k = n \), the map \( H_{A,B} \) is surjective. We show that the inclusion \( \text{Gh}_X^n(A,B) \subseteq \text{Gh}_X(A,B) = \text{Ker} H_{A,B} \) is an equality. Let \( \alpha : A \rightarrow B \) be such that \( H(\alpha) = 0 \), i.e. \( \alpha \) is \( X \)-ghost. Then \( \alpha \) factorizes through \( h^n_A : A \rightarrow \Omega X^1(A)[1] \), say via a map \( \beta : \Omega X^1_A(A)[1] \rightarrow B \). Since \( \Omega X^1_A(A)[1] \in X[1] \ast X[2] \ast \cdots \ast X[n] \), there are triangles

\[
X[i] \xrightarrow{\xi_i} M_i \xrightarrow{\xi_{i+1}} M_{i+1} \rightarrow X[i+1], \quad 1 \leq i \leq n-1, \quad M_1 = \Omega X^1_A(A)[1], \quad X_1 \in X \text{ and } M_{i+1} \in X[i+1] \ast \cdots \ast X[n]; \quad \text{in particular } M_n = X_n[n], \quad X_n \in X.
\]

Since \( \mathcal{T}(X,B[-1]) = 0 \), we have \( \xi_i \circ \beta = 0 \), and therefore \( \beta = \xi_1 \circ \beta_2 \) for some map \( \beta_2 : M_2 \rightarrow B \). Using that \( \mathcal{T}(X,B[-i]) = 0 \), \( 1 \leq i \leq n-1 \), by induction there exists a factorization \( \beta = \xi_1 \circ \xi_2 \circ \cdots \circ \xi_{n-1} \circ \beta_n \), where \( \beta_n : X_n[n] \rightarrow B \). Since \( M_i \in X[i] \ast \cdots \ast X[n] \) and since clearly any map from an object of \( X[i] \) to an object in \( X[i+1] \ast \cdots \ast X[n] \) is zero, the map \( \xi_i : M_i \rightarrow M_{i+1} \) is \( X[i] \)-ghost. In particular the map \( \xi_{n-1} \circ \beta_n : M_{n-1} \rightarrow B \)
is \( \mathcal{X}[n-1] \)-ghost. Since \( \alpha = h^0_A \circ \beta = h^0_A \circ \xi_1 \circ \xi_2 \circ \cdots \circ \xi_{n-1} \circ \beta_n \), it follows that \( \alpha \) lies in \( \text{Gh}^n_\mathcal{X}(A,B) \), hence \( \text{Gh}_\mathcal{X}(A,B) \subseteq \text{Gh}^n_\mathcal{X}(A,B) \). If in addition \( \mathcal{T}(\mathcal{X},B[-n]) = 0 \), then the map \( \beta_n \) above is zero. So \( \alpha = 0 \) and then \( \text{Gh}^n_\mathcal{X}(A,B) = 0 \). \( \square \)

Since any object \( B \in \mathcal{X}[n+1] \) satisfies the assumptions of Lemma 5.4 we have the following.

**Corollary 5.5.** Let \( \mathcal{X} \) be an \((n+1)\)-cluster tilting subcategory of \( \mathcal{T} \). Then the functor \( H : \mathcal{X}[n+1] \rightarrow \text{mod-} \mathcal{X} \) is fully faithful and for any object \( A \in \mathcal{T} \) and any object \( X \in \mathcal{X} \), we have a natural isomorphism

\[
H_{A,X[n+1]} : \mathcal{T}(A,X[n+1]) \xrightarrow{\cong} \text{Hom}(H(A),H(X[n+1]))
\]

The following result gives the implication (i) \( \Rightarrow \) (xiv) in Theorem 5.3.

**Proposition 5.6.** Let \( \mathcal{X} \) be an \((n+1)\)-cluster tilting subcategory of \( \mathcal{T} \). Then \( \text{mod-} \mathcal{X} \) has enough injectives, and the functor \( H : \mathcal{T} \rightarrow \text{mod-} \mathcal{X} \) induces an equivalence

\[
H : \mathcal{X}[n+1] \xrightarrow{\cong} \text{Inj mod-} \mathcal{X}
\]

**Proof.** Recall that by Lemma 1.5 the functor \( H : \mathcal{T} \rightarrow \text{mod-} \mathcal{X} \) is almost full, i.e. setting \( A^* = \text{Cell}_1(A) \), for any object \( A \in \mathcal{T} \), we have a canonical map \( \gamma_A^1 : A^* \rightarrow A \) such that \( H(\gamma_A^1) \) is invertible and, for any map \( \tilde{\mu} : H(A) \rightarrow H(B) \) in \( \text{mod-} \mathcal{X} \), there exists a map \( \mu : A^* \rightarrow B^* \) in \( \mathcal{T} \) and a commutative diagram

\[
\begin{array}{ccc}
H(A^*) & \xrightarrow{H(\mu)} & H(B^*) \\
\downarrow H(\gamma_A^1) & \cong & \cong \downarrow H(\gamma_B^1) \\
H(A) & \xrightarrow{\tilde{\mu}} & H(B)
\end{array}
\]

Now let \( \tilde{\mu} : H(A) \rightarrow H(B) \) be a monomorphism in \( \text{mod-} \mathcal{X} \) and let \( \tilde{\alpha} : H(A) \rightarrow H(X[n+1]) \) be a map, where \( X \in \mathcal{X} \). Clearly the map \( H(\mu) \) in the above commutative diagram is a monomorphism, so if \( C \rightarrow A^* \rightarrow B^* \rightarrow C[1] \) is a triangle in \( \mathcal{T} \), then the map \( C \rightarrow A^* \) is \( \mathcal{X} \)-ghost and therefore it factorizes through \( \mathcal{X}^\top = \mathcal{X}[1] \cdots \cdots \mathcal{X}[n] \). By Corollary 5.5, there is a map \( \alpha : A \rightarrow X[n+1] \) such that \( H(\alpha) = \tilde{\alpha} \). Since any map from an object of \( \mathcal{X}[1] \cdots \cdots \mathcal{X}[n] \) to an object of \( \mathcal{X}[n+1] \) is clearly zero, it follows that the composition \( C \rightarrow A^* \rightarrow A \rightarrow X[n+1] \) is zero and therefore \( \gamma_A^1 \circ \alpha \) factorizes through \( \mu \), i.e. \( \gamma_A^1 \circ \alpha = \mu \circ \rho \) for some map \( \rho : B^* \rightarrow X[n+1] \). Then we have \( H(\gamma_A^1) \circ H(\alpha) = H(\mu) \circ H(\rho) \) and therefore \( H(\gamma_A^1) \circ H(\alpha) = H(\gamma_A^1) \circ \tilde{\mu} \circ H(\gamma_B^1)^{-1} \circ H(\rho) \), hence \( \tilde{\alpha} = H(\alpha) = \tilde{\mu} \circ H(\gamma_B^1)^{-1} \circ H(\rho) \), i.e. \( \tilde{\alpha} \) factors through \( \tilde{\mu} \). This shows that \( H(X[n+1]) \) is injective, \( \forall X \in \mathcal{X} \).
We show that any object $H(A)$ of $\text{mod-}\mathcal{X}$ is a subobject of an object from $H(\mathcal{X}[n+1])$. From the tower of triangles $(C^*_{A[-1]})$ associated to $A[-1]$, there is a map $\omega_{A[-1]}^{n-1} : A \rightarrow \Omega^n_{\mathcal{X}}(A[-1])[n+1]$ and a triangle

$$\Omega^n_{\mathcal{X}}(A[-1]) \longrightarrow \text{Cell}_{n-1}(A[-1])[1] \longrightarrow A \longrightarrow \Omega^n_{\mathcal{X}}(A[-1])[n+1]$$

where $\text{Cell}_{n-1}(A[-1])[1]$ lies in $\mathcal{X}[1] \ast \cdots \ast \mathcal{X}[n]$. Since $\Omega^n_{\mathcal{X}}(A[-1]) \in \mathcal{X}$, we have $\Omega^n_{\mathcal{X}}(A[-1])[n+1] \in \mathcal{X}[n+1]$. Hence $H(\Omega^n_{\mathcal{X}}(A[-1])[n+1])$ is injective in $\text{mod-}\mathcal{X}$, and the map $H(\omega_{A[-1]}^{n-1}) : H(A) \longrightarrow H(\Omega^n_{\mathcal{X}}(A[-1])[n+1])$ is a monomorphism, since $H(\mathcal{X}[1] \ast \cdots \ast \mathcal{X}[n]) = 0$. Hence $\text{mod-}\mathcal{X}$ has enough injectives.

Now let $H(A)$ be an injective object of $\text{mod-}\mathcal{X}$. By the above there exists a split monomorphism $H(\mu) : H(A) \longrightarrow H(\mathcal{X}[n+1])$, where $X \in \mathcal{X}$. Hence there is a map $\tilde{\alpha} : H(\mathcal{X}[n+1]) \longrightarrow H(A)$ such that $H(\mu) \circ \tilde{\alpha} = 1_{H(A)}$. Now the map $\tilde{\alpha} \circ H(\mu)$ is an idempotent endomorphism of $H(\mathcal{X}[n+1])$ and therefore since the functor $H : \mathcal{X}[n+1] \longrightarrow \text{mod-}\mathcal{X}$ is fully faithful, there exists an idempotent endomorphism $\varepsilon : X[n+1] \longrightarrow X[n+1]$ such that $H(\varepsilon) = \tilde{\alpha} \circ H(\mu)$. Since idempotents split in $\mathcal{T}$, there exist maps $\kappa : X[n+1] \longrightarrow D$ and $\lambda : D \longrightarrow X[n+1]$ such that $\varepsilon = \kappa \circ \lambda$ and $\lambda \circ \kappa = 1_{X[n+1]}$. Plainly $D$ is of the form $X'[n+1]$, for some object $X' \in \mathcal{X}$, as a direct summand of $X[n+1]$. Clearly the map $\phi := H(\mu) \circ H(\kappa) : H(A) \longrightarrow H(X'[n+1])$ is an isomorphism with inverse the map $\psi := H(\lambda) \circ \tilde{\alpha}$. Hence the functor $H : \mathcal{X}[n+1] \longrightarrow \text{Inj \ mod-}\mathcal{X}$ is surjective on objects and therefore an equivalence. \(\square\)

Finally the next result shows the implication (xiv) \(\Rightarrow\) (i) and completes the proof of Theorem 5.3.

**Proposition 5.7.** Let $\mathcal{X}$ be a contravariantly finite $n$-rigid subcategory of $\mathcal{T}$. If the functor $H : \mathcal{X}_n^\top [n+1] \longrightarrow \text{mod-}\mathcal{X}$ has image in $\text{Inj \ mod-}\mathcal{X}$, is full and reflects isomorphisms, then $\mathcal{X}$ is $(n+1)$-cluster tilting.

**Proof.** It suffices to show that $\mathcal{X}_n^\top \subseteq \mathcal{X}$. Let $A \in \mathcal{X}_n^\top$; applying $H$ to the triangle $\Omega^1_{\mathcal{X}}(A) \longrightarrow X^n_A \longrightarrow A \longrightarrow \Omega^1_{\mathcal{X}}(A)[1]$ it follows that $\Omega^1_{\mathcal{X}}(A) \in \mathcal{X}_n^\top$, and we have a monomorphism $H(g^n_A[n+1]) : H(\Omega^1_{\mathcal{X}}(A)[n+1]) \longrightarrow H(X^n_A[n+1])$. On the other hand, since $\mathcal{X} \subseteq \mathcal{X}_n^\top$ it follows that $X^n_A[n+1] \in \mathcal{X}_n^\top[n+1]$. Since the objects $H(\Omega^1_{\mathcal{X}}(A)[n+1])$ and $H(X^n_A[n+1])$ are injective in $\text{mod-}\mathcal{X}$, the above monomorphism splits. Since $H|_{\mathcal{X}_n^\top[n+1]}$ is full and reflects isomorphisms, this implies that the map $g^n_A[n+1] : \Omega^1_{\mathcal{X}}(A)[n+1] \longrightarrow X^n_A[n+1]$, or equivalently the map $\Omega^1_{\mathcal{X}}(A)[1] \longrightarrow X_A^0[1]$, is split monic. Then the map $A \longrightarrow \Omega^1_{\mathcal{X}}(A)[1]$ is zero and therefore $A$ lies in $\mathcal{X}$ as a direct summand of $X_A^0 \in \mathcal{X}$. Hence $\mathcal{X}_n^\top = \mathcal{X}$. \(\square\)

From now on: $\mathcal{X}$ denotes an $(n+1)$-cluster tilting subcategory of $\mathcal{T}$, where $n \geq 1$. 
Corollary 5.8.

(i) The abelian category $\text{mod-}\mathcal{X}$ is Frobenius if and only if $\mathcal{X} = \mathcal{X}[n + 1]$.

(ii) For any integer $k$, there exists a torsion triple in $\mathcal{T}$:

$$(\mathcal{X}[k - n] \ast \cdots \ast \mathcal{X}[k - 2] \ast \mathcal{X}[k - 1], \mathcal{X}[k], \mathcal{X}[k + 1] \ast \mathcal{X}[k + 2] \ast \cdots \ast \mathcal{X}[k + n])$$

(iii) If $\mathcal{T}$ admits a Serre functor $\mathcal{S} : \mathcal{T} \to \mathcal{T}$, then $\mathcal{S}$ induces an equivalence: $\mathcal{S} : \mathcal{X} \xrightarrow{\cong} \mathcal{X}[n + 1]$.

Proof. Part (i) follows from Theorem 5.3, and part (iii) follows from Lemma 4.8 and Proposition 5.6.

(ii) Since $\mathcal{X}^n = \mathcal{X}$, Proposition 3.2 implies that $(\mathcal{X} \ast \mathcal{X}[1] \ast \cdots \ast \mathcal{X}[n - 1], \mathcal{X}[n])$ is a torsion pair in $\mathcal{T}$, and then so is $(\mathcal{X}[k - n] \ast \cdots \ast \mathcal{X}[k - 2] \ast \mathcal{X}[k - 1], \mathcal{X}[k]), \forall k \in \mathbb{Z}$. The proof that $(\mathcal{X}[k], \mathcal{X}[k + 1] \ast \mathcal{X}[k + 2] \ast \cdots \ast \mathcal{X}[k + n])$ is a torsion pair in $\mathcal{T}$ is dual, using cocellular towers, see Subsection 1.2. □

Let $K_0(\mathcal{X}, \oplus)$ be the Grothendieck group of the exact category $\mathcal{X}$ endowed with the split exact structure and let $K_0(\mathcal{T})$ be the Grothendieck group of the triangulated category $\mathcal{T}$. If $n = 1$, i.e. $\mathcal{X}$ is a 2-cluster tilting, and if $\mathcal{T}$ is algebraic, then Palu [53] proved that $K_0(\mathcal{T})$ is a quotient of $K_0(\mathcal{X}, \oplus)$ by a certain subgroup. For $n \geq 2$ and for arbitrary $\mathcal{T}$ we have the following. Let $G$ be the subgroup of $K_0(\mathcal{X}, \oplus)$ generated by all elements $[X], \forall X \in \mathcal{X}$, for which there exist triangles $A' \to M_1 \to A'' \to A'[1]$ in $\mathcal{T}$, $i = 1, 2$, and an isomorphism $X \oplus M_1 \cong M_2$.

Corollary 5.9. The inclusion $\mathcal{X} \to \mathcal{T}$ induces an isomorphism: $K_0(\mathcal{X}, \oplus)/G \xrightarrow{\cong} K_0(\mathcal{T})$.

Proof. Clearly the inclusion $i : \mathcal{X} \to \mathcal{T}$ induces a homomorphism $K_0(i) : K_0(\mathcal{X}, \oplus) \to K_0(\mathcal{T})$, by $K_0(i)[X] = [X]$. Let $A$ be in $\mathcal{T}$ and consider the triangle $\Omega^1_X(A) \to X^0_A \to A \to \Omega^1_X(A)[1]$. Then in $K(\mathcal{T})$ we have a relation $[A] = [X^0_A] - [\Omega^1_X(A)]$. Similarly the triangle $\Omega^2_X(A) \to X^1_A \to \Omega^1_X(A) \to \Omega^2_X(A)[1]$ gives the relation $[\Omega^2_X(A)] = [X^1_A] - [\Omega^1_X(A)]$ and therefore we have $[A] = [X^0_A] - [X^1_A] + [\Omega^2_X(A)]$. Continuing in this way we have a relation in $K_0(\mathcal{T})$: $[A] = \sum_{k=0}^{n-1}(-1)^k[X^k_A] + (-1)^n[\Omega^n_X(A)]$. By Theorem 5.3 we have $\Omega^n_X(A) := X^n_A \in \mathcal{X}$, hence $[A] = \sum_{k=0}^{n}(-1)^k[X^k_A]$, so $[A] = \sum_{k=0}^{n}(-1)^kK_0(i)(X^k_A) = K_0(i)(\sum_{k=0}^{n}(-1)^k[X^k_A]), i.e. for any object $A$ in $\mathcal{T}$, the generator $[A]$ lies in the image of $K_0(i)$. This clearly implies that $K_0(i)$ is surjective. It is easy to see that the kernel of the map $K_0(i)$ coincides with the subgroup $G$. □

Let $\mathcal{T}$ be a Hom-finite triangulated category over a field $k$, and $T$ an $(n + 1)$-cluster tilting object of $\mathcal{T}$, i.e. $\text{add} T$ is an $(n + 1)$-cluster tilting subcategory of $\mathcal{T}$, $n \geq 1$. Since $K_0(\text{add} T, \oplus)$ is free of rank the number of indecomposable direct summands of $T$, by
Corollary 5.9, \( K_0(\mathcal{T}) \) is finitely generated. Hence any Hom-finite triangulated category over a field whose Grothendieck group is not finitely generated has no cluster tilting object.

5.2. Maximal rigid subcategories

Throughout this subsection we fix an \((n + 1)\)-Calabi–Yau triangulated category \( \mathcal{T} \). We also fix a functorially finite \( n \)-rigid subcategory \( \mathcal{X} \) of \( \mathcal{T} \), \( n \geq 1 \). Note that as a direct consequence of the \((n + 1)\)-Calabi–Yau property, if \( Y \) is an object of \( \mathcal{T} \) then: \( Y \in \mathcal{X}^\perp_n \) if and only if \( Y \in \frac{1}{n} \mathcal{X} \).

The following result, which generalizes a result of Zhou and Zhu, see [61], shows that for \( n = 1 \) the functor \( \Omega^1_{\mathcal{X}} \) preserves 1-rigidity, and for \( n > 1 \) the functor \( \Omega^k_{\mathcal{X}} |_{\mathcal{X}^\perp_{n-1}} \) preserves \( n \)-rigidity for \( 0 \leq k \leq n \).

Proposition 5.10. Let \( A \) be an \( n \)-rigid object of \( \mathcal{T} \), where we assume that \( A \in \mathcal{X}^\perp_n \), if \( n > 1 \). Then the object \( \Omega^k_{\mathcal{X}}(A) \) is \( n \)-rigid, \( 1 \leq k \leq n \), and lies in \( \mathcal{X}^\perp_n \).

Proof. First assume that \( n > 1 \). For \( 0 \leq k \leq n \), consider the triangles, where \( \Omega^k_{\mathcal{X}}(A) := A \):

\[
\Omega^{k+1}_{\mathcal{X}}(A) \xrightarrow{g^k_A} X^k_A \xrightarrow{f^k_A} \Omega^k_{\mathcal{X}}(A) \xrightarrow{h^k_A} \Omega^{k+1}_{\mathcal{X}}(A)[1] \quad (T_k(A))
\]

Applying \( \mathcal{H} \) to the triangle \( T_0(A) \) and using that \( A \in \mathcal{X}^\perp_{n-1} \), hence \( \mathcal{H}(X, A[i]) = 0 \), \( 1 \leq i \leq n - 1 \), we have \( \mathcal{H}(X, \Omega^k_{\mathcal{X}}(A)[i]) = 0 \), \( 1 \leq i \leq n \), hence \( \Omega^k_{\mathcal{X}}(A) \in \mathcal{X}^\perp_n \). Using this and applying \( \mathcal{H} \) to the triangle \( T_1(A) \) we see directly that \( \Omega^k_{\mathcal{X}}(A) \in \mathcal{X}^\perp_n \). It follows by induction that \( \Omega^k_{\mathcal{X}}(A) \in \mathcal{X}^\perp_n \), \( \forall k \geq 1 \).

Applying the functor \( \mathcal{H}(\Omega^1_{\mathcal{X}}(A), -) \) to the triangle \( T_0(A) \), we have a long exact sequence

\[
\cdots \to \mathcal{H}(\Omega^1_{\mathcal{X}}(A), X^0_A[k]) \to \mathcal{H}(\Omega^1_{\mathcal{X}}(A), A[k]) \to \mathcal{H}(\Omega^1_{\mathcal{X}}(A), \Omega^1_{\mathcal{X}}(A)[k+1]) \to \mathcal{H}(\Omega^1_{\mathcal{X}}(A), X^0_A[k+1]) \to \cdots
\]

Since \( \mathcal{H}(\Omega^1_{\mathcal{X}}(A), X^0_A[k+1]) \cong \mathcal{D}(\mathcal{H}[k-n], \Omega^1_{\mathcal{X}}(A)) = \mathcal{D}(\mathcal{H}[k-n], \Omega^1_{\mathcal{X}}(A)[n-k]) = 0 \), for \( 0 \leq k \leq n - 1 \), it follows that we have an isomorphism

\[
\mathcal{H}(\Omega^1_{\mathcal{X}}(A), A[k]) \cong \mathcal{H}(\Omega^1_{\mathcal{X}}(A), \Omega^1_{\mathcal{X}}(A)[k+1]) \quad (5.1)
\]

and an exact sequence

\[
\mathcal{H}(\Omega^1_{\mathcal{X}}(A), X^0_A) \to \mathcal{H}(\Omega^1_{\mathcal{X}}(A), A) \to \mathcal{H}(\Omega^1_{\mathcal{X}}(A), \Omega^1_{\mathcal{X}}(A)[1]) \to 0 \quad (5.2)
\]

Applying the functor \( \mathcal{H}(-, A) \) to the triangle \( T_0(A) \), we have a long exact sequence
\[
\cdots \to \mathcal{T}(X_0^0[-k], A) \to \mathcal{T}(\Omega^1_X(A)[-k], A) \\
\to \mathcal{T}(A[-k-1], A) \to \mathcal{T}(X_A^0[-k-1], A) \to \cdots
\]

Since \(A\) is \(n\)-rigid and \(\mathcal{T}(X, A[k]) = 0\), for \(1 \leq k \leq n - 1\), we have \(\mathcal{T}(\Omega^1_X(A), A[k]) = 0\) and an exact sequence

\[
\mathcal{T}(A, A) \to \mathcal{T}(X_A^0, A) \to \mathcal{T}(\Omega^1_X(A), A) \to 0 \tag{5.3}
\]

Hence from (5.1) we have \(\mathcal{T}(\Omega^1_X(A), \Omega^1_X(A)[k]) = 0\), for \(2 \leq k \leq n\). On the other hand from (5.3) it follows that any map \(\alpha : \Omega^1_X(A) \to A\) is of the form \(\alpha = g^0_A \circ \rho\) for some map \(\rho : X_A^0 \to A\). Then \(\rho\) factorizes through the right \(X\)-approximation \(f^0_A : X_A^0 \to A\), i.e. there is an endomorphism \(\sigma : X_A^0 \to X_A^0\) such that \(\rho = \sigma \circ f^0_A\). Then \(\alpha = g^0_A \circ \rho = g^0_A \circ \sigma \circ f^0_A\), i.e. \(\alpha\) factorizes through \(f^0_A\) and this means that the leftmost map in (5.2) is surjective. It follows that \(\mathcal{T}(\Omega^1_X(A), \Omega^1_X(A)[1]) = 0\). We infer that \(\Omega^1_X(A)\) is \(n\)-rigid and as noted above \(\Omega^1_X(A)\) lies in \(X^+_X\). Hence replacing \(A\) with \(\Omega^1(A)\), we have that \(\Omega^2_X(A)\) is \(n\)-rigid and lies in \(X^+_X\), and then by induction we conclude that \(\Omega^n_X(A)\) is \(n\)-rigid and lies in \(X^+_X\) for \(1 \leq k \leq n\).

If \(n = 1\), then the above proof shows that \(\Omega^1_X(A)\) is \(1\)-rigid and of course \(\Omega^1_X(A)\) lies in \(X^+_X\). \(\square\)

**Definition 5.11.** A functorially finite subcategory \(X\) of \(\mathcal{T}\) is called **maximal \(n\)-rigid** if \(X\) is \(n\)-rigid and one of the following equivalent conditions is satisfied:

1. \(X\) contains any \(n\)-rigid subcategory \(Y\) such that \(Y \subseteq X^+_n\).
2. \(X\) contains any \(n\)-rigid subcategory \(Y\) such that \(Y \supseteq X_n\).

**Corollary 5.12.** Let \(X\) be a maximal \(n\)-rigid subcategory of \(\mathcal{T}\), and let \(A\) be an \(n\)-rigid object of \(\mathcal{T}\).

1. Assume that \(A \in X^+_n\), if \(n > 1\). Then \(\Omega^n_X(A) \in X\) and \(A \in X \times X[1] \times \cdots \times X[n]\).
2. Assume that \(A \in X^+_n\), if \(n > 1\). Then \(\Sigma^n_X(A) \in X\) and \(A \in X[-n] \times \cdots \times X[-1] \times X\).

In particular: \(\{A \in X^+_n \mid A \text{ is } \text{n-rigid}\} = X = \{A \in X^+_n \mid A \text{ is } \text{n-rigid}\}\).

**Proof.** (i) Let \(A \in X^+_n\). Then, as observed in Proposition 5.10, we have that \(\Omega^n_X(A)\) is \(n\)-rigid and \(\Omega^n_X(A) \in X^+_n\). Since \(X\) is maximal \(n\)-rigid, it follows that \(\Omega^n_X(A)\) lies in \(X\) and then clearly \(A\) lies in \(X \times X[1] \times \cdots \times X[n]\). Hence there is a triangle \(M \to A \to X[n] \to M[1]\), where \(M \in X \times X[1] \times \cdots \times X[n - 1]\) and \(X \in X\). If \(A \in X^+_n\), i.e. we have in addition \(\mathcal{T}(X, A[n]) = 0\), then using Calabi–Yau duality we also have \(\mathcal{T}(A, X[n]) = 0\). Hence the above triangle splits and then \(A\) lies in \(X \times X[1] \times \cdots \times X[n - 1]\) as a direct summand of \(M\). Using that \(A \in X^+_n\) and induction it follows that \(A \in X\). Hence \(\{A \in X^+_n \mid A \text{ is } \text{n-rigid}\} = X\). Part (ii) is dual. \(\square\)
The following consequence will be useful in the next section in connection with the Gorenstein property.

**Corollary 5.13.** Let $\mathcal{X}$ be a maximal $n$-rigid subcategory of $\mathcal{T}$. If $\mathcal{X}$ is $(n-1)$-crigid, then

$$\mathcal{X}[n+1] \subseteq \mathcal{X} \star \mathcal{X}[1] \supseteq \mathcal{X}[-n]$$

**Proof.** Clearly $\mathcal{X}[n+1]$ is $n$-rigid. Since $\mathcal{X}$ is $(n-1)$-crigid, it follows directly that $\mathcal{X}[n+1] \subseteq \mathcal{X}_{n-1}$ and therefore, by Corollary 5.12, $\mathcal{X}[n+1] \subseteq \mathcal{X} \star \mathcal{X}[1] \star \cdots \star \mathcal{X}[n]$. Then using that $\mathcal{X}$ is $(n-1)$-crigid, as in the proof of (i) $\Rightarrow$ (ii) in Proposition 3.12, we infer that $\mathcal{X}[n+1] \subseteq \mathcal{X} \star \mathcal{X}[1]$.

On the other hand, $\forall X \in \mathcal{X}$, the object $X[-n]$ is clearly $n$-rigid and since $\mathcal{X}$ is $(n-1)$-crigid, we have $\mathcal{X}[-n] \subseteq \mathcal{X}_{n-1}^\perp$. By Corollary 5.12 it follows that $X[-n] \in \mathcal{X} \star \mathcal{X}[1] \star \cdots \star \mathcal{X}[n]$. Hence there exists a triangle $A \rightarrow X[-n] \rightarrow B \rightarrow A[1]$, where $A \in \mathcal{X} \star \mathcal{X}[1]$ and $B \in \mathcal{X}[2] \star \cdots \star \mathcal{X}[n]$. Using that $\mathcal{X}$ is $(n-1)$-crigid and the $(n+1)$-Calabi–Yau property it is easy to see that $\mathcal{T}(\mathcal{X}[n], \mathcal{X}[2] \star \cdots \star \mathcal{X}[n]) = \mathcal{T}(\mathcal{X}, \mathcal{X}[n+2] \star \cdots \star \mathcal{X}[2n]) = 0$. Hence the map $X[-n] \rightarrow B$ is zero and therefore $X[-n]$ lies in $\mathcal{X} \star \mathcal{X}[1]$ as a direct summand of $A$. We infer that $\mathcal{X}[-n] \subseteq \mathcal{X} \star \mathcal{X}[1]$. $\square$

6. Some cluster tilting subcategories are Gorenstein. . .

Keller and Reiten proved in [46] that the cluster tilted category of a 2-cluster tilting subcategory in a 2-Calabi–Yau triangulated category is Gorenstein, see also [48] and [16]. However, by an example of Iyama, see [46], the Gorenstein property fails for higher cluster tilting subcategories. Our aim in this section is to show that a special class of $n$-cluster tilting subcategories of an arbitrary triangulated category, where $n > 2$, enjoys the property that the associated cluster tilted category is Gorenstein, and then to derive some direct consequences.

6.1. Gorenstein categories and Gorenstein-projective objects

Let $\mathcal{A}$ be an abelian category with enough projectives and enough injectives. We recall from [19] the following invariants attached to $\mathcal{A}$:

\[
\text{silp} \mathcal{A} = \sup \{ \text{id} \mathcal{P} \mid \mathcal{P} \in \text{Proj} \mathcal{A} \} \quad \& \quad \text{spli} \mathcal{A} = \sup \{ \text{pd} \mathcal{I} \mid \mathcal{I} \in \text{Inj} \mathcal{A} \}
\]

\[
\text{G-dim} \mathcal{A} := \max \{ \text{silp} \mathcal{A}, \text{spli} \mathcal{A} \}
\]

We call $\text{G-dim} \mathcal{A}$ the Gorenstein dimension of $\mathcal{A}$ and then $\mathcal{A}$ is called Gorenstein if $\text{G-dim} \mathcal{A} < \infty$. If $\text{G-dim} \mathcal{A} \leq n < \infty$, then we say that $\mathcal{A}$ is $n$-Gorenstein.

**Remark 6.1.** It is not difficult to see that if $\text{spli} \mathcal{A} < \infty$ and $\text{silp} \mathcal{A} < \infty$, then $\text{spli} \mathcal{A} = \text{silp} \mathcal{A}$; moreover $\text{G-dim} \mathcal{A} \leq \text{gl.dim} \mathcal{A}$ with equality if $\text{gl.dim} \mathcal{A} < \infty$, see [19, Proposition VII.1.3].
For future use we also recall the notion of Gorenstein-projective objects and their basic properties.

A complex of projective objects $P^\bullet: \cdots \to P^{-1} \to P^0 \to P^1 \to \cdots$ in $\mathcal{A}$ is called \textit{totally acyclic} if $P^\bullet$ and the induced complex $\mathcal{A}(P^\bullet, Q)$ are acyclic, for any projective object $Q$ of $\mathcal{A}$. Dually, a complex of injective objects $I^\bullet: \cdots \to I^{-1} \to I^0 \to I^1 \to \cdots$ is called \textit{totally acyclic} if $I^\bullet$ and the induced complex $\mathcal{A}(J, I^\bullet)$ are acyclic for any injective object $J$ of $\mathcal{A}$.

\textbf{Definition 6.2.}

(i) An object $G \in \mathcal{A}$ is called \textbf{Gorenstein-projective} if $G \cong \text{Coker}(P^{-1} \to P^0)$ for some totally acyclic complex $P^\bullet$ of projectives.

(ii) An object $G \in \mathcal{A}$ is called \textbf{Gorenstein-injective} if $G \cong \text{Ker}(I^0 \to I^1)$ for some totally acyclic complex $I^\bullet$ of injectives.

The full subcategory of Gorenstein-projective, resp. Gorenstein-injective, objects of $\mathcal{A}$ is denoted by $\text{GProj} \mathcal{A}$, resp. $\text{GInj} \mathcal{A}$. Also we denote by $\text{GProj} \mathcal{A}$, resp. $\overline{\text{GInj}} \mathcal{A}$, the stable category of $\text{GProj} \mathcal{A}$, resp. $\text{GInj} \mathcal{A}$, modulo projectives, resp. injectives.

In the following remark we remind the reader of the basic properties of Gorenstein categories which will be used in the sequel. For proofs and a more detailed discussion, we refer to [11,19].

\textbf{Remark 6.3.} Let $\mathcal{A}$ be an abelian category with enough projective and/or injective objects. In this context, we denote by $\Omega^k A$, resp. $\Sigma^k A$, $k \geq 1$, the usual $k$-syzygy, resp. $k$-cosyzygy, object of $A \in \mathcal{A}$, which is uniquely determined modulo projectives, resp. injectives. We also denote by $\Omega^k \mathcal{A}$, resp. $\Sigma^k \mathcal{A}$, the full subcategory of $\mathcal{A}$ consisting of the $k$-syzygy, resp. $k$-cosyzygy, objects of $\mathcal{A}$, and by $\text{Proj}^{<\infty} \mathcal{A}$, resp. $\text{Inj}^{<\infty} \mathcal{A}$, the full subcategory of $\mathcal{A}$ consisting of all objects with finite projective, resp. injective, dimension. If $\mathcal{C} \subseteq \mathcal{A}$ is a full subcategory of $\mathcal{A}$, then the stable category $\mathcal{C}/\text{Proj} \mathcal{A}$, resp. $\mathcal{C}/\text{Inj} \mathcal{A}$, is denoted by $\mathcal{C}$, resp. $\overline{\mathcal{C}}$, and the morphism spaces $\mathcal{C}(C_1, C_2)$, resp. $\overline{\mathcal{C}}(C_1, C_2)$, are denoted by $\text{Hom}_{\mathcal{A}}(C_1, C_2)$, resp. $\overline{\text{Hom}}_{\mathcal{A}}(C_1, C_2)$.

1. For any objects $G_1 \in \text{GProj} \mathcal{A}$, $G_2 \in \text{GInj} \mathcal{A}$, $A \in \mathcal{A}$, and any $k \geq 1$, there are isomorphisms:

$$\text{Ext}^k_{\mathcal{A}}(G_1, A) \cong \text{Hom}_{\mathcal{A}}(\Omega^k G_1, A) \quad \text{and} \quad \text{Ext}^k_{\mathcal{A}}(A, G_2) \cong \overline{\text{Hom}}_{\mathcal{A}}(A, \Sigma^k G_2)$$

2. The categories $\text{GProj} \mathcal{A}$ and $\text{GInj} \mathcal{A}$ are exact Frobenius subcategories of $\mathcal{A}$. Hence the stable categories $\text{GProj} \mathcal{A}$ and $\overline{\text{GInj}} \mathcal{A}$ are triangulated.

3. If $\mathcal{A}$ is Gorenstein then $\text{Proj}^{<\infty} \mathcal{A} = \text{Inj}^{<\infty} \mathcal{A}$; if G-dim $\mathcal{A} = d$, then $\text{Proj}^{<\infty} \mathcal{A} = \text{Proj}^{\leq d} \mathcal{A}$ (and dually $\text{Inj}^{<\infty} \mathcal{A} = \text{Inj}^{\leq d} \mathcal{A}$). Moreover we have $\text{GProj} \mathcal{A} = \Omega^d \mathcal{A}$ and $\text{GInj} \mathcal{A} = \Sigma^d \mathcal{A}$, see [11, Theorem 4.16]. It follows that if G-dim $\mathcal{A} \leq 1$, then
\( \text{GProj } \mathcal{A} = \Omega \mathcal{A} \) consists of the subobjects of the projective objects and \( \text{GInj } \mathcal{A} = \Sigma \mathcal{A} \) consists of the factors of the injectives.

4. The inclusion functor \( \text{GProj } \mathcal{A} \rightarrow \mathcal{A} \) admits the functor \( \Omega^{-d} \Omega^d : \mathcal{A} \rightarrow \text{GProj } \mathcal{A} \) as a right adjoint, and the inclusion functor \( \text{GInj } \mathcal{A} \rightarrow \mathcal{A} \) admits the functor \( \Sigma^{-d} \Sigma^d : \mathcal{A} \rightarrow \text{GInj } \mathcal{A} \) as a left adjoint.

5. If \( \mathcal{A} \) is Gorenstein, then \( \text{GProj } \mathcal{A} = \{ A \in \mathcal{A} | \text{Ext}^d_n(A, P) = 0, \forall P \in \text{Proj } \mathcal{A}, \forall n \geq 1 \} \). Dually \( \text{GInj } \mathcal{A} = \{ A \in \mathcal{A} | \text{Ext}^d_n(I, A) = 0, \forall I \in \text{Inj } \mathcal{A}, \forall n \geq 1 \} \). In this case the Gorenstein-projective objects are usually called Cohen–Macaulay objects and \( \text{GProj } \mathcal{A} \) is denoted by \( \text{CM}(\mathcal{A}) \).

### 6.2. Gorensteinness of corigid cluster tilting subcategories

Our main aim in this subsection is to prove, and discuss the consequences of, the following result. Note that part (i) is due to Keller and Reiten [46] if \( \mathcal{T} \) is 2-Calabi–Yau and to Koenig and Zhu [48] if \( \mathcal{T} \) is not necessarily Calabi–Yau.

**Theorem 6.4.** Let \( \mathcal{X} \) be an \((n + 1)\)-cluster tilting subcategory of \( \mathcal{T} \), where \( n \geq 1 \).

1. If \( n = 1 \), then \( \text{G-dim } \text{mod-} \mathcal{X} \leq 1 \).
2. \( \text{G-dim } \text{mod-} \mathcal{X} = 0 \) if and only if \( \mathcal{X} \) is \( n \)-corigid if and only if \( \mathcal{X} = \mathcal{X}[n + 1] \).
3. Assume that \( n \geq 2 \) and \( \mathcal{X} \) is \((n - k)\)-corigid, where \( 0 \leq k \leq n - 1 \). Then:

\[
0 \leq k \leq \frac{n + 1}{2} \implies \text{G-dim } \text{mod-} \mathcal{X} \leq k
\]

Moreover if \( \mathcal{X} \) is strictly \((n - k)\)-corigid, then: \( \text{G-dim } \text{mod-} \mathcal{X} = k \).

**Proof.** Since \( \mathcal{X} \) is \((n + 1)\)-cluster tilting, we have \( \mathcal{X}_{n}[n + 1] = \mathcal{X}[n + 1] \). It follows that for any object \( A \in \mathcal{T} \) we have: \( \text{cores.dim}_{\mathcal{X}_{n}[n + 1]} A = \text{cores.dim}_{\mathcal{X}[n + 1]} A = \text{id } \text{H}(A) \), and clearly \( \text{res.dim}_{\mathcal{X}} A = \text{pd } \text{H}(A) \). Hence

\[
\sup_{X \in \mathcal{X}_{n}[n + 1]} \text{cores.dim}_{\mathcal{X}_{n}[n + 1]} X = \text{silp mod-} \mathcal{X} \quad \text{and} \quad \sup_{A \in \mathcal{X}_{n}[n + 1]} \text{res.dim}_{\mathcal{X}} A = \text{spli mod-} \mathcal{X}
\]

and the assertions (i)–(iii) follow from Corollary 4.5.

It remains to show that if \( \mathcal{X} \) is strictly \((n - k)\)-corigid, then: \( \text{G-dim } \text{mod-} \mathcal{X} = k \). It suffices to show that if \( \text{pd } \text{H}(\mathcal{X}[n + 1]) \leq k - 1, \forall X \in \mathcal{X} \), then \( \mathcal{X} \) is \((n - k + 1)\)-corigid. If \( 0 \leq k \leq 1 \), then the assertion is clear. So assume that \( k \geq 2 \). We consider triangles \( (T_i) : A^t \rightarrow X^{t-1} \rightarrow A^{t-1} \rightarrow A^t[1] \), where each map \( X^{t-1} \rightarrow A^{t-1} \) is a right \( \mathcal{X} \)-approximation, \( t \geq 1 \), and \( A^0 = X[n+1] \), so that \( A^t = \Omega^{-t} \mathcal{X}(X[n+1]) \). Applying \( \text{H} \) to the triangle \( (T_1) \), we obtain an exact sequence \( 0 \rightarrow H(A^1) \rightarrow H(X^0) \rightarrow H(X[n+1]) \rightarrow 0 \) and \( T(X, A_i[-i]) = 0, 1 \leq i \leq n - k \). Using this and applying \( \text{H} \) to the triangle \( (T_2) \), we obtain an exact sequence \( 0 \rightarrow H(A^2) \rightarrow H(X^1) \rightarrow H(A^1) \rightarrow 0 \) and
\[ \mathcal{T}(X, A^2[-i]) = 0, \ 1 \leq i \leq n - k - 1. \] Continuing in this way, we finally obtain an exact sequence \( 0 \rightarrow H(A^{k-1}) \rightarrow H(X^{k-2}) \rightarrow H(A^{k-2}) \rightarrow 0 \) and \( \mathcal{T}(X, A^{k-1}[-i]) = 0, \ 1 \leq i \leq n - 2k + 2. \) Since \( \text{pd} H(X[n + 1]) \leq k - 1, \) the object \( H(A^{k-1}) \) is projective. Hence there is a map \( \alpha : X^* \rightarrow A^{k-1}, \) where \( X^* \in \mathcal{X}, \) inducing an isomorphism \( H(\alpha) : H(X^*) \cong H(A^{k-1}). \) Let \( X^* \rightarrow A^{k-1} \rightarrow B \rightarrow X^*[1] \) be a triangle. Applying \( H \) to this triangle and using that \( \mathcal{X} \) is \((n-k)\)-corigid, the fact that \( H(\alpha) \) is invertible, and the vanishing condition \( \mathcal{T}(X, A^{k-1}[-i]) = 0, \ 1 \leq i \leq n - 2k + 2, \) we infer that \( \mathcal{T}(X, B[-k + 1]) = \mathcal{T}(X, B[-k + 2]) = \cdots = \mathcal{T}(X, B[-k + n]) = 0, \) i.e. \( B[-k] \in \mathcal{X}_n^+ = \mathcal{X}. \) Hence \( B \in \mathcal{X}[k] \) and this implies that the map \( B \rightarrow X^*[1] \) is zero since it lies in \( \mathcal{T}(X, \mathcal{X}[-k + 1]) \) and this space is zero since \( 1 \leq k - 1 \leq n - k, \) \( \mathcal{X} \) is \((n-k)\)-corigid and \( n \geq 2k - 1. \) We infer that \( A^{k-1} \) admits a direct sum decomposition \( A^{k-1} \cong X^* \oplus X'[k]. \) On the other hand, since \( A^{k-1} = \Omega_{\mathcal{X}}^k(X[n + 1]), \) we know by Remark 1.4 that \( A^{k-1} \) lies in \( \mathcal{X} \ast \mathcal{X}[1] \ast \cdots \ast \mathcal{X}[k-1]. \) Since clearly any map from an object of \( \mathcal{X} \ast \mathcal{X}[1] \ast \cdots \ast \mathcal{X}[k-1] \) to an object from \( \mathcal{X}[k] \) is zero, it follows that the projection \( A^{k-1} \rightarrow \mathcal{X}'[k] \) is zero and this implies that \( A^{k-1} \cong X^* \in \mathcal{X}. \) Then \( A^{k-2} \) lies in \( \mathcal{X} \ast \mathcal{X}[1] \) and using that \( A' = \Omega_{\mathcal{X}}^k(X[n + 1]), \forall i \geq 1, \) it follows inductively that \( A'^1 \in \mathcal{X} \ast \mathcal{X}[1] \ast \cdots \ast \mathcal{X}[k-2]. \) Then \( X[n + 1] \) lies in \( \mathcal{X} \ast \mathcal{X}[1] \ast \cdots \ast \mathcal{X}[k-1], \) hence \( \mathcal{X}[n + 1] \subset \mathcal{X} \ast \mathcal{X}[1] \ast \cdots \ast \mathcal{X}[k-1]. \) Then Proposition 3.12 shows that \( \mathcal{X} \) is \((n-k+1)\)-corigid as required. \( \square \)

A special case of part (a) of the next consequence was observed independently by Iyama and Oppermann [43].

**Corollary 6.5.** Let \( \mathcal{X} \) be an \((n+1)\)-cluster tilting subcategory of \( \mathcal{T}. \)

(a) If \( \mathcal{X} \) is \((n-1)\)-corigid, then: \( \text{G-dim } \text{mod-} \mathcal{X} \leq 1. \)

(b) If \( n \geq 3 \) is odd and \( \mathcal{X} \) is strictly \((\frac{n-1}{2})\)-corigid, then: \( \text{G-dim } \text{mod-} \mathcal{X} = \frac{n+1}{2}. \)

(c) If \( n \geq 2 \) is even and \( \mathcal{X} \) is strictly \((\frac{n}{2})\)-corigid, then: \( \text{G-dim } \text{mod-} \mathcal{X} = \frac{n}{2}. \)

Using Remark 6.1 we have the following consequence.

**Corollary 6.6.** Let \( \mathcal{X} \) be an \((n+1)\)-cluster tilting subcategory of \( \mathcal{T}, \) where \( n \geq 1. \) Let \( 0 \leq k \leq n-1 \) and assume that \( \mathcal{X} \) is \((n-k)\)-corigid. If \( n \geq 2k - 1, \) then either \( \text{gl.dim } \text{mod-} \mathcal{X} = \infty \) or else \( \text{gl.dim } \text{mod-} \mathcal{X} \leq k. \) Moreover if \( \text{gl.dim } \text{mod-} \mathcal{X} < \infty \) and if \( \mathcal{X} \) is strictly \((n-k)\)-corigid, then \( \text{gl.dim } \text{mod-} \mathcal{X} = k. \)

We have seen that if \( \mathcal{X} \) is a 2-cluster tilting subcategory of \( \mathcal{T}, \) then the functor \( H: \mathcal{T} \rightarrow \text{mod-} \mathcal{X} \) is full. Generalizing [16, Theorem 5.2] we provide a partial converse and we show that, for an \((n+1)\)-cluster tilting subcategory \( \mathcal{X} \) of \( \mathcal{T}, \) fullness of \( H \) implies that \( \text{mod-} \mathcal{X} \) is 1-Gorenstein. Note that \( H \) is not always full; we refer to [16], [46] for an example of a 3-cluster tilting subcategory \( \mathcal{X} \) in a 3-Calabi–Yau triangulated category \( \mathcal{T} \) for which the functor \( H: \mathcal{T} \rightarrow \text{mod-} \mathcal{X} \) is not full and the cluster tilted category \( \text{mod-} \mathcal{X} \) is not Gorenstein.
Corollary 6.7. Let \( \mathcal{X} \) be a \((n+1)\)-cluster tilting subcategory of \( \mathcal{T} \), where \( n \geq 1 \). If the functor \( H: \mathcal{T} \rightarrow \text{mod-} \mathcal{X} \) is full, then the category \( \text{mod-} \mathcal{X} \) is 1-Gorenstein.

Proof. If \( n = 1 \), then \( \text{mod-} \mathcal{X} \) is 1-Gorenstein by Theorem 6.4(i). If \( n \geq 2 \), then by Proposition 3.11, \( \mathcal{X} \) is \((n-1)\)-corigid, and therefore \( \text{mod-} \mathcal{X} \) is 1-Gorenstein by Theorem 6.4(iii). \( \square \)

It is shown in [61] that the category of coherent functors over any maximal 1-rigid subcategory of a 2-Calabi–Yau triangulated category is 1-Gorenstein. We close this subsection by treating the case of maximal \( n \)-rigid subcategories.

Theorem 6.8. Let \( \mathcal{T} \) be an \((n+1)\)-Calabi–Yau triangulated category over a field \( k \) and \( \mathcal{X} \) a maximal \( n \)-rigid subcategory of \( \mathcal{T} \). If \( \mathcal{X} \) is an \((n-1)\)-corigid, then \( \text{mod-} \mathcal{X} \) is 1-Gorenstein.

Proof. By Lemma 4.8 the abelian category \( \text{mod-} \mathcal{X} \) has enough projectives and enough injectives; moreover we have \( \text{proj}(\mathcal{X}) = H(\mathcal{X}) \) and \( \text{inj}(\mathcal{X}) = H(\mathcal{X}[n+1]) \). By Corollary 5.13 for any object \( X \in \mathcal{X} \), there are triangles \( X^1 \rightarrow X^0 \rightarrow X[n+1] \rightarrow X^1[1] \) and \( X \rightarrow X_1[n+1] \rightarrow X_0[n+1] \rightarrow X[1] \), where the \( X^i \) and the \( X_i \) lie in \( \mathcal{X} \). Then clearly we have short exact sequences \( 0 \rightarrow H(X^1) \rightarrow H(X^0) \rightarrow H(X[n+1]) \rightarrow 0 \) and \( 0 \rightarrow H(X) \rightarrow H(X_1[n+1]) \rightarrow H(X_0[n+1]) \rightarrow 0 \) in \( \text{mod-} \mathcal{X} \). This means that \( \text{pd}(H(X[n+1])) \leq 1 \) and \( \text{id}(H(X)) \leq 1 \). We infer that \( \text{silp-} \text{mod-} \mathcal{X} \leq 1 \) and \( \text{spli-} \text{mod-} \mathcal{X} \leq 1 \). Hence \( \text{mod-} \mathcal{X} \) is 1-Gorenstein. \( \square \)

6.3. Keller–Reiten’s Morita Theorem for cluster categories

Prominent examples of \( d \)-Calabi–Yau triangulated categories over a field \( k \) are the \( d \)-cluster categories \( \mathscr{C}_H^{(d)} \), \( d \geq 1 \), associated to a finite-dimensional hereditary \( k \)-algebra \( H \) over a field \( k \), see [45]. In this context Keller and Reiten [47] proved that the \((n+1)\)-cluster categories \( \mathscr{C}_H^{(n+1)} \) are, up to a triangle equivalence, the algebraic \((n+1)\)-Calabi–Yau triangulated categories admitting an \((n-1)\)-corigid \((n+1)\)-cluster tilting object whose endomorphism algebra is hereditary. Corollary 6.5 allows us to give a slight improvement of [47, Theorem 4.2] by replacing hereditary with finite global dimension, see the comments after Theorem 4.2 in [47].

Theorem 6.9. (See [47].) Let \( \mathcal{T} \) be a \( k \)-linear triangulated category with finite-dimensional \( \text{Hom} \)-spaces over an algebraically closed field \( k \). Then for an integer \( n \geq 1 \), the following statements are equivalent.

(i) \( \mathcal{T} \) is triangle equivalent to the \((n+1)\)-cluster category \( \mathscr{C}_H^{(n+1)} \) of a finite-dimensional hereditary \( k \)-algebra \( H \).

(ii) \( \mathcal{T} \) is algebraic \((n+1)\)-Calabi–Yau and admits a \((n-1)\)-corigid \((n+1)\)-cluster tilting object \( T \) such that the endomorphism algebra \( \text{End}_\mathcal{T}(T) \) has finite global dimension.
6.4. Abelian subcategories

We show that if the cluster tilted category \( \text{mod-} \mathcal{X} \) of an \((n-k)\)-corigid \((n+1)\)-cluster tilting subcategory \( \mathcal{X} \) has finite global dimension, for some \( 0 \leq k \leq \frac{n}{2} \), then \( \text{mod-} \mathcal{X} \) can be realized as a full subcategory of \( \mathcal{T} \), in some cases via a \( \partial \)-functor \( \mathcal{T} \). Note that if \( n = k = 1 \) in the next result, then the corigidity condition is vacuous and \( \mathcal{T} \) gives the full embedding \( \text{Inj-mod-} \mathcal{X} \approx \mathcal{X}[2] \hookrightarrow \mathcal{T} \).

**Theorem 6.10.** Let \( \mathcal{X} \) be an \((n-k)\)-corigid \((n+1)\)-cluster tilting subcategory of \( \mathcal{T} \), where \( 0 \leq k \leq n - 1 \).

(i) If \( k \leq \frac{n}{2} \), then the following are equivalent:

(a) The cluster tilted category \( \text{mod-} \mathcal{X} \) has finite global dimension.

(b) The functor \( \mathcal{H} : \mathcal{T} \rightarrow \text{mod-} \mathcal{X} \) induces an equivalence:

\[ \mathcal{H} : (\mathcal{X} \star \mathcal{X}[1] \star \cdots \star \mathcal{X}[k]) \cap \mathcal{X}^+_k [k+1] \xrightarrow{\approx} \text{mod-} \mathcal{X} \]

If \( \mathcal{T} \) admits a Serre functor \( \mathcal{S} \), then the above conditions are equivalent to:

(c) The functor \( \mathcal{H} \circ \mathcal{S} : \mathcal{T} \rightarrow \text{mod-} \mathcal{X} \) induces an equivalence

\[ \mathcal{H} \circ \mathcal{S} : (\mathcal{X}[-k] \star \cdots \star \mathcal{X}[-1] \star \mathcal{X}) \cap \mathcal{X}^+[-k-1] \xrightarrow{\approx} \text{mod-} \mathcal{X} \]

In any of the above cases we have: \( \text{gl.dim mod-} \mathcal{X} \leq k \) with equality if \( \mathcal{X} \) is strictly \((n-k)\)-corigid.

(ii) If \( k = 1 \), then the induced full embedding \( \mathcal{T} : \text{mod-} \mathcal{X} \rightarrow \mathcal{T} \) is a \( \partial \)-functor, which extends uniquely to an additive functor \( \mathcal{D}^b(\text{mod-} \mathcal{X}) \rightarrow \mathcal{T} \) commuting with the shifts.

**Proof.** (i) (a) \( \Rightarrow \) (b) Since \( \mathcal{X} \) is \((n-k)\)-corigid and \( n \geq 2k \), it follows that \( \mathcal{X} \subseteq \mathcal{X}^+_{n-k}[n-k+1] \subseteq \mathcal{X}^+_k[k+1] \), i.e. \( \mathcal{X} \) is \( k \)-corigid. Then by Theorem 4.6, the functor \( \mathcal{H} \) induces an equivalence \( \mathcal{H} : (\mathcal{X} \star \mathcal{X}[1] \star \cdots \star \mathcal{X}[k]) \cap \mathcal{X}^+_k [k+1] \xrightarrow{\approx} \text{Proj}^k \text{mod-} \mathcal{X} \). By Theorem 6.4, \( \text{mod-} \mathcal{X} \) is \( k \)-Gorenstein, so finiteness of \( \text{gl.dim mod-} \mathcal{X} \) implies that \( \text{gl.dim mod-} \mathcal{X} \leq k \), and therefore \( \text{mod-} \mathcal{X} = \text{Proj}^k \text{mod-} \mathcal{X} \). The implication (b) \( \Rightarrow \) (a) and the equivalence (a) \( \Leftrightarrow \) (c) follow from Corollary 4.9 and Theorem 6.4.

(ii) Assume that \( k = 1 \), so \( n \geq 2 \), \( \mathcal{X} \) is \((n-1)\)-corigid, and \( \text{mod-} \mathcal{X} \) is hereditary. By Corollary 2.9 we have isomorphisms \( \mathcal{T}(A, B[1]) \cong \text{Ext}^1(H(A), H(B)), \forall A, B \in (\mathcal{X} \star \mathcal{X}[1]) \cap \mathcal{X}^+_1 [2] \). This implies easily that the induced fully faithful functor \( \mathcal{T} : \text{mod-} \mathcal{X} \approx (\mathcal{X} \star \mathcal{X}[1]) \cap \mathcal{X}^+_1 [2] \rightarrow \mathcal{T} \) is a \( \partial \)-functor; details are left to the reader. By a result of Amiot [1], \( \mathcal{T} \) extends uniquely to an additive functor \( \mathcal{D}^b(\text{mod-} \mathcal{X}) \rightarrow \mathcal{T} \) commuting with the shifts. \( \Box \)
6.5. Representation dimension

Recall that an additive category is called Krull–Schmidt if any of its objects is a finite coproduct of indecomposable objects and any indecomposable object has local endomorphism ring. A Krull–Schmidt category has finite representation type if there are only finitely many indecomposable objects up to isomorphism. Now let \( \mathcal{A} \) be an abelian category with enough projectives. We say that \( \mathcal{A} \) is of finite Cohen–Macaulay type, finite CM-type for short, if \( \text{GProj} \mathcal{A} \) is of finite representation type. Finally recall that the dominant dimension, \( \text{dom.dim} \mathcal{A} \), of \( \mathcal{A} \) is the largest \( n \in \mathbb{N} \cup \{\infty\} \) such that any projective object has an injective resolution whose first \( n \) terms are projective. The codominant dimension, \( \text{codom.dim} \mathcal{A} \), is defined dually. For an Artin algebra \( \Lambda \) we set:

\[
\text{dom.dim} \Lambda = \text{dom.dim} \text{ mod-}\Lambda.
\]

Our aim is to show that if \( X \) is an \( (n - 1) \)-corigid \((n + 1)\)-cluster tilting subcategory of a triangulated category \( \mathcal{T} \) and the associated cluster tilted category \( \text{mod-}\mathcal{X} \) is of finite CM-type, then \( \text{mod-}\mathcal{X} \) is equivalent to the category of finitely presented modules over a coherent ring of representation dimension \( \leq 3 \) in the sense of Auslander [3].

We begin with the following result which generalizes, and is inspired by, a result of Ringel proved in [58] in the context of finitely generated modules over an Artin algebra.

**Proposition 6.11.** Let \( \mathcal{A} \) be an abelian category with enough projectives and let \( \mathcal{U} \) be a contravariantly finite subcategory of \( \mathcal{A} \) which is closed under subobjects and contains the projectives. Also let \( \mathcal{V} \) be a contravariantly finite subcategory of \( \mathcal{A} \) which is closed under factors and set:

\[
\mathcal{E} := \mathcal{U} \oplus \mathcal{V} = \text{add} \{X \oplus Y \in \mathcal{A} \mid X \in \mathcal{U} \& Y \in \mathcal{V}\}
\]

Then \( \mathcal{E} \) is contravariantly finite in \( \mathcal{A} \) and the abelian category \( \text{mod-}\mathcal{E} \) has global dimension at most 3. If in addition \( \mathcal{A} \) has enough injectives and \( \mathcal{V} \) contains them, then \( \text{mod-}\mathcal{E} \) has dominant dimension at least 2.

**Proof.** Let \( A \) be in \( \mathcal{A} \) and let \( U_A^0 \rightarrow A \) be a right \( \mathcal{U} \)-approximation of \( A \). Since \( \mathcal{U} \) contains the projectives and is closed under subobjects, we have an exact sequence, where \( U_A^0 \) and \( \tilde{U}_A^1 \) lie in \( \mathcal{U} \):

\[
0 \rightarrow \tilde{U}_A^1 \rightarrow U_A^0 \xrightarrow{f_A} A \rightarrow 0 \tag{6.1}
\]

Let \( g_A : \tilde{V}_A^0 \rightarrow A \) be a right \( \mathcal{V} \)-approximation of \( A \) and set \( V_A^0 = \text{Im}(\tilde{V}_A^0 \rightarrow A) \). Note that \( V_A^0 \) lies in \( \mathcal{V} \) since \( \mathcal{V} \) is closed under factors. Clearly then the inclusion \( g_A : V_A^0 \rightarrow A \) is a right \( \mathcal{V} \)-approximation of \( A \). Taking the pull-back of (6.1) along the monomorphism
$g_A$ we have the following exact commutative diagram

\[
\begin{array}{cccccc}
0 & \longrightarrow & \tilde{U}_A^1 & \longrightarrow & U_A^1 & \longrightarrow & V_A^0 & \longrightarrow & 0 \\
\| & & \downarrow & & \downarrow & & g_A & & \\
0 & \longrightarrow & \tilde{U}_A^1 & \longrightarrow & U_A^0 & \overset{f_A}{\longrightarrow} & A & \longrightarrow & 0
\end{array}
\] 

(6.2)

where the middle vertical map is a monomorphism, so $U_A^1 \in \mathcal{U} \subseteq \mathcal{E}$, and (6.2) induces a short exact sequence

\[0 \rightarrow U_A^1 \rightarrow U_A^0 \oplus V_A^0 \xrightarrow{\varphi} A \rightarrow 0\]

where $\varphi = \iota(f_A, g_A)$ is easily seen to be a right $\mathcal{E}$-approximation of $A$. Hence $\mathcal{E}$ is contravariantly finite in $\mathcal{A}$ and therefore the category $\text{mod-}\mathcal{E}$ of coherent functors over $\mathcal{E}$ is abelian. Moreover the exact sequence

\[0 \rightarrow \mathcal{A}(-, U_A^1)|_{\mathcal{E}} \rightarrow \mathcal{A}(-, U_A^0 \oplus V_A^0)|_{\mathcal{E}} \xrightarrow{\varphi^*} \mathcal{A}(-, A)|_{\mathcal{E}} \rightarrow 0 \]

(6.3)

is a projective resolution in $\text{mod-}\mathcal{E}$ of the object $\mathcal{A}(-, A)|_{\mathcal{E}}$ and therefore $\text{pd}\mathcal{A}(-, A)|_{\mathcal{E}} \leq 1$. On the other hand, let $F$ be an arbitrary object of $\text{mod-}\mathcal{E}$ and choose a projective presentation $\mathcal{A}(-, W_1)|_{\mathcal{E}} \rightarrow \mathcal{A}(-, W_0)|_{\mathcal{E}} \rightarrow F \rightarrow 0$ in $\text{mod-}\mathcal{E}$, where the $W_i$ lie in $\mathcal{E}$. If $A = \text{Ker}(W_1 \rightarrow W_0)$, then setting $W_3 = U_A^1 \in \mathcal{U} \subseteq \mathcal{E}$ and $W_2 = U_A^0 \oplus V_A^0 \in \mathcal{E}$ and using (6.3) we have a projective resolution of $F$ in $\text{mod-}\mathcal{E}$ of the form:

\[0 \rightarrow \mathcal{A}(-, W_3)|_{\mathcal{E}} \rightarrow \mathcal{A}(-, W_2)|_{\mathcal{E}} \rightarrow \mathcal{A}(-, W_1)|_{\mathcal{E}} \rightarrow \mathcal{A}(-, W_0)|_{\mathcal{E}} \rightarrow F \rightarrow 0\]

Hence $\text{pd}\, F \leq 3$ and therefore $\text{gl.dim}\, \text{mod-}\mathcal{E} \leq 3$. Since $\text{Proj}\, \mathcal{A} \subseteq \mathcal{E}$, it is easy to see that the functor $\mathcal{A} \rightarrow \text{mod-}\mathcal{E}$, $A \mapsto \mathcal{A}(-, A)|_{\mathcal{E}}$ admits an exact left adjoint, see Lemma 8.5, so the above functor preserves injectives. If $\mathcal{A}$ has enough injectives and $\text{Inj}\, \mathcal{A} \subseteq \mathcal{V}$, then for any object $E \in \mathcal{E}$, let $0 \rightarrow E \rightarrow I^0 \rightarrow I^1$ be exact, where the $I^i$ are injective. Then the objects $\mathcal{A}(-, I^i)$ are projective–injective in $\text{mod-}\mathcal{E}$ and the exact sequence $0 \rightarrow \mathcal{A}(-, E)|_{\mathcal{E}} \rightarrow \mathcal{A}(-, I^0)|_{\mathcal{E}} \rightarrow \mathcal{A}(-, I^1)|_{\mathcal{E}}$ shows that the projective object $\mathcal{A}(-, E)|_{\mathcal{E}}$ admits an injective copresentation whose first two terms are projective. We infer that $\text{dom.dim}\, \text{mod-}\mathcal{E} \geq 2$. \end{proof}

\textbf{Remark 6.12.} Assume in Proposition 6.11 that $\mathcal{A}$ has enough injectives and is not semi-simple. Then it is not difficult to see that in fact $\text{dom.dim}\, \text{mod-}\mathcal{E} = 2$. Indeed if $\text{dom.dim}\, \text{mod-}\mathcal{E} \geq 3$, then as we shall see in Section 8, $\mathcal{E} = \mathcal{U} \oplus \mathcal{V}$ is a 1-cluster tilting subcategory of $\mathcal{A}$, in particular $\text{Ext}_{\mathcal{A}}^1(\mathcal{E}, \mathcal{E}) = 0$. This easily implies that $\mathcal{A}$ is semi-simple and $\mathcal{U} = \text{Proj}\, \mathcal{A} = \text{Inj}\, \mathcal{A} = \mathcal{V}$. On the other hand if $\text{gl.dim}\, \text{mod-}\mathcal{E} \leq 2$, then it is easy to see that $\mathcal{A} = \mathcal{E}$. Hence if $\mathcal{A}$ is not semi-simple and $\mathcal{E} \neq \mathcal{A}$, then $\text{gl.dim}\, \text{mod-}\mathcal{E} = 3$ and $\text{dom.dim}\, \text{mod-}\mathcal{E} = 2$. 

A full subcategory $\mathcal{C}$ of $\mathcal{A}$ is called generator, resp. cogenerator, if any object of $\mathcal{A}$ is a factor, resp. subobject, of an object from $\mathcal{C}$. Then an object $T$ of $\mathcal{A}$ is called generator, resp. cogenerator, if the full subcategory $\text{add } T$ is a generator, resp. cogenerator. Recall that an additive category is called right, resp. left, coherent if it has weak kernels, resp. weak cokernels. Then an object $T$ of $\mathcal{A}$ is called right, resp. left, coherent if the full subcategory $\text{add } T$ is right, resp. left, coherent. For instance $T \in \mathcal{A}$ is right, resp. left, coherent, if $\text{add } T$ is contravariantly, resp. covariantly, finite in $\mathcal{A}$. Of course $T$ is called coherent if $T$ is left and right coherent object. The representation dimension $\text{rep.dim } \mathcal{A}$ of $\mathcal{A}$ in the sense Auslander [3] is defined as follows.

$$\text{rep.dim } \mathcal{A} = \inf \{ \text{gl.dim mod-End}_{\mathcal{A}}(T) \mid T \text{ is a coherent generator-cogenerator of } \mathcal{A} \}$$

Note that if the object $T$ is coherent, then both the categories $\text{mod- } \text{add } T \approx \text{mod-End}_{\mathcal{A}}(T)$ and $\text{add } T\text{-mod } \approx (\text{mod-End}_{\mathcal{A}}(T)^{\text{op}})^{\text{op}}$ are abelian and $\text{gl.dim } \text{mod-End}_{\mathcal{A}}(T) = \text{w.gl.dim } \text{End}_{\mathcal{A}}(T) = \text{gl.dim } \text{mod-End}_{\mathcal{A}}(T)^{\text{op}}$.

**Corollary 6.13.** Let $\mathcal{A}$ be a 1-Gorenstein abelian category and assume that $\text{Proj } \mathcal{A}$ is covariantly finite and $\text{Inj } \mathcal{A}$ is contravariantly finite. Then $\mathcal{E} := \text{GProj } \mathcal{A} \oplus \text{GInj } \mathcal{A}$ is functorially finite in $\mathcal{A}$, and $\text{gl.dim } \text{mod- } \mathcal{E} \leq 3$ and $\text{dom.dim } \text{mod- } \mathcal{E} \geq 2$. If in addition $\mathcal{A}$ is of finite Cohen–Macaulay type, then $\text{rep.dim } \mathcal{A} \leq 3$; more precisely there exists a coherent ring $\Lambda$ with $\text{rep.dim } \Lambda \leq 3$ and an equivalence $\text{mod- } \Lambda \approx \mathcal{A}$.

**Proof.** Since $\mathcal{A}$ is Gorenstein, by [19, Theorem VII.2.2], $\text{GProj } \mathcal{A}$ and $\text{GInj } \mathcal{A}$ are functorially finite and $\text{GProj } \mathcal{A} = \Omega \mathcal{A}$ and $\text{GInj } \mathcal{A} = \Sigma \mathcal{A}$. Hence $\text{GProj } \mathcal{A}$ is closed under subobjects and $\text{GInj } \mathcal{A}$ is closed under factors. Now the first assertion follows from Proposition 6.11. Assume that $\mathcal{A}$ is of finite Cohen–Macaulay type, so $\text{GProj } \mathcal{A} = \text{add } X$, for some Gorenstein-projective object $X$. Then $X$ is coherent since $\text{add } X$ is functorially finite. By [19] there is a triangle equivalence $\text{GProj } \mathcal{A} \approx \text{GInj } \mathcal{A}$. This implies that $\text{GInj } \mathcal{A}$ is also of finite representation type, so $\text{GInj } \mathcal{A} = \text{add } Y$ for some, necessarily coherent, Gorenstein-injective object $Y$. We infer that $\mathcal{E} := \text{GProj } \mathcal{A} \oplus \text{GInj } \mathcal{A} = \text{add } (X \oplus Y)$ and the object $X \oplus Y$ is coherent. Since clearly this object is a generator–cogenerator of $\mathcal{A}$ and since $\text{mod- } \mathcal{E} = \text{mod-End}_{\mathcal{A}}(X \oplus Y)$, we infer that $\text{rep.dim } \mathcal{A} \leq 3$. Finally since $\text{Proj } \mathcal{A} \subseteq \text{GProj } \mathcal{A}$, clearly $\text{Proj } \mathcal{A}$ is of finite representation type and we have $\text{Proj } \mathcal{A} = \text{add } T$ for some, necessarily coherent, object $T \in \text{Proj } \mathcal{A}$. Then $\text{mod- } \mathcal{A} \approx \text{mod- } \Lambda$ where $\Lambda = \text{End}(T)$. \qed

For simplicity of notation from now on and throughout the rest of the paper we set:

$$\text{Proj mod- } \mathcal{X} := \text{proj } \mathcal{X}, \quad \text{Inj mod- } \mathcal{X} := \text{inj } \mathcal{X},$$

$$\text{GProj mod- } \mathcal{X} := \text{Gproj } \mathcal{X} = \text{CM}(\mathcal{X}), \quad \text{GInj mod- } \mathcal{X} := \text{Ginj } \mathcal{X}$$

and $\text{GProj mod- } \mathcal{X} := \text{Gproj } \mathcal{X} = \text{CM}(\mathcal{X})$ and $\text{GInj mod- } \mathcal{X} := \text{Ginj } \mathcal{X}$.
Lemma 6.14. Let $\mathcal{X}$ be a full subcategory of $\mathcal{T}$, and assume that $\mathcal{X}$ is $(n - 1)$-corigid if $n > 1$. Then the subcategories $\text{proj} \mathcal{X}$, $\text{Gproj} \mathcal{X}$, $\text{inj} \mathcal{X}$, $\text{Ginj} \mathcal{X}$ are functorially finite in $\text{mod} \mathcal{X}$ in the following cases:

(i) $\mathcal{X}$ is $(n + 1)$-cluster tilting.
(ii) $\mathcal{T}$ is $(n + 1)$-Calabi–Yau and $\mathcal{X}$ is maximal n-rigid.

Proof. (i) Since $\text{mod} \mathcal{X}$ has enough projectives and injectives, $\text{proj} \mathcal{X}$ is contravariantly finite and $\text{inj} \mathcal{X}$ is covariantly finite. Let $F \cong \text{H}(A)$ be in $\text{mod} \mathcal{X}$, where we may assume that $A$ lies in $\mathcal{X} \oplus \mathcal{X}[1]$. Since $\mathcal{X}$ is contravariantly finite, so is $\mathcal{X}[n+1]$; let $f : X[n+1] \to A$ be a right $\mathcal{X}[n+1]$-approximation of $A$ which induces a map $H(f) : \text{H}(X[n+1]) \to \text{H}(A)$ in $\text{mod} \mathcal{X}$. Let $\alpha = \text{H}(\alpha)$ for some map $\alpha : X'[n+1] \to A$ which then factorizes through $f$. It follows that $\text{H}(\alpha)$ factorizes through $H(f)$. Hence $H(f)$ is a right $\text{inj}(\mathcal{X})$-approximation of $H(A)$ and $\text{inj}(\mathcal{X})$ is contravariantly finite in $\text{mod} \mathcal{X}$. If $n = 1$ the above argument works using that $\mathcal{T} = \mathcal{X} \oplus \mathcal{X}[1]$ in this case. Next let $g : A \to X$ be a left $\mathcal{X}$-approximation of $A$ which induces a map $H(g) : H(A) \to H(X)$. If $\beta : H(A) \to H(X')$ is a map in $\text{mod} \mathcal{X}$, where $X' \in \mathcal{X}$, then, since $A \in \mathcal{X} \oplus \mathcal{X}[1]$, $\beta$ is induced by a map $\beta : A \to X'$ which then factorizes through $g$. It follows that $\text{H}(\beta)$ factorizes through $H(g)$. Hence $H(g)$ is a left $\text{proj}(\mathcal{X})$-approximation of $H(A)$ and $\text{proj}(\mathcal{X})$ is covariantly finite in $\text{mod} \mathcal{X}$. Case (ii) is similar using Corollary 5.13.

Finally by Corollary 6.5, the category $\text{mod} \mathcal{X}$ is 1-Gorenstein. Then by [19, Theorem VII.2.2] functorial finiteness of $\text{proj} \mathcal{X}$ and $\text{inj} \mathcal{X}$ implies that $\text{Gproj} \mathcal{X}$ and $\text{Ginj} \mathcal{X}$ are functorially finite. □

Combining Theorem 6.4, Corollary 6.13, and Lemma 6.14 we have the following consequence.

Theorem 6.15. Let $\mathcal{T}$ be a triangulated category and $\mathcal{X}$ a contravariantly finite subcategory of $\mathcal{T}$. Assume $\mathcal{X}$ is $(n - 1)$-corigid, if $n > 1$. If $\text{mod} \mathcal{X}$ is of finite CM-type, then $\text{rep.dim} \text{mod} \mathcal{X} \leq 3$ in the following cases:

(i) $\mathcal{X}$ is an $(n + 1)$-cluster tilting subcategory of $\mathcal{T}$.
(ii) $\mathcal{T}$ is $(n + 1)$-Calabi–Yau and $\mathcal{X}$ is maximal n-rigid.

More precisely $\text{mod} \mathcal{X} \cong \text{mod-End}_\mathcal{T}(T)$ for some $T \in \mathcal{X}$ and $\text{rep.dim} \text{End}_\mathcal{T}(T) \leq 3$ and $\text{dom.dim} \text{End}_\mathcal{T}(T) \geq 2$.

7. . .and stably Calabi–Yau

We have seen, in Theorem 6.4, that the cluster tilted category $\text{mod} \mathcal{X}$ associated to an $(n - k)$-corigid $(n + 1)$-cluster tilting subcategory $\mathcal{X}$ of a triangulated category
\( \mathcal{T} \) is \( k \)-Gorenstein, provided that either \( n = 1 \), or \( n > 1 \) and \( 0 \leq k \leq 1 \), or \( k \leq \frac{n+1}{2} \), if \( 2 \leq k \leq n - 1 \). In this section we show that if the triangulated category \( \mathcal{T} \) is \((n+1)\)-Calabi–Yau over a field, then the triangulated stable category modulo projectives of the Gorenstein-projective objects of \( \text{mod-} \mathcal{X} \) is \((n + 2)\)-Calabi–Yau in case \( 0 \leq k \leq 1 \), and under an additional assumption if \( 2 \leq k \leq \frac{n+1}{2} \). This generalizes a basic result of Keller and Reiten [46] who treated the case \( n = 1 \).

Throughout this section: we fix a \( k \)-linear triangulated category \( \mathcal{T} \) with split idempotents and finite-dimensional \( \text{Hom} \)-spaces over a field \( k \), and let \( \mathcal{X} \) be an \((n + 1)\)-cluster tilting subcategory of \( \mathcal{T} \), \( n \geq 1 \).

### 7.1. Serre functors

Assume that \( \mathcal{T} \) admits a Serre functor \( S \). So \( S : \mathcal{T} \rightarrow \mathcal{T} \) is a triangulated equivalence and for any objects \( A \) and \( B \) in \( \mathcal{T} \) there are natural isomorphisms

\[
\text{D}(A, B) \xrightarrow{\cong} \mathcal{T}(B, S(A))
\]

where \( \text{D} \) denotes duality with respect to the base field \( k \). For any object \( A \in \mathcal{T} \) we consider the triangles

\[
X^1_A \rightarrow X^0_A \rightarrow \text{Cell}^1(A) \rightarrow X^1_A[1] \quad \& \quad X^1_A[-1] \rightarrow \text{Cell}^1(A) \rightarrow X^0_A \rightarrow X^1_A
\]

(7.1)

where the map \( X^1_A \rightarrow X^0_A \) is the composition of the right \( \mathcal{X} \)-approximation \( X^1_A \rightarrow \Omega^1_{\mathcal{X}}(A) \) and the map \( \Omega^1_{\mathcal{X}}(A) \rightarrow X^0_A \); and the map \( X^0_A \rightarrow X^1_A \) is the composition of the map \( X^0_A \rightarrow \Sigma^1_{\mathcal{X}}(A) \) and the left \( \mathcal{X} \)-approximation \( \Sigma^1_{\mathcal{X}}(A) \rightarrow X^1_A \). Then

\[
\text{Cell}^1(A) \in \mathcal{X} \star \mathcal{X}[1] \quad \text{and} \quad \text{H}(\text{Cell}^1(A)) \cong \text{H}(A),
\]

\[
\text{Cell}^1(A) \in \mathcal{X}[-1] \star \mathcal{X} \quad \text{and} \quad \text{H}^{\text{op}}(A) \cong \text{H}(\text{Cell}^1(A))
\]

Recall that the transpose \( \text{Tr}(F) \), in the sense of Auslander and Bridger [4], of an object \( F \) in \( \text{mod-} \mathcal{X} \) is defined as follows. Let \( \text{H}(X^1) \rightarrow \text{H}(X^0) \rightarrow F \rightarrow 0 \) be a projective presentation of \( F \). Consider the duality functor \( d^r := \text{Hom}(\cdot, \text{H}(?)|_{\mathcal{X}}) : \text{mod-} \mathcal{X} \rightarrow \mathcal{X}-\text{mod} \), defined by \( d^r(F) = \text{Hom}(F, \text{H}(?)|_{\mathcal{X}}) : \mathcal{X} \rightarrow \mathfrak{A} \mathfrak{b} \), where \( \text{Hom}(F, \text{H}(?)|_{\mathcal{X}})(X) = \text{Hom}(F, \text{H}(X)) \). Similarly the dualiy functor \( d^l : \mathcal{X}-\text{mod} \rightarrow \text{mod-} \mathcal{X} \) is defined, and it is well-known that \( (d^r, d^l) : \text{mod-} \mathcal{X} \rightleftarrows \mathcal{X}-\text{mod} \) is an adjoint on the right pair of contravariant functors inducing a duality between \( \text{proj} \mathcal{X} \) and \( \text{proj} \mathcal{X}^{\text{op}} \), a duality between \( \text{Gproj} \mathcal{X} \) and \( \text{Gproj} \mathcal{X}^{\text{op}} \), and finally a duality between \( \text{Gproj} \mathcal{X} \) and \( \text{Gproj} \mathcal{X}^{\text{op}} \). Now the transpose \( \text{Tr}(F) \) of \( F \) is defined by \( \text{Tr}(F) = \text{Coker}(d^r \text{H}(X^0) \rightarrow d^r \text{H}(X^1)) \). Since \( d^r \text{H}(X) \cong \text{H}^{\text{op}}(\mathcal{X}) \), we have an exact sequence

\[
0 \rightarrow d^r(F) \rightarrow \text{H}^{\text{op}}(X^0) \rightarrow \text{H}^{\text{op}}(X^1) \rightarrow \text{Tr}(F) \rightarrow 0
\]
In this way we obtain a functor $\text{Tr} : \text{mod-}\mathcal{X} \rightarrow \text{mod-}\mathcal{X}^{\text{op}}$ which is well-known to be a duality. Note that the duality $D$ with respect to the base field $k$ acts on $\text{mod-}\mathcal{X}$ by $D(F)(x) = D(F(x))$.

Let $F \in \text{mod-}\mathcal{X}$ with presentation $H(X^1) \rightarrow H(X^0) \rightarrow F \rightarrow 0$ and $G \in \mathcal{X}\text{-mod}$ with presentation $H^{\text{op}}(X_1) \rightarrow H^{\text{op}}(X_0) \rightarrow G \rightarrow 0$. Then $F \cong H(A^s_F)$ and $G \cong H^{\text{op}}(A^G_s)$, where the objects $A^s_F$ and $A^G_s$ are defined by the following triangles in $\mathcal{T}$:

$$X^1 \rightarrow X^0 \rightarrow A^s_F \rightarrow X^1[1] \quad \text{and} \quad X_1[-1] \rightarrow A^G_s \rightarrow X_0 \rightarrow X_1 \quad (\dagger)$$

Note that if $F = H(A)$, resp. $G = H^{\text{op}}(A)$, then we may choose: $A^s_F \cong \text{Cell}_1(A)$, resp. $A^G_s = \text{Cell}^1(A)$.

**Lemma 7.1.** With the above notations, for any $F \in \text{mod-}\mathcal{X}$ and $G \in \mathcal{X}\text{-mod}$, there are isomorphisms:

$$D\text{Tr}F \cong H(SA^s_F[-1]) \quad \& \quad \text{Tr}DF \cong H(\text{Cell}^1(S^{-1}A^s_F)[1]) \quad (7.2)$$

$$D\text{Tr}G \cong H^{\text{op}}(S^{-1}A^G_s)[1]) \quad \& \quad \text{Tr}DG \cong H^{\text{op}}(\text{Cell}_1(SA^G_s)[-1]) \quad (7.3)$$

**In particular:** $D\text{Tr}H(A) \cong H(S\text{Cell}_1(A)[-1])$ and $\text{Tr}DH(A) \cong H(\text{Cell}_1(S^{-1}\text{Cell}_1(A))[1])$.

**Proof.** Applying the $\mathcal{X}$-dual functor $D^* = \text{Hom}(-, H(\cdot)|\mathcal{X}) : \text{mod-}\mathcal{X} \rightarrow \mathcal{X}\text{-mod}$ to the projective presentation of $F$, we have an exact sequence $0 \rightarrow D^*F \rightarrow H^{\text{op}}(X^0) \rightarrow H^{\text{op}}(X^1) \rightarrow \text{Tr}F \rightarrow 0$. Then applying the cohomological functor $H^{\text{op}}$ to the first triangle in (\dagger) and using that $\mathcal{X}$ is rigid, we see directly that we have an exact sequence $H^{\text{op}}(X^0) \rightarrow H^{\text{op}}(X^1) \rightarrow H^{\text{op}}(A^s_F[-1]) \rightarrow 0$. Hence $\text{Tr}F \cong H^{\text{op}}(A^s_F[-1])$.

Using Serre duality it follows that: $D\text{Tr}(F) \cong DH^{\text{op}}(A^s_F[-1]) \cong D\mathcal{T}(A^s_F[-1], -)|x \cong \mathcal{T}(S^{-1}A^s_F[-1])|x = H(SA^s_F[-1])$.

On the other hand by using Serre duality we have: $DF \cong DH(A^s_F) = D\mathcal{T}(-, A^s_F)|x \cong \mathcal{T}(A^s_F, S)|x = \mathcal{T}(S^{-1}A^s_F, -)|x = H^{\text{op}}(S^{-1}A^s_F)$. Set $B := S^{-1}A^s_F$, so we have $H^{\text{op}}(B) = H^{\text{op}}(S^{-1}A^s_F) \cong H^{\text{op}}(\text{Cell}^1(B))$. From the triangle $(+) : X^B_1[-1] \rightarrow \text{Cell}^1(B) \rightarrow X^B_0 \rightarrow X^B_1$ we have a projective presentation $H^{\text{op}}(X^B_1) \rightarrow H^{\text{op}}(X^B_0) \rightarrow H^{\text{op}}(\text{Cell}^1(B)) \rightarrow 0$ inducing an exact sequence $H(X^B_1) \rightarrow H(X^B_0) \rightarrow \text{Tr}H^{\text{op}}(\text{Cell}^1(B)) \rightarrow 0$. Applying $H$ to the triangle $(+)$ and using that $\mathcal{X}$ is rigid, we have an isomorphism $\text{Tr}H^{\text{op}}(\text{Cell}^1(B)) \cong H(\text{Cell}^1(B)[1])$. Putting things together we have isomorphisms $\text{Tr}DF \cong \text{Tr}H^{\text{op}}(S^{-1}A^s_F) \cong H(\text{Cell}^1(S^{-1}A^s_F)[1])$.

The isomorphisms (7.3) are proved similarly. \qed

**Lemma 7.2.** The cluster tilted category $\text{mod-}\mathcal{X}$ has Auslander–Reiten sequences. In particular $\text{AR}$-formula
\[ D\text{Hom}(H(A), H(B)) \cong \text{Ext}^1(H(B), \text{DTr}(H(A))) \cong \text{Ext}^1(H(B), H(\text{SCell}_1(A)[-1])) \quad (7.4) \]

holds for any objects $H(A)$ and $H(B)$ in $\text{mod} \cdot X$. Further if $A \in X \star \mathcal{X}[1]$, then:

\[ D\text{Hom}(H(A), H(B)) \cong \text{Ext}^1(H(B), H(\text{S}A[-1])) \]

**Proof.** Since $\mathcal{X}$ is $(n+1)$-cluster tilting, it follows that $\mathcal{X}$ is functorially finite in $\mathcal{T}$, so $\mathcal{X}$ has weak kernels and weak cokernels. On the other hand by using Serre duality ($\ast$), we have isomorphisms, $\forall X \in \mathcal{X}$:

\[ DH(X) \cong D\mathcal{T}(X, X) \cong \mathcal{T}(X, \mathcal{S}X) \cong \mathcal{T}(\mathcal{S}^{-1}X, \mathcal{X}) \cong \mathcal{H}^{op}(\mathcal{S}^{-1}X) \]

\[ DH^{op}(X) \cong D\mathcal{T}(X, X) \cong \mathcal{T}(X, \mathcal{S}X) \cong \mathcal{H}(\mathcal{S}X) \]

Hence $k$-duals of contravariant or covariant representable functors over $\mathcal{X}$ are coherent. This clearly implies that $\mathcal{X}$, and therefore $\text{mod} \cdot \mathcal{X}$, is a dualizing $k$-variety, see [5]. In particular $\text{mod} \cdot \mathcal{X}$ has Auslander–Reiten sequences and AR-formula (7.4) holds. The last isomorphism holds since $\text{Cell}_1(A) = A$, if $A \in X \star \mathcal{X}[1]$. \qed

### 7.2. $(n+1)$-Calabi–Yau categories

From now on we assume that: $\mathcal{T}$ is $(n+1)$-Calabi–Yau, i.e. there is an isomorphism of triangulated functors: $\mathcal{S}(?)^\ast \cong (?)[n+1]$. Then we have natural isomorphisms

\[ D\text{Hom}_T(A, B) \cong \text{Hom}_T(B, A[n+1]) \quad (7.5) \]

**Lemma 7.3.** Let $A$ be an object in $X \star \mathcal{X}[1]$. Then $\text{DTr}(A) \cong H(A[n])$, and for any Gorenstein-projective object $H(B)$ of $\text{mod} \cdot X$, there is a natural isomorphism:

\[ D\text{Hom}(H(A), H(B)) \cong \text{Hom}(\Omega^1 H(B), H(A[n])) \]

**Proof.** Since $A \in X \star \mathcal{X}[1]$, we may take $A = \text{Cell}_1(A)$ and then by Lemma 7.1, we have an isomorphism $\text{DTr}(H(A)) \cong H(A[n])$. By Lemma 7.2 we have an isomorphism $D\text{Hom}(H(A), H(B)) \cong \text{Ext}^1(H(B), H(A[n]))$. Since $H(B) \in \text{Gproj} X$, by Remark 6.3 we have an isomorphism $\text{Ext}^1(H(B), H(A[n])) \cong \text{Hom}(\Omega^1 H(B), H(A[n]))$ and the last assertion follows. \qed

To proceed further we need the following two preliminary results.

**Lemma 7.4.** Let $n \geq 2$, and $A$ be an object in $X \star \mathcal{X}[1]$ such that $H(A)$ is Gorenstein-projective. Let

\[ C \rightarrow A \rightarrow B \rightarrow C[1] \]
be a triangle in $\mathcal{T}$. If $C$ lies in $\mathcal{X}[-n] \ast \mathcal{X}[-n+1] \ast \cdots \ast \mathcal{X}[-1]$, then there exists a short exact sequence

$$0 \rightarrow H(A) \rightarrow H(B) \rightarrow H(C[1]) \rightarrow 0$$

which remains exact after the application of the functor $\text{Hom}(-, H(X))$, $\forall X \in \mathcal{X}$. Moreover if $H(C[1])$ is Gorenstein-projective, then so is $H(B)$. The converse holds if $\text{mod-}\mathcal{X}$ is Gorenstein.

**Proof.** We have $A[1] \in \mathcal{X}[1] \ast \mathcal{X}[2]$, and this implies that $H(A[1]) = 0$ since $n \geq 2$. Applying the homological functor $H$ to the triangle $C \rightarrow A \rightarrow B \rightarrow C[1]$, we have therefore an exact sequence $H(C) \rightarrow H(A) \rightarrow H(B) \rightarrow H(C[1]) \rightarrow 0$. We show that the map $H(C) \rightarrow H(A)$ is zero. Since $H(A)$ is Gorenstein-projective, there is a monomorphism $H(A) \rightarrow H(X)$, where $X \in \mathcal{X}$. Since $A$ lies in $\mathcal{X} \ast \mathcal{X}[1]$, this map is induced by a map $A \rightarrow X$. Using that $\mathcal{X}$ is $n$-rigid, it follows that the composition $C \rightarrow A \rightarrow X$ is zero, since $C$ lies in $\mathcal{X}[-n] \ast \mathcal{X}[-n+1] \ast \cdots \ast \mathcal{X}[-1]$ and $X \in \mathcal{X}$. As a consequence the composition $H(C) \rightarrow H(A) \rightarrow H(X)$ is zero and then the map $H(C) \rightarrow H(A)$ is zero since $H(A) \rightarrow H(X)$ is a monomorphism. We infer that the map $H(A) \rightarrow H(B)$ is a monomorphism and therefore the sequence (7.6) is exact.

As the above argument shows, any map $H(A) \rightarrow H(X)$, $X \in \mathcal{X}$, factorizes through $H(B)$. It follows that the exact sequence (7.6) remains exact after the application of the functor $\text{Hom}(-, H(X))$, $\forall X \in \mathcal{X}$. If $H(C[1])$ is Gorenstein-projective, then so is $H(B)$, since $\text{Gproj}\mathcal{X}$ is well-known to be closed under extensions. Conversely let $\text{mod-}\mathcal{X}$ be Gorenstein, and let $H(B)$ be Gorenstein-projective. Applying $\text{Hom}(-, H(X))$, $\forall X \in \mathcal{X}$, to the exact sequence (7.6) we have trivially $\text{Ext}^k(H(C[1]), H(X)) = 0$, $\forall X \in \mathcal{X}$, $\forall k \geq 1$. It follows that $H(C[1])$ is Gorenstein-projective by Remark 6.3. 

**Lemma 7.5.** Assume that the cluster tilted category $\text{mod-}\mathcal{X}$ is $k$-Gorenstein, $k \geq 0$, and for any object $A \in \mathcal{X} \ast \mathcal{X}[1]$ such that $H(A)$ is Gorenstein-projective, there is a natural isomorphism in $\text{Gproj}\mathcal{X}$:

$$\Omega^{-(n+1)}H(A) \xrightarrow{\cong} \Omega^{-k}\Omega^kH(A[n]), \quad \text{or}$$

$$H(A) \xrightarrow{\cong} \Omega^{n+1}H(A[n]) \quad \text{if } k \leq n+1 \quad (\dagger\dagger)$$

Then $\text{Gproj}\mathcal{X}$ is $(n+2)$-Calabi-Yau.

**Proof.** Since $\text{mod-}\mathcal{X}$ is $k$-Gorenstein, it follows from Remark 6.3 that $\Omega^kH(C) \in \text{Gproj}\mathcal{X}$, $\forall C \in \mathcal{T}$, and moreover the functor $\Omega^{-k}\Omega^k : \text{mod-}\mathcal{X} \rightarrow \text{Gproj}\mathcal{X}$ is a right adjoint of the inclusion $\text{mod-}\mathcal{X} \rightarrow \text{Gproj}\mathcal{X}$. Then by Lemma 7.3 we have natural isomorphisms

$$\text{D Hom} \ (H(A), H(B)) \xrightarrow{\cong} \text{Hom} \ (\Omega^kH(B), H(A[n]))$$

$$\xrightarrow{\cong} \text{Hom} \ (\Omega^kH(B), \Omega^{-k}\Omega^kH(A[n]))$$
\[
\cong \rightarrow \text{Hom} (\Omega^1 H(B), \Omega^{-(n+1)} H(A)) \\
\cong \rightarrow \text{Hom} (H(B), \Omega^{-(n+2)} H(A))
\]

for any Gorenstein-projective objects \( H(B) \) and \( H(A) \), with \( A \in \mathcal{X} \star \mathcal{X}[1] \). Since any object \( F = H(C) \) of \( \text{mod} \mathcal{X} \) is isomorphic to an object of the form \( H(A) \), where \( A \in \mathcal{X} \star \mathcal{X}[1] \), namely \( A = \text{Cell}_1(C) \), the above natural isomorphisms hold for all Gorenstein-projective objects. Hence the functor \( \Omega^{-(n+2)} : \text{Gproj} \mathcal{X} \rightarrow \text{Gproj} \mathcal{X} \) serves as a Serre functor in \( \text{Gproj} \mathcal{X} \), i.e. \( \text{Gproj} \mathcal{X} \) is \((n+2)\)-Calabi–Yau. Finally if \( n+1 \geq k \), then since \( \Omega^k \text{mod} \mathcal{X} = \text{Gproj} \mathcal{X} \) and since \( \Omega^{-k} \Omega^k |_{\text{Gproj} \mathcal{X}} \cong \text{id}_{\text{Gproj} \mathcal{X}} \), we have an isomorphism of functors \( \Omega^{n+1} \Omega^{-k} \Omega^k \cong \Omega^{n+1} \). This implies that the existence of an isomorphism \( \Omega^{-(n+1)} H(A) \cong \Omega^{-k} \Omega^k H(A[n]) \) is equivalent to the existence of an isomorphism \( H(A) \xrightarrow{\cong} \Omega^{n+1} H(A[n]) \). \( \square \)

Now we can prove the main result of this section. Note that the case \( n = 1 \) in Theorem 7.6 is due to Keller and Reiten [46, Theorem 3.3]. We give below a short proof of their result for the convenience of the reader.

**Theorem 7.6.** Let \( \mathcal{T} \) be an \((n+1)\)-Calabi–Yau triangulated category over a field \( k \). Let \( \mathcal{X} \) be an \((n+1)\)-cluster tilting subcategory of \( \mathcal{T} \) and assume that \( \mathcal{X} \) is \((n-k)\)-corigid, where \( n \geq 2k-1 \), if \( n \geq 2 \).

If any object \( H(C) \), where \( C \in \mathcal{X}[-n+1] \star \cdots \star \mathcal{X}[-n+k] \), has finite projective or injective dimension, then the stable triangulated category \( \text{Gproj} \mathcal{X} \) is \((n+2)\)-Calabi–Yau.

**Proof.** First note that, since by Theorem 6.4 the cluster tilted category \( \text{mod} \mathcal{X} \) is \( k \)-Gorenstein, from Remark 6.3 it follows that \( \Omega^k H(A) \) lies in \( \text{Gproj} \mathcal{X} \), \( \forall A \in \mathcal{T} \). We use throughout that if an object of \( \text{mod} \mathcal{X} \) has finite projective dimension, then its projective dimension is at most \( k \). By Lemma 7.5 it suffices to show the existence of a natural isomorphism \((\dagger\dagger)\) for any object \( A \in \mathcal{X} \star \mathcal{X}[1] \) such that \( H(A) \) is Gorenstein-projective.

1. **Case** \( n = 1 \). Then \( \mathcal{T} = \mathcal{X} \star \mathcal{X}[1] \), and \( \mathcal{X} \) is a 2-cluster tilting subcategory of the 2-Calabi–Yau category \( \mathcal{T} \). By Lemma 6.14, there exists a triangle \( X^1_{A[1]} [-1] \rightarrow X^0_{A[1]} [-1] \rightarrow A \rightarrow X^1_{A[1]} \), where the last map is a left \( \mathcal{X} \)-approximation of \( A \) and the \( X^i_{A[1]} \) lie in \( \mathcal{X} \), and moreover the map \( H(A) \rightarrow H(X^1_{A[1]}) \) is a left \( \text{proj} \mathcal{X} \)-approximation of \( H(A) \). Since \( H(A) \) is Gorenstein-projective, it follows that \( H(A) \) is a subobject of a projective and this implies that the map \( H(A) \rightarrow H(X^1_{A[1]}) \) is a monomorphism. Hence applying \( H \) to the above triangle and using that \( H(\mathcal{X}, \mathcal{X}[1]) = 0 \), we have an exact sequence

\[
0 \rightarrow H(A) \rightarrow H(X^1_{A[1]}) \rightarrow H(X^0_{A[1]}) \rightarrow H(A[1]) \rightarrow 0
\]
which shows that we have an isomorphism: $\mathbb{H}(A) \cong \Omega^2 \mathbb{H}(A[1])$ in $\text{mod}-\mathcal{X}$.

Since $\Omega^1 \mathbb{H}(A[1])$ is Gorenstein-projective, we have an isomorphism $\Omega^{-2} \mathbb{H}(A) \cong \Omega^{-1} \Omega^1 \mathbb{H}(A[1])$ as required.

2. **Case** $n \geq 2$. We divide the proof into several steps.

**Step 1:** Any object $\mathbb{H}(C)$, where $C \in \mathcal{X}[-n+1] \ast \cdots \ast \mathcal{X}[-1]$, has finite projective dimension.

**Proof.** Indeed we have a triangle $A \rightarrow C \rightarrow B \rightarrow A[1]$, where $A \in \mathcal{X}[-n+1] \ast \cdots \ast \mathcal{X}[-n+k]$ and $B \in \mathcal{X}[-n+k+1] \ast \cdots \ast \mathcal{X}[-1]$. Hence $B[-1] \in \mathcal{X}[-n+k] \ast \cdots \ast \mathcal{X}[-2]$ and by hypothesis, $\mathbb{H}(A)$ has finite projective dimension. Since $\mathcal{X}$ is $(n-k)$-corigid, it follows that $\mathbb{H}(B) = 0 = \mathbb{H}(B[-1])$. It follows from the above triangle that we have an isomorphism $\mathbb{H}(A) \cong \mathbb{H}(C)$ and therefore $\mathbb{H}(C)$ has finite projective dimension.

**Step 2:** There are isomorphisms: $\Omega^k \mathbb{H}(A[n]) \cong \mathbb{H}(\Omega^k_{\mathcal{X}}(A[n]))$.

**Proof.** We use the triangles, $1 \leq t \leq n$:

$$
\Omega^t_{\mathcal{X}}(A[n]) \rightarrow X_{A[n]}^{t-1} \rightarrow \Omega^t_{\mathcal{X}}(A[n]) \rightarrow \Omega^t_{\mathcal{X}}(A[n])[1] \quad (T^t_{A[n]})
$$

where $\Omega^t_{\mathcal{X}}(A[n]) \in \mathcal{X}$ and $\Omega^0_{\mathcal{X}}(A[n]) = A[n]$. Using that $\mathcal{T}(\mathcal{X}, A[i]) = 0$, $1 \leq i \leq n-1$, and $\mathcal{T}(\mathcal{X}, \mathcal{X}[-i]) = 0$, $1 \leq i \leq n-k$, from the triangle $(T^t_{A[n]})$ we deduce a short exact sequence:

$$
0 \rightarrow \mathbb{H}(\Omega^k_{\mathcal{X}}(A[n])) \rightarrow \mathbb{H}(X_{A[n]}^0) \rightarrow \mathbb{H}(A[n]) \rightarrow 0
$$

and isomorphisms:

$$
\mathbb{H}(\Omega^t_{\mathcal{X}}(A[n])[-t]) = 0, \quad 1 \leq t \leq n-k
$$

Using this and applying $\mathbb{H}$ to the triangle $(T^2_{A[n]})$, we have an exact sequence

$$
0 \rightarrow \mathbb{H}(\Omega^2_{\mathcal{X}}(A[n])) \rightarrow \mathbb{H}(X_{A[n]}^1) \rightarrow \mathbb{H}(\Omega^1_{\mathcal{X}}(A[n])) \rightarrow 0
$$

and isomorphisms:

$$
\mathbb{H}(\Omega^2_{\mathcal{X}}(A[n])[-k]) = 0, \quad 1 \leq k \leq n-k-1
$$

Continuing in this way, and finally applying $\mathbb{H}$ to the triangle $(T^{k-1}_{A[n]})$, we have an exact sequence

$$
0 \rightarrow \mathbb{H}(\Omega^{k-1}_{\mathcal{X}}(A[n])) \rightarrow \mathbb{H}(X_{A[n]}^{k-2}) \rightarrow \mathbb{H}(\Omega^{k-2}_{\mathcal{X}}(A[n])) \rightarrow 0
$$

and an isomorphism:

$$
\mathbb{H}(\Omega^{k-1}_{\mathcal{X}}(A[n])[-1]) = 0
$$
Finally applying $H$ to the triangle $(T_{A[n]}^k)$, we have an exact sequence

$$0 \to H(\Omega^k_X(A[n])) \to H(X_{A[n]}^{k-1}) \to H(\Omega^{k-1}_X(A[n])) \to 0$$

From the above exact sequences we deduce that:

$$\Omega^k H(A[n]) \xrightarrow{\cong} H(\Omega^k_X(A[n])) \quad (7.7)$$

**Step 3:** There are isomorphisms: $\Omega^k H(A) \xrightarrow{\cong} \Omega^{k+2} H(\Omega^{n-1}_X(A[n])) \xrightarrow{\cong} \Omega^{k+3} H(\Omega^{n-2}_X(A[n]))$.

**Proof.** Consider the following triangles, for $1 \leq t \leq n$:

$$\Omega^t_X(A[n])[t-1] \to \operatorname{Cell}_{t-1}(A[n]) \to A[n] \to \Omega^t_X(A[n])[t] \quad (C^t_{A[n]})$$

arising from the tower of triangles $(C^t_{A[n]})$ associated to $A[n]$. Applying $H$ and using that $H(A[i]) = 0$, $1 \leq i \leq n-1$, and, by Proposition 3.2, $H(\Omega^t_X(A[n])[i]) = 0$, $1 \leq i \leq t$, we deduce an exact sequence

$$H(\operatorname{Cell}_{t-1}(A[n])[-n]) \to H(A) \to H(\Omega^t_X(A[n])[-n+t]) \to H(\operatorname{Cell}_{t-1}(A[n])[-n+1]) \to 0$$

Since $\operatorname{Cell}_{t-1}(A[n])[-n]$ lies in $(X \star X[1] \star \cdots \star X[t-1])[-n] = X[-n] \star X[-n+1] \star \cdots \star X[-t+1]$ which is contained in $X[-n] \star X[-t] \star \cdots \star X[-1]$, since $t \leq n$, by Lemma 7.4 we deduce short exact sequences

$$0 \to H(A) \to H(\Omega^t_X(A[n])[-n+t]) \to H(\operatorname{Cell}_{t-1}(A[n])[-n+1]) \to 0 \quad (7.8)$$

for $2 \leq t \leq n$. Moreover we infer isomorphisms:

$$H(\Omega^t_X(A[n])[-n+t+1]) \xrightarrow{\cong} H(\operatorname{Cell}_{t-1}(A[n])[-n+2])$$

$$H(\Omega^t_X(A[n])[-n+t+2]) \xrightarrow{\cong} H(\operatorname{Cell}_{t-1}(A[n])[-n+3])$$

$$\vdots$$

$$H(\Omega^t_X(A[n])[-t+2]) \xrightarrow{\cong} H(\operatorname{Cell}_{t-1}(A[n])[-1]) \quad (7.9)$$

Consider the short exact sequence (7.8) for $t = n-1$. Since $\operatorname{Cell}_{n-2}(A[n])[-n+1]$ lies in $X[-n+1] \star \cdots X[-1]$, by Step 1 it follows that $\operatorname{pd} H(\operatorname{Cell}_{n-2}(A[n])[-n+1]) \leq k$. Then (7.8) gives us an isomorphism in $\text{mod-}X$:

$$\Omega^k H(A) \xrightarrow{\cong} \Omega^k H(\Omega^{n-1}_X(A[n])[-1]) \quad (7.10)$$
From the triangle \((T_{A[n]}^n)\), we have an exact sequence
\[
0 \to H(\Omega_{X}^{n-1}(A[n])[-1]) \to H(\Omega_{X}^{n}(A[n])) \to H(X_{A[n]}^{n-1})
\]
\[
\to H(\Omega_{X}^{n-1}(A[n])) \to 0
\]

Since \(\Omega_{X}^{n}(A[n]) := X_{A[n]}^{n}\) lies in \(X\), it follows that
\[
H(\Omega_{X}^{n-1}(A[n])[-1]) \xrightarrow{\cong} \Omega^{2}H(\Omega_{X}^{n-1}(A[n]))
\]
(7.11)

Consider the short exact sequence
\[
0 \to H(\Omega_{X}^{n-2}(A[n])[-1]) \to H(\Omega_{X}^{n-1}(A[n]))
\]
\[
\to H(X_{A[n]}^{n-2}) \to H(\Omega_{X}^{n-2}(A[n])) \to 0
\]
induced from the triangle \((T_{A[n]}^{n-2})\). Then we have a short exact sequence
\[
0 \to H(\Omega_{X}^{n-2}(A[n])[-1]) \to H(\Omega_{X}^{n-1}(A[n]))
\]
\[
\to \Omega H(\Omega_{X}^{n-2}(A[n])) \to 0
\]
(7.12)

Setting \(t = n - 2\) in (7.9), we have an isomorphism \(H(\Omega_{X}^{n-2}(A[n])[-1]) \cong H(\text{Cell}_{n-3}(A[n])[-n+2])\). Since \(\text{Cell}_{n-3}(A[n])[-n+2]\) lies in \(X[-n+2]*\cdots*X[-1]\), as above we have \(\text{pd } H(\Omega_{X}^{n-2}(A[n])[-1]) \leq k\). Hence from (7.12) we get an isomorphism \(\Omega^{k+1}H(\Omega_{X}^{n-1}(A[n])) \xrightarrow{\cong} \Omega^{k+2}H(\Omega_{X}^{n-2}(A[n]))\). Since \(\Omega^{k}\text{mod}-X = \text{Gproj } X\), combining (7.10) and (7.11) gives us the desired isomorphisms:
\[
\Omega^{k}H(A) \xrightarrow{\cong} \Omega^{k}H(\Omega_{X}^{n-1}(A[n])[-1]) \xrightarrow{\cong} \Omega^{k+2}H(\Omega_{X}^{n-1}(A[n]))
\]
\[
\cong \Omega^{k+3}H(\Omega_{X}^{n-2}(A[n]))
\]
(7.13)

**Step 4:** There are isomorphisms: \(\Omega^{k}H(A) \xrightarrow{\cong} \Omega^{n+1}H(\Omega_{X}^{k}(A[n]))\).

**Proof:** Consider the exact sequence
\[
0 \to H(\Omega_{X}^{n-3}(A[n])[-1]) \to H(\Omega_{X}^{n-2}(A[n])) \to H(X_{A[n]}^{n-3})
\]
\[
\to H(\Omega_{X}^{n-3}(A[n])) \to 0
\]
induced from the triangle \((T_{A[n]}^{n-3})\). Then we have a short exact sequence
\[
0 \to H(\Omega_{X}^{n-3}(A[n])[-1]) \to H(\Omega_{X}^{n-2}(A[n])) \to \Omega H(\Omega_{X}^{n-3}(A[n])) \to 0
\]
Setting \(t = n - 3\) in (7.9), we have an isomorphism \(H(\Omega_{X}^{n-3}(A[n])[-1]) \cong H(\text{Cell}_{n-4}(A[n])[-n+3])\). Since \(\text{Cell}_{n-4}(A[n])[-n+3]\) lies in \(X[-n+3]*\cdots*X[-1]\),
by Step 1 we have $\text{pd} H(\Omega_X^{n-2}(A[n])[-1]) \leq k$. Hence from the above exact sequence we obtain as above an isomorphism:

$$\Omega^k H(\Omega_X^{n-2}(A[n])) \cong \Omega^{k+1} H(\Omega_X^{n-3}(A[n]))$$  (7.14)

Combining (7.13) and (7.14) we arrive at an isomorphism:

$$\Omega^k H(A) \cong \Omega^{k+3} H(\Omega_X^{n-2}(A[n])) \cong \Omega^{k+4} H(\Omega_X^{n-3}(A[n]))$$

Continuing in this way we obtain inductively isomorphisms:

$$\Omega^k H(A) \cong \Omega^{k+3} H(\Omega_X^{n-2}(A[n])) \cong \Omega^{k+4} H(\Omega_X^{n-3}(A[n])) \cong \cdots \cong \Omega^{n+1} H(\Omega_X(A[n]))$$

**Step 5: There is an isomorphism:** $H(A) \cong \Omega^{n+1} H(A[n])$.

**Proof:** By Step 4 we have an isomorphism $\Omega^k H(A) \cong \Omega^{n+1} H(\Omega_X(A[n]))$ and by Step 2 we have an isomorphism $H(\Omega^k_X(A[n])) \cong \Omega^k H(A[n])$. Combining the two, we have an isomorphism $\Omega^k H(A) \cong \Omega^{n+1} \Omega^k H(A[n]) \cong \Omega^k \Omega^{n+1} H(A[n])$. However since $\Omega^k \text{mod-}X = \text{Gproj}(X)$ and $n + 1 \geq k$, the object $\Omega^{n+1} H(A[n])$ is Gorenstein-projective. Since $H(A)$ is also Gorenstein-projective, we infer an isomorphism in $\text{mod-}X$:

$$H(A) \cong \Omega^{n+1} H(A[n])$$

By Step 5 and Lemma 7.5 the assertion follows. ☐

If $k = 1$ in Theorem 7.6, then clearly $H(C) = 0$, for any $C \in \mathcal{X}[-n+1] \star \cdots \star \mathcal{X}[-1]$. As a consequence we have the following result, case $k = 1$ of which was obtained independently by Iyama and Oppermann, see [43, Theorem 5.11].

**Corollary 7.7.** Let $\mathcal{X}$ be an $(n-k)$-corigid $(n+1)$-cluster tilting subcategory of $\mathcal{T}$, $n \geq 2$, $0 \leq k \leq 1$. If $\mathcal{T}$ is $(n+1)$-Calabi–Yau, then the stable triangulated category $\text{Gproj}(\mathcal{X})$ is $(n+2)$-Calabi–Yau.

**Corollary 7.8.** Let $\mathcal{T}$ be an $(n+1)$-Calabi–Yau triangulated category over a field $k$. Let $\mathcal{X}$ be an $(n+1)$-cluster tilting subcategory of $\mathcal{T}$ and assume that $\mathcal{X}$ is $(n-k)$-corigid, where $n \geq 2k - 1$, if $n \geq 2$. If $\mathcal{X}[-n+1] \star \cdots \star \mathcal{X}[-n+k] \subseteq (\mathcal{X} \star \mathcal{X}[1] \star \cdots \star \mathcal{X}[k]) \cap \mathcal{X}_k[k+1]$ then $\text{Gproj}(\mathcal{X})$ is $(n+2)$-Calabi–Yau.

**Proof.** Since $n \geq 2k - 1$, it follows $\mathcal{X}$ is $(k-1)$-corigid and $k$-rigid. Then by Corollary 4.9, the functor $H(C)$ has finite projective dimension (in fact $\text{pd} H(C) \leq k$) for any object $C$.
Let \( \mathcal{T} \) be an \((n+1)\)-Calabi–Yau triangulated category over a field \( k \), for instance \( \mathcal{T} = \mathcal{C}_{H}^{(n+1)} \) the \((n+1)\)-cluster category of a finite-dimensional hereditary \( k \)-algebra \( H \). We also fix a full subcategory \( \mathcal{X} \) of \( \mathcal{T} \) satisfying the conditions of Theorem 7.6. Recall that the triangulated category of singularities, \( \mathbf{D}^{b}_{\text{sing}}(\Lambda) \), in the sense of Orlov, see [52], associated to a finite-dimensional \( k \)-algebra \( \Lambda \), is the Verdier quotient \( \mathbf{D}^{b}(\text{mod-}\Lambda)/\mathbf{K}^{b}(\text{proj} \Lambda) \) of the bounded derived category of finite-dimensional \( \Lambda \)-modules by the thick subcategory of perfect complexes.

**Corollary 7.9.**

(i) The triangulated category \( \mathbf{Gproj} \mathcal{X} \) has AR-triangles with AR-translation

\[
\tau = \Omega^{-(n+1)} : \mathbf{Gproj} \mathcal{X} \xrightarrow{\cong} \mathbf{Gproj} \mathcal{X},
\]

\[
\tau \mathcal{H}(A) = \Omega^{-(n+1)} \mathcal{H}(A) = \Omega^{-k} \Omega^{k} \mathcal{H}(\text{Cell}_{1}(A)[n])
\]

(ii) The stable category \( \text{mod-} \mathbf{Gproj} \mathcal{X} \) of coherent functors over the stable category \( \mathbf{Gproj} \mathcal{X} \) of Gorenstein-projective coherent functors over \( \mathcal{X} \) is a triangulated category which is \((3n+5)\)-Calabi–Yau.

(iii) If \( \mathcal{X} = \text{add} T \) for some object \( T \in \mathcal{T} \), then the triangulated category of singularities \( \mathbf{D}^{b}_{\text{sing}}(\text{End}_{\mathcal{T}}(T)) \) associated to the cluster-tilted \( k \)-algebra \( \text{End}_{\mathcal{T}}(T) \) is \((n+2)\)-Calabi–Yau.

**Proof.** Part (i) follows from Theorem 7.6 and the fact that the AR-translation in a triangulated category with Serre functor \( S \) is given by \( \tau = S[-1] \), see [57]. For (ii) note that the category \( \text{mod-} \mathbf{Gproj} \mathcal{X} \) of coherent functors over \( \mathbf{Gproj} \mathcal{X} \) is Frobenius and therefore its stable category \( \text{mod-} \mathbf{Gproj} \mathcal{X} \) is triangulated. Then the assertion follows from Theorem 7.6 and a result of Keller which says that if a triangulated category \( \mathcal{C} \) is \( d \)-Calabi–Yau, then the stable category \( \text{mod-} \mathcal{C} \) of coherent functors over \( \mathcal{C} \) is \((3d-1)\)-Calabi–Yau, see [45]. Part (iii) follows from the fact that, since by Theorem 6.4 the endomorphism algebra \( \text{End}_{\mathcal{T}}(T) \) is Gorenstein, the triangulated category of singularities \( \mathbf{D}^{b}_{\text{sing}}(\text{End}_{\mathcal{T}}(T)) \) of \( \text{End}_{\mathcal{T}}(T) \) is triangle equivalent to the stable category \( \mathbf{Gproj} \text{End}_{\mathcal{T}}(T) \) of finitely generated Gorenstein-projective modules over \( \text{End}_{\mathcal{T}}(T) \), so Theorem 7.6 applies.

**Remark 7.10.** Using Serre duality, it is easy to see that a \( d \)-Calabi–Yau triangulated category \( \mathcal{T} \), \( 1 \leq d \leq 2n \), may contain a non-trivial \((n+1)\)-cluster tilting subcategory, only if \( d \geq n+1 \), and may contain a non-trivial \((n-k)\)-corigid \((n+1)\)-cluster tilting subcategory, only if \( d = n+1 \).
Remark 7.11. Let $\mathcal{X}$ be an $(n+1)$-cluster tilting subcategory of a triangulated category $\mathcal{T}$. In the special case where $\mathcal{X}$ satisfies $\mathcal{X} = \mathcal{X}[n+1]$, more pleasant results can be obtained. First observe that by Corollary 5.8, the category $\text{mod-}\mathcal{X}$ of coherent functors over $\mathcal{X}$ is Frobenius so the stable category $\text{mod-}\mathcal{X}$ is triangulated. It follows then from the work of Geiss, Keller and Oppermann in [36], that if $\mathcal{T}$ is $d$-Calabi–Yau, where $d = k(n + 1)$, $k \in \mathbb{Z}$, which is easily seen to be the only possible value of $d$ for $\mathcal{T}$ to be $d$-Calabi–Yau, then the stable category $\text{mod-}\mathcal{X}$ is $(d + 2k - 1)$-Calabi–Yau. The proof uses that in this case $\mathcal{X}$ carries a natural $(n + 3)$-angulated structure in the sense of [36]. For $k = 0, 1$ this also follows from Corollary 7.7.

Part 3. Categorified higher Auslander correspondence

In this part we develop, in the context of abelian categories, a categorified version of higher Auslander correspondence between $n$-Auslander algebras and $n$-cluster tilting modules, which is due to Auslander for $n = 0$ and to Iyama for $n > 0$. We also study cluster tilting subcategories in this setting and we investigate homological properties of the associated stable Auslander categories. We also give applications to (infinitely generated) cluster tilting modules over any ring and to Cohen–Macaulay modules over an Artin algebra.

8. Categorified Auslander correspondence

Auslander in the early seventies proved the following remarkable result which played a fundamental role for much of the later developments in the representation theory of Artin algebras:

Theorem 8.1. (See [3].) There is bijective correspondence between Morita equivalence classes of Artin algebras $\Lambda$ of finite representation type and Artin algebras $\Gamma$ such that $\text{gl.dim } \Gamma \leq 2 \leq \text{dom.dim } \Gamma$.

A categorified version of Auslander’s correspondence was proved in [13]. First recall that the free abelian category of an additive category $\mathcal{C}$ is an abelian category $\mathfrak{F}(\mathcal{C})$ together with an additive functor $F: \mathcal{C} \to \mathfrak{F}(\mathcal{C})$ satisfying the following universal property: for any additive functor $G: \mathcal{C} \to \mathcal{A}$ to an abelian category $\mathcal{A}$, there exists a unique exact functor $G^*: \mathfrak{F}(\mathcal{C}) \to \mathcal{A}$ such that $G^* \circ F = G$. The free abelian category $\mathfrak{F}(\mathcal{C})$ of any additive category $\mathcal{C}$ always exists, it is unique up to equivalence, and it admits the following descriptions: $\mathfrak{F}(\mathcal{C}) \approx \text{mod- } (\mathcal{C} \text{-mod})^{\text{op}} \approx ((\text{mod- } \mathcal{C}) \text{-mod})^{\text{op}}$; we refer to [35], [13] for details and more information. We call an abelian category $\mathcal{F}$ with enough projectives and injectives an Auslander category if $\text{gl.dim } \mathcal{F} \leq 2 \leq \text{dom.dim } \mathcal{F}$.

The following result recovers and gives a categorified version of Auslander’s correspondence.
Theorem 8.2. (See [13, Theorems 6.1 and 6.6; Corollary 6.9].) 1. An abelian category \( \mathcal{F} \) is free if and only if \( \mathcal{F} \) is an Auslander category. Moreover the maps

\[ \mathcal{C} \leftrightarrow \mathfrak{F}(\mathcal{C}) \quad \text{and} \quad \mathcal{F} \leftrightarrow \text{Proj}\mathcal{F} \cap \text{Inj}\mathcal{F} \]

give, up to equivalence, mutually inverse bijections between:

(I) Additive categories \( \mathcal{C} \) (with split idempotents).

(II) Auslander categories \( \mathcal{F} \).

2. The maps

\[ \mathfrak{A} \leftrightarrow \text{mod-}\mathfrak{A} \quad \text{and} \quad \mathcal{F} \leftrightarrow \text{Proj}\mathcal{F} \]

give, up to equivalence, mutually inverse bijections between

(a) (Representation finite) Abelian categories \( \mathfrak{A} \) with enough injectives.

(b) Auslander categories \( \mathcal{F} \) such that \( \text{Proj}\mathcal{F} \) is abelian (and representation finite).

(c) Free abelian categories \( \mathcal{F} \) such that \( \text{Proj}\mathcal{F} \cap \text{Inj}\mathcal{F} \) is left coherent (and \( \text{Proj}\mathcal{F} \) is representation finite).

3. The maps

\[ \mathfrak{B} \leftrightarrow (\mathfrak{B}-\text{mod})^{\text{op}} \quad \text{and} \quad \mathcal{F} \leftrightarrow \text{Inj}\mathcal{F} \]

give, up to equivalence, mutually inverse bijections between

(d) (Representation finite) Abelian categories \( \mathfrak{B} \) with enough projectives.

(e) Auslander categories \( \mathcal{F} \) such that \( \text{Inj}\mathcal{F} \) is abelian (and representation finite).

(f) Free abelian categories \( \mathcal{F} \) such that \( \text{Proj}\mathcal{F} \cap \text{Inj}\mathcal{F} \) is right coherent (and \( \text{Inj}\mathcal{F} \) is representation finite).

Recently Iyama proved, as one of the main result of [40], the following far reaching generalization of Auslander’s correspondence which in addition supports interesting connections with cluster tilting theory in module categories. Recall that a finitely generated module \( M \) over a finite-dimensional algebra \( \Lambda \) is called an \( n \)-cluster tilting module if

\[ \{X \in \text{mod-}\Lambda \mid \text{Ext}^k_\Lambda(M, X) = 0, 1 \leq k \leq n\} = \text{add} M = \{X \in \text{mod-}\Lambda \mid \text{Ext}^k_\Lambda(X, M) = 0, 1 \leq k \leq n\} \]

see [40,41], where the terminology \((n+1)\)-cluster tilting or maximal \( n \)-orthogonal module is used.

Theorem 8.3. (See [40].) For any \( n \geq 0 \), the map \( M \mapsto \text{End}_\Lambda(M) \) gives a bijection between equivalence classes of \( n \)-cluster tilting \( \Lambda \)-modules \( M \) for finite-dimensional
algebras $\Lambda$ and Morita-equivalence classes of finite-dimensional algebras $\Gamma$ such that $\text{gl.dim} \Gamma \leq n + 2 \leq \text{dom.dim} \Gamma$.

Our aim in this section is to refine and generalize Theorems 8.2, 8.3 in the context of abelian categories. In fact, we shall prove the following categorification of Theorem 8.3, referring to [14] for a relative version, which shows that (part 1. of) Theorem 8.2 is the zero case of a natural correspondence at a higher level. For the notions of $n$-cluster tilting objects or subcategories and $n$-Auslander categories, we refer to Subsection 8.3 below.

**Theorem 8.4 (Generalized Auslander correspondence).** For any integer $n \geq 0$, the map $M \mapsto \text{mod-}M$ gives, up to equivalence, a bijective correspondence between:

(I) $n$-cluster tilting subcategories $\mathcal{M}$ in abelian categories $\mathcal{A}$ with enough injectives.

(II) $n$-Auslander categories $\mathcal{B}$.

Note that the abelian category $\mathcal{A}$ in the above correspondence is uniquely determined, up to equivalence, by the $n$-cluster tilting subcategory $\mathcal{M}$ it contains: in fact $\mathcal{A}$ is equivalent to the dual of the category of coherent covariant functors over the full subcategory of projective–injective objects of $\text{mod-}M$, see Proposition 8.12.

Specializing Theorem 8.4 to $M = \text{add} M$ for an $n$-cluster tilting $\Lambda$-module $M$ over a finite-dimensional algebra $\Lambda$, we obtain Iyama’s Theorem 8.3. The proof of (a more precise form of) Theorem 8.4 will be given in Theorem 8.23 after developing the necessary tools and giving definitions of all unexplained terms.

### 8.1. From cluster tilting subcategories to Auslander categories

Throughout this subsection we fix an abelian category $\mathcal{A}$ with enough injectives. We also fix a contravariantly finite subcategory $\mathcal{M}$ of $\mathcal{A}$ which is closed under direct summands and isomorphisms. Then the category $\text{mod-}M$ of coherent functors over $\mathcal{M}$ is abelian and we have a left exact functor

$$\mathbb{H} : \mathcal{A} \rightarrow \text{mod-}M, \quad \mathbb{H}(A) = \mathcal{A}(-, A)|_\mathcal{M}$$

From now on: we assume that any right $\mathcal{M}$-approximation in $\mathcal{A}$ is an epimorphism. Note that if $\mathcal{A}$ has enough projectives, then this happens if and only if $\text{Proj} \mathcal{A} \subseteq \mathcal{M}$.

**Lemma 8.5.** There is an adjoint pair of functors

$$(\mathbb{R}, \mathbb{H}) : \text{mod-}M \rightleftarrows \mathcal{A}$$

where the functor $\mathbb{R}$ is exact and the functor $\mathbb{H}$ is fully faithful and induces an equivalence $\mathcal{M} \cong \text{Proj mod-}M$. 
**Proof.** Define a functor \( \mathbb{R}: \text{mod-M} \to \mathcal{A} \) as follows. If \( F \in \text{mod-M} \) admits a presentation \((- , M^1) \to (- , M^0) \to F \to 0\), where \( f: M^1 \to M^0 \) is a map in \( \mathcal{M} \), then \( \mathbb{R}(F) = \text{Coker } f \), i.e. \( \mathbb{R} \) is defined by the exact sequence \( M^1 \to M^0 \to \mathbb{R}(F) \to 0 \). We leave to the reader the easy proof that in this way we obtain a left adjoint of \( \mathbb{H} \). Since right \( \mathcal{M} \)-approximations are epics, it follows directly that the natural map \( \mathbb{R} \mathbb{H} \to \text{Id}_{\mathcal{A}} \) is invertible, so \( \mathbb{H} \) is fully faithful. By Yoneda, \( \mathbb{H} \) induces an equivalence \( \mathcal{M} \cong \text{Proj mod-M} \) since \( \mathcal{M} \) is closed under direct summands. Let \( A_F = \text{Ker}(M^1 \to M^0) \) in the presentation of \( F \) above and let \( M_{A_F} \to A_F \) be a right \( \mathcal{M} \)-approximation. Then we have an exact sequence \( \mathbb{H}(M_{A_F}) \to \mathbb{H}(M^1) \to \mathbb{H}(M^0) \to F \to 0 \) which is part of a projective resolution of \( F \). Since the map \( M_{A_F} \to A_F \) is an epimorphism, applying \( \mathbb{R} \) we have an exact sequence \( M_{A_F} \to M^1 \to M^0 \). This means that \( L_1 \mathbb{R}(F) = 0 \). We infer that \( \mathbb{R} \) is exact. \( \square \)

By Lemma 8.5, Ker \( \mathbb{R} := \text{mod-M} \) is a localizing subcategory of \( \text{mod-M} \), so

\[
0 \to \text{mod-M} \to \text{mod-M} \to \mathcal{A} \to 0
\]  

(8.1)

is a short exact sequence of abelian categories and \( \text{mod-M} \) consists of all coherent functors \( F : \mathcal{M}^{\text{en}} \to \mathfrak{A} \) with a projective presentation \( \mathbb{H}(M^1) \to \mathbb{H}(M^0) \to F \to 0 \) where the map \( M^1 \to M^0 \) is epic in \( \mathcal{A} \).

Let \( F \in \text{mod-M} \), and fix throughout a projective presentation \( \mathbb{H}(M^1) \to \mathbb{H}(M^0) \to F \to 0 \) of \( F \). Then we have an exact sequence

\[
0 \to \mathbb{H}(A_F) \to \mathbb{H}(M^1) \to \mathbb{H}(M^0) \to F \to 0
\]  

(8.2)

where \( A_F = \text{Ker}(M^1 \to M^0) \) and then \( \Omega(F) = \text{Im}(\mathbb{H}(M^1) \to \mathbb{H}(M^0)) \) and \( \mathbb{H}(A_F) = \Omega^2(F) \).

Let \( A \in \mathcal{A} \). Since right \( \mathcal{M} \)-approximations are epics, there exists an exact sequence

\[
\cdots \to M^n_A \to M^{n-1}_A \to \cdots \to M^1_A \to M^0_A \to A \to 0
\]  

(8.3)

called an \( \mathcal{M} \)-resolution of \( A \), such that its image under \( \mathbb{H} \) is a projective resolution of \( \mathbb{H}(A) \):

\[
\cdots \to \mathbb{H}(M^n_A) \to \mathbb{H}(M^{n-1}_A) \to \cdots \to \mathbb{H}(M^1_A)
\to \mathbb{H}(M^0_A) \to \mathbb{H}(A) \to 0
\]  

(8.4)

This is defined as the Yoneda composition of short exact sequences \( 0 \to K^{i+1}_A \to M_i^i \to K^i_A \to 0 \) constructed inductively, where each map \( M^i_A \to K^i_A \) is a right \( \mathcal{M} \)-approximation of \( K^i \), \( \forall i \geq 0 \), and \( K^0_A = A \). It follows from (8.2) and (8.4) that any coherent functor \( F \in \text{mod-M} \) admits a projective resolution of the form
\[
\cdots \longrightarrow \mathbb{H}(M_{A_F}^n) \longrightarrow \mathbb{H}(M_{A_F}^{n-1}) \longrightarrow \cdots \longrightarrow \mathbb{H}(M_{A_F}^0)
\]
\[
\longrightarrow \mathbb{H}(M^1) \longrightarrow \mathbb{H}(M^0) \longrightarrow F \longrightarrow 0
\]  

(8.5)

Now in analogy with the triangulated case we define

\[
M_n^+ = \{ A \in \mathcal{A} \mid \text{Ext}^k(M, A) = 0, \ 1 \leq k \leq n \} \quad \text{and} \quad
\]

\[
\frac{M}{n} = \{ A \in \mathcal{A} \mid \text{Ext}^k(A, M) = 0, \ 1 \leq k \leq n \}
\]

Then \( M \) is called \( n \)-rigid if \( M \subseteq M_n^+ \) or equivalently \( M \subseteq \frac{M}{n} \).

**Lemma 8.6.** For any object \( F \in \text{mod-}M \): \( F \in \widehat{\text{mod-}M} \) if and only if \( \text{Hom}(F, \mathbb{H}(B)) = 0 \), \( \forall B \in \mathcal{A} \). If \( F \in \text{mod-}M \), then \( \text{Ext}^1(F, \mathbb{H}(B)) = 0 \), \( \forall B \in \mathcal{A} \). If in addition \( M \subseteq M_n^+ \), then there are isomorphisms:

\[
F \cong \text{Ext}^1(-, A_F)|_M \cong \text{Ext}^2(-, K_{A_F}^1)|_M \cong \cdots \cong \text{Ext}^{n-1}(-, K_{A_F}^{n-2})|_M \cong \text{Ext}^n(-, K_{A_F}^{n-1})|_M
\]  

(*)

**Proof.** The first part follows from the adjunction isomorphism \( \text{Hom}(F, \mathbb{H}(B)) \cong \text{Hom}(\mathbb{R}(F), B) \) and the fact that \( \text{Ker} \mathbb{R} = \widehat{\text{mod-}M} \). Let \( F \in \text{mod-}M \); applying the exact functor \( \mathbb{R} \) to (8.2) we have an isomorphism \( \mathbb{R}(\Omega F) \cong \mathbb{R}(\mathbb{H}(M^0)) = M^0 \). Then the exact sequence \( (\mathbb{H}(M^0), \mathbb{H}(B)) \longrightarrow (\Omega F, \mathbb{H}(B)) \longrightarrow \text{Ext}^1(F, \mathbb{H}(B)) \longrightarrow 0 \) is isomorphic to \( (M^0, B) \longrightarrow (\mathbb{R}(\Omega F), B) \longrightarrow \text{Ext}^1(F, \mathbb{H}(B)) \longrightarrow 0 \), where the first map is invertible. Hence \( \text{Ext}^1(F, \mathbb{H}(B)) = 0 \), \( \forall B \in \mathcal{A} \). If \( M \subseteq M_n^+ \), then from the extension \( 0 \longrightarrow A_F \longrightarrow M^1 \longrightarrow M^0 \longrightarrow 0 \) we have \( F \cong \text{Ext}^1(-, A_F)|_M \), since \( \text{Ext}^1(-, M^1)|_M = 0 \). Considering the short exact sequences \( 0 \longrightarrow K_{A_F}^{i+1} \longrightarrow M_{A_F}^i \longrightarrow K_{A_F}^i \longrightarrow 0 \), \( i \geq 0 \), where \( K_{A_F}^0 = A_F \), and using that \( M \subseteq M_n^+ \), as above the isomorphisms (*) follow easily by induction. \( \square \)

**Lemma 8.7.** For any object \( B \in \mathcal{A} \), and any \( n \geq 1 \), the following are equivalent:

(i) \( B \in M_n^+ \).

(ii) The functor \( \mathbb{H} \) induces isomorphisms:

\[
\text{Ext}^k(-, B) \cong \text{Ext}^k(\mathbb{H}(-), \mathbb{H}(B)), \quad 1 \leq k \leq n
\]

**Proof.** If (ii) holds, then using that \( \mathbb{H}(M) \) is projective in \( \text{mod-}M \), for any \( M \in \mathcal{M} \), we infer that \( \text{Ext}^k(M, B) = 0 \), \( 1 \leq k \leq n \). Hence \( B \) lies in \( M_n^+ \). For the converse, considering an \( M \)-resolution of \( A \) and using that \( M \) is \( n \)-rigid and that the left exact functor \( \mathbb{H} \) is fully faithful and induces an equivalence between \( M \) and \( \text{Proj mod-}M \), it follows directly by induction that the hypothesis \( B \in M_n^+ \) implies that the naturally induced maps \( \text{Ext}^k(A, B) \longrightarrow \text{Ext}^k(\mathbb{H}(A), \mathbb{H}(B)) \), for \( 1 \leq k \leq n \), are invertible, \( \forall A \in \mathcal{A} \). \( \square \)
For any object $M \in \mathcal{M}$, consider the functor $(-, M): \mathcal{M}^{\text{op}} \rightarrow \mathfrak{Ab}$, $M' \mapsto \text{Hom}(M', M)$. If $\mathcal{A}$ has enough projectives and $0 \rightarrow \Omega M \rightarrow P_M \rightarrow M \rightarrow 0$ is an exact sequence in $\mathcal{A}$ where $P_M$ is projective, then it is easy to see that we have an exact sequence of functors

$$0 \rightarrow \mathbb{H}(\Omega M) \rightarrow \mathbb{H}(P_M) \rightarrow \mathbb{H}(M) \rightarrow (-, M) \rightarrow 0 \quad (\dagger)$$

Hence, for any $M \in \mathcal{M}$, the functor $(-, M)$ lies in $\text{mod-} \mathcal{M}$.

Observe that since for $F = (-, M)$ we have $A_F = \Omega M$, it follows that we have isomorphisms, $\forall M' \in \mathcal{M}, \forall k \geq 1$:

$$\text{Ext}^{k+2}((- M), (M')) \overset{\cong}{\rightarrow} \text{Ext}^k(\mathbb{H}(\Omega M), \mathbb{H}(M')) \quad (\ddagger)$$

We use the family of functors $(-, M)$, $M \in \mathcal{M}$, to give the following characterization of rigidity:

\textbf{Lemma 8.8.} If $n \geq 1$ then for the following statements:

(i) $\mathcal{M}$ is $n$-rigid;

(ii) For any $F \in \text{mod-} \mathcal{M} \subseteq \text{mod-} \mathcal{M}$, we have: $\text{Ext}^k(F, \mathbb{H}(M)) = 0$, $0 \leq k \leq n + 1$,

we have (i) $\Rightarrow$ (ii). If $\mathcal{A}$ has enough projectives, then the statements (i) and (ii) are equivalent.

\textbf{Proof.} (i) $\Rightarrow$ (ii) Since $\mathcal{M} \subseteq \mathcal{M}_n^+$, from (8.2) and Lemma 8.7 we have $\text{Ext}^{k+2}(F, \mathbb{H}(M)) \cong \text{Ext}^k(\mathbb{H}(A_F), \mathbb{H}(M)) \cong \text{Ext}^k(A_F, M)$, $1 \leq k \leq n$. Since $F \in \text{mod-} \mathcal{M}$, we have a short exact sequence $0 \rightarrow A_F \rightarrow M^l \rightarrow M^0 \rightarrow 0$. Applying $\mathcal{A}(-, X)$, with $X \in \mathcal{M}$, the induced long exact sequence gives $\text{Ext}^k(A_F, X) = 0$, $1 \leq k \leq n - 1$. Hence $\text{Ext}^{k+2}(F, \mathbb{H}(X)) \cong \text{Ext}^k(A_F, X) = 0$, $1 \leq k \leq n - 1$, and then by Lemma 8.6: $\text{Ext}^k(F, \mathbb{H}(M)) = 0$, $0 \leq k \leq n$. Applying $(-, \mathbb{H}(X))$ to the short exact sequence $0 \rightarrow \mathbb{H}(A_F) \rightarrow \mathbb{H}(M^1) \rightarrow \Omega(F) \rightarrow 0$ we have an exact sequence $(\mathbb{H}(M^1), \mathbb{H}(X)) \rightarrow (\mathbb{H}(A_F), \mathbb{H}(X)) \rightarrow \text{Ext}^2(F, \mathbb{H}(X)) \rightarrow 0$. Since $(\mathbb{H}(M^1), \mathbb{H}(X)) \rightarrow (\mathbb{H}(A_F), \mathbb{H}(X))$ is isomorphic to the map $(M^1, X) \rightarrow (A_F, X)$ which is epic since $\text{Ext}^1(M^0, X) = 0$, we infer that $\text{Ext}^2(F, \mathbb{H}(X)) = 0$. Hence $\text{Ext}^k(F, \mathbb{H}(X)) = 0$, $0 \leq k \leq n + 1$, as required.

(ii) $\Rightarrow$ (i) Assume that $\mathcal{A}$ has enough projectives. For $M \in \mathcal{M}$ we consider the functor $(-, M) \in \text{mod-} \mathcal{M}$. Let $G = \text{Im}(\mathbb{H}(P_M) \rightarrow \mathbb{H}(M))$. Then $0 = \text{Ext}^2((- M), \mathbb{H}(M)) \cong \text{Ext}^1(G, \mathbb{H}(M)) \cong \text{Im}([\mathcal{A}(P_M, M) \rightarrow \mathcal{A}(\Omega M, M)]) \cong \text{Ext}^1(M, M)$, hence $M \subseteq \mathcal{M}_n^+$.

Using this, the isomorphism $(\dagger)$ and Lemma 8.7, we have: $0 = \text{Ext}^2((- M), \mathbb{H}(M)) \cong \text{Ext}^1(\mathbb{H}(\Omega M), \mathbb{H}(M)) \cong \text{Ext}^1(\Omega M, M) \cong \text{Ext}^2(M, M)$. Hence $M \subseteq \mathcal{M}_n^+$. Continuing in this way and using Lemma 8.7 and the isomorphism $(\ddagger)$ we have $M \subseteq \mathcal{M}_n^+$ as required. \hfill \Box

To proceed further we need the following observation.
Lemma 8.9. Let $\mathcal{C}$ be an abelian category with enough projectives, and $C \in \mathcal{C}$. Then: $C = 0$ if and only if $C$ has finite projective dimension and $\text{Ext}^k(C, P) = 0$, $\forall P \in \text{Proj}\mathcal{C}$, $0 \leq k \leq \text{pd} C$.

Proof. Let $\text{pd} C = n$ and let $0 \longrightarrow P^n \longrightarrow P^{n-1} \longrightarrow \cdots \longrightarrow P^0 \longrightarrow C \longrightarrow 0$ be a projective resolution of $C$. Then $0 = \text{Ext}^n(C, P^n) \cong \text{Ext}^1(\Omega^{n-1}C, P^n)$, so the extension $0 \longrightarrow P^n \longrightarrow P^{n-1} \longrightarrow \Omega^{n-1}C \longrightarrow 0$ splits and therefore $\Omega^{n-1}C$ is projective. Then $\text{Ext}^1(\Omega^{n-2}C, \Omega^{n-1}C) \cong \text{Ext}^{n-1}(C, \Omega^{n-1}C) = 0$, so the extension $0 \longrightarrow \Omega^{n-1}C \longrightarrow P^{n-2} \longrightarrow \Omega^{n-2}C \longrightarrow 0$ splits and therefore $\Omega^{n-2}C$ is projective. Continuing in this way we see that $C$ is projective. Then $C = 0$ since $\mathcal{C}(C, C) = 0$. □

Proposition 8.10. If $M \neq \text{Proj} \mathcal{A}$ and $M$ is n-rigid, i.e. $M \subseteq M^n_n$, then $\text{gl.dim mod-M} \geq n + 2$. Moreover:

$$M^n_n = M \neq \text{Proj} \mathcal{A} \quad \Longrightarrow \quad \frac{1}{n}M = M \neq \text{Inj} \mathcal{A} \quad \& \quad \text{gl.dim mod-M} = n + 2$$

Proof. Since $M \neq \text{Proj} \mathcal{A}$, we have $\text{mod-M} \neq 0$. Indeed if $\text{mod-M} = 0$ then the functor $\mathbb{R}$ induces an equivalence $\text{mod-M} \cong \mathcal{A}$ and this implies that $\mathcal{A}$ has enough projectives and $M = \text{Proj} \mathcal{A}$. This contradiction shows that there exists $0 \neq F \in \text{mod-M}$. If $\text{gl.dim mod-M} \leq n + 1$, then by Lemma 8.7 we have $\text{Ext}^k(F, \mathbb{R}(X)) = 0$, $0 \leq k \leq n + 1$, and then $F = 0$ by Lemma 8.9. This contradiction shows that $\text{gl.dim mod-M} \geq n + 2$. Now assume that $M^n_n = M$. Then clearly $\text{Inj} \mathcal{A} \subseteq M^n_n = M$. To show that $\text{gl.dim mod-M} = n + 2$, it suffices to show that $\text{gl.dim mod-M} \leq n + 2$. By using the exact sequences (8.2), (8.4) this holds if $\text{pd} \mathbb{R}(A) \leq n$, $\forall A \in \mathcal{A}$. Since the exact sequence (8.3) becomes a projective resolution of $\mathbb{R}(A)$ after applying $\mathbb{R}$, it suffices to show that $\mathbb{R}(K^n_A)$ is projective, or equivalently that $K^n_A \in \mathcal{M}$. Since $M^n_n = M$, it suffices to show that $K^n_A \in M^n_n$. Since $M$ is n-rigid, we have $\text{Ext}^1(M, K^n_A) = 0$, $1 \leq i \leq n$. In particular $\text{Ext}^1(M, K^n_A) = 0$ and we have an isomorphism $\text{Ext}^2(M, K^n_A) \cong \text{Ext}^1(M, K^{n-1}_A) = 0$, hence $K^n_A \in \mathcal{M}_2$. Continuing in this way we have finally isomorphisms $\text{Ext}^n(M, K^n_A) \cong \text{Ext}^{n-1}(M, K^{n-1}_A) \cong \cdots \cong \text{Ext}^1(M, K^1_A) = 0$. Hence $K^n_A \in M^n_n = M$.

Now let $A \in \frac{1}{n}M$. Applying $\mathcal{A}(-, M)$ to the extension $0 \longrightarrow K^1_A \longrightarrow M^0_A \longrightarrow A \longrightarrow 0$, we have $\text{Ext}^i(K^1_A, M) = 0$, $1 \leq i \leq n - 1$. Using this and applying $\mathcal{A}(-, M)$ to the extension $0 \longrightarrow K^2_A \longrightarrow M^1_A \longrightarrow K^1_A \longrightarrow 0$, we have $\text{Ext}^i(K^2_A, M) = 0$, $1 \leq i \leq n - 2$. Continuing in this way we finally have $\text{Ext}^i(K^{n-1}_A, M) = 0$. Since $\text{pd} \mathbb{R}(A) \leq n$, we have $K^n_A \in M$ and therefore the extension $0 \longrightarrow K^n_A \longrightarrow M^{n-1}_A \longrightarrow K^{n-1}_A \longrightarrow 0$ splits, hence $K^{n-1}_A \in M$. Since $\text{Ext}^1(K^{n-2}_A, M) = 0$, the extension $0 \longrightarrow K^{n-1}_A \longrightarrow M^{n-2}_A \longrightarrow K^{n-2}_A \longrightarrow 0$ splits and therefore $K^{n-2}_A \in M$. Continuing in this way we see that the objects $K^i_n$ lie in $M$ and therefore since $\text{Ext}^1(A, M) = 0$, we infer that the extension $0 \longrightarrow K^n_A \longrightarrow M^0_A \longrightarrow A \longrightarrow 0$ splits. Then $A \in M$ as a direct summand of $M^0$. We conclude that $A \in M$, i.e. $\frac{1}{n}M = M$. If $\text{Inj} \mathcal{A} = M$, then clearly $M = \mathcal{A}$ and therefore $\mathcal{A}$ is semisimple. It follows that $\mathcal{A} = M = \text{Proj} \mathcal{A}$ and this is not the case. Hence $\text{Inj} \mathcal{A} \subsetneq M$. □
Remark 8.11. The following observations will be useful later.

1. Assume that $\mathcal{M}^{\perp}_n = \mathcal{M}$. Then $\mathcal{M}^{\perp}_{n+1} = \text{Inj} \mathcal{A}$. Indeed, if $A \in \mathcal{M}^{\perp}_{n+1}$, then clearly $A \in \mathcal{M}$ and $\Sigma A \in \mathcal{M}^{\perp}_n = \mathcal{M}$. Hence any extension $0 \to A \to I \to \Sigma A \to 0$, where $I \in \text{Inj} \mathcal{A}$, splits and therefore $A$ is injective. Hence $\mathcal{M}^{\perp}_{n+1} \subseteq \text{Inj} \mathcal{A}$ and therefore $\mathcal{M}^{\perp}_{n+1} = \text{Inj} \mathcal{A}$ since always $\text{Inj} \mathcal{A} \subseteq \mathcal{M}^{\perp}_{n+1}$. If $\mathcal{A}$, in addition, has enough projectives, then since $\mathcal{M}^{\perp}_n = \mathcal{M}$ by Proposition 8.10, we have similarly that $\mathcal{M}^{\perp}_{n+1} = \mathcal{M} = \text{Proj} \mathcal{A}$. It follows directly from this that $\mathcal{M} = \text{Proj} \mathcal{A} = \text{Inj} \mathcal{A}$, in particular $\mathcal{A}$ is Frobenius.

2. It follows from Lemmas 8.7 and 8.8, and Proposition 8.10 that the following are equivalent:
   (a) $\mathcal{M}$ is $n$-rigid.
   (b) grade$_{\text{mod-}\mathcal{M}} F \geq n + 2$, $\forall F \in \text{mod-}\mathcal{M}$, $F \neq 0$, with equality in (b) if $\mathcal{M}^{\perp}_n = \mathcal{M}$. This generalizes a result of Buchweitz [25, Theorem 2.13].

   We recall that if $\mathcal{C}$ is an abelian category, then the grade of $X$ of $X \in \mathcal{C}$ is defined by grade$_{\mathcal{C}} X = \inf \{ i \geq 0 \mid \text{Ext}^i(X, P) \neq 0, \forall P \in \text{Proj} \mathcal{C} \}$.

3. If $\mathcal{M}^{\perp}_n = \mathcal{M}$, then for any $F \in \text{mod-}\mathcal{M}$ we have: Ext$_{\text{mod-}\mathcal{M}}^k(F, (-, \mathcal{M})) = 0$, $\forall k \geq 0$, except possibly for $k = n + 2$. In this case Ext$_{\text{mod-}\mathcal{M}}^{n+2}(F, (-, \mathcal{M})) \cong \text{Ext}^n_{\mathcal{M}}((-A F) |_{\mathcal{M}}, (-, \mathcal{M})) \cong \text{Ext}^n_{\mathcal{A}}(A F, \mathcal{M})$. In particular for $F = (-, \mathcal{M})$, $\mathcal{M} \in \mathcal{M}$, we have: Ext$_{\text{mod-}\mathcal{M}}^{n+2}((-\mathcal{M}), (-, \mathcal{M})) \cong \text{Ext}^n_{\mathcal{A}}(\mathcal{M}, \mathcal{M})$.

4. If $\mathcal{M}^{\perp}_n = \mathcal{M}$, then any object $A \in \mathcal{A}$ admits an $\mathcal{M}$-resolution

$$0 \to M^n_A \to M^{n-1}_A \to \cdots \to M^1_A \to M^0_A \to A \to 0$$

Indeed as in the proof of Proposition 8.10 we have $\text{pd} \mathcal{H}(A) \leq n$. Using the projective resolution of $\mathcal{H}(A)$ which arises by applying $\mathcal{H}$ to the $\mathcal{M}$-resolution (8.3), we infer that the image of the map $\mathcal{H}(M^n_A) \to \mathcal{H}(M^{n-1}_A)$ is projective. Then $\text{Im}(M^n_A \to M^{n-1}_A)$ in (8.3) lies in $\mathcal{M}$ and our claim follows.

Proposition 8.12. Let $\mathcal{A}$ be an abelian category with enough injectives and $\mathcal{M}$ a contravariantly finite subcategory of $\mathcal{A}$ such that any right $\mathcal{M}$-approximation is epic. If $\text{Proj} \mathcal{A} \neq \mathcal{M} = \mathcal{M}^{\perp}_n$ for some $n \geq 1$, then:

1. We have equalities:

$$\text{gl.dim } \text{mod-}\mathcal{M} = n + 2 = \text{dom.dim } \text{mod-}\mathcal{M} \tag{8.6}$$

2. $\text{Inj} \mathcal{A} = \mathcal{M} = \mathcal{M}^{\perp}_n$, and $\mathcal{A}$, $\text{Inj} \mathcal{A}$, and $\mathcal{M}$ can be recovered from $\mathcal{B} := \text{mod-}\mathcal{M}$ as follows:

$$\mathcal{A} \cong (\mathcal{U}\text{-mod})^{\text{op}} \text{ and } \mathcal{M} \cong \text{Proj} \mathcal{B} \text{ and } \text{Inj} \mathcal{A} \cong \mathcal{U} \tag{8.7}$$
where \( \mathcal{U} \) denotes the full subcategory of projective–injective objects of \( \text{mod} \)-\( \mathcal{M} \) and is covariantly finite.

3. \( \mathcal{M} \) is covariantly finite in \( \mathcal{A} \) if and only if \( \text{Proj} \mathcal{B} \) is covariantly finite in \( \mathcal{B} = \text{mod} \)-\( \mathcal{M} \).

**Proof.** 1., 2. In view of Proposition 8.10, it remains to show that \( \text{dom.dim} \text{mod} \)-\( \mathcal{M} = n + 2 \) and the existence of the equivalences (8.7). Let \( M \in \mathcal{M} \) and consider an injective resolution of \( M \):

\[
0 \to M \to I^0 \to I^1 \to \cdots \to I^n \to I^{n+1} \to \cdots
\]

Applying the left exact functor \( \mathbb{H} \) and using that \( M \) is \( n \)-rigid, we have an exact sequence:

\[
0 \to \mathbb{H}(M) \to \mathbb{H}(I^0) \to \mathbb{H}(I^1) \to \cdots
\]

\[
\to \mathbb{H}(I^{n-1}) \to \mathbb{H}(I^n) \to \mathbb{H}(I^{n+1}) \quad (\dagger)
\]

Since any projective object of \( \text{mod} \)-\( \mathcal{M} \) is of the form \( \mathbb{H}(M) \) for some \( M \in \mathcal{M} \) and since the functor \( \mathbb{H} \) preserves injective objects (since its left adjoint \( \mathbb{R} \) is exact), it follows that \((\dagger)\) is part of an injective resolution of the projective object \( \mathbb{H}(M) \). Since \( \text{Inj} \mathcal{A} \subseteq \mathcal{M} \), the objects \( \mathbb{H}(I^k) \) above are projective–injective. Hence \( \text{dom.dim} \text{mod} \)-\( \mathcal{M} \geq n + 2 \). If \( \text{dom.dim} \text{mod} \)-\( \mathcal{M} \geq n + 3 \), then for any \( M \in \mathcal{M} \) there is an exact sequence \( 0 \to \mathbb{H}(M) \to \mathbb{H}(I^0) \to \cdots \to \mathbb{H}(I^n) \to \mathbb{H}(I^{n+1}) \to F \to 0 \) where the \( I^k \) are injective in \( \mathcal{A} \). Clearly then \( \text{pd} F = n + 3 \) and this is impossible since \( \text{gl.dim} \text{mod} \)-\( \mathcal{M} = n + 2 \) by Proposition 8.10. We have a fully faithful functor \( \mathbb{H} : \text{Inj} \mathcal{A} \to \mathcal{U} \). If \( F \) lies in \( \mathcal{U} \), then \( F = \mathbb{H}(M) \) for some \( M \in \mathcal{M} \) since \( F \) is projective. Let \( M \to J \) be monic, where \( J \) is injective; clearly the induced monic \( \mathbb{H}(M) \to \mathbb{H}(J) \) splits and then \( M \) is injective as a direct summand of \( J \). We infer that the functor \( \mathbb{H} : \text{Inj} \mathcal{A} \to \mathcal{U} \) is essentially surjective, hence an equivalence. Let \( F \) be in \( \text{mod} \)-\( \mathcal{M} \) and let \( \mathbb{R}(F) \to I \) be a monomorphism, where \( I \) is injective in \( \mathcal{A} \). Then \( \mathbb{H}(I) \) lies in \( \mathcal{U} \) and it is easy to see that the composition \( F \to \mathbb{R}(F) \to \mathbb{H}(I) \) is a left \( \mathcal{U} \)-approximation of \( F \). Hence \( \mathcal{U} \) is covariantly finite in \( \text{mod} \)-\( \mathcal{M} \). Finally we have \( \mathcal{A} \approx (\text{Inj} \mathcal{A} \text{-mod})^\text{op} \approx (\mathcal{U} \text{-mod})^\text{op} \), the first equivalence being true for any abelian category with enough injectives.

3. If \( \mathcal{M} \) is covariantly finite in \( \mathcal{A} \), let \( F \in \mathcal{B} \) and let \( \mathbb{R}(F) \to M^{\mathbb{R}(F)} \) be a left \( \mathcal{M} \)-approximation of \( \mathbb{R}(F) \). We leave to the reader as an easy exercise to check that the composite map \( F \to \mathbb{R}(F) \to \mathbb{H}(M^{\mathbb{R}(F)}) \) is a left \( \text{Proj} \mathcal{B} \)-approximation of \( F \), where the first map is the unit of the adjoint pair \((\mathbb{R}, \mathbb{H})\). Hence \( \text{Proj} \mathcal{B} \) is covariantly finite. If this holds, let \( B \in \mathcal{B} \) and \( \mathbb{H}(B) \to \mathbb{H}(M) \) be a left \( \text{Proj} \mathcal{B} \)-approximation of \( \mathbb{H}(B) \). This gives us a map \( B \to M \) which clearly is a left \( \mathcal{M} \)-approximation of \( B \). Hence \( \mathcal{M} \) is covariantly finite. \( \square \)

In Section 9 we shall prove that any full subcategory \( \mathcal{M} \) of \( \mathcal{A} \) satisfying the conditions of Proposition 8.12 is covariantly finite in \( \mathcal{A} \).
8.2. From Auslander categories to cluster tilting subcategories

Throughout this subsection, we fix an abelian category $\mathcal{B}$ with enough projectives. We assume that: the full subcategory

$$\mathcal{U} := \text{Proj} \mathcal{B} \cap \text{Inj} \mathcal{B}$$

consisting of all projective–injective objects of $\mathcal{B}$ is covariantly finite.

**Remark 8.13.** $\text{Proj} \mathcal{C} \cap \text{Inj} \mathcal{C}$ is covariantly finite in any abelian category $\mathcal{C}$ satisfying: (α) $\text{Proj} \mathcal{C}$ is covariantly finite, and (β) $\text{dom.dim} \mathcal{C} \geq 1$. Indeed consider the composition $A \rightarrow P \rightarrow J$, where $A \rightarrow P$ is a left $\text{Proj} \mathcal{C}$-approximation of $A$, and $P \rightarrow J$ is monic, where $J$ is projective–injective, which exists since $\text{dom.dim} \mathcal{C} \geq 1$. Then it is easy to see that the map $A \rightarrow J$ is a left $(\text{Proj} \mathcal{C} \cap \text{Inj} \mathcal{C})$-approximation of $A$.

It follows that the category of coherent functors $\mathcal{U}\text{-mod}$ is abelian with enough projectives. Then setting

$$\mathcal{A} := (\mathcal{U}\text{-mod})^{\text{op}}$$

we obtain an abelian category with enough injectives.

**Lemma 8.14.** There is an adjoint pair of functors

$$(\mathbb{R}^{\text{op}}, \mathbb{H}^{\text{op}}): \mathcal{B} \leftrightarrow \mathcal{A}$$

where the functor $\mathbb{H}^{\text{op}}$ is fully faithful and the functor $\mathbb{R}^{\text{op}}$ is exact and induces an equivalence $\mathcal{U} \xrightarrow{\approx} \text{Inj} \mathcal{A}$.

**Proof.** First let $B \in \mathcal{B}$. Since $\mathcal{U}$ is covariantly finite, there exists a left $\mathcal{U}$-approximation $g_0^B: B \rightarrow U_0^B$. Let $L_1^B = \text{Coker} g_0^B$ and let $g_1^B: U_0^B \rightarrow U_1^B$ be the composition of $U_0^B \rightarrow L_1^B$ with a left $\mathcal{U}$-approximation $L_1^B \rightarrow U_1^B$. Then we obtain a complex

$$B \xrightarrow{g_0^B} U_0^B \xrightarrow{g_1^B} U_1^B$$

in $\mathcal{B}$ with the property that the induced complex $\mathcal{B}(U_1^B, -)|_{\mathcal{U}} \rightarrow \mathcal{B}(U_0^B, -)|_{\mathcal{U}} \rightarrow \mathcal{B}(B, -)|_{\mathcal{B}} \rightarrow 0$ is exact in $\mathcal{U}\text{-mod}$. We set $\mathbb{R}^{\text{op}}(B) = \mathcal{B}(B, -)|_{\mathcal{U}}, \forall B \in \mathcal{B}$, so that in $\mathcal{A} = (\mathcal{U}\text{-mod})^{\text{op}}$ we have an exact sequence

$$0 \rightarrow \mathbb{R}^{\text{op}}(B) \rightarrow \mathbb{R}^{\text{op}}(U_0^B) \rightarrow \mathbb{R}^{\text{op}}(U_1^B)$$

(8.9)

It is easy to see that the assignment $B \rightarrow \mathbb{R}^{\text{op}}(B)$ extends to a functor $\mathbb{R}^{\text{op}}: \mathcal{B} \rightarrow \mathcal{A}$ which clearly is exact since $\mathcal{U}$ consists of injective objects of $\mathcal{B}$. We define a functor
\( \mathbb{H}^{\text{op}} : \mathcal{A} \to \mathcal{B} \) as follows: if \( F \) lies in \( \mathcal{A} \), then in \( \mathcal{U}\text{-mod}, F \) admits a presentation \((U_1, -)|_{\mathcal{U}} \to (U_0, -)|_{\mathcal{U}} \to F \to 0\). Then we have a map \( f_F : U_0 \to U_1 \) in \( \mathcal{B} \) where the \( U_i \) lie in \( \mathcal{U} \). It is easy to see that the assignment \( F \mapsto \mathbb{H}^{\text{op}}(F) = \text{Ker} f_F \):

\[
0 \to \mathbb{H}^{\text{op}}(F) \to U^0 \to U^1
\]

defines an additive functor \( \mathbb{H}^{\text{op}} : \mathcal{A} \to \mathcal{B} \). We show that \( (\mathbb{R}^{\text{op}}, \mathbb{H}^{\text{op}}) \) is an adjoint pair. First observe that in (8.10) the maps \( \mathbb{H}^{\text{op}}(F) \to U_0 \) and \( \text{Im}(U_0 \to U_1) \to U_1 \) are left \( \mathcal{U}\)-approximations since both maps are monics and \( \mathcal{U} \) consists of injective objects. As a consequence applying to (8.10) the exact functor \( \mathbb{R}^{\text{op}} \) we have an isomorphism \( \mathbb{R}^{\text{op}} \mathbb{H}^{\text{op}}(F) \xrightarrow{\simeq} F \). Now let \( B \in \mathcal{B} \) and consider a diagram as in (8.8). By construction, the naturally induced map \( B \to \text{Ker} g_1^B \) is the map \( B \to \mathbb{H}^{\text{op}} \mathbb{R}^{\text{op}}(B) \) which is the evaluation at \( B \) of a natural map \( \text{Id}_{\mathcal{B}} \to \mathbb{H}^{\text{op}} \mathbb{R}^{\text{op}} \). It is easy to see that the isomorphism \( \mathbb{R}^{\text{op}} \mathbb{H}^{\text{op}} \xrightarrow{\simeq} \text{Id}_{\mathcal{A}} \) and the natural map \( \text{Id}_{\mathcal{B}} \to \mathbb{H}^{\text{op}} \mathbb{R}^{\text{op}} \) are the counit and the unit respectively of an adjoint pair \( (\mathbb{R}^{\text{op}}, \mathbb{H}^{\text{op}}) \), where the functor \( \mathbb{H}^{\text{op}} \) is fully faithful, since the counit is invertible. Clearly then \( \mathbb{R}^{\text{op}} \) induces an equivalence \( \mathcal{U} \approx \text{Inj}(\mathcal{U}\text{-mod})^{\text{op}} \approx \text{Inj} \mathcal{A} \). □

**Lemma 8.15.** If dom.dim \( \mathcal{B} \geq 2 \), then the functor \( \mathbb{R}^{\text{op}} \) induces an equivalence

\[
\text{Proj} \mathcal{B} \xrightarrow{\simeq} \text{Im} \mathbb{R}^{\text{op}}
\]

**Proof.** We have to show that the functor \( \mathbb{R}^{\text{op}} \) is fully faithful when restricted to Proj \( \mathcal{B} \). So let \( \alpha : P \to Q \) be a map in Proj \( \mathcal{B} \). Choosing copresentations of \( P \) and \( Q \) as in (8.8) and using that the maps \( g_0^P \) and \( g_0^Q \) are monomorphisms since dom.dim \( \mathcal{B} \geq 2 \), we have the following exact commutative diagram

\[
\begin{array}{cccc}
0 & \to & P & \xrightarrow{g_0^P} & U^P_0 & \xrightarrow{g_1^P} & U^P_1 \\
\downarrow{\alpha} & & \downarrow{\beta} & & \downarrow{\gamma} & \\
0 & \to & Q & \xrightarrow{g_0^Q} & U^Q_0 & \xrightarrow{g_1^Q} & U^Q_1 \\
\end{array}
\]

(\ast)

where the maps \( \beta \) and \( \gamma \) exist since \( U^P_0 \) and \( U^P_1 \) are left \( \mathcal{U}\)-approximations of \( P \) and \( \text{Coker} g_0^P \) respectively, and \( U^Q_0, U^Q_1 \) lie in \( \mathcal{U} \). Applying the exact functor \( \mathbb{R}^{\text{op}} \), where \( \mathbb{R}^{\text{op}}(B) = \mathcal{B}(B, -)|_{\mathcal{U}} \), to the diagram (\ast) and setting \( \mu = \mathbb{R}^{\text{op}}(g_0^P) \) and \( \nu = \mathbb{R}^{\text{op}}(g_0^Q) \), we have the following exact commutative diagram:

\[
\begin{array}{cccc}
0 & \to & \mathbb{R}^{\text{op}}(P) & \xrightarrow{\mu} & (U^P_0, -)|_{\mathcal{U}} & \xrightarrow{(g_1^P, -)|_{\mathcal{U}}} & (U^P_1, -)|_{\mathcal{U}} \\
\downarrow{\mathbb{R}^{\text{op}}(\alpha)} & & \downarrow{(\beta, -)|_{\mathcal{U}}} & & \downarrow{(\gamma, -)|_{\mathcal{U}}} & \\
0 & \to & \mathbb{R}^{\text{op}}(Q) & \xrightarrow{\nu} & (U^Q_0, -)|_{\mathcal{U}} & \xrightarrow{(g_1^Q, -)|_{\mathcal{U}}} & (U^Q_1, -)|_{\mathcal{U}} \\
\end{array}
\]

(\ast\ast)
If \( \mathbb{R}^{\text{op}}(\alpha) = 0 \), then using the injectivity of the objects \((U, -)|_{\mathcal{U}} \in \mathscr{A}\), it follows from diagram (***) that there is a map \( \delta: U_I^P \rightarrow U_0^Q \) such that \( g_I^P \circ \delta = \beta \). Then \( g_I^P \circ \beta = 0 \) and this implies that \( \alpha = 0 \) since \( g_0^Q \) is a monic. Hence \( \mathbb{R}^{\text{op}}|_{\text{Proj}} \mathscr{B} \) is faithful. If \( \tilde{\alpha}: \mathbb{R}^{\text{op}}(P) \rightarrow \mathbb{R}^{\text{op}}(Q) \) is map in \( \mathscr{A} \), then injectivity of the objects \((U, -)|_{\mathcal{U}} \) implies that there are maps \((\beta, -)|_{\mathcal{U}}: (U_0^P, -)|_{\mathcal{U}} \rightarrow (U_0^Q, -)|_{\mathcal{U}} \) and \((\gamma, -)|_{\mathcal{U}}: (U_1^P, -)|_{\mathcal{U}} \rightarrow (U_1^Q, -)|_{\mathcal{U}} \) such that the induced diagram (***) with \( \mathbb{R}^{\text{op}}(\alpha) \) deleted, is commutative. Clearly then \( \beta \) and \( \gamma \) make the right square of diagram (*) commutative and this implies the existence of a map \( P \rightarrow Q \) making the whole diagram (*) commutative. It follows then that \( \mathbb{R}^{\text{op}}(\alpha) = \tilde{\alpha} \) and this means that \( \mathbb{R}^{\text{op}}|_{\text{Proj}} \mathscr{B} \) is full. \( \square \)

From now on we set \( \mathcal{M} := \mathbb{R}^{\text{op}}(\text{Proj} \mathscr{B}) \subseteq \mathscr{A} \). Then by Lemma 8.15, the functor \( \mathbb{R}^{\text{op}} \) induces an equivalence

\[
\mathbb{R}^{\text{op}}: \text{Proj} \mathscr{B} \cong \mathcal{M}, \quad \text{if dom.dim } \mathcal{B} \geq 2
\]

**Proposition 8.16.** Let \( \mathcal{B} \) be an abelian category with enough projectives. Assume that \( \text{gl.dim } \mathcal{B} = n + 2 = \text{dom.dim } \mathcal{B} \), where \( n \geq 1 \), and let \( \mathcal{U} = \text{Proj} \mathcal{B} \cap \text{Inj} \mathcal{B} \). If \( \text{Proj} \mathcal{B} \) is covariantly finite in \( \mathcal{B} \), then \( \mathcal{U} = \text{Proj} \mathcal{B} \cap \text{Inj} \mathcal{B} \) is covariantly finite in \( \mathcal{B} \), and \( \mathcal{M} \) is functorially finite in \( \mathcal{A} \).

If \( \mathcal{U} \) is covariantly finite, then setting \( \mathcal{A} = (\mathcal{U}\text{-mod})^{\text{op}} \) we have the following:

(a) \( \mathcal{M} \) is contravariantly finite in \( \mathcal{A} \) and any right \( \mathcal{M} \)-approximation is an epimorphism.

(b) \( \mathcal{M}^{\perp\mathcal{U}} = \mathcal{M} \neq \text{Proj } \mathcal{A} \).

Moreover \( \mathcal{B}, \text{Proj } \mathcal{B}, \) and \( \mathcal{U} \) can be recovered from \( \mathcal{A} \) and \( \mathcal{M} \) as follows:

\[
\mathcal{B} \cong \text{mod-}\mathcal{M} \quad \text{and} \quad \mathcal{U} \cong \text{Inj } \mathcal{A} \quad \text{and} \quad \text{Proj } \mathcal{B} \cong \mathcal{M}
\]

**Proof.** The first assertion follows from Lemma 8.15 and Remark 8.13. Assume that \( \mathcal{U} \) is covariantly finite.

(\( \alpha \)) Let \( F \in \mathcal{A} \). Since \( \mathcal{B} \) has enough projectives, there is an epimorphism \( P \rightarrow \mathbb{R}^{\text{op}}(F) \), where \( P \) is projective in \( \mathcal{B} \). Applying the exact functor \( \mathbb{R}^{\text{op}} \) and using that \( \mathbb{R}^{\text{op}} \mathbb{R}^{\text{op}} \cong \text{Id}_{\mathcal{A}} \), we then have an epimorphism \( M_F \rightarrow F \), where \( M_F := \mathbb{R}^{\text{op}}(P) \). Clearly this map is a right \( M \)-approximation of \( F \).

(\( \beta \)) We first show that \( \mathcal{M} \) is \( n \)-rigid, i.e. \( \text{Ext}^k_{\mathcal{A}}(\mathcal{M}, \mathcal{M}) = 0 \), \( 1 \leq k \leq n \). Let \( M_i \) be in \( \mathcal{M} \), \( i = 1, 2 \). Then there are projective objects \( P \) and \( Q \) in \( \mathcal{B} \) such that \( M_1 = \mathbb{R}^{\text{op}}(P) \) and \( M_2 = \mathbb{R}^{\text{op}}(Q) \). Since \( \text{dom.dim } \mathcal{B} = n + 2 \), there exists an exact sequence in \( \mathcal{B} \):

\[
0 \rightarrow Q \rightarrow U^0 \rightarrow U^1 \rightarrow \cdots \rightarrow U^n \rightarrow U^{n+1}
\]

where the \( U^k \) lie in \( \mathcal{U} \). Applying the exact functor \( \mathbb{R}^{\text{op}} \) we then have an exact sequence:
which is part of an injective resolution of $M_2$. Hence for $0 \leq k \leq n$, the $k$th-cohomology of the complex:

\[ 0 \rightarrow (M_1, M_2) \rightarrow (M_1, (U^0, -)|u) \rightarrow (M_1, (U^1, -)|u) \rightarrow \cdots \rightarrow (M_1, (U^n, -)|u) \rightarrow (M_1, (U^{n+1}, -)|u) \]

gives $\text{Ext}^k(M_1, M_2)$. Using the adjoint pair $(\mathbb{R}^{\text{op}}, \mathbb{H}^{\text{op}})$ the above complex is isomorphic to the complex

\[ 0 \rightarrow (M_1, M_2) \rightarrow (P, U^0) \rightarrow (P, U^1) \rightarrow \cdots \rightarrow (P, U^n) \rightarrow (P, U^{n+1}) \]

which is exact since $P$ is projective. We infer that $\text{Ext}^k_{\mathcal{A}}(M_1, M_2) = 0$, $1 \leq k \leq n$. Hence $\mathcal{M}$ is $n$-rigid.

Now let $F \in \mathcal{A}$ be in $\mathcal{M}^1_n$, i.e. $\text{Ext}^k_{\mathcal{A}}(M, F) = 0$, $1 \leq k \leq n$. Since from (8.10) we have an exact sequence $0 \rightarrow \mathbb{H}^{\text{op}}(F) \rightarrow U^0 \rightarrow U^1$ in $\mathcal{B}$, where the $U^i$ are projective–injective, and since $\text{gl.dim } \mathcal{B} = n + 2$, it follows that $\text{pd } \mathbb{H}^{\text{op}}(F) \leq n$. Hence there exists a projective resolution of $\mathbb{H}^{\text{op}}(F)$ in $\mathcal{B}$ of the form:

\[ 0 \rightarrow P^n \rightarrow P^{n-1} \rightarrow \cdots \rightarrow P^1 \rightarrow P^0 \rightarrow \mathbb{H}^{\text{op}}(F) \rightarrow 0 \]

Applying the exact functor $\mathbb{R}^{\text{op}}$ we have an exact resolution of $F \cong \mathbb{R}^{\text{op}}\mathbb{H}^{\text{op}}(F)$ by objects of $\mathcal{M}$:

\[ 0 \rightarrow M^n \rightarrow M^{n-1} \rightarrow \cdots \rightarrow M^1 \rightarrow M^0 \rightarrow F \rightarrow 0 \]

Applying $(\mathcal{M}, -)$ to the above exact sequence, we have $K^i \in \mathcal{M}^1_n$, for $1 \leq i \leq n-1$, where $K^i = \text{im}(M^i \rightarrow M^{i-1})$. Applying $(\mathcal{M}, -)$ to the extension $0 \rightarrow K^1 \rightarrow M^0 \rightarrow F \rightarrow 0$ and using that $F \in \mathcal{M}^1_n$ we see directly that $K^1 \in \mathcal{M}^1_n$. Using this and applying $(\mathcal{M}, -)$ to the extension $0 \rightarrow K^2 \rightarrow M^1 \rightarrow K^1 \rightarrow 0$ we see that $K^2 \in \mathcal{M}^1_n$. Continuing in this way we see finally that $K^i \in \mathcal{M}^1_n$, $1 \leq i \leq n-1$. It follows that the extension $0 \rightarrow M^n \rightarrow M^{n-1} \rightarrow K^{n-1} \rightarrow 0$ splits and therefore $K^{n-1} \in \mathcal{M}$. Using this and the fact that $K^i \in \mathcal{M}^1_n$, $1 \leq i \leq n-1$, we see finally that $K^1 \in \mathcal{M}$. Then the extension $0 \rightarrow K^1 \rightarrow M^0 \rightarrow F \rightarrow 0$ splits and therefore $F \in \mathcal{M}$. We infer that $\mathcal{M}^1_n = \mathcal{M}$. Since $\text{gl.dim } \mathcal{B} = n + 2$ it follows that $\mathcal{M} \neq \text{Proj } \mathcal{A}$.  

Finally, by Lemma 8.14, the functor $\mathbb{R}^{\text{op}}$ induces an equivalence between $\mathcal{U}$ and $\text{Inj } \mathcal{A}$. Since, by Lemma 8.15, the functor $\mathbb{R}^{\text{op}}$ induces an equivalence between $\text{Proj } \mathcal{B}$ and $\mathcal{M} = \mathbb{R}^{\text{op}}(\text{Proj } \mathcal{B})$ and since $\mathcal{B}$ has enough projectives, it follows that there is induced an equivalence between $\mathcal{B} = \text{mod-} \text{Proj } \mathcal{B}$ and $\text{mod-} \mathcal{M}$. □
8.3. Generalized Auslander correspondence

To state and prove the main result of this section we need the following definitions which generalize, and are inspired by, Iyama’s [39].

Definition 8.17. Let \( \mathcal{A} \) be an abelian category. A full subcategory \( \mathcal{M} \) of \( \mathcal{A} \) is called an \( n \)-cluster tilting, resp. \( n \)-cluster cotilting, subcategory, \( n \geq 1 \), if:

(i) \( \mathcal{M} \) is contravariantly, resp. covariantly, finite in \( \mathcal{A} \).
(ii) Any right, resp. left, \( \mathcal{M} \)-approximation is an epimorphism, resp. monomorphism.
(iii) \( \mathcal{M}_n^\perp = \mathcal{M} \), resp. \( \mathcal{M}^\perp_n = \mathcal{M} \).

We consider \( \mathcal{M} = \mathcal{A} \) as, the unique, 0-cluster tilting, resp. 0-cluster cotilting, subcategory of \( \mathcal{A} \).

Specializing, we have the following notions. An object \( M \in \mathcal{A} \) is called an \( n \)-cluster tilting, resp. \( n \)-cluster cotilting, object, if \( \text{add} M \) is an \( n \)-cluster tilting, resp. \( n \)-cluster cotilting, subcategory of \( \mathcal{A} \). If \( \mathcal{A} \) has all small coproducts, resp. products, then \( M \) is called an \( n \)-cluster tilting, resp. \( n \)-cluster cotilting, object, if \( \text{Add} M \), resp. \( \text{Prod} M \), is an \( n \)-cluster tilting, resp. \( n \)-cluster cotilting, subcategory.

Remark 8.18. Let \( \mathcal{A} \) be an abelian category with small coproducts, resp. products, and let \( M \in \mathcal{A} \).

(i) \( \text{Add} M \) is contravariantly finite in \( \mathcal{A} \), resp. \( \text{Prod} M \) is covariantly finite in \( \mathcal{A} \).
(ii) If \( \mathcal{A} \) has enough projectives and \( M \) is a generator, then \( M \) is an \( n \)-cluster tilting object if and only if \( M_n^\perp = \text{Add} M \).
(iii) If \( \mathcal{A} \) has enough injectives and \( M \) is a cogenerator, then \( M \) is an \( n \)-cluster cotilting object if and only if \( M_n^\perp = \text{Prod} M \).

Definition 8.19. An abelian category \( \mathcal{B} \) is called \( n \)-Auslander, resp. \( n \)-coAuslander, category, \( n \geq 0 \), if:

(i) \( \mathcal{B} \) has enough projectives, resp. injectives.
(ii) The full subcategory \( \text{Proj} \mathcal{B} \cap \text{Inj} \mathcal{B} \) is covariantly, resp. contravariantly, finite in \( \mathcal{B} \).
(iii) \( \text{gl.dim} \mathcal{B} \leq n + 2 \leq \text{dom.dim} \mathcal{B} \), resp. \( \text{gl.dim} \mathcal{B} \leq n + 2 \leq \text{codom.dim} \mathcal{B} \).

A ring \( \Lambda \) is called a (right) \( n \)-Auslander ring, if \( \text{Mod}\Lambda \) is an \( n \)-Auslander category.

Remark 8.20. It should be noted that if \( \mathcal{A} \) is an abelian category with enough projectives and enough injectives, then \( \text{dom.dim} \mathcal{A} \geq n \) if and only if \( \text{codom.dim} \mathcal{A} \geq n \), see [38, Lemma 2.1] for the case of rings, the general case being similar. Also observe that:
(α) \(M\) is \(n\)-cluster tilting in \(\mathcal{A}\) if and only if \(M^{\text{op}}\) is \(n\)-cluster cotilting in \(\mathcal{A}^{\text{op}}\), and (β) \(\mathcal{B}\) is an \(n\)-Auslander category if and only if \(\mathcal{B}^{\text{op}}\) is an \(n\)-coAuslander category. Clearly an \(n\)-Auslander category \(\mathcal{B}\) is either semisimple or else satisfies \(\text{gl.dim } \mathcal{B} = n + 2 = \text{dom.dim } \mathcal{B}\).

**Example 8.21.** Let \(\Lambda\) be an Artin algebra. Then \(\Lambda\) is an Auslander algebra, see [8], if and only if \(\Lambda\) is a 0-Auslander ring. Also it is easy to see that \(\Lambda\) is an \(n\)-Auslander ring if and only if \(\text{mod-}\Lambda\) is an \(n\)-Auslander category. On the other hand let \(\mathcal{P}\) be a dualizing \(R\)-variety over a commutative Artin ring, see [5], for instance \(\mathcal{P} = \text{proj}\Lambda\) for an Artin algebra \(\Lambda\). Then the category \(\text{mod-}\mathcal{P}\) is \(n\)-Auslander if and only if \(\text{mod-}\mathcal{P}\) is \(n\)-coAuslander.

From now on we consider pairs \(\{\mathcal{A}, M\}\) consisting of an abelian category \(\mathcal{A}\) with enough injectives and an \(n\)-cluster tilting subcategory \(M\) of \(\mathcal{A}\). We call two such pairs \(\{\mathcal{A}, M\}\) and \(\{\mathcal{A}', M'\}\) **cluster equivalent**, if there is an equivalence \(M \approx M'\).

By the following lemma, we infer that the pair \(\{\mathcal{A}, M\}\) is completely determined, up to equivalence, by \(M\).

**Lemma 8.22.** If \(\{\mathcal{A}, M\}\) and \(\{\mathcal{A}', M'\}\) are cluster equivalent pairs, then the categories \(\mathcal{A}\) and \(\mathcal{A}'\) are equivalent.

**Proof.** Any equivalence \(M \approx M'\) extends to an equivalence \(\text{mod-}M \approx \text{mod-}M'\) and therefore to an equivalence \(\mathcal{U} \approx \mathcal{U}'\), where \(\mathcal{U}\), resp. \(\mathcal{U}'\), is the full subcategory of projective–injective objects of \(\text{mod-}M\), resp. \(\text{mod-}M'\). Since, by Proposition 8.12, \(\mathcal{A} \approx (\mathcal{U}\text{-}\text{mod})^{\text{op}}\) and \(\mathcal{A}' \approx (\mathcal{U}'\text{-}\text{mod})^{\text{op}}\), it follows that if the pairs \(\{\mathcal{A}, M\}\) and \(\{\mathcal{A}', M'\}\) are cluster equivalent, then the categories \(\mathcal{A}\) and \(\mathcal{A}'\) are equivalent. \(\Box\)

Now we can prove the following main result of this section which gives a categorified version of higher Auslander–Iyama correspondence.

**Theorem 8.23 (Generalized Auslander correspondence).** For any integer \(n \geq 0\), the map \(M \mapsto \text{mod-}M\) induces, up to (cluster) equivalence, a bijective correspondence between:

1. Pairs \(\{\mathcal{A}, M\}\) consisting of an abelian category \(\mathcal{A}\) with enough injectives and an \(n\)-cluster tilting subcategory \(M\) of \(\mathcal{A}\).
2. \(n\)-Auslander categories \(\mathcal{B}\).

The mutually inverse bijections are given as follows:

\[
\{\mathcal{A}, M\} \leftrightarrow \text{mod-}M \quad \text{and} \quad \mathcal{B} \leftrightarrow \{\mathcal{U}\text{-}\text{mod})^{\text{op}}, N\}
\]

where \(\mathcal{U} = \text{Proj}\mathcal{B} \cap \text{Inj}\mathcal{B}\) and \(N = \mathcal{B}(\text{Proj}\mathcal{B}, -)|_{\mathcal{U}}\). Under the above correspondence: \(M\) is covariantly finite in \(\mathcal{A}\) if and only if \(\text{Proj}\mathcal{B}\) is covariantly finite in \(\mathcal{B}\).
Proof. For $n \geq 1$ the assertion follows from Propositions 8.12 and 8.16. It remains to treat the case $n = 0$. Let $\mathcal{M} = \mathcal{A}$ be 0-cluster tilting, so $\mathcal{A}$ is abelian with enough injectives. Then $\mathcal{B} := \text{mod-}\mathcal{A} \cong \text{mod-} (\text{lnj } \mathcal{A}-\text{mod})^{\text{op}}$ is the free abelian category of the additive category $\text{lnj } \mathcal{A}$, so by Theorem 8.2, $\mathcal{B}$ has enough projectives and $\text{gl.dim } \mathcal{B} \leq 2 \leq \text{dom.dim } \mathcal{B}$. Next we show that $\text{Proj } \mathcal{B} \cap \text{lnj } \mathcal{B} \cong \text{lnj } \mathcal{A} = \mathcal{U}$ is covariantly finite in $\mathcal{B}$. Indeed let $F$ be in $\mathcal{B} = \text{mod-}\mathcal{A}$ and let $\mathcal{A}(-, A) \xrightarrow{f} \mathcal{A}(-, B) \xrightarrow{\varepsilon} F \rightarrow 0$ be a presentation of $F$, where $f_* = \mathcal{A}(-, f)$. If $A \xrightarrow{J} B \xrightarrow{C} C \rightarrow 0$ is exact in $\mathcal{A}$, then clearly there exists a unique map $\tau : F \rightarrow \mathcal{A}(-, C)$ such that $\varepsilon \circ \tau = \mathcal{A}(-, c)$. It is easy to see that the map $\tau$ is a left $\text{Proj } \mathcal{B}$-approximation of $F$, so $\text{Proj } \mathcal{B} \cong \mathcal{A}$ is covariantly finite in $\mathcal{B}$. Using Remark 8.13, it follows that $\text{Proj } \mathcal{B} \cap \text{lnj } \mathcal{B} \cong \text{lnj } \mathcal{A} = \mathcal{U}$ is covariantly finite in $\mathcal{B}$. We infer that $\mathcal{B}$ is a 0-Auslander category and $\mathcal{A} = (\text{lnj } \mathcal{A}-\text{mod})^{\text{op}} = (\mathcal{U}-\text{mod})^{\text{op}}$. Conversely let $\mathcal{B}$ be a 0-Auslander category. By Theorem 8.2, there is an equivalence $\mathcal{B} \cong \text{mod-} (\mathcal{U}-\text{mod})^{\text{op}}$ and the subcategory $\mathcal{U} = \text{Proj } \mathcal{B} \cap \text{lnj } \mathcal{B}$ is covariantly finite in $\mathcal{B}$. Then $\mathcal{U}$ is left coherent, so $\mathcal{A} := (\mathcal{U}-\text{mod})^{\text{op}}$ is abelian with enough injectives, i.e. it is a 0-cluster tilting subcategory of itself. Clearly the maps $\mathcal{A} \hookrightarrow \text{mod-}\mathcal{A}$ and $\mathcal{B} \hookrightarrow (\mathcal{U}-\text{mod})^{\text{op}}$, where $\mathcal{U} = \text{Proj } \mathcal{B} \cap \text{lnj } \mathcal{B}$, are mutually inverse bijections. For the last assertion see part 3. of Proposition 8.12. \(\square\)

Note that, as we shall see in Proposition 9.2 of the next section, any $n$-cluster tilting subcategory of $\mathcal{A}$ is covariantly finite. Hence in the last assertion of Theorem 8.23 the full subcategory of projectives of the associated $n$-Auslander category is always covariantly finite in $\mathcal{B}$.

For the purpose of the next direct consequence of our previous results, we call a triangulated category $\mathcal{T}$ strongly algebraic if $\mathcal{T}$ is triangle equivalent to the stable category $\mathcal{A}$ modulo projectives of a Frobenius abelian category $\mathcal{A}$.

Corollary 8.24. Under the correspondence of Theorem 8.23: $\mathcal{A}$ is Frobenius if and only if $\mathcal{U}$ is Frobenius, for $\mathcal{U} = \text{Proj } \mathcal{B} \cap \text{lnj } \mathcal{B}$. In this case $\mathcal{M}$ is an $(n+1)$-cluster tilting subcategory of $\mathcal{A}$, and conversely any $(n+1)$-cluster tilting subcategory of a strongly algebraic triangulated category arises in this way.

Our next result gives the dual version of Theorem 8.23. The proof is dual and is left to the reader.

Theorem 8.25. For any integer $n \geq 0$, the map $\mathcal{M} \mapsto (\mathcal{M}-\text{mod})^{\text{op}}$ gives, up to equivalence, a bijective correspondence between:

(I) $n$-cluster cotilting subcategories $\mathcal{M}$ in abelian categories $\mathcal{A}$ with enough projectives.
(II) $n$-coAuslander categories $\mathcal{B}$.
The following result gives some information on the (higher) K-theory of $n$-Auslander categories. Recall that an abelian category is a length category if any of its objects has finite (composition) length.

**Proposition 8.26.** Let $\mathcal{B}$ be an $n$-Auslander category.

(i) There is a 0-Auslander category $\mathcal{F}$, an abelian category $\mathcal{C}$ with $\operatorname{gl.dim} \mathcal{C} \leq 3n - 1$, and a short exact sequence of abelian categories

$$0 \to \mathcal{C} \to \mathcal{F} \to \mathcal{B} \to 0$$

which induces isomorphisms in K-theory: $K_i(\mathcal{F}) \cong K_i(\mathcal{B}) \oplus K_i(\mathcal{C})$, $\forall i \geq 0$.

(ii) If the abelian category $\mathcal{A} = \operatorname{Proj} \mathcal{F}$ is a length category with enough projectives and injectives and $\operatorname{Proj} \mathcal{B}$ is covariantly finite in $\mathcal{B}$, then there exists a short exact sequence of abelian categories

$$0 \to \mathcal{D} \to \mathcal{B} \to \mathcal{A} \to 0$$

which induces isomorphisms in K-theory: $K_i(\mathcal{B}) \cong K_i(\mathcal{A}) \oplus K_i(\mathcal{D})$, $\forall i \geq 0$.

**Proof.** (i) By Theorem 8.23, there is an abelian category $\mathcal{A}$ with enough injectives and an $n$-cluster tilting subcategory $\mathcal{M}$ of $\mathcal{A}$ such that $\mathcal{B} = \operatorname{mod-}\mathcal{M}$. Since $\mathcal{A} = (\operatorname{Inj} \mathcal{A}\text{-mod})^{\text{op}}$, it follows by Theorem 8.2 that $\mathcal{F} := \operatorname{mod-}\mathcal{A} = \operatorname{mod-} (\operatorname{Inj} \mathcal{A}\text{-mod})^{\text{op}}$ is a 0-Auslander category and $\mathcal{A} = \operatorname{Proj} \mathcal{F}$. Now the inclusion $\mathcal{M} \subseteq \mathcal{A}$ induces a short exact sequence of abelian categories $0 \to \mathcal{C} \to \mathcal{F} \to \mathcal{B} \to 0$, where the functor $\mathcal{F} \to \mathcal{B}$ is the restriction $\mathcal{mod-}\mathcal{A} \to \mathcal{mod-}\mathcal{M}$, and as easily seen $\mathcal{C} := \mathcal{mod-} (\mathcal{A}/\mathcal{M})$ is a Serre subcategory of $\mathcal{F}$ identified with the full subcategory of $\mathcal{F}$ consisting of all coherent functors $\mathcal{A}^{\text{op}} \to \mathcal{Ab}$ vanishing on $\mathcal{M}$. Since $\mathcal{M}$ is contravariantly finite in $\mathcal{A}$, it follows that $\mathcal{A}/\mathcal{M}$ is a left triangulated category with loop functor $\Omega_{\mathcal{M}}: \mathcal{A}/\mathcal{M} \to \mathcal{A}/\mathcal{M}$ given by $\Omega_{\mathcal{M}}(A) = \ker$ of a right $\mathcal{M}$-approximation of $A$; we refer to [18] for details and more information on left triangulated categories. Now let $F: (\mathcal{A}/\mathcal{M})^{\text{op}} \to \mathcal{Ab}$ be a coherent functor with presentation $(-, B) \to (-, C) \to F \to 0$ and let $\Omega_{\mathcal{M}}(C) \to A \to B \to C$ be a left triangle in $\mathcal{A}/\mathcal{M}$. Then the exact sequence of functors $\cdots \to (-, \Omega_{\mathcal{M}}(C)) \to (-, \Omega_{\mathcal{M}}(A)) \to (-, \Omega_{\mathcal{M}}(B)) \to (-, \Omega_{\mathcal{M}}(C)) \to (-, A) \to (-, B) \to (-, C) \to F \to 0$ is a projective resolution of $F$ in $\mathcal{mod-} \mathcal{A}/\mathcal{M}$. If $A \in \mathcal{A}$, then by part 4. of Remark 8.11, there exists an $\mathcal{M}$-resolution of $A$ of the form $0 \to M^0_A \to M^{n-1} \to \cdots \to M^0_A \to A \to 0$ and $\Omega_{\mathcal{M}}(A) = \text{Im}(M^k_A \to M^{k-1}_A)$, $\forall k \geq 1$. Since $M^0_A = \Omega_{\mathcal{M}}(A)$ lies in $\mathcal{M}$, we have $\Omega_{\mathcal{M}}(A) = 0$ in $\mathcal{A}/\mathcal{M}$. It follows that $\Omega_{\mathcal{M}} = 0$ and from the projective resolution of $F \in \mathcal{mod-} (\mathcal{A}/\mathcal{M})$ it follows that $\text{pd} F \leq 3n - 1$. Hence $\operatorname{gl.dim} \mathcal{C} \leq 3n - 1$. Finally since $\operatorname{gl.dim} \mathcal{B} \leq n + 2 < \infty$, the last assertion follows from [6, Proposition 2.4].

(ii) Under the assumption of part (ii), the $n$-cluster tilting subcategory $\mathcal{M}$ of $\mathcal{A}$ is $n$-cluster cotilting, see Theorem 9.6 below. In particular $\operatorname{Proj} \mathcal{A} \subseteq \mathcal{M}$ and then the
restriction functor $\mathcal{B} = \text{mod-} \mathcal{M} \rightarrow \mathcal{A} = \text{mod-} \text{Proj} \mathcal{A}$ induces a short exact sequence $0 \rightarrow \mathcal{D} \rightarrow \mathcal{B} \rightarrow \mathcal{A} \rightarrow 0$ of abelian categories, where $\mathcal{D} = \text{mod-} \mathcal{M}$. Since any object of $\mathcal{A}$ has finite length, the assertion follows from [6, Proposition 2.4]. □

In particular any 1-Auslander category $\mathcal{B}$, is a Gabriel quotient of a 0-Auslander category $\mathcal{F}$ by an abelian category $\mathcal{C}$ with $\text{gl.dim} \mathcal{C} \leq 2$.

The bijective correspondence below is due to Iyama [40]. We consider pairs $\{\Lambda, M\}$, where $\Lambda$ is an Artin algebra and $M$ is a finitely generated $n$-cluster (co)tilting module over $\Lambda$. Two such pairs $\{\Lambda, M\}$ and $\{\Lambda', M'\}$, are cluster equivalent, if there is an equivalence $\text{add} M \approx \text{add} M'$. Then as before we have an equivalence $\text{mod-} \Lambda \approx \text{mod-} \Lambda'$, so the algebras $\Lambda$ and $\Lambda'$ are Morita equivalent.

**Corollary 8.27.** The map $\{\Lambda, M\} \mapsto \text{End}_\Lambda(M)$ gives a bijection between cluster/Morita equivalence classes of:

(i) Pairs $\{\Lambda, M\}$, where $\Lambda$ is an Artin algebra and $M$ is a finitely generated $n$-cluster (co)tilting $\Lambda$-module.

(ii) $n$-Auslander algebras $\Gamma$.

Moreover: (α) $M$ is an $n$-cluster (co)tilting $\Lambda$-module if and only if $M$ is an $n$-rigid (co)generator and $n+2 = \text{gl.dim} \text{End}_\Lambda(M)$, and (β) $\text{rep.dim} \Lambda \leq n+2$ and $K_*(\text{add} M, \oplus) \cong K_*(\text{mod-} \Lambda) \oplus K_*(\text{mod-} \text{End}_\Lambda(M))$.

8.4. Universality and functoriality

As observed in Theorem 8.2, the 0-Auslander categories are precisely the free abelian categories, so they are characterized by a universal property. More precisely let $\text{Abel}$ be the (large) category of abelian categories and exact functors, and $\text{Add}$ the (large) category of additive categories and additive functors. Then Theorem 8.2 implies that the forgetful functor $\mathfrak{U} : \text{Abel} \rightarrow \text{Add}$ admits a left adjoint $\mathfrak{F} : \text{Add} \rightarrow \text{Abel}$, where $\mathfrak{F}(\mathcal{C})$ is the 0-Auslander category of the additive category $\mathcal{C}$. In other words for any additive category $\mathcal{C}$ and any abelian category $\mathcal{A}$, the canonical functor $\mathcal{C} \rightarrow \mathfrak{F}(\mathcal{C})$ induces a natural isomorphism

$$\text{Hom}_{\text{Abel}}(\mathfrak{F}(\mathcal{C}), \mathcal{A}) \cong \text{Hom}_{\text{Add}}(\mathcal{C}, \mathfrak{U}(\mathcal{A}))$$

However it is not clear if the class of $n$-Auslander categories are characterized by an analogous universal property. We close this section by analyzing briefly this problem.

If $\mathcal{M} \subseteq \mathcal{A}$ is a full subcategory of an abelian category $\mathcal{A}$, then an additive functor $F : \mathcal{A} \rightarrow \mathcal{C}$ is called right $\mathcal{M}$-exact if for any map $f : A \rightarrow B$ in $\mathcal{A}$ which is $\mathcal{M}$-epic in $\mathcal{A}$, i.e. $\mathcal{A}(M, f)$ is surjective, $\forall M \in \mathcal{M}$, the map $F(f)$ is epic in $\mathcal{C}$. If $\mathcal{C}$ is an abelian category, then let $\mathcal{L}ex_{\mathcal{M}-ex}(\mathcal{A}, \mathcal{C})$ be the category of left exact and right $\mathcal{M}$-exact functors $\mathcal{A} \rightarrow \mathcal{C}$, and let $\mathcal{E}x(\text{mod-} \mathcal{M}, \mathcal{C})$ be the category of exact functors $\text{mod-} \mathcal{M} \rightarrow \mathcal{C}$. 


First we show that the restricted Yoneda functor $\mathbb{H}: \mathcal{A} \to \text{mod-} \mathcal{M}$ satisfies a universal property:

**Proposition 8.28.** Let $\mathcal{M} \subseteq \mathcal{A}$ be a contravariantly finite full subcategory of an abelian category $\mathcal{A}$ and let $\mathbb{H}: \mathcal{A} \to \text{mod-} \mathcal{M}$, $\mathbb{H}(A) = \mathcal{A}(\mathcal{M})$ be the restricted Yoneda functor. Then $\mathbb{H}$ is left exact and right $\mathcal{M}$-exact, and if $S: \mathcal{A} \to \mathcal{C}$ is a left exact and right $\mathcal{M}$-exact functor to an abelian category $\mathcal{C}$, then there exists a unique exact functor $S^*: \text{mod-} \mathcal{M} \to \mathcal{C}$ such that $S^* \circ \mathbb{H} = S$. In other words, $\mathbb{H}$ induces an equivalence:

$$- \circ \mathbb{H} : \text{Ext}(\text{mod-} \mathcal{M}, \mathcal{C}) \xrightarrow{\cong} \text{Ext}_{\text{M-ex}}(\mathcal{A}, \mathcal{C}), \quad F \mapsto F \circ \mathbb{H}$$

**Proof.** First observe that $\text{mod-} \mathcal{M}$ is abelian since $\mathcal{M}$ is contravariantly finite in $\mathcal{A}$. Clearly the functor $\mathbb{H}$ is left exact and right $\mathcal{M}$-exact. As in the proof of Proposition 8.26, the inclusion $\mathcal{M} \subseteq \mathcal{A}$ induces by restriction an exact functor $\text{mod-} \mathcal{A} \to \text{mod-} \mathcal{M}$ and we have a short exact sequence of abelian categories

$$0 \to \text{mod-} (\mathcal{A}/M) \to \text{mod-} \mathcal{A} \to \text{mod-} \mathcal{M} \to 0$$

where the category $\text{mod-} (\mathcal{A}/M)$ of coherent functors over the stable category $\mathcal{A}/M$ is identified with the full subcategory of $\text{mod-} \mathcal{A}$ consisting of all coherent functors $\mathcal{A}^{\text{op}} \to \textbf{Ab}$ admitting a presentation $\mathcal{A}(-, B) \to \mathcal{A}(-, C) \to F \to 0$, where the map $B \to C$ is $\mathcal{M}$-epic. Now the functor $S: \mathcal{A} \to \mathcal{C}$ admits a unique right exact extension $S^1: \text{mod-} \mathcal{A} \to \mathcal{C}$ by setting $S^1(F) = \text{Coker} S(f)$, where $\mathcal{A}(-, B) \to \mathcal{A}(-, C) \to F \to 0$ is a presentation of $F$ in $\text{mod-} \mathcal{A}$. Since $S$ is left exact, it is easy to see that $S^1$ is exact. On the other hand, since $S$ is right $\mathcal{M}$-exact, it is easy to see that $S^1$ kills the objects of the Serre subcategory $\text{mod-} (\mathcal{A}/M) \subseteq \text{mod-} \mathcal{A}$ and therefore it factorizes through the exact quotient functor $\text{mod-} \mathcal{A} \to \text{mod-} \mathcal{M}$ yielding an exact functor $S^*: \text{mod-} \mathcal{M} \to \mathcal{C}$. Clearly we have $S^* \circ \mathbb{H} = S$ and the exact functor $S^*$ is unique with this property. \[\square\]

As a direct consequence we have the following.

**Corollary 8.29.** Let $\mathcal{M}$ be an $n$-cluster tilting subcategory of an abelian category $\mathcal{A}$ and let $\mathcal{B} = \text{mod-} \mathcal{M}$ be the associated $n$-Auslander category. For any abelian category $\mathcal{C}$, the functor $\mathbb{H}: \mathcal{A} \to \mathcal{B}$ induces an equivalence

$$- \circ \mathbb{H} : \text{Ext}(\mathcal{B}, \mathcal{C}) \to \text{Ext}_{\text{M-ex}}(\mathcal{A}, \mathcal{C}), \quad F \mapsto F \circ \mathbb{H}$$

Our aim now is to show that for any $n \geq 0$, appropriately defined, (large) categories of $n$-cluster tilting categories and $n$-Auslander categories are equivalent:

- Let $\text{AuslCat}(n)$ be the (large) category with objects $n$-Auslander categories. If $\mathcal{B}$ and $\mathcal{B}'$ are $n$-Auslander categories, then a morphism between them in $\text{AuslCat}(n)$ is an exact functor $G: \mathcal{B} \to \mathcal{B}'$ preserving projectives and projective-injective objects.

Let ClustTilt\((n)\) be the (large) category with object pairs \(\{\mathcal{A}, \mathcal{M}\}\) consisting of an abelian category \(\mathcal{A}\) with enough injectives and an \(n\)-cluster tilting subcategory \(\mathcal{M}\) of \(\mathcal{A}\). A morphism \(\{\mathcal{A}, \mathcal{M}\} \to \{\mathcal{A}', \mathcal{M}'\}\) in ClustTilt\((n)\) is a left exact functor \(F: \mathcal{A} \to \mathcal{A}'\) such that: \((\alpha)\) \(F(\mathcal{M}) \subseteq \mathcal{M}'\) and \(F(\text{Inj } \mathcal{A}) \subseteq \text{Inj } \mathcal{A}'\), and \((\beta)\) \(F\) sends \(\mathcal{M}\)-epics in \(\mathcal{A}\) to \(\mathcal{M}'\)-epics in \(\mathcal{A}'\).

**Lemma 8.30.** If \(F: \{\mathcal{A}, \mathcal{M}\} \to \{\mathcal{A}', \mathcal{M}'\}\) is a morphism in ClustTilt\((n)\), then there exists a unique exact functor \(F^\ast: \text{mod-}\mathcal{M} \to \text{mod-}\mathcal{M}'\) which preserves projectives and projective-injectives, such that the following diagram commutes:

\[
\begin{array}{ccc}
\mathcal{A} & \xrightarrow{F} & \mathcal{A}' \\
\downarrow & & \downarrow \\
\text{mod-}\mathcal{M} & \xrightarrow{F^\ast} & \text{mod-}\mathcal{M}'
\end{array}
\]

**Proof.** Since \(F\) is left exact and sends \(\mathcal{M}\)-epics to \(\mathcal{M}'\)-epics and since \(\mathcal{H}'\) is left exact and sends \(\mathcal{M}'\)-epics to epics, the composite functor \(\mathcal{H}' \circ F: \mathcal{A} \to \text{mod-}\mathcal{M}'\) is left exact and sends \(\mathcal{M}\)-epics to epics. Then by Proposition 8.28, there exists a unique exact functor \(F^\ast: \text{mod-}\mathcal{M} \to \text{mod-}\mathcal{M}'\) such that \(F^\ast \circ \mathcal{H} = \mathcal{H}' \circ F\). In fact if \(\mathcal{H}(\mathcal{M}^1) \to \mathcal{H}(\mathcal{M}^0) \to X \to 0\) is a finite presentation of \(X\) in \(\text{mod-}\mathcal{M}\), then \(F^\ast(X)\) is defined by the exact sequence \(\mathcal{H}'(F(\mathcal{M}^1)) \to \mathcal{H}'(F(\mathcal{M}^0)) \to F^\ast(X) \to 0\). Since \(F(\mathcal{M}) \subseteq \mathcal{M}'\) and \(F^\ast \mathcal{H}(\mathcal{M}) = \mathcal{H}'F(\mathcal{M})\), it follows that \(F^\ast\) preserves projectives. On the other hand, since \(F(\text{Inj } \mathcal{A}) \subseteq \text{Inj } \mathcal{A}'\) and \(F^\ast \mathcal{H}(\text{Inj } \mathcal{A}) = \mathcal{H}'F(\text{Inj } \mathcal{A})\), and since the functors \(\mathcal{H}\) and \(\mathcal{H}'\) induce equivalences between \(\text{Inj } \mathcal{A}\), resp. \(\text{Inj } \mathcal{A}'\), and the full subcategory of projective–injective objects of \(\text{mod-}\mathcal{M}\), resp. \(\text{mod-}\mathcal{M}'\), it follows that \(F^\ast\) preserves projective-injectives. \(\square\)

**Lemma 8.30** allows us to define a functor

\[\mathcal{A}: \text{ClustTilt}(n) \to \text{AuslCat}(n), \quad \mathcal{A}(\{\mathcal{A}, \mathcal{M}\}) = \text{mod-}\mathcal{M} \quad \& \quad \mathcal{A}(F) = F^\ast\]

**Lemma 8.31.** Let \(G: \mathcal{B} \to \mathcal{B}'\) be a morphism in AuslCat\((n)\). Let \(\mathcal{A} = (\mathcal{U}\text{-mod})^{\text{op}}\) and \(\mathcal{A}' = (\mathcal{U}'\text{-mod})^{\text{op}}\), where \(\mathcal{U}\) and \(\mathcal{U}'\) are the projective–injective objects of \(\mathcal{B}\) and \(\mathcal{B}'\) respectively, and let \(\mathcal{M} = \mathcal{B}(\text{Proj } \mathcal{B}, -)|_{\mathcal{U}}\) and \(\mathcal{M}' = \mathcal{B}'(\text{Proj } \mathcal{B}', -)|_{\mathcal{U}}\). Then \(G\) induces a left exact functor \(G^\uparrow: \mathcal{A} \to \mathcal{A}'\) such that \(G^\uparrow(\mathcal{U}) \subseteq \mathcal{U}'\) and \(G^\uparrow(\mathcal{M}) \subseteq \mathcal{M}'\), and \(G^\uparrow\) sends \(\mathcal{M}\)-epics to \(\mathcal{M}'\)-epics.

**Proof.** By Theorem 8.23 we may assume that \(\mathcal{B} = \text{mod-}\mathcal{M}\) and \(\mathcal{B}' = \text{mod-}\mathcal{M}'\), where \(\mathcal{M}\) and \(\mathcal{M}'\) are \(n\)-cluster tilting subcategories in the abelian categories \(\mathcal{A} = \mathcal{U}\text{-mod}\) and \(\mathcal{A}' = \mathcal{U}'\text{-mod}\). Then \(G: \text{mod-}\mathcal{M} \to \text{mod-}\mathcal{M}'\) is an exact functor such that \(G(\text{Proj mod-}\mathcal{M}) \subseteq \text{Proj mod-}\mathcal{M}'\) and \(G(\mathcal{U}) \subseteq \mathcal{U}'\). The induced functor \(G: \mathcal{U} \to \mathcal{U}'\) extends uniquely to a left exact functor \(G^\uparrow: \mathcal{A} \to \mathcal{A}'\) and clearly \(G^\uparrow\) preserves injectives and sends \(\mathcal{M}\) to \(\mathcal{M}'\).
It remains to show that $G^\dagger$ sends $\mathcal{M}$-epics to $\mathcal{M}'$-epics. Consider, as in Lemma 8.30, the left exact functors $\mathbb{H}: \mathcal{A} \rightarrow \text{mod-}\mathcal{M}$ and $\mathbb{H}' : \mathcal{A}' \rightarrow \text{mod-}\mathcal{M}'$. By our assumptions for $G$ it follows easily that $G \circ \mathbb{H} = \mathbb{H}' \circ G^\dagger$. Let $f : B \rightarrow C$ be an $\mathcal{M}$-epic in $\mathcal{A}$. Then $\mathbb{H}'(G^\dagger(f))$ is epic since $\mathbb{H}$ sends $\mathcal{M}$-epics to epics and $G$ is exact. This means that the map $G^\dagger(f)$ is $\mathcal{M}'$-epic. □

Lemma 8.31 allows us to define a functor

$$\mathbb{C} : \text{AuslCat}(n) \rightarrow \text{ClustTilt}(n), \quad \mathbb{C}(\mathcal{B}) = \{\{\mathcal{U}\text{-mod})^\dagger, \mathcal{M}\}\} \quad \& \quad \mathbb{C}(G) = G^\dagger$$

where $\mathcal{U} = \text{Proj } \mathcal{B} \cap \text{Inj } \mathcal{B}$ and $\mathcal{M} = \mathcal{B}(\text{Proj } \mathcal{B}, -)|_{\mathcal{U}}$.

**Theorem 8.32.** The functors $\mathbb{A}$ and $\mathbb{C}$ define mutually inverse equivalences

$$\mathbb{A} : \text{ClustTilt}(n) \xrightarrow{\sim} \text{AuslCat}(n) : \mathbb{C}$$

**Proof.** Using Propositions 8.12 and 8.16 or Theorem 8.23, for any pair $\{\mathcal{A}, \mathcal{M}\}$ in $\text{ClustTilt}(n)$, we have:

$$\mathbb{C} \mathbb{A}(\{\mathcal{A}, \mathcal{M}\}) = \mathbb{C}(\text{mod-}\mathcal{M}) = \{\{\mathcal{U}\text{-mod})^\dagger, \mathcal{N}\}\} \approx \{\mathcal{A}, \mathcal{M}\}$$

since $\mathcal{N} := \mathcal{B}(\text{Proj } \mathcal{B}, -)|_{\mathcal{U}} \approx \mathcal{M}$ and $\mathcal{U} := \text{Proj } \text{mod-}\mathcal{M} \cap \text{Inj } \text{mod-}\mathcal{M}$, so that $\mathcal{A} \approx (\mathcal{U}\text{-mod})^\dagger$.

Similarly for any $n$-Auslander category $\mathcal{B}$ we have:

$$\mathbb{A} \mathbb{C}(\mathcal{B}) = \mathbb{A}(\{\mathcal{A}, \mathcal{M}\}) = \text{mod-}\mathcal{M} \approx \mathcal{B}$$

where: $\mathcal{M} := \mathcal{B}(\text{Proj } \mathcal{B}, -)|_{\mathcal{U}}$ and $\mathcal{A} := (\mathcal{U}\text{-mod})^\dagger$, where $\mathcal{U} = \text{Proj } \mathcal{B} \cap \text{Inj } \mathcal{B}$.

If $F, G : \{\mathcal{A}, \mathcal{M}\} \rightarrow \{\mathcal{A}', \mathcal{M}'\}$ are morphisms in $\text{ClustTilt}(n)$ such that $\mathbb{A}(F) \equiv \mathbb{A}(G)$, so $F^\dagger \equiv G^\dagger$, then using Lemma 8.30, we have $\mathbb{H}' F^\dagger \equiv G^\dagger = \mathbb{H}' G$. Since $\mathbb{H}' \mathbb{H}^\dagger \equiv \text{Id}_{\mathcal{A}'},$ we have obviously $F \equiv G$. On the other hand if $\tilde{F} : \mathbb{A}(\{\mathcal{A}, \mathcal{M}\}) = \text{mod-}\mathcal{M} \rightarrow \mathbb{A}(\{\mathcal{A}', \mathcal{M}'\}) = \text{mod-}\mathcal{M}'$ is a morphism in $\text{AuslCat}(n)$, then it is easy to see that $\tilde{F} \equiv \mathbb{A}(\tilde{F}^\dagger) = \mathbb{A}(\tilde{F}^\dagger) = \tilde{F}^\dagger$. Indeed, since $\mathbb{B}$ has enough projectives, the functors $\tilde{F}$ and $\tilde{F}^\dagger$ are isomorphic since both are (right) exact and clearly agree on the projective objects of $\mathcal{B}$. Hence $\mathbb{A}$ is an equivalence of large categories with quasi-inverse $\mathbb{C}$. □

We leave to the reader the interpretation of the above results in the context of module over rings and in particular in the context of finitely generated modules over an Artin algebra.

9. Cluster tilting versus cluster cotilting

Our definition of $n$-cluster tilting subcategories in abelian or triangulated categories in Sections 8 and 5 respectively are slightly different than those of Iyama in [39–41].
and [44]. Recall from [41] that a full subcategory $\mathcal{M}$ of an abelian category $\mathcal{A}$, say with enough projectives and/or injectives, is called $n$-cluster tilting (also $(n-1)$-cluster tilting or maximal $n$-orthogonal in the terminology used in [39]) if: (A1) $\mathcal{M}$ is contravariantly finite and $\mathcal{M}_n^\perp = \mathcal{M}$, and (A2) $\mathcal{M}$ is covariantly finite and $\mathcal{M} = \mathcal{M}_n^\perp$. Also recall from [44] that a full subcategory $\mathcal{X}$ of a triangulated category $\mathcal{T}$ is called $(n+1)$-cluster tilting if: (T1) $\mathcal{X}$ is contravariantly finite and $\mathcal{X}_n^\perp = \mathcal{X}$, and (T2) $\mathcal{X}$ is covariantly finite and $\mathcal{X} = \mathcal{X}_n^\perp$.

In Section 5 we proved that the conditions (T1) and (T2) are equivalent, so only half of the conditions in [44] are needed in the definition of cluster tilting subcategories in triangulated categories. Our aim in this section is to show that the conditions (A1) and (A2) are also equivalent in the abelian case. As a consequence $n$-cluster tilting and $n$-cluster cotilting are equivalent notions. This improves on related results of Iyama in [39–41] and has some interesting consequences on (infinitely generated) $n$-cluster tilting modules.

We begin our analysis by fixing some notation. For a full subcategory $\mathcal{M}$ of $\mathcal{A}$ and any integer $k \geq 0$, we denote by $\mathcal{M}_{\leq k}$, resp. $\tilde{\mathcal{M}}_{\leq k}$ the full subcategory of $\mathcal{A}$ consisting of all objects $A$ admitting an exact sequence $0 \rightarrow M_k \rightarrow \cdots \rightarrow M_1 \rightarrow M_0 \rightarrow A \rightarrow 0$, where the $M_i \in \mathcal{M}$, $0 \leq i \leq k$, resp. and the sequence remains exact after applying the functor $\mathcal{A}(\mathcal{M},-)$.

For instance if $\mathcal{M}$ is an $n$-cluster tilting subcategory of $\mathcal{A}$, then by part 4. of Remark 8.11 we have $\mathcal{A} = \tilde{\mathcal{M}}_{\leq n}$. We also denote by $\mathcal{M}_{\leq k}$, resp. $\tilde{\mathcal{M}}_{\leq k}$, the full subcategory of $\mathcal{A}$ consisting of all objects $A$ such that there exists an exact sequence $0 \rightarrow A \rightarrow M^0 \rightarrow M^1 \rightarrow \cdots \rightarrow M^k \rightarrow 0$, where the $M^i \in \mathcal{M}$, $0 \leq i \leq k$, resp. and the sequence remains exact after applying the functor $\mathcal{A}(-,\mathcal{M})$.

Clearly for any $t \geq 0$ we have $\tilde{\mathcal{M}}_{\leq t} \subseteq \mathcal{M}_{\leq t}$ and $\tilde{\mathcal{M}}_{\leq t} \subseteq \mathcal{M}_{\leq t}$.

**Lemma 9.1.** Let $\mathcal{A}$ be an abelian category with enough injectives and $\mathcal{M}$ a contravariantly finite subcategory of $\mathcal{A}$ such that right $\mathcal{M}$-approximations are epic and $\mathcal{M}_{n}^\perp = \mathcal{M}$. Then for $0 \leq t \leq n-1$ we have:

$$(\alpha) \quad \mathcal{M}_{\leq t} = \tilde{\mathcal{M}}_{\leq t} \subseteq \frac{1}{n-t} \mathcal{M} \quad \text{and} \quad (\beta) \quad \mathcal{M}_{\leq t} = \tilde{\mathcal{M}}_{\leq t} \subseteq \mathcal{M}_{n-t}^\perp$$

**Proof.** We only prove (α) since (β) is dual. If $t = 0$, then clearly $\mathcal{M}_{\leq 0} = \mathcal{M} = \tilde{\mathcal{M}}_{\leq 0}$. By Proposition 8.10 we have $\mathcal{M} = \frac{1}{n} \mathcal{M}$, so (α) is true for $t = 0$. Assume that $t = 1$ and let $A \in \mathcal{M}_{\leq 1}$. Then there exists an exact sequence $0 \rightarrow A \rightarrow M^0 \rightarrow M^1 \rightarrow 0$. Applying the functor $\mathcal{A}(-,\mathcal{M})$ to this exact sequence and using that $\mathcal{M}$ is $n$-rigid, we see directly that $\text{Ext}^k(\mathcal{M},A) = 0$ for $1 \leq k \leq n-1$. Hence $A \in \frac{1}{n-1} \mathcal{M} \cap \tilde{\mathcal{M}}_{\leq 1}$ and therefore $\mathcal{M}_{\leq 1} \subseteq \frac{1}{n-1} \mathcal{M} \cap \tilde{\mathcal{M}}_{\leq 1}$. Hence we have $\mathcal{M}_{\leq 1} = \tilde{\mathcal{M}}_{\leq 1} \subseteq \frac{1}{n-1} \mathcal{M}$ and (α) holds for $t = 1$. Assume that $\mathcal{M}_{\leq t-1} = \tilde{\mathcal{M}}_{\leq t-1} \subseteq \frac{1}{n-t+1} \mathcal{M}$, for $0 \leq t \leq n-1$, and let $A \in \mathcal{M}_{\leq t}$. Then there exists a short exact sequence $0 \rightarrow A \rightarrow M^0 \rightarrow B \rightarrow 0$, where $B \in \mathcal{M}_{\leq t-1}$. By hypothesis then we have $B \in \frac{1}{n-t+1} \mathcal{M}$. Using this and $n$-rigidity of $\mathcal{M}$, application of the functor $\mathcal{A}(-,\mathcal{M})$ to this exact sequence gives that the sequence remains exact and
\[ \text{Ext}^k(M, A) = 0, \text{for } 1 \leq k \leq n - t. \] This implies that \( A \in \hat{M}^{\leq t} \cap \frac{1}{n-t}M. \) We infer that \( M^{\leq t} = \hat{M}^{\leq t} \subseteq \frac{1}{n-t}M. \)

Setting for convenience \( \hat{0}M = \mathcal{A}, \) we have the following:

**Proposition 9.2.** Let \( \mathcal{A} \) be an abelian category with enough injectives and \( M \) a contravariantly finite subcategory of \( \mathcal{A}. \) If any right \( M \)-approximation is epic and \( M^{\perp} _{\hat{n}} = M, \) then \( M \) is covariantly finite and

\[ M^{\leq t} = \hat{M}^{\leq t} = \frac{1}{n-t}M, \quad 0 \leq t \leq n \]

**Proof.** Let \( A \) be an object of \( \mathcal{A}, \) consider an injective resolution

\[ 0 \rightarrow A \rightarrow I^0 \rightarrow I^1 \rightarrow \cdots \rightarrow I^n \rightarrow I^{n+1} \rightarrow \cdots \]

of \( A, \) and let \( \Sigma^t A = \text{Im}(I^{t-1} \rightarrow I^t), \forall i \geq 1, \) be the cosyzygy objects of \( A. \) We shall construct a left \( M \)-approximation of \( A \) in two steps. The first one is the proof of the following:

1. **Claim:** For any \( 1 \leq t \leq n, \) there exist objects \( X^t \in \mathcal{A}, K^{1}_{X^t} \in \hat{M}^{\leq t-1} \cap M^{\perp} _{1} \) and \( Z^t \in M^{\leq n-t}, \) and an exact commutative diagram:

\[ \begin{array}{cccccccc}
0 & 0 & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
& & K^{1}_{X^t} & K^{1}_{X^t} & & & & & \\
& & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \\
0 & \Sigma^{t-1} A & X^{t-1} & Z^t & 0 & \downarrow & \downarrow & \downarrow & \downarrow \\
0 & \Sigma^{t-1} A & I^{t-1} & \Sigma^t A & 0 & & & & \\
& & & & & 0 & 0 & & \\
\end{array} \] (9.1)

Note that in this case, since \( K^{1}_{X^t} \in M^{\perp} _{1} \) and since \( \text{Inj} \mathcal{A} \subseteq M, \) we have \( \text{Ext}^1(I^{t-1}, K^{1}_{X^t}) = 0. \) Therefore the second vertical short exact sequence splits and we have a direct sum decomposition \( X^{t-1} = K^{1}_{X^t} \oplus I^{t-1} \) and then clearly \( X^{t-1} \in \hat{M}^{\leq t-1} \cap M^{\perp} _{1}. \)

**Proof of the Claim:** By Remark 8.11, any object \( X \in \mathcal{A} \) admits an \( M \)-resolution

\[ 0 \rightarrow M^n_X \rightarrow \cdots \rightarrow M^0_X \rightarrow X \rightarrow 0, \] such that \( K^i_X := \text{Im}(M^i_X \rightarrow M^{i-1}_X) \in \hat{M}_{n-i}^{\leq i}. \)
and clearly $K^i_X \in \mathcal{M}^1_i$, $\forall i \geq 1$. Let $Z^n := M^{0}_{\Sigma^n A} \rightarrow \Sigma^n A$ be a right $\mathcal{M}$-approximation of $\Sigma^n A$ with kernel $K^1_{\Sigma^n A} := K^1_{\Sigma^n A}$, and consider the pull-back diagram

\[
\begin{array}{ccccccccc}
0 & \downarrow & 0 & \downarrow & & & & & \\
& & K^1_{\Sigma^n A} & \rightarrow & K^1_{\Sigma^n A} & \rightarrow & & & \\
& & & & & & & & \\
0 & \rightarrow & \Sigma^{n-1} A & \rightarrow & X^{n-1} & \rightarrow & M^{0}_{\Sigma^n A} & \rightarrow & 0 \\
& \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
0 & \rightarrow & \Sigma^{n-1} A & \rightarrow & I^{n-1} & \rightarrow & \Sigma^n A & \rightarrow & 0 \\
& 0 & & 0 & & 0 & & 0 & \\
\end{array}
\]

which defines the object $X^{n-1}$. Then $K^1_{\Sigma^n A} \in \hat{\mathcal{M}}_{\leq n-1} \cap \mathcal{M}^1_1$. Since $\mathcal{M}$ contains the injectives, the middle vertical short exact sequence splits and therefore $X^{n-1} = K^1_{\Sigma^n A} \oplus I^{n-1}$ lies in $\hat{\mathcal{M}}_{\leq n-1} \cap \mathcal{M}^1_1$. In particular there exists a short exact sequence $0 \rightarrow K^1_{X^{n-1}} \rightarrow M^0_{X^{n-1}} \rightarrow X^{n-1} \rightarrow 0$ where $K^1_{X^{n-1}} \in \hat{\mathcal{M}}_{\leq n-2} \cap \mathcal{M}^1_1$.

Consider the following exact commutative diagram induced by the composition of epimorphisms $M^0_{X^{n-1}} \rightarrow X^{n-1} \rightarrow M^0_{\Sigma^n A}$, which defines the object $Z^{n-1}$:

\[
\begin{array}{ccccccccc}
0 & \downarrow & 0 & \downarrow & & & & & \\
& & K^1_{X^{n-1}} & \rightarrow & K^1_{X^{n-1}} & \rightarrow & & & \\
& & & & & & & & \\
0 & \rightarrow & Z^{n-1} & \rightarrow & M^0_{X^{n-1}} & \rightarrow & M^0_{\Sigma^n A} & \rightarrow & 0 \\
& \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
0 & \rightarrow & \Sigma^{n-1} A & \rightarrow & X^{n-1} & \rightarrow & M^0_{\Sigma^n A} & \rightarrow & 0 \\
& 0 & & 0 & & 0 & & 0 & \\
\end{array}
\]
We observe that $Z^{n-1} \in M^{\leq 1}$. Next we form the pull-back of the short exact sequence $0 \rightarrow \Sigma^{-2} A \rightarrow I^{n-2} \rightarrow \Sigma^{-1} A \rightarrow 0$ along the epimorphism $Z^{n-1} \rightarrow \Sigma^{-1} A$:}

\[
\begin{array}{ccccccccc}
0 & 0 \\
\downarrow & & & \downarrow \\
K^1_{X^{n-1}} & \longrightarrow & K^1_{X^{n-1}} \\
\downarrow & & & \downarrow \\
0 & \rightarrow & \Sigma^{-2} A & \rightarrow & X^{n-2} & \rightarrow & Z^{n-1} & \rightarrow & 0 \\
\downarrow & & & \downarrow & & \downarrow & & \downarrow \\
0 & \rightarrow & \Sigma^{-2} A & \rightarrow & I^{n-2} & \rightarrow & \Sigma^{-1} A & \rightarrow & 0 \\
\downarrow & & & \downarrow & & \downarrow & & \downarrow \\
0 & & & & 0 & & & & 0
\end{array}
\]

which defines the object $X^{n-2}$. Since $K^1_{X^{n-1}}$ lies in $M^1_1$ and $I^{n-2} \in M$, it follows that the middle vertical sequence splits and therefore $X^{n-2} = K^1_{X^{n-1}} \oplus I^{n-2}$. In particular we have $X^{n-2} \in \widehat{M}^{\leq n-2} \cap M^1_1$.

Continuing in this way we construct, for $1 \leq t \leq n$, the exact commutative diagram (9.1) as desired.

2. In particular for $t = 1$, we have the following pull-back diagram

\[
\begin{array}{ccccccccc}
0 & 0 \\
\downarrow & & & \downarrow \\
M^1_{X^1} & \longrightarrow & M^1_{X^1} \\
\downarrow & & & \downarrow \\
0 & \rightarrow & A & \rightarrow & M^0_A & \rightarrow & Z^1 & \rightarrow & 0 \\
\downarrow & & & \downarrow & & \downarrow & & \downarrow \\
0 & \rightarrow & A & \rightarrow & I^0 & \rightarrow & \Sigma A & \rightarrow & 0 \\
\downarrow & & & \downarrow & & \downarrow & & \downarrow \\
0 & & & & 0 & & & & 0
\end{array}
\]

which defines the object $M^0_A = X^0$, and where $M^1_{X^1} \in \widehat{M}^{\leq 0} \cap M^1_1$, hence $M^1_{X^1} \in M$, and $Z^1 \in M^{\leq n-1}$. Since $M^1_{X^1} \in M$ and $I^0 \in \text{Inj} \mathcal{A} \subseteq M$, as before the middle vertical sequence splits and therefore $M^0_A = M^1_{X^1} \oplus I^0 \in M$. 
We claim that the map $A \to M_0^A$ is a left $\mathcal{M}$-approximation of $A$, i.e. any map $A \to M$, where $M \in \mathcal{M}$, factorizes through $M_0^A$. To this end, clearly it suffices to show that $\text{Ext}^1_{\mathcal{A}}(Z^1, M) = 0$, i.e. $Z^1 \in \frac{1}{n} \mathcal{M}$. Since $Z^1$ lies in $\mathcal{M}^{\perp n-1}$, it follows from Lemma 9.1 that indeed $Z^1 \in \frac{1}{n} \mathcal{M}$. We infer that $A \to M_0^A$ is a left $\mathcal{M}$-approximation of $A$, and therefore $\mathcal{M}$ is covariantly finite in $\mathcal{A}$.

It remains to show that the inclusion $\mathcal{M}_{\leq t} \subseteq \frac{1}{n-t} \mathcal{M}$, $0 \leq t \leq n$, of Lemma 9.1 is an equality. First let $t = n$, so we have to show that $\mathcal{M}_{\leq n} = \mathcal{A} := \frac{1}{0} \mathcal{M}$. Let $A$ be an object of $\mathcal{A}$; since $\mathcal{M}$ is covariantly finite in $\mathcal{A}$ and contains the injectives, there exists an $\mathcal{M}$-coresolution $0 \to A \to M^0 \to M^1 \to \cdots \to M^k \to M^{k+1} \to \cdots$ of $A$. Setting $A^i = \text{Im}(M^{i-1} \to M^i)$, $\forall i \geq 0$, and using that $\mathcal{M}$ is $n$-rigid, it is easy to see that $A^k \in \frac{1}{n} \mathcal{M}$, $\forall k \geq 1$, and in particular $A^n \in \frac{1}{n} \mathcal{M} = \mathcal{M}$. It follows then that $A \in \mathcal{M}_{\leq n}$ and therefore $\mathcal{M}_{\leq n} = \mathcal{A}$. On the other hand, let $A \in \frac{1}{n-t} \mathcal{M}$, where $0 \leq t \leq n - 1$. Then applying the functor $\mathcal{A}(-, \mathcal{M})$ successively to the short exact sequences $0 \to A^{i-1} \to M^{i-1} \to A^i \to 0$, we see easily by induction that $A^i \in \frac{1}{n-t+i} \mathcal{M}$. In particular we have $A^t \in \frac{1}{n-t} \mathcal{M} = \mathcal{M}$. Then clearly $A \in \mathcal{M}_{\leq t}$. Hence $\mathcal{M}_{\leq t} = \mathcal{M}_{\leq t} = \frac{1}{n-t} \mathcal{M}$, for $0 \leq t \leq n$. □

Setting $\mathcal{A} := \mathcal{M}_0^1$, by duality we have the following.

**Proposition 9.3.** Let $\mathcal{A}$ be an abelian category with enough projectives and $\mathcal{M}$ a covariantly finite subcategory of $\mathcal{A}$. If any left $\mathcal{M}$-approximation is monic and $\frac{1}{n} \mathcal{M} = \mathcal{M}$, then $\mathcal{M}$ is contravariantly finite and

$$\mathcal{M}_{\leq t} = \hat{\mathcal{M}}_{\leq t} = \mathcal{M}_{n-t}^1, \quad 0 \leq t \leq n$$

Combining Propositions 9.2, 9.3 and Proposition 8.10 and its dual, we have the following consequence which shows that $n$-cluster tilting and $n$-cluster cotilting are equivalent notions.

**Corollary 9.4.** Let $\mathcal{A}$ be an abelian category with enough projective and injective objects. For a full subcategory $\mathcal{M}$ of $\mathcal{A}$, the following are equivalent:

(i) $\mathcal{M}$ is contravariantly finite, $\text{Proj} \mathcal{A} \subseteq \mathcal{M}$, and $\mathcal{M}_{\leq} = \mathcal{M}$.

(ii) $\mathcal{M}$ is covariantly finite, $\text{Inj} \mathcal{A} \subseteq \mathcal{M}$, and $\frac{1}{n} \mathcal{M} = \mathcal{M}$.

In other words: $\mathcal{M}$ is an $n$-cluster tilting subcategory if and only if $\mathcal{M}$ is an $n$-cluster cotilting subcategory. Moreover we have for $0 \leq t \leq n - 1$:

$$\mathcal{M}_{\leq t} = \hat{\mathcal{M}}_{\leq t} = \mathcal{M}_{n-t}^1 \quad \& \quad \mathcal{M}_{\leq t} = \hat{\mathcal{M}}_{\leq t} = \frac{1}{n-t} \mathcal{M} \quad \& \quad \hat{\mathcal{M}}_{\leq n} = \mathcal{A} = \hat{\mathcal{M}}_{\leq n}$$

**Remark 9.5.** Iyama and Oppermann in [43] call an object $M$ in an abelian category $\mathcal{A}$ a right, resp. left, $(n + 1)$-cluster tilting object if $\text{add} M = \frac{1}{n} \mathcal{M}$, resp. $\text{add} M = M_{n}^1$. It follows from Corollary 9.4 that if $\mathcal{A}$ has enough projectives and $\text{add} M$ contains the
projectives, then: $M$ is a right $(n+1)$-cluster tilting object $\iff$ $M$ is a left $(n+1)$-cluster tilting object.

The following summarizes some of the main results of this and the previous section.

**Theorem 9.6.** Let $\mathcal{A}$ be a non-semisimple abelian category with enough projectives and enough injectives. For a full subcategory $\mathcal{M}$ of $\mathcal{A}$ and any integer $n \geq 1$, the following are equivalent.

(i) $\mathcal{M}$ is an $n$-cluster tilting subcategory of $\mathcal{A}$.
(ii) $\mathcal{M}$ is an $n$-cluster cotilting subcategory of $\mathcal{A}$.
(iii) mod-$\mathcal{M}$ is an $n$-Auslander category.
(iv) $\mathcal{M}$-mod is an $n$-Auslander category.
(v) $\mathcal{M}$ is contravariantly finite $n$-rigid, Proj $\mathcal{A} \subseteq \mathcal{M}$, and gl.dim mod-$\mathcal{M} = n + 2$.
(vi) $\mathcal{M}$ is covariantly finite $n$-rigid, lnj $\mathcal{A} \subseteq \mathcal{M}$, and gl.dim mod-$\mathcal{M} = n + 2$.

If (i) holds, then: $\mathcal{M}^\leq n = \mathcal{A} = \mathcal{M}^{< n}$. Moreover: dom.dim mod-$\mathcal{M} = \text{codom.dim M-mod} = \text{gl.dim } \mathcal{M}$-mod $= n + 2$. If in addition $\mathcal{M}$ is of finite representation type, then: rep.dim $\mathcal{A} \leq n + 2$.

**Proof.** The implications (i) $\iff$ (iii) $\iff$ (v) and (ii) $\iff$ (iv) $\Rightarrow$ (vi) follow from Proposition 8.10, its dual, and Theorems 8.23 and 8.25. The equivalence (i) $\iff$ (ii) follows from Corollary 9.4. It remains to prove the implication (vi) $\Rightarrow$ (ii), since the implication (v) $\Rightarrow$ (i) is dual.

(v) $\Rightarrow$ (ii) Let $A \in \frac{1}{n} \mathcal{M}$. Since $\mathcal{M}$ is covariantly finite and contains the injectives, there exists an exact sequence $0 \rightarrow A \rightarrow M_1^A \rightarrow M_0^A$, where the $M_i^A$ lie in $\mathcal{M}$. Then we have an exact sequence $0 \rightarrow \mathbb{H}(A) \rightarrow \mathbb{H}(M_1^A) \rightarrow \mathbb{H}(M_0^A) \rightarrow F \rightarrow 0$ in mod-$\mathcal{M}$. Since gl.dim mod-$\mathcal{M} = n + 2$, it follows that pd $F \leq n + 2$ and therefore pd $\mathbb{H}(A) \leq n$. Hence we have an $\mathcal{M}$-resolution $0 \rightarrow M_A^1 \rightarrow M_A^{n-1} \rightarrow \cdots \rightarrow M_A^1 \rightarrow M_A^0 \rightarrow A \rightarrow 0$ of $A$. Applying to the extension $0 \rightarrow K_A^1 \rightarrow M_A^0 \rightarrow A \rightarrow 0$ the functor $\mathcal{A}(-, \mathcal{M})$, we see that $K_A^1 \in \frac{1}{n-1} \mathcal{M}$ and then by induction $K_A^{n-i} \in \frac{i}{n} \mathcal{M}, 1 \leq i \leq n - 1$. In particular since $K_A^{n-1} \in \frac{1}{n} \mathcal{M}$, the extension $0 \rightarrow M_A^n \rightarrow M_A^{n-1} \rightarrow K_A^{n-1} \rightarrow 0$ splits and therefore $K_A^{n-1} \in \mathcal{M}$. Since $K_A^{n-2} \in \frac{1}{n-2} \mathcal{M}$, it follows that the extension $0 \rightarrow K_A^{n-2} \rightarrow M_A^{n-2} \rightarrow K_A^{n-2} \rightarrow 0$ splits and therefore $K_A^{n-2} \in \mathcal{M}$. Continuing in this way we see that the extension $0 \rightarrow K_A^1 \rightarrow M_A^0 \rightarrow A \rightarrow 0$ splits, hence $A \in \mathcal{M}$. We infer that $\frac{1}{n} \mathcal{M} = \mathcal{M}$.

Assume now that (i) holds. By part 4. of Remark 8.11 we have $\mathcal{A} = \mathcal{M}^\leq n$. By Proposition 9.2, with $t = n$, we have $\mathcal{A} = \mathcal{M}^\leq n$. On the other hand since $\mathcal{M}$ is functorially finite in $\mathcal{A}$ it follows that $\mathcal{M}$ is coherent and then by [13] we have gl.dim mod-$\mathcal{M} = n + 2$. Finally if $\mathcal{M} = \text{add } M$, for some $M \in \mathcal{A}$, then clearly $M$ is a coherent generator–cogenerator of $\mathcal{A}$. Since mod-End$\mathcal{A}(M) \approx \text{mod-} \mathcal{M}$, it follows that gl.dim mod-End$\mathcal{A}(M) = n + 2$ and therefore rep.dim $\mathcal{A} \leq n + 2$. □
Example 9.7. (i) Let $\mathcal{A}$ be a cocomplete abelian category with enough projectives and injectives, and let $M$ be a self-compact generator such that $M^+_n = \text{Add} M$. Recall that $M$ is self-compact if the functor $\mathcal{A}(M, -) \colon \text{Add} M \to \text{Ab}$ preserves coproducts. Our interest for self-compact objects, stems from the fact that $M$ is self-compact if and only if the canonical functor $\text{mod-Add} M \to \text{Mod-End}_{\mathcal{A}}(M)$, $F \mapsto F(M)$ is an equivalence, see \cite[Proposition 8.4]{12}. It follows by Remark 8.18 that $M$ is an $n$-cluster tilting object, i.e. $\text{Add} M$ is an $n$-cluster tilting subcategory, and the category $\text{mod-Add} T \cong \text{Mod-End}_{\mathcal{A}}(M)$ is an $n$-Auslander category, i.e. $\text{End}_{\mathcal{A}}(M)$ is an $n$-Auslander ring. Also by Proposition 9.2, the subcategory $\text{Add} M$ is $n$-cluster cotilting, so $\text{Add} M$ is covariantly finite and in particular closed under all small products, i.e. $M$ is by definition product-complete.

(ii) Let $\mathcal{A}$ be an abelian $R$-linear category with enough injectives over a commutative ring $R$ and let $M$ be an object of $\mathcal{A}$ such that the $R$-module $\mathcal{A}(M, X)$ is finitely generated for any $X \in \mathcal{A}$. If $M^+_n = \text{add} M$ and $\text{add} M$ contains a projective generator of $\mathcal{A}$, then $M$ is an $n$-cluster tilting object and $\text{mod-End}_{\mathcal{A}}(M)$ is an $n$-Auslander category. This follows from the fact that $\text{add} M$ is contravariantly finite in $\mathcal{A}$.

The following observation shows that any finitely generated $n$-cluster tilting module $M$ is product-complete and in particular $\Sigma$-pure-injective, i.e. any coproduct of copies of $M$ is pure-injective.

Corollary 9.8. Let $\Lambda$ be a ring and $M$ a finitely generated $n$-cluster tilting $\Lambda$-module, $n \geq 1$. Then $\text{End}_{\Lambda}(M)$ is a left coherent and right perfect $n$-Auslander ring and $M$ is a product-complete, in particular $\Sigma$-pure-injective, $\Lambda$-module which is finitely presented as an $\text{End}_{\Lambda}(M)$-module.

Proof. Since finitely generated modules are self-compact, as in Example 9.7, $M$ is product-complete and $\text{End}_{\Lambda}(M)$ is a left coherent and right perfect $n$-Auslander ring. By a result of Krause and Saorín \cite{49}, product-completeness of $M$ implies that $M$ is $\Sigma$-pure-injective and finitely presented over $\text{End}_{\Lambda}(M)$. 

We say that pairs $\{\mathcal{A}, M\}$ and $\{\mathcal{A}', M'\}$, consisting of Grothendieck categories $\mathcal{A}$ and $\mathcal{A}'$ and $n$-cluster tilting objects $M \in \mathcal{A}$ and $M' \in \mathcal{A}'$, are cluster equivalent if the pairs $\{\mathcal{A}, \text{Add} M\}$ and $\{\mathcal{A}', \text{Add} M'\}$ are cluster equivalent in the sense of Section 8. Note that in this case the Grothendieck categories $\mathcal{A}$ and $\mathcal{A}'$ are equivalent.

Theorem 9.9. For any $n \geq 0$, the map

$$\{\mathcal{A}, M\} \mapsto \Lambda = \text{End}_{\mathcal{A}}(M)$$

gives a bijective correspondence between (cluster/Morita) equivalence classes of

(I) Pairs $\{\mathcal{A}, M\}$ where $\mathcal{A}$ is a Grothendieck category and $M \in \mathcal{A}$ is a self-compact $n$-cluster tilting object.

(II) Left coherent and right perfect $n$-Auslander rings $\Lambda$. 
Under this bijection we have an equivalence

$$\mathcal{A} \xrightarrow{\sim} (\mathcal{U}\text{-mod})^{\text{op}}$$

where $\mathcal{U} = \text{Proj} \Lambda \cap \text{Inj} \Lambda$ and $\Lambda = \text{End}_\mathcal{A}(M)$.

**Proof.** (I) $\leftrightarrow$ (II) If $\mathcal{A}$ is a Grothendieck category and $M$ is a self-compact $n$-cluster tilting object of $\mathcal{A}$, then by Example 9.7, $\text{End}_\mathcal{A}(M)$ is an $n$-Auslander ring. Since $M$ is self-compact, the product-preserving functor $F: \mathcal{A} \rightarrow \text{Mod-End}_\mathcal{A}(M), F(A) = \text{Hom}_\mathcal{A}(M, A)$, induces an equivalence between $\text{Add} M$ and $\text{Proj End}_\mathcal{A}(M)$. Let $\{P_i\}_{i \in I}$ be a set of projective $\text{End}_\mathcal{A}(M)$-modules. Then $P_i = F(X_i)$ for some $X_i \in \text{Add} M, \forall i \in I$, and then $\prod_{i \in I} P_i = \prod_{i \in I} F(X_i) \cong F(\prod_{i \in I} X_i)$. By Proposition 9.2, $\text{Add} M$ is covariantly finite, in particular closed under products, in $\mathcal{A}$. Then $\prod_{i \in I} X_i \in \text{Add} M$ and $\prod_{i \in I} P_i \in F(\text{Add} M) = \text{Proj End}_\mathcal{A}(M)$. Therefore $\text{Proj End}_\mathcal{A}(M)$ is closed under products, hence $\text{End}_\mathcal{A}(M)$ is left coherent and right perfect by Chase’s Theorem [28].

(II) $\rightarrow$ (I) Let $\Lambda$ be a left coherent and right perfect $n$-Auslander ring, so $\text{Mod-\Lambda}$ is an $n$-Auslander category. By Theorem 8.23 there is an abelian category $\mathcal{A}$ with enough injectives and an $n$-cluster tilting subcategory $\mathcal{M}$ such that $\text{mod-\mathcal{M}} \cong \text{Mod-\Lambda}$ and the short exact sequence of abelian categories (8.1) takes the form

$$0 \rightarrow \widehat{\text{mod-\mathcal{M}}} \rightarrow \text{Mod-\Lambda} \xrightarrow{R} \mathcal{A} \rightarrow 0$$

Then $\mathcal{A}$ is a quotient of $\text{Mod-\Lambda}$ by the localizing subcategory $\widehat{\text{mod-\mathcal{M}}}$ and this implies that $\mathcal{A}$ is a Grothendieck category. Since $\text{Proj} \Lambda = \text{Add} \Lambda$, the equivalence $\text{Proj} \Lambda \cong \text{Proj mod-\mathcal{M}} \approx \mathcal{M}$ induced by $R$ shows that $\mathcal{M} = \text{Add} M$, where $M = R(\Lambda)$, and then $\Lambda \cong \text{End}_\mathcal{A}(M)$. Hence $M$ is an $n$-cluster tilting object of $\mathcal{A}$ and, since the functor $\text{mod-\mathcal{M}} \rightarrow \text{Mod-End}_\mathcal{A}(M) = \text{Mod-\Lambda}, F \mapsto F(M)$ is an equivalence, we infer that $M$ is self-compact.

By Proposition 8.12 and Theorem 8.23 the map $\{\mathcal{A}, M\} \mapsto \Lambda = \text{End}_\mathcal{A}(M)$ on cluster/Morita equivalence classes (I) and (II) is a bijection, and the last part follows from Proposition 8.12. \qed

**Remark 9.10.** It is shown in [14] that any cocomplete abelian category with a set of self-compact generators is a Grothendieck category. Since any $n$-cluster tilting object is clearly a generator, it follows that the condition “$\mathcal{A}$ is Grothendieck” in (I) above can be replaced by “$\mathcal{A}$ is cocomplete abelian”.

**Remark 9.11.** Let us call an object $M$ in an additive category $\mathcal{A}$ with infinite coproducts an *additive generator* if $\mathcal{A} = \text{Add} M$. Then the case $n = 0$ of Theorem 9.9 gives a bijective correspondence between Grothendieck categories with a self-compact additive generator and left coherent and right perfect 0-Auslander rings. As in Remark 9.10, in this correspondence Grothendieck categories can be replaced by cocomplete abelian categories.
For instance if $\Lambda$ is a ring, then $\mathcal{A} = \text{Proj}\Lambda$ contains $\Lambda$ as a self-compact additive generator. Hence $\text{Proj}\Lambda$ is a Grothendieck category if and only if $\Lambda$ is a left coherent and right perfect 0-Auslander ring. Since left coherent and right perfect rings are semiprimary, see [50], we deduce easily the well-known result of Tachikawa [60] characterizing rings whose projective modules form a Grothendieck category: $\text{Proj}\Lambda$ is a Grothendieck category if and only if $\Lambda$ is a semiprimary QF-3 ring such that $\text{gl.dim} \Lambda \leq 2 \leq \text{dom.dim} \Lambda$.

10. Stable Auslander categories

Let $\mathcal{M}$ be an $n$-cluster tilting subcategory in an abelian category $\mathcal{A}$ and let $\text{mod-}\mathcal{M}$ and $\mathcal{M}\text{-mod}$ be the corresponding $n$-Auslander categories of $\mathcal{M}$. Our aim in this section is to study homological properties of the stable (co-)Auslander categories $\text{mod-}\mathcal{M}$ and $\mathcal{M}\text{-mod}$ associated to $\mathcal{M}$. This includes the case $n = 0$, where $\mathcal{A}$ is abelian with injectives considered as a 0-cluster tilting subcategory of itself.

Throughout this section: we fix an abelian category $\mathcal{A}$ with enough projective and injective objects. We also fix a functorially finite subcategory $\mathcal{M}$ of $\mathcal{A}$ and assume that $\mathcal{M}$ contains the projectives and the injectives.

10.1. Recollements and duality

Under the above assumptions both the categories $\text{mod-}\mathcal{M}$ and $\mathcal{M}\text{-mod}$ are abelian and we have the fully faithful left exact functors

$$
\mathbb{H} : \mathcal{A} \rightarrow \text{mod-}\mathcal{M}, \quad \mathbb{H}(A) = \mathcal{A}(-, A)\big|_\mathcal{M} \quad & \quad \\
\mathbb{H}^{\text{op}} : \mathcal{A} \rightarrow (\text{mod-}\mathcal{M})^{\text{op}}, \quad \mathbb{H}^{\text{op}}(A) = \mathcal{A}(A, -)\big|_\mathcal{M}
$$

Consider the stable category $\overline{\mathcal{M}}$ modulo projectives and the stable category $\overline{\text{mod-}\mathcal{M}}$ modulo injectives. We identify the category of coherent functors $\overline{\text{mod-}\mathcal{M}}$ with the full subcategory of $\text{mod-}\mathcal{M}$ consisting of functors $F$ admitting a presentation $\mathbb{H}(M^1) \rightarrow \mathbb{H}(M^0) \rightarrow F \rightarrow 0$, where the map $M^1 \rightarrow M^0$ is epic. Similarly we identify the category $\overline{\text{mod-}\mathcal{M}}$ with the full subcategory of $\mathcal{M}\text{-mod}$ consisting of all functors $F$ admitting a presentation $\mathbb{H}^{\text{op}}(M_0) \rightarrow \mathbb{H}^{\text{op}}(M_1) \rightarrow F \rightarrow 0$, where the map $M_1 \rightarrow M_0$ is monic. Both categories $\overline{\text{mod-}\mathcal{M}}$ and $\overline{\text{mod-}\mathcal{M}}$ are abelian because the category $\overline{\mathcal{M}}$ has weak kernels and the category $\overline{\mathcal{M}}$ has weak cokernels. In fact we have the following short exact sequences of abelian categories

$$
0 \rightarrow \text{mod-}\overline{\mathcal{M}} \rightarrow \text{mod-}\mathcal{M} \rightarrow \mathcal{A} \rightarrow 0 \quad & \quad \\
0 \rightarrow (\overline{\text{mod-}\mathcal{M}})^{\text{op}} \rightarrow (\text{mod-}\mathcal{M})^{\text{op}} \rightarrow \mathcal{A} \rightarrow 0
$$

induced by the exact adjoints $\mathbb{R} : \text{mod-}\mathcal{M} \rightarrow \mathcal{A} \leftarrow (\text{mod-}\mathcal{M})^{\text{op}} : \mathbb{R}^{\text{op}}$ of the Yoneda restriction functors $\mathbb{H}$ and $\mathbb{H}^{\text{op}}$ respectively. By Lemma 8.6 we have $F \in \text{mod-}\overline{\mathcal{M}}$ if
and only if $\text{Hom}(F, \mathbb{H}(A)) = 0$, for any object $A \in \mathbb{A}$. Since $\mathbb{M}$ contains the injectives and $\mathbb{M}$ is covariantly finite, for any object $A \in \mathbb{A}$ there exists a monic left $\mathbb{M}$-approximation $A \rightarrowtail M^A$, and then clearly $\text{Hom}(F, \mathbb{H}(A)) = 0$, $\forall A \in \mathbb{A}$, if and only if $\text{Hom}(F, \mathbb{H}(M)) = 0$, $\forall M \in \mathbb{M}$. We infer that if $\mathbb{M}$ contains the injectives, then $F \in \text{mod-}\mathbb{M}$ if and only if $\text{Hom}(F, \mathbb{H}(\mathbb{A})) = 0$ if and only if $\text{Hom}(F, \mathbb{H}(\mathbb{M})) = 0$ if and only if $\text{Hom}(F, \mathbb{H}(\text{Inj-}\mathbb{A})) = 0$. Dually since $\mathbb{M}$ contains the projectives, then $F \in \text{mod-}\mathbb{M}$ if and only if $\text{Hom}(F, \mathbb{H}(\mathbb{A})) = 0$ if and only if $\text{Hom}(F, \mathbb{H}(\text{Proj-}\mathbb{M})) = 0$ if and only if $\text{Hom}(F, \mathbb{H}(\text{Proj-}\mathbb{A})) = 0$.

If $\mathbb{U}$ is a full subcategory of an abelian category $\mathbb{C}$, then $\text{Fac } \mathbb{U}$, resp. $\text{Sub } \mathbb{U}$, denotes the full subcategory of $\mathbb{C}$ consisting of all factors, resp. subobjects, of an object from $\mathbb{U}$. Recall that a triple $(\mathbb{U}, \mathbb{V}, \mathbb{W})$ of full subcategories of $\mathbb{C}$ is called a $\text{TTF-triple}$ and $\mathbb{V}$ a $\text{TTF-class}$, if $(\mathbb{U}, \mathbb{V})$ and $(\mathbb{V}, \mathbb{W})$ are torsion pairs in $\mathbb{C}$. In this case the TTF-class $\mathbb{V}$ is an abelian category, the inclusion $\mathbb{V} \rightarrowtail \mathbb{C}$ is exact and admits a left and right adjoint, see [19, Chapter I] for details on torsion pairs and TTF-triples in abelian categories. Recall that the diagram (†) below consisting of abelian categories and additive functors is a recollement diagram, if:

\[
\begin{array}{ccc}
\mathbb{A} & \xrightarrow{i} & \mathbb{B} \\
\mathbb{C} & \xleftarrow{r} & \\
\mathbb{B} & \xrightarrow{e} & \mathbb{C}
\end{array}
\]

(i) $(l, e, r)$ is an adjoint triple.
(ii) $(q, i, p)$ is an adjoint triple.
(iii) The functors $i$, $l$, and $r$ are fully faithful.
(iv) $\text{Im } i = \text{Ker } e$.

To proceed further we need the following result which is of independent interest.

**Proposition 10.1.** Let $U: \mathbb{B} \rightarrowtail \mathbb{A}$ be an exact functor between abelian categories.

1. Assume that $\mathbb{A}$ and $\mathbb{B}$ have enough projectives. If $U$ admits a fully faithful right adjoint $R$ and $R(\text{Proj } \mathbb{A}) \subseteq \text{Proj } \mathbb{B} \subseteq \text{Im } R$, then $U$ admits a fully faithful left adjoint $L$ and there exists a recollement situation

\[
\begin{array}{ccc}
\mathbb{C} & \xrightarrow{i^*} & \mathbb{B} \\
\mathbb{B} & \xleftarrow{i_*} & \mathbb{A}
\end{array}
\]

where $\mathbb{C} = \text{Ker } U$. Moreover we have $L = L_0 R$ and there exists a TTF-triple in $\mathbb{B}$:

$$(\text{Fac } R(\text{Proj } \mathbb{A}), \mathbb{C}, \text{Sub } R(\mathbb{A}))$$
2. Assume that $\mathcal{A}$ and $\mathcal{B}$ have enough injectives. If $U$ admits a fully faithful left adjoint $L$ and $L(\text{Inj} \mathcal{A}) \subseteq \text{Inj} \mathcal{B} \subseteq \text{Im} L$, then $U$ admits a fully faithful right adjoint $R$ and there exists a recollement situation

$$
\begin{array}{ccc}
\mathcal{C} & \xrightarrow{j^*} & \mathcal{B} \\
& j^* & \\
& & \uparrow j^' \\
\mathcal{L} & \xrightarrow{U} & \mathcal{A} \\
& U & \\
& & \downarrow U \\
\mathcal{R} & \xrightarrow{R} & \\
\end{array}
$$

where $\mathcal{C} = \text{Ker} U$. Moreover we have $R = R^0L$ and there exists a TTF-triple in $\mathcal{B}$:

$$
( \text{Fac} L(\mathcal{A}), \mathcal{C}, \text{Sub} L(\text{Inj} \mathcal{A}))
$$

**Proof.** We prove only part 1, since the proof of 2. is completely dual.

We define $L = L_0R: \mathcal{A} \to \mathcal{B}$ to be the 0th left derived functor of $R$, so $L(A)$ is defined by the exact sequence $R(P^1) \to R(P^0) \to L(A) \to 0$, where $P^1 \to P^0 \to A \to 0$ is a projective presentation of $A$ in $\mathcal{A}$. We show that $L$ is a left adjoint of $U$. Let $A$ be in $\mathcal{A}$ as above and $B$ be in $\mathcal{B}$. Since $\mathcal{B}$ has enough projectives and $\text{Proj} \mathcal{B} \subseteq \text{Im} R$, there exists a projective presentation $R(Y) \to R(X) \to B \to 0$, where $X, Y \in \mathcal{A}$. Applying $\mathcal{B}(R(P^i), -)$ and using that $R$ preserves projectives, we have an exact sequence $\mathcal{B}(R(P^i), R(Y)) \to \mathcal{B}(R(P^i), R(X)) \to \mathcal{B}(R(P^i), B) \to 0$ which, since $R$ is fully faithful, is isomorphic to the exact sequence $\mathcal{A}(P^i, Y) \to \mathcal{A}(P^i, X) \to \mathcal{A}(P^i, U(B)) \to 0$. Since the functor $U$ is exact and the functor $R$ is fully faithful we have $UR = \text{Id}_\mathcal{A}$ and an exact sequence $Y \to X \to U(B) \to 0$ in $\mathcal{A}$. Since the $P^i$ are projective we have an exact sequence $\mathcal{A}(P^i, Y) \to \mathcal{A}(P^i, X) \to \mathcal{A}(P^i, U(B)) \to 0$. This shows that we have an isomorphism $\mathcal{B}(R(P^i), B) \cong \mathcal{A}(P^i, U(B))$. Using this isomorphism, the exact sequence $0 \to \mathcal{B}(L(A), B) \to \mathcal{B}(R(P^0), B) \to \mathcal{B}(R(P^1), B)$ is isomorphic to the exact sequence $0 \to \mathcal{A}(L(A), B) \to \mathcal{A}(P^0, U(B)) \to \mathcal{A}(P^1, U(B))$. Since the kernel of the last map is isomorphic to $\mathcal{A}(A, U(B))$ we infer that we have an isomorphism $\mathcal{B}(L(A), B) \cong \mathcal{A}(A, U(B))$. Clearly this isomorphism is natural in $A \in \mathcal{A}$ and $B \in \mathcal{B}$, so we have an adjoint pair $(L, U)$ and therefore an adjoint triple $(L, U, R)$.

Since $R$ is fully faithful, standard arguments show that so is $L$. Denote by $i_*: \mathcal{C} \to \mathcal{B}$ the inclusion of $\text{Ker} U := \mathcal{C}$ in $\mathcal{B}$. We define functors $i^*$ and $i^!$ by the exact sequences of functors:

$$
L \to \text{Id}_\mathcal{B} \xrightarrow{i^*} 0 \quad \& \quad 0 \xrightarrow{i^!} \text{Id}_\mathcal{B} \to RU
$$

induced from the counit $L \to \text{Id}_\mathcal{B}$ of the adjoint pair $(L, U)$ and the unit $\text{Id}_\mathcal{B} \to RU$ of the adjoint pair $(U, R)$. It is easy to see that in this way we obtain additive functors $i^*: \mathcal{B} \to \mathcal{C}$ and $i^!: \mathcal{B} \to \mathcal{C}$ and an adjoint triple $(i^*, i_*, i^!)$. Hence the diagram in the statement represents a recollement situation. By general facts on recollements there is a TTF-triple $(\text{Ker} i^*, \mathcal{C} = \text{Ker} U, \text{Ker} i^!)$ in $\mathcal{B}$. We show that

$$
\text{Fac} R(\text{Proj} \mathcal{A}) = \text{Ker} i^* \quad \& \quad \text{Sub} R(\mathcal{A}) = \text{Ker} i^!
$$
Clearly if $B$ lies in $\text{Ker}^{i}$, then the unit $B \to R(U(B))$ is monic and therefore $B$ lies in $\text{Sub} R(\mathcal{A})$. Conversely if $B \to R(A)$ is a monomorphism, then since the unit $B \to R(U(B))$ is the reflection of $B$ in $\im R$, it follows that $B \to R(A)$ factors through the unit and therefore the latter is monic. Then its kernel $i^{i}(B)$ is zero and therefore $B \in \text{Ker}^{i}$. We infer that $\text{Sub} R(\mathcal{A}) = \text{Ker}^{i}$.

Now let $B \in \text{Ker}^{i^{s}}$. Then we have an epimorphism $L(U(B)) \to B \to 0$. Since by construction we have an exact sequence $R(P^{1}) \to R(P^{0}) \to L(U(B)) \to 0$, where $P^{1} \to P^{0} \to U(B) \to 0$ is a projective presentation of $U(B)$ in $\mathcal{A}$, it follows that we have an epimorphism $R(P^{0}) \to B \to 0$, so $B$ lies in $\text{Fac} R(\text{Proj} \mathcal{A})$. Conversely if $R(P) \to B \to 0$ is exact, where $P$ is projective in $\mathcal{A}$, then applying the right exact functor $LU$ we have an epimorphism $LU(P) \to LU(B) \to 0$ and therefore an epimorphism $L(P) \to LU(B) \to 0$ since $UR = \text{Id}_{\mathcal{A}}$. Since by naturality the composition $L(P) \to LU(B) \to B$ is equal to the composition $L(P) \to R(P) \to B$ and since clearly the map $L(P) \to R(P)$ is invertible because $P$ is projective, we infer that $L(P) \to LU(B) \to B$ is epic and therefore the counit $LU(B) \to B$ is epic. This means by construction that $i^{s}(B) = 0$, so $B \in \text{Ker}^{i^{s}}$. We infer that $\text{Fac} R(\text{Proj} \mathcal{A}) = \text{Ker}^{i^{s}}$. □

Returning to our default situation, let $\mathcal{A}$ be an abelian category with enough projectives and injectives and $\mathcal{M}$ a functorially finite subcategory of $\mathcal{A}$ containing the projectives and the injectives. Then considering the abelian categories $\text{mod-} \mathcal{M}$ and $(\text{M-mod})^{\text{op}}$, it is clear that the assumptions of Proposition 10.1 hold for the exact left adjoint $R: \text{mod-} \mathcal{M} \to \mathcal{A}$ of the fully faithful functor $\mathbb{H}$ and for the exact right adjoint $\mathbb{H}^{\text{op}}: (\text{M-mod})^{\text{op}} \to \mathcal{A}$ of the fully faithful functor $\mathbb{H}^{\text{op}}$. Hence by Proposition 10.1 we have the following consequence.

**Proposition 10.2.** Let $\mathcal{A}$ be an abelian category with enough projective and injective objects and $\mathcal{M}$ a functorially finite subcategory of $\mathcal{A}$ containing the projectives and the injectives. Then there are recollements:

![Rercollement Diagram](image)

and $TTF$-triples in $\text{mod-} \mathcal{M}$ and in $(\text{M-mod})^{\text{op}}$ respectively:

$$\left( \text{Fac} \mathbb{H}(\text{Proj} \mathcal{A}), \text{mod-} \mathcal{M}, \text{Sub} \mathbb{H}(\mathcal{A}) \right) \quad \& \quad \left( \text{Fac} \mathbb{H}^{\text{op}}(\mathcal{A}), (\text{M-mod})^{\text{op}}, \text{Sub} \mathbb{H}^{\text{op}}(\text{Inj} \mathcal{A}) \right)$$
Remark 10.3. We give more explicit descriptions of the functors $\Pi^*, \Pi^! : \text{mod-}\mathcal{M} \to \text{mod-}\mathcal{M}$. Fix a projective presentation $\mathbb{H}(M^1) \to \mathbb{H}(M^0) \to F \to 0$ of $F \in \text{mod-}\mathcal{M}$.

- Let $P_{M^0} \to M^0$ be an epimorphism, where $P_{M^0}$ is projective. Then the induced map $M^1 \oplus P_{M^0} \to M^0$ is an epimorphism and it is easy to see that there is an exact sequence

$$\mathbb{H}(M^1 \oplus P_{M^0}) \to \mathbb{H}(M^0) \to \Pi^*(F) \to 0$$

If $K$ is defined by the exact sequence $0 \to K \to M^1 \oplus P_{M^0} \to M^0 \to 0$ and if $\mathcal{M}$ is 1-rigid, then clearly

$$\Pi^*(F) = R^1\mathbb{H}(K) = \mathbf{Ext}^1_{\mathcal{M}}(-, K)|_{\mathcal{M}}$$

In particular if $M \in \mathcal{M}$, then $\Pi^*\mathbb{H}(M) = \mathcal{M}(-, M)$ and if $\mathcal{M}$ is 1-rigid, then $\Pi^*\mathbb{H}(M) = R^1\mathbb{H}(\Omega M) \cong \mathcal{M}(-, M)$.

- Consider the exact sequence $0 \to A_F \to M^1 \to M^0 \to \mathbb{R}(F) \to 0$ and let $L = \text{Im}(M^1 \to M^0)$. By diagram chasing it is not difficult to see that there exists an exact sequence

$$0 \to \text{Coker}[\mathbb{H}(M^1) \to \mathbb{H}(L)] \to F \to \mathbb{H}\mathbb{R}(F)$$

$$\to \text{Ker}[R^1\mathbb{H}(L) \to R^1\mathbb{H}(M^0)] \to 0$$

Hence $\Pi^1(F) = \text{Coker}[\mathbb{H}(M^1) \to \mathbb{H}(L)]$. If $\mathcal{M}$ is 1-rigid, then clearly $\text{Coker}[\mathbb{H}(M^1) \to \mathbb{H}(L)] = R^1\mathbb{H}(A_F)$ and $\text{Ker}[R^1\mathbb{H}(L) \to R^1\mathbb{H}(M^0)] = R^1\mathbb{H}(L)$. Hence in this case we have an isomorphism and an exact sequence

$$\Pi^1(F) = R^1\mathbb{H}(A_F)$$

$$0 \to \Pi^1(F) \to F \to \mathbb{H}\mathbb{R}(F) \to R^1\mathbb{H}(L) \to 0$$

which describes the kernel and the cokernel of the unit of the adjoint pair $(\mathbb{R}, \mathbb{H})$. Note that if $\mathcal{M}$ is 2-rigid, then the exact sequence $0 \to A_F \to M^1 \to L \to 0$ shows that $R^1\mathbb{H}(L) \cong R^2\mathbb{H}(A_F) = \mathbf{Ext}^2_{\mathcal{M}}(-, A_F)|_{\mathcal{M}}$.

Remark 10.4. It is not difficult to see that for the left adjoint $T = L_0\mathbb{H} : \mathcal{A} \to \text{mod-}\mathcal{M}$ of the exact functor $\mathbb{R} : \text{mod-}\mathcal{M} \to \mathcal{A}$, we have the following:

$$L_mT : \mathcal{A} \to \text{mod-}\mathcal{M}, \quad L_mT(A) = \text{Hom}(-, \Omega^{m+1}(A))|_{\mathcal{M}}, \quad \forall m \geq 1$$

In particular $\text{Im} L_mT \subseteq \text{mod-}\mathcal{M}, \forall m \geq 1$. If $\mathcal{M}$ is 1-rigid, then $\mathbf{Ext}^1_{\mathcal{A}}(-, \Omega^{m+2}(?))_{\mathcal{M}} = R^1\mathbb{H}\Omega^{m+2}(?) \cong L_mT(?), \forall m \geq 1$, and there exists an exact sequence:

$$0 \to R^1\mathbb{H}(\Omega^2A) \to T(A) \to \mathbb{H}(A) \to R^1\mathbb{H}(\Omega A) \to 0$$
As a consequence $T$ is exact if and only if $L_1T = 0$ if and only if $L_1T(A) = \text{Hom}(-, \Omega^2 A)_M = 0$, $\forall A \in \mathcal{A}$, if and only if for any object $A \in \mathcal{A}$, any map $M \rightarrow \Omega^2 A$, where $M \in \mathcal{M}$, factorizes through a projective object. In particular $T$ is exact if $\text{gl.dim} \mathcal{A} \leq 2$. We leave to the reader to check that the natural map $T \rightarrow \mathbb{H}$ is a monomorphism when evaluated at objects $A$ with $\text{pd} A \leq 1$, and is an isomorphism if in addition $\text{Ext}^1(M, P) = 0$, $\forall M \in \mathcal{M}$ and $P \in \text{Proj} \mathcal{A}$, for instance if $\mathcal{M}$ is 1-rigid.

Hence if $\mathcal{M}$ is functorially finite and $\text{Proj} \mathcal{A} \subseteq \mathcal{M} \supseteq \text{Inj} \mathcal{A}$, then there are identifications of TTF classes:

\[
\text{mod-} \mathcal{M} = \text{Tors}(\text{mod-} \mathcal{M}) \quad \& \quad \text{mod-} \mathcal{M} = \text{Tors}(\text{M-mod}) \tag{10.1}
\]

where

\[
\text{Tors}(\text{mod-} \mathcal{M}) = \{ F \in \text{mod-} \mathcal{M} \mid \text{Hom}(F, \mathbb{H}(\mathcal{M})) = 0 \} \quad \& \quad \text{Tors}(\text{M-mod}) = \{ F \in \text{M-mod} \mid \text{Hom}(F, \mathbb{H}^{\text{op}}(\mathcal{M})) = 0 \}
\]

are the full subcategories of $\text{mod-} \mathcal{M}$ and $\text{M-mod}$ respectively consisting of the torsion functors, i.e. those functors annihilated by the $\mathcal{M}$-dual functors $\text{Hom}(-, \mathbb{H}(\mathcal{M}))$ and/or $\text{Hom}(-, \mathbb{H}^{\text{op}}(\mathcal{M}))$, $\forall M \in \mathcal{M}$.

**Definition 10.5.** Let $\mathcal{A}$ be an abelian category with enough projective and injective objects and $\mathcal{M}$ an $n$-cluster tilting subcategory of $\mathcal{A}$. The TTF-class $\text{mod-} \mathcal{M}$ in the Auslander category $\text{mod-} \mathcal{M}$ is called the stable $n$-Auslander category of $\mathcal{A}$ or $\mathcal{M}$, and the TTF-class $\text{mod-} \mathcal{M}$ in the coAuslander category $\text{M-mod}$ is called the stable $n$-coAuslander category of $\mathcal{A}$ or $\mathcal{M}$.

For instance if $M$ is an $n$-cluster tilting module over an Artin algebra $\Lambda$, then the stable Auslander category $\text{mod-} \mathcal{M}$, where $\mathcal{M} = \text{add} M$, is the category $\text{mod-} \text{End}_\Lambda(M)$ of finitely generated modules over the stable $n$-Auslander algebra $\text{End}_\Lambda(M)$ and dually $(\text{M-mod})^{\text{op}} \approx \text{mod-} \text{End}_\Lambda(M)^{\text{op}}$.

To proceed further we shall need the following useful result which shows that there is a duality between the stable $n$-Auslander category and the stable $n$-coAuslander category. This result is due to Iyama and proved in [39, Theorem 3.6.1] in a more restrictive context, see also Buchweitz’s treatment in [26] for a special case and [34, Section 3]. For the convenience of the reader we give the proof in our context.

**Proposition 10.6.** Let $\mathcal{A}$ be an abelian category with enough projective and injective objects and let $\mathcal{M}$ be an $n$-cluster tilting subcategory of $\mathcal{A}$. Then the adjoint on the right pair $(d_r, d_t)$ of $\mathcal{M}$-dual functors

\[
d_r = \text{Hom}(-, \mathbb{H}(-)) : \text{mod-} \mathcal{M} \rightarrow \text{M-mod} \quad \& \quad d_t = \text{Hom}(-, \mathbb{H}^{\text{op}}(-)) : \text{M-mod} \rightarrow \text{mod-} \mathcal{M}
\]
induce quasi-inverse dualities between the stable $n$-(co)Auslander categories:

$$R^{n+2}d_r : \text{mod-}\mathcal{M} \xrightarrow{\cong} (\text{mod-}\mathcal{M})^{\text{op}} : R^{n+2}d_l$$

**Proof.** Let $\mathbb{H}(M^1) \to \mathbb{H}(M^0) \to F \to 0$ be a presentation of $F \in \text{mod-}\mathcal{M}$, where the sequence $0 \to A_F \to M^1 \to M^0 \to 0$ is exact in $\mathcal{A}$. Since $\mathcal{M}$ is $n$-cluster tilting, there exists an $\mathcal{M}$-resolution $0 \to M^n_{A_F} \to M^{n-1}_{A_F} \to \cdots \to M^0_{A_F} \to A_F \to 0$ of $A_F$. Applying $\mathbb{H}$ we have a deleted projective resolution

$$0 \to \mathbb{H}(M^n_{A_F}) \to \mathbb{H}(M^{n-1}_{A_F}) \to \cdots$$

$$\to \mathbb{H}(M^0_{A_F}) \to \mathbb{H}(M^1) \to \mathbb{H}(M^0) \to 0$$

of $F$. Applying $d_r$, and using that $d_r \mathbb{H} = \mathbb{H}^{\text{op}}$, we have a complex

$$0 \to \mathbb{H}^{\text{op}}(M^0) \to \mathbb{H}^{\text{op}}(M^1) \to \mathbb{H}^{\text{op}}(M^0_{A_F}) \to \cdots$$

$$\to \mathbb{H}^{\text{op}}(M^{n-1}_{A_F}) \to \mathbb{H}^{\text{op}}(M^n_{A_F}) \to 0$$

whose cohomology gives the derived functors $R^k d_r (F)$. Since $\mathcal{M}$ is $n$-rigid and $d_r (F) = 0$, it follows that the above complex is acyclic everywhere except in the last position on the right which corresponds to $(n+2)$-degree. Hence the above complex is a deleted projective resolution of $R^{n+2}d_r (F)$. This functor is coherent and lies in $\overline{\mathcal{M}}$-mod since it has a presentation $\mathbb{H}^{\text{op}}(M^{n-1}_{A_F}) \to \mathbb{H}^{\text{op}}(M^n_{A_F}) \to R^{n+2}d_r (F) \to 0$ where the map $M^n_{A_F} \to M^{n-1}_{A_F}$ is a monomorphism. Hence we have a contravariant functor $R^{n+2}d_r : \text{mod-}\mathcal{M} \to \overline{\mathcal{M}}$-mod. Applying $d_l$ to the deleted projective resolution of $R^{n+2}d_r (F)$ constructed above, we obtain the original deleted projective resolution of $F$ and this means that we have a natural isomorphism $F \cong R^{n+2}d_l R^{n+2}d_r (F)$. By symmetry we have a natural isomorphism $G \cong R^{n+2}d_r R^{n+2}d_l (G)$, $\forall G \in \overline{\mathcal{M}}$-mod. Hence $R^{n+2}d_r$ is a duality with quasi-inverse $R^{n+2}d_l$. □

In the sequel we shall use the fact that if $\mathcal{M}$ is $n$-rigid and $\text{Proj} \ A \subseteq \mathcal{M} \supseteq \text{Inj} \ A$, then we have in particular $\text{Ext}^k_{\mathcal{A}} (M, P) = 0 = \text{Ext}^k_{\mathcal{A}} (I, M), 1 \leq k \leq n$ for $M \in \mathcal{M}, P \in \text{Proj} \ A$ and $I \in \text{Inj} \ A$. As a consequence of [4, Proposition 2.46] and its dual, we have canonical isomorphisms, $\forall M \in \mathcal{M}$ and $\forall X \in \mathcal{A}$:

$$\text{Ext}^k_{\mathcal{A}} (M, X) \xrightarrow{\cong} \mathcal{A}(\Omega^k M, X) \quad \&$$

$$\mathcal{A}(M, X) \xrightarrow{\cong} \mathcal{A}(\Omega^k M, \Omega^k X), \quad 1 \leq k \leq n \quad (10.2)$$

$$\text{Ext}^k_{\mathcal{A}} (X, M) \xrightarrow{\cong} \mathcal{A}(X, \Sigma^k M) \quad \&$$

$$\mathcal{A}(X, M) \xrightarrow{\cong} \mathcal{A}(\Sigma^k X, \Sigma^k M), \quad 1 \leq k \leq n \quad (10.3)$$

On the other hand if $\mathcal{M}$ is an $n$-cluster tilting subcategory of $\mathcal{A}$, then any object $A \in \mathcal{A}$ has an $\mathcal{M}$-resolution and an $\mathcal{M}$-coresolution of length $\leq n$:
10.2. The subcategories \( \hat{\mathcal{M}} \leq 1 \) and \( \hat{\mathcal{M}} \leq 1 \)

Let \( \mathcal{M} \) be a functorially finite 1-rigid subcategory of \( \mathcal{A} \). We analyze the structure of the full subcategories \( \hat{\mathcal{M}} \leq 1 \) and \( \hat{\mathcal{M}} \leq 1 \) of \( \mathcal{A} \) appearing in Propositions 9.2 and 9.3 in connection with the categories \( \text{mod}\mathcal{M} \) and \( \overline{\mathcal{M}}\text{-mod} \).

Recall that a functor \( F \) in \( \text{mod}\mathcal{M} \), resp. in \( \mathcal{M}\text{-mod} \), is called torsion, if \( d_r(F) = 0 \), resp. \( d_l(F) = 0 \). We denote by \( \text{Tors}(\text{mod}\mathcal{M}) \), resp. \( \text{Tors}(\mathcal{M}\text{-mod}) \), the full subcategory of \( \text{mod}\mathcal{M} \), resp. \( \mathcal{M}\text{-mod} \), consisting of the torsion functors. By an observation of Auslander and Reiten, see [7, Lemma 4.1], and using that any map between torsion functors which factorizes through a projective functor is clearly zero, it follows that the Auslander–Bridger transpose duality functors \( \text{Tr}: \mathcal{M}\text{-mod} \to \text{mod}\mathcal{M} \) and \( \text{Tr}: \text{mod}\mathcal{M} \to \mathcal{M}\text{-mod} \), see Subsection 7.1, induce dualities

\[
\text{Proj}^{\leq 1}\mathcal{M}\text{-mod} \cong \text{Tors}(\text{mod}\mathcal{M})^{\text{op}} \quad \& \quad \text{Proj}^{\leq 1}\text{mod}\mathcal{M} \cong \text{Tors}(\mathcal{M}\text{-mod})^{\text{op}} \tag{10.6}
\]

Combining (10.1) and (10.6), we have dualities

\[
(\text{Proj}^{\leq 1}\mathcal{M}\text{-mod})^{\text{op}} \cong \text{mod}\mathcal{M} \quad \& \quad (\text{Proj}^{\leq 1}\text{mod}\mathcal{M})^{\text{op}} \cong \overline{\mathcal{M}}\text{-mod} \tag{10.7}
\]

Next observe that the functors \( \mathbb{H}^{\text{op}} \) and \( \mathbb{H} \) clearly induce equivalences \( \hat{\mathcal{M}} \leq 1 \cong \text{Proj}^{\leq 1}\mathcal{M}\text{-mod}^{\text{op}} \) and \( \hat{\mathcal{M}} \leq 1 \cong \text{Proj}^{\leq 1}\text{mod}\mathcal{M} \) respectively, which in turn induce equivalences

\[
\hat{\mathcal{M}} \leq 1 \cong \text{Proj}^{\leq 1}\mathcal{M}\text{-mod}^{\text{op}} \quad \& \quad \hat{\mathcal{M}} \leq 1 \cong \text{Proj}^{\leq 1}\text{mod}\mathcal{M} \tag{10.8}
\]

Summarizing the above observations, we have the following result, part (i) of which gives an alternative description of the stable \( n \)-Auslander categories. Note also that part (i) gives, in our context, a different and shorter proof of a result due to Demonet and Liu, see [30, Theorem 3.2] in the setting of exact categories, and part (ii) follows from (i) and Proposition 10.6.

**Theorem 10.7.** Let \( \mathcal{A} \) be an abelian category with enough projective and injective objects and \( \mathcal{M} \) a functorially finite \( n \)-rigid subcategory of \( \mathcal{A} \) containing the projectives and the injectives.

(i) The functors \( \mathbb{H}: \mathcal{A} \to \text{mod}\mathcal{M} \) and \( \mathbb{H}^{\text{op}}: \mathcal{A} \to (\mathcal{M}\text{-mod})^{\text{op}} \) and the duality \( \text{Tr}: \text{mod}\mathcal{M} \to \mathcal{M}\text{-mod} \) induce equivalences:

\[
0 \to M_A^n \to M_A^{n-1} \to \cdots \to M_A^1 \to M_A^0 \to A \to 0 \tag{10.4}
\]

\[
0 \to A \to M_A^0 \to M_A^1 \to \cdots \to M_A^{n-1} \to M_A^n \to 0 \tag{10.5}
\]
\[
\begin{align*}
\text{Tr} \circ \mathbb{H}^{\text{op}} : \hat{\mathcal{M}}_{\leq 1}^{\text{op}} / \mathcal{M} & \xrightarrow{\approx} \text{mod-} \mathcal{M} \quad & \& \\
\text{Tr} \circ \mathbb{H} : \hat{\mathcal{M}}_{n-1}^{\perp} / \mathcal{M} & \xrightarrow{\approx} (\overline{\mathcal{M}} \text{-mod})^{\text{op}}
\end{align*}
\]

In particular the category $\hat{\mathcal{M}}_{\leq 1} / \mathcal{M}$ is abelian with enough projectives and the category $\hat{\mathcal{M}}_{\leq 1} / \mathcal{M}$ is abelian with enough injectives.

(ii) If $\mathcal{M}$ is an $n$-cluster tilting subcategory of $\mathcal{A}$, then there are equivalences

\[
\mathbb{R} \circ \text{Tr} \circ \mathbb{R}^{n+2} d_r \circ \text{Tr} \circ \mathbb{H}^{\text{op}} : \frac{\hat{\mathcal{M}}_{\leq 1}}{\mathcal{M}} = \frac{\hat{\mathcal{M}}_{n-1}^{\perp}}{\mathcal{M}} \xrightarrow{\approx} \mathbb{R}^{\text{op}} \circ \text{Tr} \circ \mathbb{R}^{n+2} d_r \circ \text{Tr} \circ \mathbb{H}
\]

and the involved abelian categories have enough projectives and enough injectives.

It should be noted that in case $\mathcal{M}$ is an $n$-cluster tilting subcategory of $\mathcal{A}$, then the equivalences of Theorem 10.7 can be suitably generalized to include the subcategories $\mathcal{M}^{\leq t}$ and $\hat{\mathcal{M}}_{\leq t}$, for $2 \leq t \leq n$, by using, in the Frobenius case, Proposition 2.13 and [44, Theorem 6.3]. Details are left to the interested reader.

10.3. Projective objects in $\text{mod-} \mathcal{M}$ and $\overline{\mathcal{M}} \text{-mod}$

Let $\mathcal{M}$ be a functorially finite 1-rigid subcategory of $\mathcal{A}$. We are interested in the description of the projective objects of $\text{mod-} \mathcal{M}$, resp. $\overline{\mathcal{M}} \text{-mod}$, inside the category $\text{mod-} \mathcal{M}$, resp. $\overline{\mathcal{M}} \text{-mod}$.

Let $F \in \text{mod-} \mathcal{M}$ with presentation $\mathbb{H}(M^1) \rightarrow \mathbb{H}(M^0) \rightarrow F \rightarrow 0$, where the map $M^1 \rightarrow M^0$ is an epimorphism in $\mathcal{M}$. Applying the left exact functor $\mathbb{H}$ to the short exact sequence $0 \rightarrow A_F \rightarrow M^1 \rightarrow M^0 \rightarrow 0$ and using that $\mathcal{M}$ is 1-rigid, so that $R^1\mathbb{H}(\mathcal{M}) = \text{Ext}^1(\mathcal{M}^\perp, \mathcal{M})|_{\mathcal{M}} = 0$, we have the exact sequence

\[
0 \rightarrow \mathbb{H}(A_F) \rightarrow \mathbb{H}(M^1) \rightarrow \mathbb{H}(M^0) \rightarrow R^1\mathbb{H}(A_F) \rightarrow 0 \tag{10.9}
\]

and therefore $F = R^1\mathbb{H}(A_F)$. Hence any object $F \in \text{mod-} \mathcal{M}$ is of the form $R^1\mathbb{H}(A_F)$ for some object $A_F$ which is a kernel of an epimorphism in $\mathcal{M}$. Now let $M \in \mathcal{M}$ and let $0 \rightarrow \Omega M \rightarrow P_M \rightarrow M \rightarrow 0$ be exact, where $P_M$ is projective. Then we have the following exact sequence

\[
0 \rightarrow \mathbb{H}(\Omega M) \rightarrow \mathbb{H}(P_M) \rightarrow \mathbb{H}(M) \rightarrow R^1\mathbb{H}(\Omega M) \rightarrow 0 \tag{10.10}
\]

showing that the functor $R^1\mathbb{H}(\Omega M)$ lies in $\text{mod-} \mathcal{M}$ and, using (10.2), we have isomorphisms
\[ R^1 \mathbb{H}(\Omega M) = \text{Ext}^1(-, \Omega M)|_\mathcal{M} \cong \mathcal{A}(\Omega(-), \Omega M)|_\mathcal{M} \cong \mathcal{A}(-, M)|_\mathcal{M} = \mathcal{M}(-, M) \]

showing that the functor \( R^1 \mathbb{H}(\Omega M) \) is projective in \( \text{mod-} \mathcal{M} \) and any projective in \( \text{mod-} \mathcal{M} \) is of this form. Note that by Remark 10.3 we have \( \Pi^* \mathbb{H}(M) = R^1 \mathbb{H}(\Omega M), \forall M \in \mathcal{M}. \)

Let \( F = R^1 \mathbb{H}(A_F) \) be an object in \( \text{mod-} \mathcal{M} \). Let \( 0 \rightarrow \Omega M^0 \rightarrow P_{M^0} \rightarrow M^0 \rightarrow 0 \) be an exact sequence, where \( P_{M^0} \) is projective, and consider the pull-back short exact sequence \( 0 \rightarrow \Omega M^1 \rightarrow \Omega M^0 \rightarrow A_F \rightarrow 0 \) of the projective presentation of \( M^0 \) along the monomorphism \( A_F \rightarrow M^0 \). Then applying the left exact functor \( \mathbb{H} \) to this short exact sequence we have a long exact sequence:

\[
0 \rightarrow \mathbb{H}(\Omega M^1) \rightarrow \mathbb{H}(\Omega M^0) \rightarrow \mathbb{H}(A_F) \rightarrow R^1 \mathbb{H}(\Omega M^1) \\
\rightarrow R^1 \mathbb{H}(\Omega M^0) \rightarrow R^1 \mathbb{H}(A_F) \rightarrow R^2 \mathbb{H}(\Omega M^1) \rightarrow \cdots \tag{10.11}
\]

Clearly the map \( R^1 \mathbb{H}(A_F) \rightarrow R^2 \mathbb{H}(\Omega M^1) \) factorizes through \( R^1 \mathbb{H}(M^1) = \text{Ext}^1(-, M^1)|_\mathcal{M} \) which is zero since \( \mathcal{M} \) is 1-rigid. Therefore (10.11) becomes an exact sequence

\[
0 \rightarrow \mathbb{H}(\Omega M^1) \rightarrow \mathbb{H}(\Omega M^0) \rightarrow \mathbb{H}(A_F) \\
\rightarrow R^1 \mathbb{H}(\Omega M^1) \rightarrow R^1 \mathbb{H}(\Omega M^0) \rightarrow R^1 \mathbb{H}(A_F) \rightarrow 0 \tag{10.12}
\]

and as noted above the objects \( R^1 \mathbb{H}(\Omega M^i) \) are projective in \( \text{mod-} \mathcal{M} \). This proves the following.

**Lemma 10.8.** The abelian category \( \text{mod-} \mathcal{M} \) has enough projectives and:

\[
\text{Proj } \text{mod-} \mathcal{M} = \{ \text{Ext}^1_{\mathcal{A}}(-, \Omega M)|_\mathcal{M} \mid M \in \mathcal{M} \} \\
= \{ R^1 \mathbb{H}(\Omega M) \mid M \in \mathcal{M} \} = \{ \mathcal{M}(-, M)|_\mathcal{M} \mid M \in \mathcal{M} \} = \Pi^* \mathbb{H}(\mathcal{M})
\]

If \( F \in \text{mod-} \mathcal{M} \) and \( \mathbb{H}(M^1) \rightarrow \mathbb{H}(M^0) \rightarrow F \rightarrow 0 \) is a finite projective presentation of \( F \) in \( \text{mod-} \mathcal{M} \), where the sequence \( 0 \rightarrow A_F \rightarrow M^1 \rightarrow M^0 \rightarrow 0 \) is exact in \( \mathcal{A} \), then the exact sequence

\[ R^1 \mathbb{H}(\Omega M^1) \rightarrow R^1 \mathbb{H}(\Omega M^0) \rightarrow F \rightarrow 0 \]

is a projective presentation of \( F \) in \( \text{mod-} \mathcal{M} \).

Working with covariant coherent functors, by duality we have the following

**Lemma 10.9.** The abelian category \( \overline{\mathcal{M}}\text{-mod} \) has enough projectives and:

\[
\text{Proj } \overline{\mathcal{M}}\text{-mod} = \{ \text{Ext}^1_{\mathcal{A}^\text{op}}(\Sigma M, -)|_\mathcal{M} \mid M \in \mathcal{M} \} \\
= \{ R^1 \mathbb{H}^\text{op}(\Sigma M) \mid M \in \mathcal{M} \} = \{ \overline{\mathcal{M}}(M, -)|_\mathcal{M} \mid M \in \mathcal{M} \}
\]
If $F \in \overline{\mathcal{M}}$-mod and $H^{op}(M_0) \rightarrow H^{op}(M_1) \rightarrow F \rightarrow 0$ is a projective presentation of $F$ in $\mathcal{M}$-mod, where the sequence $0 \rightarrow M_1 \rightarrow M_0 \rightarrow A^F \rightarrow 0$ is exact in $\mathcal{A}$, then the exact sequence

$$R^1H^{op}(\Sigma M_0) \rightarrow R^1H^{op}(\Sigma M_1) \rightarrow F \rightarrow 0$$

is a projective presentation of $F = R^1H^{op}(A^F)$ in $\overline{\mathcal{M}}$-mod.

10.4. Injective objects in mod-$\mathcal{M}$ and in $\overline{\mathcal{M}}$-mod

From now on and until the rest of this section, we assume that $\mathcal{M}$ is an $n$-cluster tilting subcategory of $\mathcal{A}$. We are interested in the description of the injective objects of mod-$\mathcal{M}$, resp. $\overline{\mathcal{M}}$-mod, inside the category mod-$\mathcal{M}$, resp. $\overline{\mathcal{M}}$-mod.

Let $M$ be in $\mathcal{M}$ and let $0 \rightarrow M \rightarrow I_M \rightarrow \Sigma M \rightarrow 0$ be exact, where $I_M$ is injective. Consider, as in (10.5), an $\mathcal{M}$-coresolution $0 \rightarrow \Sigma M \rightarrow M_0^{\Sigma M} \rightarrow M_1^{\Sigma M} \rightarrow \cdots \rightarrow M_n^{\Sigma M} \rightarrow 0$ of $\Sigma M$, i.e. each $M_k^{\Sigma M}$ is a left $\mathcal{M}$-approximation of $\text{Im}(M_{k-1}^{\Sigma M} \rightarrow M_k^{\Sigma M})$. Hence we have an exact sequence

$$0 \rightarrow M \rightarrow I_M \rightarrow M_0^{\Sigma M} \rightarrow M_1^{\Sigma M} \rightarrow \cdots \rightarrow M_n^{\Sigma M} \rightarrow 0 \quad (10.13)$$

Applying the left exact functor $\mathbb{H}$ and using that $\mathcal{M}$ is $n$-rigid, it follows that we have an exact sequence

$$0 \rightarrow \mathbb{H}(M) \rightarrow \mathbb{H}(I_M) \rightarrow \mathbb{H}(M_0^{\Sigma M}) \rightarrow \cdots \rightarrow \mathbb{H}(M_n^{\Sigma M}) \rightarrow R^{n+1}\mathbb{H}(M) \rightarrow 0 \quad (10.14)$$

which is a projective resolution of $R^{n+1}\mathbb{H}(M)$ in mod-$\mathcal{M}$. Since the map $M_n^{\Sigma M} \rightarrow M_n^{\Sigma M}$ is an epimorphism, we infer that $R^{n+1}\mathbb{H}(M)$ lies in mod-$\mathcal{M}$. We show that this functor is injective in mod-$\mathcal{M}$, by invoking Proposition 10.6. Indeed by Proposition 10.6 it follows that:

- The functor $R^{n+2}_d: \text{Inj mod-}\mathcal{M} \rightarrow \text{Proj } \overline{\mathcal{M}}$-mod is an equivalence with quasi-inverse $R^{n+2}_d$.
- The functor $R^{n+2}_d: \text{Inj } \overline{\mathcal{M}}$-mod $\rightarrow \text{Proj mod-}\overline{\mathcal{M}}$ is an equivalence with quasi-inverse $R^{n+2}_d$.

Therefore, using Lemma 10.9, any injective object of mod-$\mathcal{M}$ is of the form $R^{n+2}_dR^1\mathbb{H}^{op}(\Sigma M)$, $M \in \mathcal{M}$. In order to compute $R^{n+2}_dR^1\mathbb{H}^{op}(\Sigma M)$, we apply $\mathbb{H}^{op}$ to the exact sequence (10.13) to obtain the complex

$$0 \rightarrow \mathbb{H}^{op}(M_n^{\Sigma M}) \rightarrow \mathbb{H}^{op}(M_{n-1}^{\Sigma M}) \rightarrow \cdots \rightarrow \mathbb{H}^{op}(M_0^{\Sigma M}) \rightarrow \mathbb{H}^{op}(I_M) \rightarrow \mathbb{H}^{op}(M) \rightarrow 0$$
which is a deleted projective resolution of $R^1\mathbb{H}^{\text{op}}(\Sigma M)$. Applying $d_i$ we obtain the exact complex (10.14), so

$$R^{n+2}d_i(R^1\mathbb{H}^{\text{op}}(\Sigma M)) \cong R^{n+1}\mathbb{H}(M)$$

This proves the following result.

**Lemma 10.10.** The abelian category $\text{mod-}\mathcal{M}$ has enough injectives and

$$\text{Inj mod-}\mathcal{M} = \{\text{Ext}^n_{\mathcal{A}}(\_ , M)|_\mathcal{M} \mid M \in \mathcal{M}\} = \{R^n\mathbb{H}(M) \mid M \in \mathcal{M}\} = \{\mathcal{M}(\Omega^n(\_ ), M)|_\mathcal{M} \mid M \in \mathcal{M}\}$$

If $F \in \text{mod-}\mathcal{M}$ and $\mathbb{H}(M^1) \to \mathbb{H}(M^0) \to F \to 0$ is a projective presentation of $F$ in $\text{mod-}\mathcal{M}$, where the sequence $0 \to A_F \to M_1 \to M_0 \to 0$ is exact in $\mathcal{A}$, then the exact sequence

$$0 \to F \to R^{n+1}\mathbb{H}(M^n_{A_F}) \to R^{n+1}\mathbb{H}(M^{n-1}_{A_F})$$

is an injective copresentation of $F = R^1\mathbb{H}(A_F)$ in $\text{mod-}\mathcal{M}$.

By duality we have the following.

**Lemma 10.11.** The abelian category $\overline{\mathcal{M}}\text{-mod}$ has enough injectives and

$$\text{Inj } \overline{\mathcal{M}}\text{-mod} = \{\text{Ext}^n_{\mathcal{A}}(M, \_)|_\mathcal{M} \mid M \in \overline{\mathcal{M}}\} = \{R^n\mathbb{H}^{\text{op}}(M) \mid M \in \mathcal{M}\} = \{\overline{\mathcal{M}}(\Sigma^n(\_ ), M)|_\mathcal{M} \mid M \in \mathcal{M}\}$$

If $F \in \overline{\mathcal{M}}\text{-mod}$ and $\mathbb{H}^{\text{op}}(M_0) \to \mathbb{H}^{\text{op}}(M_1) \to F \to 0$ is a projective presentation of $F$ in $\overline{\mathcal{M}}\text{-mod}$, where the sequence $0 \to M_1 \to M_0 \to A^F \to 0$ is exact in $\mathcal{A}$, then the exact sequence

$$0 \to F \to R^{n+1}\mathbb{H}^{\text{op}}(M^{A_F}_n) \to R^{n+1}\mathbb{H}^{\text{op}}(M^{A_F}_{n-1})$$

is an injective copresentation of $F = R^1\mathbb{H}^{\text{op}}(A^F)$ in $\overline{\mathcal{M}}\text{-mod}$.

10.5. $\text{mod-}\mathcal{M}$ as an Auslander category

Let $M \in \mathcal{M}$ and $0 \to M \to I^0 \to \cdots \to I^{n+1} \to \cdots$ be an injective resolution of $M$ in $\mathcal{A}$. Since $\mathcal{M}$ is $n$-rigid, the sequence $0 \to \mathbb{H}(I) \to \mathbb{H}(I^0) \to \cdots \to \mathbb{H}(I^n) \to \mathbb{H}(I^{n+1})$ is exact. We set $E_M := \text{Coker}(\mathbb{H}(I^n) \to \mathbb{H}(I^{n+1}))$, so $E_M = \Sigma^{n+2}\mathbb{H}(M)$, and we have an exact sequence

$$0 \to \mathbb{H}(M) \to \mathbb{H}(I^0) \to \cdots \to \mathbb{H}(I^n) \to \mathbb{H}(I^{n+1}) \to E_M \to 0 \ (10.15)$$
Considering the full subcategory $\mathcal{E}_M = \{E_M \in \text{mod-}\mathcal{M} \mid M \in \mathcal{M}\}$ of $\text{mod-}\mathcal{M}$ consisting of all objects $E_M$ arising in this way, we have inclusions:

$$\mathcal{E}_M \subseteq \mathcal{E}_M \oplus \mathbb{H}(\text{Inj} \mathcal{A}) \subseteq \text{Inj} \mathcal{B}, \quad \text{where} \quad \mathcal{B} = \text{mod-}\mathcal{M}$$

Indeed let $E_M$ be in $\mathcal{E}_M$, so there is an $M \in \mathcal{M}$ and an exact sequence (10.15). Since each $\mathbb{H}(I^k)$ is injective and $\text{gl.dim} \mathcal{B} \leq n + 2$, it follows that $E_M$ is injective in $\mathcal{B}$. Hence $\text{add}\{\mathbb{H}(I) \oplus E_M \mid I \in \text{Inj} \mathcal{B} \& E_M \in \mathcal{E}_M\} := \mathcal{E}_M \oplus \mathbb{H}(\text{Inj} \mathcal{A}) \subseteq \text{Inj} \mathcal{B}$ since $\mathbb{H}(I)$ is projective–injective in $\mathcal{B}$, $\forall I \in \text{Inj} \mathcal{A}$.

**Remark 10.12.** 1. If $\mathcal{B} = \text{mod-}\mathcal{M}$ is an $n$-Auslander category and $\mathcal{B}$ has enough injectives, then using that $\text{dom.dim} \mathcal{B} \geq m$ if and only if $\text{codom.dim} \mathcal{B} \geq m$, see [38, Lemma 2.1], it follows that $\text{Inj} \mathcal{B} = \mathcal{E}_M \oplus \mathbb{H}(\text{Inj} \mathcal{A})$.

Indeed let $E$ be an injective object of $\text{mod-}\mathcal{M}$ and assume that $E$ is non-projective, so $E \notin \text{Im} \mathbb{H}$. Since $\text{dom.dim} \mathcal{B} \geq n + 2$, by Remark 8.20 it follows that $\text{dom.dim} \mathcal{B} \geq n + 2$, and this means that there exists an exact sequence $P^{n+1} \rightarrow P^n \rightarrow \cdots \rightarrow P^1 \rightarrow P^0 \rightarrow E \rightarrow 0$ in $\mathcal{B}$, where each $P^k$ is projective–injective, hence of the form $\mathbb{H}(I^k)$ for some $I^k \in \text{Inj} \mathcal{A}$. Since $\text{gl.dim} \mathcal{B} \leq n + 2$, the kernel of $P^{n+1} \rightarrow P^n$ is projective, hence of the form $\mathbb{H}(M)$, for some $M \in \mathcal{M}$. Summarizing we have an exact sequence

$$0 \rightarrow \mathbb{H}(M) \rightarrow \mathbb{H}(I^{n+1}) \rightarrow \cdots \rightarrow \mathbb{H}(I^1) \rightarrow \mathbb{H}(I^0) \rightarrow E \rightarrow 0$$

Since $\mathbb{H}$ is fully faithful and $M$ is $n$-rigid, we have that the sequence

$$0 \rightarrow M \rightarrow I^{n+1} \rightarrow \cdots \rightarrow I^1 \rightarrow I^0$$

is exact. This means that $E = E_M$ for $M \in \mathcal{M}$ and therefore $E \in \mathcal{E}_M$. On the other hand if $E$ is projective–injective, then $E = \mathbb{H}(I)$ for some $I \in \text{Inj} \mathcal{A}$. We infer that $\text{Inj} \mathcal{B} = \mathcal{E}_M \oplus \mathbb{H}(\text{Inj} \mathcal{A})$.

2. Assume that $\mathcal{B} = \text{mod-}\mathcal{M}$ has injective envelopes. This is the case, for instance, if $\mathcal{A}$ is cocomplete and $\mathcal{M}$ consists of self-compact objects since then $\text{mod-}\mathcal{M} \approx \text{Mod-}\mathcal{M}$. Since $\text{mod-}\mathcal{M}$ is a localizing subcategory of $\text{mod-}\mathcal{M}$ by [54] it follows that any injective object of $\text{mod-}\mathcal{M}$ is of the form $E = E_1 \oplus E_2$ where $E_1$ is the injective envelope $E(F)$ of the largest subobject $F$ of $E$ which lies in $\text{mod-}\mathcal{M}$ and $E_2$ lies in $\text{Sub} \mathbb{H}(\mathcal{A})$. However the largest subobject of $E$ lying in $\text{mod-}\mathcal{M}$ is the functor $\Pi^1(E)$ which is injective in $\text{mod-}\mathcal{M}$ since $\Pi^1$ preserves injectives as a right adjoint of the exact functor $\Pi_*$. Then Lemma 10.10 gives $\Pi^1(E) = R^{n+1}\mathbb{H}(M)$ for some $M \in \mathcal{M}$ and then $E_1 = E(R^{n+1}\mathbb{H}(M))$. Since clearly $E_2$ is of the form $\mathbb{H}(I)$ for some injective object $I$ of $\mathcal{A}$ we infer that

$$\text{Inj} \text{mod-}\mathcal{M} = E(R^{n+1}\mathbb{H}(M)) \oplus \mathbb{H}(\text{Inj} \mathcal{A}) = \{E(R^{n+1}\mathbb{H}(M)) \oplus \mathbb{H}(I) \in \text{mod-}\mathcal{M} \mid M \in \mathcal{M} \& I \in \text{Inj} \mathcal{A}\}$$

In particular if the hereditary torsion pair $(\text{mod-}\mathcal{M}, \text{Sub}(\mathcal{A}))$ is stable, that is $\text{mod-}\mathcal{M}$ is closed under injective envelopes in $\text{mod-}\mathcal{M}$, then $\text{Inj} \text{mod-}\mathcal{M} = R^{n+1}\mathbb{H}(M) \oplus \mathbb{H}(\text{Inj} \mathcal{A})$. 
The following generalizes a basic result of Iyama, see [42, Proposition 4.2].

**Theorem 10.13.** The map \( \{\mathcal{A}, M\} \rightarrow \text{mod-}M \) gives a bijection between (cluster) equivalence classes of:

(i) Pairs \( \{\mathcal{A}, M\} \) where \( \mathcal{A} \) is an abelian category with \( \text{gl.dim} \mathcal{A} \leq n + 1 \) and \( M \) is an \( n \)-cluster tilting subcategory of \( \mathcal{A} \).

(ii) Auslander categories \( \mathcal{B} \) such that

\[
\text{Ext}^k_{\mathcal{A}}(E, P) = 0, \quad 1 \leq k \leq n + 1, \quad \forall E \in \mathcal{E}_M \quad \& \quad \forall P \in \text{Proj} \mathcal{B}
\]

Under this bijection the \( n \)-Auslander category \( \mathcal{B} = \text{mod-}M \) has enough injectives, \( \mathcal{E}_M = \mathbb{R}^{n+1}\mathbb{H}(M) \), and

\[
\text{Inj} \mathcal{B} = \mathbb{R}^{n+1}\mathbb{H}(M) \oplus \mathbb{H}(\text{Inj} \mathcal{A})
\]

**Proof.** By Theorem 8.23, Lemmas 10.8 and 10.10, in order to establish the bijective correspondence between (i) and (ii), it suffices to show that if \( M \) is an \( n \)-cluster tilting subcategory of \( \mathcal{A} \), then, \( \forall M, M' \in M \):

\[
\text{gl.dim} \mathcal{A} \leq n + 1 \iff \text{Ext}^k(E_M, \mathbb{H}(M')) = 0, \quad 1 \leq k \leq n + 1
\]

“\( \Rightarrow \)” Assume that \( \text{gl.dim} \mathcal{A} \leq n + 1 \). Let \( M \in M \) and consider an injective resolution of \( M \):

\[
0 \rightarrow M \rightarrow I^0 \rightarrow I^1 \rightarrow \cdots \rightarrow I^n \rightarrow I^{n+1} \rightarrow 0
\]

Applying the functor \( \mathbb{H} \) and using that \( M \) is \( n \)-rigid, we have the following exact complex

\[
0 \rightarrow \mathbb{H}(M) \rightarrow \mathbb{H}(I^0) \rightarrow \mathbb{H}(I^1) \rightarrow \cdots \rightarrow \mathbb{H}(I^n) \rightarrow \mathbb{H}(I^{n+1}) \rightarrow E_M \rightarrow 0 \quad (10.16)
\]

which is a projective resolution of \( E_M \) in \( \text{mod-}M \). Note that the object \( E_M \) is torsion since the map \( I^n \rightarrow I^{n+1} \) is an epimorphism, in fact \( E_M \cong \mathbb{R}^{n+1}\mathbb{H}(M) \). Applying to this resolution the functor \( \text{Hom}(\cdot, \mathbb{H}(M')) \), \( M' \in M \), we have the following complex

\[
0 \rightarrow (\mathbb{H}(I^{n+1}), \mathbb{H}(M')) \rightarrow \cdots \rightarrow (\mathbb{H}(I^1), \mathbb{H}(M')) \rightarrow (\mathbb{H}(I^0), \mathbb{H}(M')) \rightarrow (\mathbb{H}(M), \mathbb{H}(M')) \rightarrow 0
\]

which, using that \( \mathbb{H} \) is fully faithful, is isomorphic to the complex

\[
0 \rightarrow \mathcal{A}(I^{n+1}, M') \rightarrow \mathcal{A}(I^n, M') \rightarrow \cdots \rightarrow \mathcal{A}(I^1, M') \\
\quad \rightarrow \mathcal{A}(I^0, M') \rightarrow \mathcal{A}(M, M') \rightarrow 0 \quad (10.17)
\]
whose cohomologies give \( \text{Ext}^*(E_M, \mathbb{H}(M')) \). Since \( E_M \) is torsion, \( \text{Inj} \mathcal{A} \subseteq \mathcal{M} \) and \( \mathcal{M} \) is \( n \)-rigid, this complex is acyclic everywhere in the last position on the right which corresponds to \((n + 2)\)-degree where the cohomology is \( \text{Ext}^1(\Sigma M, M') \). We infer that 
\[
\text{Ext}^k(E_M, \mathbb{H}(M')) = 0, \ 1 \leq k \leq n + 1, \ M, M' \in \mathcal{M}.
\]

"\( \Longleftrightarrow \)" Assume the vanishing condition: \( \text{Ext}^k(E_M, \mathbb{H}(M')) = 0, \ 1 \leq k \leq n + 1, \ \forall M, M' \in \mathcal{M} \).

Consider an injective resolution of \( M' \):
\[
0 \rightarrow M' \rightarrow J^0 \rightarrow J^1 \rightarrow \cdots \rightarrow J^n \rightarrow J^{n+1} \rightarrow \cdots
\]

Then, as before, \( n \)-rigidity of \( \mathcal{M} \) implies that the following complex is exact
\[
0 \rightarrow \mathbb{H}(M') \rightarrow \mathbb{H}(J^0) \rightarrow \mathbb{H}(J^1) \rightarrow \cdots
\]
\[
\rightarrow \mathbb{H}(J^n) \rightarrow \mathbb{H}(J^{n+1}) \rightarrow E_{M'} \rightarrow 0
\]  
(10.18)

and therefore it is an injective resolution of \( \mathbb{H}(M') \) in \( \text{mod-M} \), since \( E_{M'} \) is injective and the objects \( \mathbb{H}(J^k) \) are projective–injective. Applying the functor \( \text{Hom}(E_M, -) \) to (10.18) we have the following complex
\[
0 \rightarrow \text{Hom}(E_M, \mathbb{H}(J^0)) \rightarrow \text{Hom}(E_M, \mathbb{H}(J^1)) \rightarrow \cdots
\]
\[
\rightarrow \text{Hom}(E_M, \mathbb{H}(J^{n+1})) \rightarrow \text{Hom}(E_M, E_{M'}) \rightarrow 0
\]  
(10.19)

whose \( k \)th cohomology gives \( \text{Ext}^k(E_M, \mathbb{H}(M')), \ 0 \leq k \leq n+1 \). By hypothesis the complex (10.19) is acyclic, except possibly in degrees 0 and \( n + 2 \), where the cohomologies are \( \text{Hom}(E_M, \mathbb{H}(M')) \) and \( \text{Ext}^{n+2}(E_M, \mathbb{H}(M')) \) respectively. Using the adjoint pair \( (\mathbb{R}, \mathbb{H}) \), it follows that the complex (10.19) is isomorphic to the complex
\[
0 \rightarrow \text{Hom}(\mathbb{R}(E_M), J^0) \rightarrow \text{Hom}(\mathbb{R}(E_M), J^1) \rightarrow \cdots
\]
\[
\rightarrow \text{Hom}(\mathbb{R}(E_M), J^{n+1}) \rightarrow \text{Hom}(E_M, E_{M'}) \rightarrow 0
\]  
(10.20)

Clearly the \( k \)th cohomology, for \( 0 \leq k \leq n \), of the complex (10.20) computes \( \text{Ext}^k(\mathbb{R}(E_M), M') \). Applying the exact functor \( \mathbb{R} \) to (10.15) and using that \( \mathbb{R}\mathbb{H} = \text{Id}_{\mathcal{A}} \), we have an exact sequence \( I^n \rightarrow I^{n+1} \rightarrow \mathbb{R}(E_M) \rightarrow 0 \) which shows that \( \mathbb{R}(E_M) = \Sigma^{n+2}M \). We infer that \( \text{Ext}^k(\Sigma^{n+2}M, M') = 0, \ 1 \leq k \leq n \). Hence \( \Sigma^{n+2}M \in \perp \mathcal{M} = \mathcal{M} \).

On the other hand \( \text{Ext}^{n+1}(E_M, \mathbb{H}(M')) = \text{Ext}^1(E_M, \Sigma^n\mathbb{H}(M')) = 0 \), so the complex (10.19) is exact in degree \( n + 1 \), and this implies by diagram chasing that the sequence
\[
0 \rightarrow \text{Hom}(E_M, \mathbb{H}(\Sigma^n(M'))) \rightarrow \text{Hom}(E_M, \mathbb{H}(J^n)) \rightarrow \text{Hom}(E_M, \mathbb{H}(\Sigma^{n+1}(M'))) \rightarrow 0
\]

is exact. Using the adjoint pair \( (\mathbb{R}, \mathbb{H}) \) and the fact that \( \mathbb{R}(E_M) = \Sigma^{n+2}M \), we have the following exact commutative diagram
0 \longrightarrow \text{Hom}(E_M, \mathbb{H}(\Sigma^n(M'))) \longrightarrow \text{Hom}(E_M, \mathbb{H}(J^n))
\cong \quad \cong
0 \longrightarrow \mathcal{A}(\Sigma^{n+2}M, \Sigma^n(M')) \longrightarrow \mathcal{A}(\Sigma^{n+2}M, J^n)
\longrightarrow \text{Hom}(E_M, \mathbb{H}(\Sigma^{n+1}(M'))) \longrightarrow 0
\cong
\longrightarrow \mathcal{A}(\Sigma^{n+2}M, \Sigma^{n+1}(M')) \longrightarrow \text{Ext}^1_{\mathcal{A}}(\Sigma^{n+2}M, \Sigma^nM') \longrightarrow 0

which shows that \( \text{Ext}^1(\Sigma^{n+2}M, \Sigma^nM') \cong \text{Ext}^{n+1}(\Sigma^{n+2}M, M') = 0 \). Since this happens for any \( M' \in M \) and since \( \Sigma^{n+2}M \) already lies in \( \frac{1}{n+1}M \), we infer that \( \Sigma^{n+2}M \in \frac{1}{n+1}M \).

Since by Remark 8.11, \( \frac{1}{n+1}M = \text{Proj} \mathcal{A} \), we infer that \( \Sigma^{n+2}M \) is projective in \( \mathcal{A} \). Then the exact sequence \( 0 \longrightarrow \Sigma^{n+1}M \longrightarrow I^{n+1} \longrightarrow \Sigma^{n+2}M \longrightarrow 0 \) splits and therefore \( \Sigma^{n+1}M \) is injective as a direct summand of \( I^{n+1} \). Hence \( \text{id} M \leq n + 1 \), for any object \( M \) of \( M \). Finally let \( A \in \mathcal{A} \) and consider an \( \mathcal{M} \)-resolution \( 0 \longrightarrow M^n_A \longrightarrow M^{n-1}_A \longrightarrow \cdots \longrightarrow M^1_A \longrightarrow M^0_A \longrightarrow A \longrightarrow 0 \) of \( A \). Since \( \text{id} M \leq n + 1 \), we have directly by dimension shifting that \( \text{id} A \leq n + 1 \) and therefore \( \text{gl.dim} \mathcal{A} \leq n + 1 \).

We saw in the proof of the direction “\( \longrightarrow \)” that if \( \text{gl.dim} \mathcal{A} \leq n + 1 \), then the injective object \( E_M \) is isomorphic to the object \( \mathbb{R}^{n+1}\mathbb{H}(M) \) which lies, and is injective, in \( \text{mod-}M \) by Lemma 10.8. Now let \( F \in \text{mod-}M \). Since, by Proposition 10.2, \( (\text{mod-}M, \text{Sub} \mathbb{H}(\mathcal{A})) \) is a torsion pair in \( \text{mod-}M \), there exists an exact sequence \( 0 \longrightarrow X_F \longrightarrow F \longrightarrow Y^F \longrightarrow 0 \), where \( X_F \) lies in \( \text{mod-}M \) and there is a monomorphism \( Y^F \longrightarrow \mathbb{H}(A) \) for some \( A \in \mathcal{A} \).

If \( A \longrightarrow I \) is a monomorphism, where \( I \) is injective in \( \mathcal{A} \), then \( Y^F \longrightarrow \mathbb{H}(I) \) is a monomorphism from \( Y^F \) to a (projective-)injective in \( \text{mod-}M \). Since \( \text{mod-}M \) has enough injectives and any injective object of \( \text{mod-}M \) is of the form \( \mathbb{R}^{n+1}\mathbb{H}(M) \), for \( M \in \mathcal{M} \), there exists a monomorphism \( X_F \longrightarrow \mathbb{R}^{n+1}\mathbb{H}(M) \) and as noted above \( \mathbb{R}^{n+1}\mathbb{H}(M) \) is injective in \( \text{mod-}M \). Then by standard arguments we may construct a monomorphism \( F \longrightarrow \mathbb{R}^{n+1}\mathbb{H}(M) \oplus \mathbb{H}(I) \) and \( \mathbb{R}^{n+1}\mathbb{H}(M) \oplus \mathbb{H}(I) \) is injective in \( \text{mod-}M \). Hence \( \text{mod-}M \) has enough injectives and any injective of \( \text{mod-}M \) is a direct summand of \( \mathbb{R}^{n+1}\mathbb{H}(M) \oplus \mathbb{H}(I) \), where \( M \in \mathcal{M} \) and \( I \in \text{Inj} \mathcal{A} \).

**Theorem 10.14.** Let \( \mathcal{A} \) be an abelian category with enough projective and injective objects and let \( \mathcal{M} \) be an \( n \)-cluster tilting subcategory of \( \mathcal{A} \). Then we have the following. 

(i) \( \text{gl.dim} \mathcal{A} \leq n + 1 \) \( \implies \) \( \text{gl.dim} \text{mod-}M \leq n + 2 \).

(ii) \( \text{gl.dim} \mathcal{A} \leq n + 1 \leq \text{dom.dim} \mathcal{A} \) \( \implies \) \( \text{gl.dim} \text{mod-}M \leq n + 2 \leq \text{dom.dim} \text{mod-}M \).

In particular let \( \mathcal{A} \) be an \( (n-1) \)-Auslander category and \( M \) an \( n \)-cluster tilting subcategory \( \mathcal{M} \). If \( \text{Proj} \mathcal{A} \) is covariantly finite in \( \mathcal{A} \), then the stable \( n \)-Auslander category \( \text{mod-}M \) is an \( n \)-Auslander category.
Proof. (i) Since $\text{Inj mod-}\mathcal{M} = R^{n+1}\mathbb{H}(\mathcal{M})$ and since by (the proof of) Theorem 10.13 any object of the form $R^{n+1}\mathbb{H}(M)$, $M \in \mathcal{M}$, is isomorphic to $E_M \in \text{Inj mod-}\mathcal{M}$, we infer that $\text{Inj mod-}\mathcal{M} \subseteq \text{Inj mod-}\mathcal{M}$. This means that the inclusion $\Pi_* : \text{mod-}\mathcal{M} \to \text{mod-}\mathcal{M}$ preserves injectives. Since $\text{mod-}\mathcal{M}$ has enough injectives, this implies that its left adjoint $\Pi^*: \text{mod-}\mathcal{M} \to \text{mod-}\mathcal{M}$ is exact. Let $F \in \text{mod-}\mathcal{M}$ and consider a projective resolution

$$0 \to \mathbb{H}(M^{n+2}) \to \mathbb{H}(M^{n+1}) \to \cdots \to \mathbb{H}(M^1) \to \mathbb{H}(M^0) \to \Pi_*(F) \to 0$$

of $\Pi_*(F)$ in $\text{mod-}\mathcal{M}$. Applying $\Pi^*$ and using that $\Pi^*\Pi_* = \text{Id}_{\text{mod-}\mathcal{M}}$ and $\Pi^*$ preserves projectives since its right adjoint $\Pi_*$ is exact, we obtain a projective resolution

$$0 \to \Pi^*\mathbb{H}(M^{n+2}) \to \Pi^*\mathbb{H}(M^{n+1}) \to \cdots$$

$$\to \Pi^*\mathbb{H}(M^1) \to \Pi^*\mathbb{H}(M^0) \to F \to 0$$

of $F$ in $\text{mod-}\mathcal{M}$ and therefore $\text{pd} F \leq n + 2$. We infer that $\text{gl.dim mod-}\mathcal{M} \leq n + 2$.

(ii) Now assume in addition that $\text{dom.dim } \mathcal{A} \geq n + 1$. Let $F$ be a projective object of $\text{mod-}\mathcal{M}$, so $F = R^1\mathbb{H}(\Omega M) = M(-, M) = \Pi^*\mathbb{H}(M)$, for some $M \in \mathcal{M}$. Since $\text{dom.dim mod-}\mathcal{M} \geq n + 2$, there exists an exact sequence in $\text{mod-}\mathcal{M}$, where $E_M \in \text{mod-}\mathcal{M}$:

$$0 \to \mathbb{H}(M) \to \mathbb{H}(I^0) \to \cdots \to \mathbb{H}(I^n) \to \mathbb{H}(I^{n+1}) \to E_M \to 0$$

where each $I^k$ is injective in $\mathcal{A}$. Applying the exact functor $\Pi^*$ we have an exact sequence

$$0 \to F \to \Pi^*\mathbb{H}(I^0) \to \cdots$$

$$\to \Pi^*\mathbb{H}(I^n) \to \Pi^*\mathbb{H}(I^{n+1}) \to E_M \to 0$$

\((*)\)

We show that the functor $\Pi^*\mathbb{H}(I) = \text{Ext}^1_{\mathcal{A}}(\cdot, \Omega I)|_M \cong M(-, I)$ is (projective--)injective in $\text{mod-}\mathcal{M}$, for any injective object $I \in \mathcal{A}$. Since $\text{dom.dim } \mathcal{A} \geq n + 1$, it follows by [38, Lemma 2.1] that $\text{codom.dim } \mathcal{A} \geq n + 1$ and this means that for the injective object $I$ there exists an exact sequence

$$0 \to Q \to P^n \to P^{n-1} \to \cdots \to P^1 \to P^0 \to I \to 0$$

where the $P^k$ are projective–injective in $\mathcal{A}$, $0 \leq k \leq n$. This means that $\Omega I = \Sigma^n Q$ (up to injective summands). Since $\text{gl.dim } \mathcal{A} \leq n + 1$, it follows that $Q$ is projective, in particular $Q \in \mathcal{M}$, and then

$$\Pi^*\mathbb{H}(I) = \text{Ext}^1_{\mathcal{A}}(\cdot, \Omega I)|_M \cong \text{Ext}^1_{\mathcal{A}}(\cdot, \Sigma^n Q)|_M \cong \text{Ext}^{n+1}_{\mathcal{A}}(\cdot, Q)|_M = R^{n+1}\mathbb{H}(Q)$$

Hence $\Pi^*\mathbb{H}(I)$ is projective–injective in $\text{mod-}\mathcal{M}$. Then the exact sequence \((*)\) shows that $\text{dom.dim mod-}\mathcal{M} \geq n + 2$. In order to prove that $\text{mod-}\mathcal{M}$ is an $n$–Auslander category, it remains to show that $\mathcal{U} := \text{Proj mod-}\mathcal{M} \cap \text{Inj mod-}\mathcal{M}$ or equivalently by Remark 8.13,
Proj mod-\(\mathcal{M}\) is covariantly finite in mod-\(\mathcal{M}\). Let \(F\) be in mod-\(\mathcal{M}\) with projective presentation \(\mathcal{M}(−, M^1) \to \mathcal{M}(−, M^0) \to F \to 0\). Consider the map \(M^1 \to M^0\) in \(\mathcal{A}\) and let \(M^1 \to P\) be a left projective approximation of \(M^1\) in \(\mathcal{A}\). Then consider the exact sequence \(M^1 \to M^0 \oplus P \to X \to 0\) and let \(X \to M^X\) be a left \(\mathcal{M}\)-approximation of \(X\). Then it is easy to see that the map \(M^0 \to M^X\) is a weak cokernel of \(M^1 \to M^0\) in \(\mathcal{M}\). Since the composition \(\mathcal{M}(−, M^1) \to \mathcal{M}(−, M^0) \to \mathcal{M}(−, M^X)\) is clearly zero in mod-\(\mathcal{M}\), it follows that the map \(\mathcal{M}(−, M^0) \to \mathcal{M}(−, M^X)\) factorizes uniquely through \(F\), say via a map \(F \to \mathcal{M}(−, M^X)\). It is easy to see that this map is in fact a left Proj mod-\(\mathcal{M}\)-approximation of \(F\), so Proj mod-\(\mathcal{M}\) is covariantly finite in mod-\(\mathcal{M}\). We infer that mod-\(\mathcal{M}\) is an \(n\)-Auslander category.

**Remark 10.15.** Keeping the assumptions of Theorem 10.14, it follows from its proof that any object of the form \(\mathcal{M}(−, I)\), \(I \in \text{Inj } \mathcal{A}\), is projective–injective in mod-\(\mathcal{M}\). Now let \(G\) be a projective–injective object of mod-\(\mathcal{M}\). By projectivity \(G = \mathcal{M}(−, M) = R^1\mathbb{H}(\Omega M)\) for some \(M \in \mathcal{M}\). Let \(M \to I\) be a monomorphism in \(\mathcal{A}\) where \(I\) is injective. Then we have a monomorphism \(\mathbb{H}(M) \to \mathbb{H}(I)\) in mod-\(\mathcal{M}\) and therefore a monomorphism \(\Pi^*\mathbb{H}(M) \to \Pi^*\mathbb{H}(I)\) in mod-\(\mathcal{M}\) since the functor \(\Pi^*\) is exact. Since this monomorphism is isomorphic to \(\mathcal{M}(−, M) \to \mathcal{M}(−, I)\) and the object \(\mathcal{M}(−, M)\) is injective, it follows that \(G\) is a direct summand of \(\mathcal{M}(−, I)\). This implies that \(M\) is a direct summand of \(I\) in \(\mathcal{A}\) and then clearly \(M \in \text{Inj } \mathcal{A}\). We infer that \(\mathcal{U} := \text{Proj mod-\(\mathcal{M}\) \cap Inj mod-\(\mathcal{M}\)} = \{\mathcal{M}(−, I) \in \text{mod-\(\mathcal{M}\)} \mid I \in \mathcal{A}\} = \Pi^*\mathbb{H}(\text{Inj } \mathcal{A})\) and we have an equivalence

\[
\text{Inj } \mathcal{A} \xrightarrow{\cong} \mathcal{U} = \text{Proj mod-\(\mathcal{M}\) \cap Inj mod-\(\mathcal{M}\), \quad I \mapsto M(−, I)\}
\]

Let \(\mathcal{A}^I\) be the abelian category (\(\text{U-mod}\))^\op \cong (\text{Inj } \mathcal{A} \text{-mod})^\op and set \(\mathcal{N} := \text{mod-\(\mathcal{M}\)}(\text{Proj mod-\(\mathcal{M}\)}, −)|_\mathcal{U} \cong (\mathcal{M})|_\text{Inj } \mathcal{A}\). By Theorem 8.23, \(\mathcal{N}\) is an \(n\)-cluster tilting subcategory of \(\mathcal{A}^I\) and its \(n\)-Auslander category is equivalent to mod-\(\mathcal{N}\) \cong mod-\(\mathcal{M}\) in particular \(\mathcal{N} \cong \mathcal{M}\). Note that the projection \(\text{Inj } \mathcal{A} \to \text{Inj } \mathcal{A}\) induces by restriction a full exact embedding of abelian categories (\(\text{Inj } \mathcal{A} \text{-mod})^\op = \mathcal{A}^I \to \mathcal{A} = (\text{Inj } \mathcal{A} \text{-mod})^\op).

### 10.6. The Gorenstein property

We close this section by studying the Gorenstein property of stable \(n\)-Auslander categories mod-\(\mathcal{M}\) associated to an \(n\)-cluster tilting subcategory \(\mathcal{M}\) of \(\mathcal{A}\).

We begin with the following.

**Proposition 10.16.** Let \(\mathcal{A}\) be an abelian category with enough projective and injective objects, and let \(\mathcal{M}\) be an \(n\)-cluster tilting subcategory of \(\mathcal{A}\). If \(\text{G-dim } \mathcal{A} \leq n\), then \(\mathcal{A}\) is Frobenius and:
(i) If \( n = 1 \), then \( \text{G-dim} \text{mod-} \overline{\mathcal{M}} \leq 1 \).

(ii) If \( n \geq 2 \) and \( \text{Ext}^1_{\mathcal{A}}(\mathcal{M}, \Omega^i \mathcal{M}) = 0 \), where \( 2 \leq i \leq n \) and \( 0 \leq k \leq \frac{n+1}{2} \), then

\[
\text{G-dim} \text{mod-} \overline{\mathcal{M}} \leq k
\]

**Proof.** Since \( \text{G-dim} \mathcal{A} \leq n \), it follows that \( \mathcal{A} \) is Gorenstein and, by Remark 6.3, \( \text{GProj} \mathcal{A} \) consists of all objects \( A \) such that \( \text{Ext}^k_{\mathcal{A}}(A, P) = 0 \), for any projective object \( P \) and for any \( k \geq 1 \), or equivalently for \( 1 \leq k \leq n \) since the injective dimension of any projective object is at most \( n \). Since \( \mathcal{M} \) is \( n \)-rigid and \( \mathcal{M} \) contains the projectives of \( \mathcal{A} \), it follows that \( \mathcal{M} \subseteq \text{GProj} \mathcal{A} \). Then clearly \( \overline{\mathcal{M}} \) is an \( (n+1) \)-cluster tilting subcategory of the triangulated category \( \text{GProj} \mathcal{A} \). Since any injective object has projective dimension at most \( n \) and since any injective is Gorenstein-projective because \( \text{Inj} \mathcal{A} \subseteq \mathcal{M} \subseteq \text{GProj} \mathcal{A} \), it follows that \( \text{Inj} \mathcal{A} \subseteq \text{Proj}^{<\infty} \mathcal{A} \cap \text{GProj} \mathcal{A} = \text{Proj} \mathcal{A} \), hence any injective is projective. If \( P \) is projective, let \( 0 \rightarrow P \rightarrow I \rightarrow \Sigma P \rightarrow 0 \) be exact where \( I \) is injective. Since \( P \) has finite injective dimension (bounded by \( n \)), we have that also \( \Sigma P \) has finite injective dimension. Since any injective is projective, it follows that \( \text{Ext}^k_{\mathcal{A}}(E, \Sigma P) = 0 \), \( \forall k \geq 1 \) and for any injective object \( E \). It follows that \( \Sigma P \) is Gorenstein-injective, and then it is injective since it has finite injective dimension. Then the above sequence splits and \( P \) is injective as a direct summand of \( I \). Hence any projective is injective and \( \mathcal{A} \) is Frobenius.

If \( n = 1 \), then \( \overline{\mathcal{M}} \) is a 2-cluster tilting subcategory of \( \text{GProj} \mathcal{A} \) and therefore \( \text{G-dim} \text{mod-} \overline{\mathcal{M}} \leq 1 \), by Theorem 6.4(i). If \( n \geq 2 \), then the vanishing condition \( \text{Ext}^1(\mathcal{M}, \Omega^i \mathcal{M}) = 0 \), where \( 2 \leq i \leq n \) and \( 1 \leq k \leq n \), implies that the \((n+1)\)-cluster tilting subcategory \( \overline{\mathcal{M}} \) of \( \mathcal{A} \) is \((n-k)\)-corigid, since \( \text{Ext}^1_{\mathcal{A}}(\mathcal{M}, \Omega^i \mathcal{M}) \cong \mathcal{A}(\Omega \mathcal{M}, \Omega^i \mathcal{M}) \cong \mathcal{A}(\mathcal{M}, \Omega^{-i+1} \mathcal{M}) = \mathcal{A}(\mathcal{M}, \Sigma^{-i+1} \mathcal{M}) \). Then by Theorem 6.4(iii) we have \( \text{G-dim} \text{mod-} \overline{\mathcal{M}} \leq k \). \( \square \)

**Theorem 10.17.** Let \( \mathcal{A} \) be an abelian category with enough projective and injective objects, and let \( \mathcal{M} \) be an \( n \)-cluster tilting subcategory of \( \mathcal{A} \). Assume the following vanishing conditions:

(i) \( \text{Ext}^{n+1}_{\mathcal{M}}(\mathcal{M}, \Omega \mathcal{M}) = 0 = \text{Ext}^{n+1}_{\mathcal{M}}(\Sigma \mathcal{M}, \mathcal{M}) \).

(ii) \( \text{Ext}^{k}_{\mathcal{M}}(\mathcal{M}, \mathcal{M}) = 0 \), for \( n+2 \leq k \leq 2n \).

Then the stable Auslander category \( \text{mod-} \overline{\mathcal{M}} \) is Gorenstein and \( \text{G-dim} \text{mod-} \overline{\mathcal{M}} \leq n \).

**Proof.** We show that \( \text{spl} \text{mod-} \overline{\mathcal{M}} \leq n \), i.e. the projective dimension of any injective object is at most \( n \), and \( \text{silp} \text{mod-} \overline{\mathcal{M}} \leq n \), i.e. the injective dimension of any projective object is at most \( n \). By Lemmas 10.8 and 10.10, we have to show that:

\[
\text{id} R^1 \mathbb{H}(\Omega \mathcal{M}) \leq n \quad \& \quad \text{pd} R^{n+1} \mathbb{H}(\mathcal{M}) \leq n
\]

Using the duality \( R^{n+2} d_l : \text{mod-} \overline{\mathcal{M}} \rightarrow \text{mod-} \mathcal{M} \) with quasi-inverse \( R^{n+2} d_l \), established in Proposition 10.6, in order to show that any injective object \( F \) in \( \text{mod-} \overline{\mathcal{M}} \) has projective
dimension at most \( n \), it suffices to show that any projective object \( G \) in \( \mathbb{M} \text{-mod} \) has injective dimension at most \( n \). Since by Lemma 10.11, any projective object \( G \) of \( \mathbb{M} \text{-mod} \) is of the form \( G = R^1 \mathbb{H}^{\text{op}}(\Sigma M) \), for \( M \in \mathbb{M} \), it suffices to show that:

\[
\text{id} R^1 \mathbb{H}(\Omega M) \leq n \quad \& \quad \text{id} R^1 \mathbb{H}^{\text{op}}(\Sigma M) \leq n
\]

1. Let \( F = R^1 \mathbb{H}(\Omega M) \in \text{mod-\mathbb{M}} \) be a projective object, where \( M \in \mathbb{M} \). Then we have exact sequences

\[
0 \to M^n_{\Omega M} \to M^{n-1}_{\Omega M} \to \cdots \to M^1_{\Omega M} \to M^0_{\Omega M} \to \Omega M \to 0 \quad \& \\
0 \to \Omega M \to P \to M \to 0
\]

As usual we set \( K^i_{\Omega M} = \text{im}(M^i_{\Omega M} \to M^{i-1}_{\Omega M}) \), \( 1 \leq i \leq n-1 \). We shall construct inductively an injective resolution of \( F \) of length at most \( n \) of the following form:

\[
0 \to F \to R^{n+1} \mathbb{H}(M^n_{\Omega M}) \to R^{n+1} \mathbb{H}(M^{n-1}_{\Omega M}) \to \cdots \\
\to R^{n+1} \mathbb{H}(M^1_{\Omega M}) \to R^{n+1} \mathbb{H}(M^0_{\Omega M}) \to 0
\]

Setting \( K^n_{\Omega M} := M^n_{\Omega M} \), we apply the left exact functor \( \mathbb{H} \) successively to the short exact sequences:

\[
0 \to K^t_{\Omega M} \to M^{t-1}_{\Omega M} \to K^{t-1}_{\Omega M} \to 0, \quad 1 \leq t \leq n \quad \& \\
0 \to \Omega M \to P \to M \to 0 \quad (10.21)
\]

and use \( n \)-rigidity of \( \mathbb{M} \) and the vanishing conditions \( \text{Ext}^{n+1}(\mathbb{M}, \Omega M) = 0 \) and \( \text{Ext}^k(M, M) = 0, \ n + 2 \leq k \leq 2n \):

\((A_n)\) For \( t = n \) in \( (10.21) \), we have isomorphisms:

\[
R^k \mathbb{H}(K^{n-1}_{\Omega M}) = 0, \quad 1 \leq k \leq 2n - 1, \quad k \neq n, n + 1
\]

and an exact sequence

\[
0 \to R^n \mathbb{H}(K^{n-1}_{\Omega M}) \to R^{n+1} \mathbb{H}(M^n_{\Omega M}) \to R^{n+1} \mathbb{H}(M^{n-1}_{\Omega M}) \\
\to R^{n+1} \mathbb{H}(K^{n-1}_{\Omega M}) \to 0
\]

\((A_{n-1})\) For \( t = n - 1 \) in \( (10.21) \), we have isomorphisms:

\[
R^k \mathbb{H}(K^{n-2}_{\Omega M}) \cong R^{k+1} \mathbb{H}(K^{n-1}_{\Omega M}), \quad 1 \leq k \leq 2n - 1, \quad k \neq n, n + 1
\]

and an exact sequence
\[ 0 \to \mathbb{R}^n \mathbb{H}(K_{\Omega M}^{n-2}) \to \mathbb{R}^{n+1} \mathbb{H}(K_{\Omega M}^{n-1}) \to \mathbb{R}^{n+1} \mathbb{H}(M_{\Omega M}^{n-2}) \to 0 \]
\[ \to \mathbb{R}^{n+1} \mathbb{H}(K_{\Omega M}^{n-2}) \to \mathbb{R}^{n+2} \mathbb{H}(K_{\Omega M}^{n-1}) \to 0 \]

\((A_{n-2})\) For \( t = n - 2 \) in (10.21), we have isomorphisms:
\[ \mathbb{R}^k \mathbb{H}(K_{\Omega M}^{n-3}) \cong \mathbb{R}^{k+1} \mathbb{H}(K_{\Omega M}^{n-2}), \quad 1 \leq k \leq 2n - 1, \quad k \neq n, n + 1 \]

and an exact sequence:
\[ 0 \to \mathbb{R}^n \mathbb{H}(K_{\Omega M}^{n-3}) \to \mathbb{R}^{n+1} \mathbb{H}(K_{\Omega M}^{n-2}) \to \mathbb{R}^{n+1} \mathbb{H}(M_{\Omega M}^{n-3}) \to 0 \]
\[ \to \mathbb{R}^{n+1} \mathbb{H}(K_{\Omega M}^{n-3}) \to \mathbb{R}^{n+2} \mathbb{H}(K_{\Omega M}^{n-2}) \to 0 \]

\((A_2)\) For \( t = 2 \) in (10.21), we have isomorphisms:
\[ \mathbb{R}^k \mathbb{H}(K_{\Omega M}^1) \cong \mathbb{R}^{k+1} \mathbb{H}(K_{\Omega M}^2), \quad 1 \leq k \leq 2n - 1, \quad k \neq n, n + 1 \]

and an exact sequence:
\[ 0 \to \mathbb{R}^n \mathbb{H}(K_{\Omega M}^1) \to \mathbb{R}^{n+1} \mathbb{H}(K_{\Omega M}^2) \to \mathbb{R}^{n+1} \mathbb{H}(M_{\Omega M}^1) \to 0 \]
\[ \to \mathbb{R}^{n+1} \mathbb{H}(K_{\Omega M}^1) \to \mathbb{R}^{n+2} \mathbb{H}(K_{\Omega M}^2) \to 0 \]

\((A_1)\) For \( t = 1 \) in (10.21), we have \( \mathbb{R}^{n+2} \mathbb{H}(K_{\Omega M}^1) = 0 \) and isomorphisms:
\[ \mathbb{R}^k \mathbb{H}(\Omega M) \cong \mathbb{R}^{k+1} \mathbb{H}(K_{\Omega M}^1), \quad 1 \leq k \leq 2n - 1, \quad k \neq n, n + 1 \]

and an exact sequence:
\[ 0 \to \mathbb{R}^n \mathbb{H}(\Omega M) \to \mathbb{R}^{n+1} \mathbb{H}(K_{\Omega M}^1) \to \mathbb{R}^{n+1} \mathbb{H}(M_{\Omega M}^0) \to 0 \]

\((A_0)\) Finally from the second short exact sequence in (10.21), we have
\[ \mathbb{R}^k \mathbb{H}(\Omega M) = 0, \quad 2 \leq k \leq 2n, \quad k \neq n + 2 \]

and an exact sequence:
\[ 0 \to \mathbb{R}^{n+1} \mathbb{H}(P) \to \mathbb{R}^{n+1} \mathbb{H}(M) \to \mathbb{R}^{n+2} \mathbb{H}(\Omega M) \to 0 \]

Combining the isomorphisms in \((A_1)-(A_n)\), we have isomorphisms:
\[ F = \mathbb{R}^1 \mathbb{H}(\Omega M) \cong \mathbb{R}^2 \mathbb{H}(K_{\Omega M}^1) \cong \mathbb{R}^3 \mathbb{H}(K_{\Omega M}^2) \cong \cdots \cong \mathbb{R}^{n-1} \mathbb{H}(K_{\Omega M}^{n-2}) \cong \mathbb{R}^n \mathbb{H}(K_{\Omega M}^{n-1}) \]
On the other hand using the isomorphisms and the exact sequences in \((A_0)- (A_n)\), we obtain:

\[
\text{R}^n \mathbb{H}(K_{\Omega M}^{n-2}) = \text{R}^n \mathbb{H}(K_{\Omega M}^{n-2}) = \cdots = \text{R}^n \mathbb{H}(K_{\Omega M}^{1}) = \text{R}^n \mathbb{H}(\Omega M) = 0
\]

and

\[
\text{R}^{n+2} \mathbb{H}(K_{\Omega M}^{n-1}) = \text{R}^{n+2} \mathbb{H}(K_{\Omega M}^{n-2}) = \cdots = \text{R}^{n+2} \mathbb{H}(K_{\Omega M}^{2}) = 0
\]

As a consequence the exact sequences in \((A_0)-(A_n)\) become respectively an exact sequence

\[
0 \rightarrow F \rightarrow \text{R}^{n+1} \mathbb{H}(M_{\Omega M}^{n}) \rightarrow \text{R}^{n+1} \mathbb{H}(M_{\Omega M}^{n-1}) \rightarrow \text{R}^{n+1} \mathbb{H}(K_{\Omega M}^{n-1}) \rightarrow 0 \quad (E_{n-1})
\]

short exact sequences

\[
\begin{align*}
0 & \rightarrow \text{R}^{n+1} \mathbb{H}(K_{\Omega M}^{n-1}) \rightarrow \text{R}^{n+1} \mathbb{H}(M_{\Omega M}^{n-2}) \rightarrow \text{R}^{n+1} \mathbb{H}(K_{\Omega M}^{n-2}) \rightarrow 0 \quad (E_{n-2}) \\
0 & \rightarrow \text{R}^{n+1} \mathbb{H}(K_{\Omega M}^{n-2}) \rightarrow \text{R}^{n+1} \mathbb{H}(M_{\Omega M}^{n-3}) \rightarrow \text{R}^{n+1} \mathbb{H}(K_{\Omega M}^{n-3}) \rightarrow 0 \quad (E_{n-3}) \\
& \vdots \\
0 & \rightarrow \text{R}^{n+1} \mathbb{H}(K_{\Omega M}^{2}) \rightarrow \text{R}^{n+1} \mathbb{H}(M_{\Omega M}^{1}) \rightarrow \text{R}^{n+1} \mathbb{H}(K_{\Omega M}^{1}) \rightarrow 0 \quad (E_1)
\end{align*}
\]

and an isomorphism

\[
\text{R}^{n+1} \mathbb{H}(K_{\Omega M}^{1}) \cong \text{R}^{n+1} \mathbb{H}(M_{\Omega M}^{0}) \quad (E_0)
\]

Forming the Yoneda composition of the exact sequences \((E_i), 1 \leq i \leq n-1\) and the isomorphism \((E_0)\) we arrive at an exact sequence

\[
0 \rightarrow F \rightarrow \text{R}^{n+1} \mathbb{H}(M_{\Omega M}^{n}) \rightarrow \text{R}^{n+1} \mathbb{H}(M_{\Omega M}^{n-1}) \rightarrow \cdots
\]

\[
\rightarrow \text{R}^{n+1} \mathbb{H}(M_{\Omega M}^{1}) \rightarrow \text{R}^{n+1} \mathbb{H}(M_{\Omega M}^{0}) \rightarrow 0
\]

which is an injective resolution of the projective object \(F\). Hence \(\text{id} F \leq n\) and therefore \(\text{spl mod-M} \leq n\).

2. Working in the category of covariant coherent functors \(\underline{\text{M-mod}}\) and using the second vanishing condition \(\text{Ext}^{n+1}_{\Omega}(\Sigma M, M) = 0\) of (i) and the vanishing condition \(\text{Ext}^{k}_{\Omega}(M, M) = 0, n+2 < k < 2n\), in (ii), we have in a dual way that \(\text{id} \text{R}^{1} \mathbb{H}^{\text{op}}(\Sigma M) \leq n\). As explained before this is equivalent to say that \(\text{pd} G \leq n\) for any injective object \(G\) of \(\text{mod-M}\). We infer that \(\text{spl mod-M} \leq n\). \(\square\)

If \(n = 1\) in Theorem 10.17, then the second condition (ii) is vacuous. Hence we have the following.
Corollary 10.18. Let $\mathcal{A}$ be an abelian category with enough projective and injective objects, and let $\mathcal{M}$ be a 1-cluster tilting subcategory of $\mathcal{A}$. If $\text{Ext}^2_{\mathcal{A}}(\mathcal{M}, \Omega \mathcal{M}) = 0 = \text{Ext}^2_{\mathcal{A}}(\Sigma \mathcal{M}, \mathcal{M})$, then the stable Auslander category $\text{mod-}\overline{\mathcal{M}}$ is Gorenstein and $\text{G-dim mod-}\overline{\mathcal{M}} \leq 1$.

Remark 10.19. If $\mathcal{A}$ is Frobenius, then $\text{Ext}^2_{\mathcal{A}}(\mathcal{M}, \Omega \mathcal{M}) \cong \mathcal{A}(\Omega^2 \mathcal{M}, \Omega \mathcal{M}) \cong \text{Ext}^2_{\mathcal{A}}(\mathcal{M}, \mathcal{M}) = 0$ and similarly $\text{Ext}^2_{\mathcal{A}}(\Sigma \mathcal{M}, \mathcal{M}) = 0$. Hence the condition $\text{Ext}^2_{\mathcal{A}}(\mathcal{M}, \Omega \mathcal{M}) = 0 = \text{Ext}^2_{\mathcal{A}}(\Sigma \mathcal{M}, \mathcal{M})$ of Corollary 10.18 holds always in the Frobenius case and $\overline{\mathcal{M}}$ is a 2-cluster tilting subcategory of $\mathcal{A}$. It follows that the bound $\text{G-dim mod-}\overline{\mathcal{M}} \leq 1$ of Corollary 10.18 generalizes Theorem 6.4(i) to the non-stable case.

It is easy to see that in case $n = 1$ the conditions $R^2 \mathbb{H}(P) = 0 = R^2 \mathbb{H}^\text{op}(I)$, $P \in \text{Proj } \mathcal{A}$, and $I \in \text{Inj } \mathcal{A}$, imply the conditions of Corollary 10.18. Hence we have the following consequence.

Corollary 10.20. Let $\mathcal{A}$ be an abelian category with enough projective and injective objects, and let $\mathcal{M}$ be a 1-cluster tilting subcategory of $\mathcal{A}$. If $\text{Ext}^2_{\mathcal{A}}(\mathcal{M}, P) = 0 = \text{Ext}^2_{\mathcal{A}}(I, M)$, $\forall M \in \mathcal{M}$, $P \in \text{Proj } \mathcal{A}$, and $I \in \text{Inj } \mathcal{A}$, then the stable Auslander category $\text{mod-}\overline{\mathcal{M}}$ is Gorenstein and $\text{G-dim mod-}\overline{\mathcal{M}} \leq 1$.

More generally for $n > 1$ it is easy to see that the conditions $\text{Ext}^{n+1}_{\mathcal{A}}(\mathcal{M}, P) = 0 = \text{Ext}^{n+1}_{\mathcal{A}}(I, M)$, $\forall M \in \mathcal{M}$, $P \in \text{Proj } \mathcal{A}$, and $I \in \text{Inj } \mathcal{A}$, which clearly hold in case $\mathcal{A}$ is a Frobenius category or more generally in case $\text{G-dim } \mathcal{A} \leq n$, imply the conditions (i) in Theorem 10.17, and then we have the following.

Corollary 10.21. Let $\mathcal{A}$ be an abelian category with enough projective and injective objects, and let $\mathcal{M}$ be an $n$-cluster tilting subcategory of $\mathcal{A}$. Assume the following vanishing conditions:

$$\text{Ext}^{n+1}_{\mathcal{A}}(\mathcal{M}, \text{Proj } \mathcal{A}) = 0 = \text{Ext}^{n+1}_{\mathcal{A}}(\text{Inj } \mathcal{A}, \mathcal{M})$$
$$\text{Ext}^k_{\mathcal{A}}(\mathcal{M}, \mathcal{M}) = 0,$$  
$n + 2 \leq k \leq 2n$  \hspace{2cm} (†)

(1) The stable Auslander category $\text{mod-}\overline{\mathcal{M}}$ is Gorenstein and $\text{G-dim mod-}\overline{\mathcal{M}} \leq n$.

(2) If $\text{G-dim } \mathcal{A} \leq 2n$, then $\mathcal{A}$ is Frobenius and $\overline{\mathcal{M}}$ is an $(n+1)$-cluster tilting subcategory of $\mathcal{A}$.

(3) If $\text{G-dim } \mathcal{A} \leq 2n$ and the triangulated category $\mathcal{A}$ is $(n+1)$-Calabi–Yau, then the $(n+1)$-cluster tilting subcategory $\overline{\mathcal{M}}$ of $\mathcal{A}$ is $(n-1)$-corigid and $\text{G-dim mod-}\overline{\mathcal{M}} \leq 1$.

Proof. (1) This follows from Theorem 10.17.

(2) The vanishing conditions (†) imply that $\text{Ext}^k_{\mathcal{A}}(M, P) = 0$, for $1 \leq k \leq 2n$, for any object $M \in \mathcal{M}$ and any projective object $P$ of $\mathcal{A}$. It follows that if $\text{G-dim } \mathcal{A} \leq 2n$, then $\mathcal{M} \subseteq \text{GProj } \mathcal{A}$ and then as in Proposition 10.16 we infer that $\mathcal{A}$ is Frobenius. Clearly then $\overline{\mathcal{M}}$ is an $(n+1)$-cluster tilting subcategory of $\mathcal{A}$. 
(3) If in addition $\mathcal{A}$ is $(n+1)$-Calabi–Yau, then for $n+2 \leq k \leq 2n$:

$$0 = \Ext^k_{\mathcal{A}}(\mathcal{M}, \mathcal{M}) \cong \mathcal{A}(\mathcal{M}, \Sigma^k \mathcal{M}) \cong \mathcal{A}(\mathcal{M}, \Sigma^{n+1} \Sigma^{k-n-1} \mathcal{M})$$

$$\cong \mathcal{D}(\mathcal{M}, \Sigma^{k-n-1} \mathcal{M}) \cong \mathcal{D}(\mathcal{M}, \Sigma^{n-k+1} \mathcal{M})$$

Hence the $(n+1)$-cluster tilting subcategory $\mathcal{M}$ of $\mathcal{A}$ is $(n-1)$-corigid. Then by Corollary 6.5(a) we have $\Gdim \mod \mathcal{M} \leq 1$. □

10.7. The module-theoretic interpretation

We close this section by summarizing our results in this section in the context of categories of modules over an associative ring $\Lambda$. Below $\Gdim \Lambda$ means $\Gdim \Mod \Lambda$, i.e. the Gorenstein dimension of the category of right $\Lambda$-modules.

**Theorem 10.22.** Let $M$ be a self-compact $n$-cluster tilting $\Lambda$-module over a ring $\Lambda$ and let $\Gamma = \End_\Lambda(M)$ be, as in Theorem 9.9, the associated left coherent and right perfect $n$-Auslander ring.

(i) The $\Gamma$-dual functor $\Hom_{\Gamma}(-, \Gamma)$ induces a duality $\Ext^{n+2}_{\Gamma}(-, \Gamma) : \mod \End_\Lambda(M) \cong (\End_\Lambda(M) - \text{mod})^{\op}$.

(ii) $\r.gl. \dim \Lambda \leq n+1 \iff \Ext^k_{\Gamma}(E, \Gamma) = 0$, $1 \leq k \leq n+1$, for any injective $\Gamma$-module $E$.

(iii) If $\Lambda$ is an $(n-1)$-Auslander ring, then:

(a) The stable $n$-Auslander ring $\End_\Lambda(M)$ is an $n$-Auslander ring.

(b) $M$ is an $n$-cluster tilting object in $(\Inj \Lambda - \text{mod})^{\op}$.

(c) If $\Lambda$ is an Artin algebra, then $M$ is an $n$-cluster tilting object in $\mod \End_\Lambda(D \Lambda)^{\op}$ with endomorphism algebra the stable $n$-Auslander algebra $\End_\Lambda(M)$.

(iv) $\Gdim \Lambda \leq n \iff \Lambda$ is QF. In this case $M$ is finitely generated and:

(a) If $n = 1$, then $\Gdim \End_\Lambda(M) \leq 1$.

(b) If $n \geq 2$ and $\Ext^i_{\Lambda}(M, \Omega^i \mathcal{M}) = 0$, where $2 \leq i \leq n-k+1$ and $0 \leq k \leq \frac{n+1}{2}$, then $\Gdim \End_\Lambda(M) \leq k$.

(v) If $n = 1$ and $\Ext^2_{\Lambda}(M, \mathcal{P}) = 0 = \Ext^3_{\Lambda}(J, \mathcal{M})$, $\forall P \in \Proj \Lambda$, $\forall J \in \Inj \Lambda$, then: $\Gdim \End_\Lambda(M) \leq 1$. 

(vi) If $n \geq 2$, and assuming that: $\Ext^{n+1}_{\Lambda}(M, \mathcal{P}) = 0 = \Ext^{n+1}_{\Lambda}(J, \mathcal{M})$, $\forall P \in \Proj \Lambda$, $\forall J \in \Inj \Lambda$, and $\Ext^k_{\Lambda}(M, \mathcal{M}) = 0$, for $n+2 \leq k \leq 2n$, then:

(a) $\Gdim \End_\Lambda(M) \leq n$.

(b) $\Gdim \Lambda \leq 2n \iff \Lambda$ is QF; in this case $M$ is a compact $(n+1)$-cluster tilting object in $\Mod \Lambda$.

(c) If $\Gdim \Lambda \leq 2n$ and the triangulated category $\mod \Lambda$ is $(n+1)$-Calabi–Yau, then the $(n+1)$-cluster tilting object $M$ of $\mod \Lambda$ is $(n-1)$-corigid and $\Gdim \End_\Lambda(M) \leq 1$. 

11. Cluster tilting subcategories of Cohen–Macaulay objects

In this section we are interested in the homological behavior of the category of coherent functors over rigid subcategories of an abelian category induced by cluster-tilting subcategories of the triangulated category of Gorenstein-projective (or Cohen–Macaulay) objects.

Throughout this section: we fix an abelian category $\mathcal{A}$ and assume that $\mathcal{A}$ has enough projectives. We also fix a contravariantly finite subcategory $\mathcal{M}$ of $\mathcal{A}$ and assume that $\mathcal{M}$ contains the projectives.

Then, as in the previous sections, right $\mathcal{M}$-approximations of objects of $\mathcal{A}$ are epics and we have an adjoint pair $(R,H): \text{mod-}\mathcal{M} \rightleftarrows \mathcal{A}$, where the functor $H: \mathcal{A} \to \text{mod-}\mathcal{M}$, $H(A) = A(-,A)|_{\mathcal{M}}$ is fully faithful and the functor $R$ is exact. Moreover $\ker R = \text{mod-}\mathcal{M}$ is equivalent to the category $\text{mod-}\mathcal{M}$ of coherent functors over the stable category $\mathcal{M}$ of $\mathcal{M}$ modulo projectives, and we have an exact sequence:

$$0 \to \text{mod-}\mathcal{M} \to \text{mod-}\mathcal{M} \to \mathcal{A} \to 0$$  \hspace{1cm} (11.1)

We consider the stable category $\text{GProj} \mathcal{A}$ of Gorenstein-projective objects of $\mathcal{A}$ modulo projectives as a triangulated category with suspension functor $\Omega^{-1}$. For a full subcategory $X$ of $\text{GProj} \mathcal{A}$, we denote by $M = \pi^{-1}X_0$ the pre-image of $X$ under the projection functor $\pi: \text{GProj} \mathcal{A} \to \text{GProj} \mathcal{A}$, so that $\mathcal{M} = X$. Note that then $\text{Proj} \mathcal{A} \subseteq \mathcal{M} \subseteq \text{GProj} \mathcal{A}$.

In what follows the operations $\pi X \leftarrow X \rightarrow X^\pi$ are taken inside the triangulated category $\text{GProj} \mathcal{A}$, while the operations $\pi^\perp M \leftarrow M \rightarrow M^\perp$ are taken inside the abelian category $\mathcal{A}$.

The following observation is easy and is left to the reader noting that if $\mathcal{A}$ is Gorenstein, then $\text{GProj} \mathcal{A}$ is contravariantly finite, and also covariantly finite if in addition $\text{Proj} \mathcal{A}$ is covariantly finite, see [11].

**Lemma 11.1.** $X$ is contravariantly, resp. covariantly, finite in $\text{GProj} \mathcal{A}$ if and only if $\mathcal{M}$ is contravariantly, resp. covariantly, finite in $\text{GProj} \mathcal{A}$. If $\mathcal{A}$ is Gorenstein, then $X$ is contravariantly finite in $\text{GProj} \mathcal{A}$ if and only if $\mathcal{M}$ is contravariantly finite in $\mathcal{A}$; if $\text{Proj} \mathcal{A}$ is covariantly finite in $\mathcal{A}$, then $X$ is covariantly finite in $\text{GProj} \mathcal{A}$ if and only if $\mathcal{M}$ is covariantly finite in $\mathcal{A}$. Finally $X^\pi = M^\perp \cap \text{GProj} \mathcal{A}$ and $\pi^\perp X = \pi^\perp M \cap \text{GProj} \mathcal{A}$.

We call the abelian category $\mathcal{A}$ (projectively) Gorenstein if $\mathcal{A}$ has enough projectives and there exists $n \geq 0$, such that any object of $\mathcal{A}$ admits an exact resolution of length $\leq n$ by Gorenstein-projective objects. The minimum such $n$ is called the Gorenstein projective dimension $G\text{-dim}_{\mathcal{P}} \mathcal{A}$ of $\mathcal{A}$. Dually one defines the Gorenstein injective dimension $G\text{-dim}_{\mathcal{I}} \mathcal{A}$ of $\mathcal{A}$ and when $\mathcal{A}$ is (injectively) Gorenstein. Note that if $\mathcal{A}$ has enough projectives and enough injectives, then these dimensions coincide with the Gorenstein dimension $G\text{-dim} \mathcal{A} = \max\{\text{slp} \mathcal{A}, \text{spli} \mathcal{A}\}$ as defined in Section 6, so: $G\text{-dim}_{\mathcal{P}} \mathcal{A} = G\text{-dim} \mathcal{A} = G\text{-dim}_{\mathcal{I}} \mathcal{A}$, see [19].
Lemma 11.2. Let $\mathcal{A}$ be an abelian category with enough projectives. Then $\mathcal{A}$ is Gorenstein if and only if $\mathcal{A}$ contains a contravariantly finite subcategory $\mathcal{M}$ such that $\text{Proj} \mathcal{A} \subseteq \mathcal{M} \subseteq \text{GProj} \mathcal{A}$ and: $\text{gl.dim mod-} \mathcal{M} < \infty$.

**Proof.** If $\mathcal{A}$ is Gorenstein, then $\mathcal{M} := \text{GProj} \mathcal{A}$ is contravariantly finite by [19]. Hence $\text{mod-} \mathcal{M}$ is abelian. Since $\mathcal{M}$ is resolving, by the arguments of [15, Corollaries 6.8 and 6.13] it follows that $\text{gl.dim mod-} \mathcal{M} < \infty$. Conversely if $\text{gl.dim mod-} \mathcal{M} = t < \infty$, then $\text{pd} \mathcal{H}(A) \leq t$, $\forall A \in \mathcal{A}$, and since $\mathcal{M}$ contains the projectives, this implies that any object $A$ of $\mathcal{A}$ admits a finite $\mathcal{M}$-resolution of length $\leq t$. Since $\mathcal{M}$ consists of Gorenstein-projectives, it follows that $\text{G-dim} \mathcal{A} \leq t$, hence $\mathcal{A}$ is Gorenstein by [19]. \qed

Let $\mathcal{U}$, $\mathcal{V}$ be full subcategories of $\mathcal{A}$. Then the *Gabriel product* $\mathcal{U} \diamond \mathcal{V}$ is defined to be the full subcategory

$$\mathcal{U} \diamond \mathcal{V} = \text{add}\{A \in \mathcal{A} \mid \exists \text{ an exact sequence } : 0 \rightarrow U \rightarrow A \rightarrow V \rightarrow 0,$$

where $U \in \mathcal{U}$ and $V \in \mathcal{V}\}$

Inductively we define $\mathcal{U}_1 \diamond \mathcal{U}_2 \diamond \cdots \diamond \mathcal{U}_n$, $\forall n \geq 1$, for full subcategories $\mathcal{U}_i$ of $\mathcal{A}$. It is easy to see that the operation $\diamond$ is associative and clearly $\mathcal{U}_1 \diamond \mathcal{U}_2 \diamond \cdots \diamond \mathcal{U}_n$ coincides with the full subcategory $\text{Filt}(\mathcal{U}_1, \cdots, \mathcal{U}_n)$ of $\mathcal{A}$ consisting of direct summands of objects $A$ which admit a finite filtration

$$0 = A_0 \subseteq A_1 \subseteq A_2 \subseteq \cdots \subseteq A_{n-1} \subseteq A_n = A$$

such that $A_k/A_{k-1} \in \mathcal{U}_k$, $1 \leq k \leq n$.

If $\mathcal{M}$ is contained in $\text{GProj} \mathcal{A}$, then we denote by $\Omega^{-1} \mathcal{M}$ the full subcategory of $\text{GProj} \mathcal{A}$ consisting of all direct summands of objects $A$ for which there exists an exact sequence $0 \rightarrow M \rightarrow P \rightarrow A \rightarrow 0$, where $M \in \mathcal{M}$ and $P$ is projective. Then $\Omega^{-k} \mathcal{M}$ is defined inductively for $k \geq 2$ by $\Omega^{-k} \mathcal{M} = \Omega^{-1} \Omega^{-k+1} \mathcal{M}$. Now we are ready to prove the main result of this section which generalizes, and is inspired by, some results of Iyama [40,39].

**Theorem 11.3.** Let $\mathcal{A}$ be an abelian category with enough projectives.

1. (a) The map $\chi \mapsto M = \chi^{-1} \chi$ gives a bijective correspondence between:
   (i) $(n+1)$-cluster tilting subcategories $\chi$ of $\text{GProj} \mathcal{A}$.
   (ii) Contravariantly finite subcategories $\mathcal{M}$ of $\text{GProj} \mathcal{A}$ such that:
   $$\mathcal{M}_{n+1} \cap \text{GProj} \mathcal{A} = \mathcal{M}$$
   (iii) Covariantly finite subcategories $\mathcal{M}$ of $\text{GProj} \mathcal{A}$ such that:
   $$\mathcal{M}_{n} \cap \text{GProj} \mathcal{A} = \mathcal{M}$$
Proof. 1. (b) Under the above correspondences:

(i) \( \mathcal{A} \) is Gorenstein if and only if \( \mathcal{M} \) is contravariantly finite in \( \mathcal{A} \) and \( \text{gl.dim mod-}\mathcal{M} < \infty \).

(ii) If \( \mathcal{A} \) is Gorenstein and \( \text{Proj } \mathcal{A} \) is covariantly finite, then \( \mathcal{M} \) is functorially finite in \( \mathcal{A} \).

2. Let \( \mathcal{A} \) be Gorenstein, \( 0 \neq \mathcal{X} \) be an \((n+1)\)-cluster tilting subcategory of \( \text{GProj } \mathcal{A} \), and \( \mathcal{M} = \pi^{-1} \mathcal{X} \).

(a) \( \text{pd } \mathbb{H}(\mathcal{G}) \leq n \), \( \forall \mathcal{G} \in \text{GProj } \mathcal{A} \), and setting \( d := \text{G.dim } \mathcal{A} \) there are equalities:

\[
\mathcal{A} = \mathcal{M} \circ \Omega^{-1} \mathcal{M} \circ \cdots \circ \Omega^{-n} \mathcal{M} \circ \text{Proj}^\leq d \mathcal{A} = \text{Filt}(\mathcal{M}, \Omega^{-1} \mathcal{M}, \cdots \Omega^{-n} \mathcal{M}, \text{Proj}^\leq d \mathcal{A})
\]  
\hspace{1cm} (11.2)

(b) \( \text{pd}_{\text{mod-}\mathcal{M}} F = n + 2 \), \( \forall F \in \text{mod-}\mathcal{X} \), \( F \neq 0 \), and

\[
\text{gl.dim mod-}\mathcal{M} = \max \{n + 2, \text{G.dim } \mathcal{A}\}
\]  
\hspace{1cm} (11.3)

In particular, if \( \text{G.dim } \mathcal{A} \leq n + 2 \), then: \( \text{gl.dim mod-}\mathcal{M} = n + 2 \).

(c) If \( \mathcal{A} \) has enough injectives, then any projective object \( \mathcal{Q} \) of \( \text{mod-}\mathcal{M} \) admits an injective resolution

\[
0 \longrightarrow \mathcal{Q} \longrightarrow J^0 \longrightarrow J^1 \longrightarrow \cdots \longrightarrow J^n \longrightarrow J^{n+1} \longrightarrow \cdots
\]

where: \( \text{pd } J^k \leq \text{G.dim } \mathcal{A} \), \( 0 \leq k \leq n + 1 \).

Proof. 1. Parts (a) and (b)(ii) follow from Lemmas 11.1, 11.2 and Theorem 5.3. For (b)(i) note that if \( \mathcal{M} \) is contravariantly finite in \( \mathcal{A} \) and \( \text{gl.dim mod-}\mathcal{M} < \infty \), then \( \mathcal{A} \) is Gorenstein by Lemma 11.2. If \( \mathcal{A} \) is Gorenstein, then \( \mathcal{M} \) is contravariantly finite in \( \mathcal{A} \) by Lemma 11.1. Finiteness of \( \text{gl.dim mod-}\mathcal{M} \) will be shown in 2. below.

2. (a) By Theorem 5.3 we have: \( \text{GProj } \mathcal{A} = \mathcal{X} \star \Omega^{-1} \mathcal{X} \star \cdots \star \Omega^{-n} \mathcal{X} \), and by parts (ii) and (iii) of Proposition 2.13 for any Gorenstein-projective object \( \mathcal{G} \) there exists an \( \mathcal{M} \)-resolution of \( \mathcal{G} \) of the form

\[
0 \longrightarrow \mathcal{M}^n \longrightarrow \mathcal{M}^{n-1} \longrightarrow \cdots \longrightarrow \mathcal{M}^1 \longrightarrow \mathcal{M}^0 \longrightarrow \mathcal{G} \longrightarrow 0
\]  
\hspace{1cm} (11.4)

Applying \( \mathbb{H} \) to (11.4) and using that \( \mathcal{M} \) is \( n \)-rigid, we have an exact sequence

\[
0 \longrightarrow \mathbb{H}(\mathcal{M}^n) \longrightarrow \mathbb{H}(\mathcal{M}^{n-1}) \longrightarrow \cdots \longrightarrow \mathbb{H}(\mathcal{M}^1) \longrightarrow \mathbb{H}(\mathcal{M}^0) \longrightarrow \mathbb{H}(\mathcal{G}) \longrightarrow 0
\]

which is a projective resolution of \( \mathbb{H}(\mathcal{G}) \). Hence \( \text{pd } \mathbb{H}(\mathcal{G}) \leq n \), \( \forall \mathcal{G} \in \text{GProj } \mathcal{A} \).

Since \( \mathcal{A} \) is Gorenstein, by [19], for any object \( \mathcal{A} \in \mathcal{A} \), there is an exact sequence \( 0 \longrightarrow \mathcal{Y}_A \longrightarrow \mathcal{G}_A \longrightarrow \mathcal{A} \longrightarrow 0 \), where \( \mathcal{G}_A \) is Gorenstein-projective and \( \mathcal{Y}_A \) has finite projective dimension; then \( \text{pd } \mathcal{Y}_A \leq d \) since \( \text{Proj}^\leq d \mathcal{A} = \text{Proj}^\leq d \mathcal{A} \), see Remark 6.3. Since \( \mathcal{G}_A \) is Gorenstein-projective, there exists an extension \( 0 \longrightarrow \mathcal{G}_A \longrightarrow \mathcal{P} \longrightarrow \Omega^{-1} \mathcal{G}_A \longrightarrow 0 \),...
where $P$ is projective and $\Omega^{-1}G_A$ is Gorenstein-projective. Then the composition $Y_A \rightarrow G_A \rightarrow P$ induces a short exact sequence $0 \rightarrow Y_A \rightarrow P \rightarrow Y^A \rightarrow 0$ and clearly $Y^A$ has finite projective dimension. By diagram chasing then it is easy to see that there exists a short exact sequence $0 \rightarrow G_A \rightarrow A \oplus P \rightarrow Y^A \rightarrow 0$. Hence $A \in \text{GProj}\mathscr{A} \circ \text{Proj}^{<\infty}\mathscr{A}$. Using that $G\text{Proj}\mathscr{A} = \mathcal{X} \ast \Omega^{-1}\mathcal{X} \ast \cdots \ast \Omega^{-n}\mathcal{X}$, it follows that $G\text{Proj}\mathscr{A} = M \circ \Omega^{-1}M \circ \cdots \circ \Omega^{-n}M$. We infer that $\mathscr{A} = G\text{Proj}\mathscr{A} \circ \text{Proj}^{<\infty}\mathscr{A} = M \circ \Omega^{-1}M \circ \cdots \circ \Omega^{-n}M \circ \text{Proj}^{<\infty}\mathscr{A}$.

2.(b) Since $\mathcal{X} \neq 0$ we have $M \neq \text{Proj}\mathscr{A}$. Then $\text{gl.dim mod-M} \geq n+2$ by Proposition 8.10. If $0 \neq F \in \text{mod-}\mathcal{X}$, then $F$ admits a presentation $0 \rightarrow \mathbb{H}(A_F) \rightarrow \mathbb{H}(M^1) \rightarrow \mathbb{H}(M^0) \rightarrow F \rightarrow 0$, where the map $M^1 \rightarrow M^0$ is epic. Since the $M^i$ are Gorenstein-projective and $G\text{Proj}\mathscr{A}$ is closed under kernels of epimorphisms, it follows that $A_F$ is Gorenstein-projective. Then by 2.(a) we have $\text{pd } \mathbb{H}(A_F) \leq n$ and therefore $\text{pd } F \leq n+2$. Since by Lemma 8.8, $\text{Ext}^k(F, \mathbb{H}(M)) = 0$, $0 \leq k \leq n+1$, and $F \neq 0$, we infer by Lemma 8.9 that $\text{pd } F = n+2$.

To show the equality (11.3), set $d := \text{G-dim } \mathscr{A}$. Let $F \in \text{mod-}M$ and consider the exact sequence $0 \rightarrow \mathbb{H}(A_F) \rightarrow \mathbb{H}(M^1) \rightarrow \mathbb{H}(M^0) \rightarrow F \rightarrow 0$. Consider, as in (8.3), an $M$-resolution of $A_F$:

$$
\cdots \rightarrow M^n_{A_F} \rightarrow M^{n-1}_{A_F} \rightarrow \cdots \rightarrow M^1_{A_F} \rightarrow M^0_{A_F} \rightarrow A_F \rightarrow 0 \quad (11.5)
$$

Then clearly $K^i_{A_F} \in M^i_F$, $\forall i \geq 1$. On the other hand composing the exact sequence

$$
0 \rightarrow A_F \rightarrow M^1 \rightarrow M^0 \rightarrow R(F) \rightarrow 0 \quad (11.6)
$$

with the exact sequence (11.5) we have an exact sequence

$$
\cdots \rightarrow M^n_{A_F} \rightarrow M^{n-1}_{A_F} \rightarrow \cdots \\
\rightarrow M^0_{A_F} \rightarrow M^1 \rightarrow M^0 \rightarrow R(F) \rightarrow 0 \quad (11.7)
$$

Assume first that $d \geq 3$. Since $G\text{Proj}\mathscr{A}$ is resolving and $\text{G-dim } R(F) \leq d$, we have clearly $K^{d-2}_{A_F} \in G\text{Proj}\mathscr{A}$, see [4, Lemma 3.12]. Hence $K^{d-2}_{A_F} \in M^{d-2}_{F} \cap G\text{Proj}\mathscr{A}$. If $d-2 > n$, then $K^{d-2}_{A_F} \in M^d_F \cap G\text{Proj}\mathscr{A} = M$. This implies that $\text{pd } \mathbb{H}(A_F) \leq d-2$ and therefore $\text{pd } F \leq d$. If $d-2 \leq n$, then using that $G\text{Proj}\mathscr{A}$ is resolving and $K^{d-2}_{A_F} \in G\text{Proj}\mathscr{A}$ it follows that $K^n_{A_F}$ is Gorenstein-projective and therefore lies in $M^d_F \cap G\text{Proj}\mathscr{A} = M$. This implies that $\mathbb{H}(A_F) \leq n$ and therefore $\text{pd } F \leq n+2$. On the other hand if $d \leq 2$, then clearly $d < n+2$ and $\text{G-dim } R(F) \leq 2$ and therefore (11.6) shows that $A_F$ lies in $G\text{Proj}\mathscr{A}$. It follows that $K^n_{A_F} \in M^d_F \cap G\text{Proj}\mathscr{A} = M$ and as above we have $\text{pd } F \leq n+2$. Hence $\text{pd } F \leq \max\{n+2, d\}$, for any $F \in \text{mod-M}$, and therefore $\text{gl.dim mod-M} \leq \max\{n+2, d\}$.

Summarizing we have shown that $n+2 \leq \text{gl.dim mod-M} \leq \max\{n+2, d\}$.

- If $d \leq n+2$, then we have $n+2 \leq \text{gl.dim mod-M} \leq n+2$, hence: $\text{gl.dim mod-M} = n+2 = \max\{n+2, d\}$.
• Assume that \( n + 2 \leq d \), so \( \text{gl.dim mod-}\mathcal{M} \leq d \). If \( \text{gl.dim mod-}\mathcal{M} \neq d \), then for any object \( A \in \mathcal{A} \) we have \( \text{pd } \mathbb{H}(A) \leq d - 1 \) and therefore there exists an \( \mathcal{M} \)-resolution \( 0 \rightarrow M^{d-1} \rightarrow M^{d-2} \rightarrow \cdots \rightarrow M^0 \rightarrow A \rightarrow 0 \) of \( A \) of length \( \leq d - 1 \). Since \( \mathcal{M} \) consists of Gorenstein-projective objects it follows that \( \text{G-dim } A \leq d - 1 \) and therefore \( d = \text{G-dim } \mathcal{A} \leq d - 1 \). This contradiction shows that \( \text{gl.dim mod-}\mathcal{M} = d = \max\{n + 2, d\} \).

2.(c) For any \( M \in \mathcal{M} \) consider an injective resolution \( 0 \rightarrow M \rightarrow I^0 \rightarrow I^1 \rightarrow \cdots \). Applying the functor \( \mathbb{H} \), and using that \( \mathcal{M} \) is \( n \)-rigid, we have an exact sequence in \( \text{mod-}\mathcal{M} \)

\[ 0 \rightarrow Q \rightarrow J^0 \rightarrow J^1 \rightarrow \cdots \rightarrow J^n \rightarrow J^{n+1} \]

where \( Q = \mathbb{H}(M) \) and \( J^k = \mathbb{H}(I^k) \), \( 0 \leq k \leq n + 1 \). Since the left adjoint \( \mathbb{R} \) of \( \mathbb{H} \) is exact it follows that the \( J^k \) are injective in \( \text{mod-}\mathcal{M} \). On the other hand, since \( \mathcal{A} \) is Gorenstein, we have \( \text{pd } E \leq \text{spli } \mathcal{A} = \text{G-dim } \mathcal{A} := d \) for any injective object \( E \in \mathcal{A} \). It follows that, for any \( k \geq 0 \), there exists a projective resolution

\[ 0 \rightarrow P^k_d \rightarrow P^k_{d-1} \rightarrow \cdots \rightarrow P^k_1 \rightarrow P^k_0 \rightarrow I^k \rightarrow 0 \]

Since \( \mathcal{M} \) consists of Gorenstein-projectives, we have \( \text{Ext}^m(\mathcal{M}, P) = 0 \), \( \forall m \geq 1 \), and for any projective object \( P \in \mathcal{A} \). Hence applying \( \mathbb{H} \) to the above resolution we obtain an exact sequence

\[ 0 \rightarrow \mathbb{H}(P^k_d) \rightarrow \mathbb{H}(P^k_{d-1}) \rightarrow \cdots \rightarrow \mathbb{H}(P^k_1) \rightarrow \mathbb{H}(P^k_0) \rightarrow J^k \rightarrow 0 \]

which is a projective resolution in \( \text{mod-}\mathcal{M} \) of the injective object \( J^k \). Hence \( \text{pd } J^k \leq d \).

\[ \square \]

**Corollary 11.4.** Let \( \mathcal{A} \) be a Gorenstein abelian category, \( \mathcal{X} \) an \( (n + 1) \)-cluster tilting subcategory of \( \text{GProj } \mathcal{A} \), and set \( \mathcal{M} = \pi^{-1}(\mathcal{X}) \subseteq \mathcal{A} \). If \( \text{G-dim } \mathcal{A} \leq n + 2 \), then the following are equivalent.

(i) \( \text{dom.dim mod-}\mathcal{M} \geq n + 2 \).

(ii) \( \mathcal{A} \) is Frobenius.

(iii) \( \text{mod-}\mathcal{M} \) is an \( n \)-Auslander category.

(iv) \( \mathcal{M} \) is an \( n \)-cluster (co)tilting subcategory of \( \mathcal{A} \).

Moreover if \( \mathcal{A} \) is a length category, then the short exact sequence of abelian categories

\[ 0 \rightarrow \text{mod-}\mathcal{X} \rightarrow \text{mod-}\mathcal{M} \rightarrow \mathcal{A} \rightarrow 0 \]

induces isomorphisms in \( K \)-theory: \( K_*(\mathcal{M}, \oplus) \cong K_*(\mathcal{A}) \oplus K_*(\text{mod-}\mathcal{X}) \).

**Proof.** (i) \( \iff \) (iii) \( \iff \) (iv) Since \( \text{G-dim } \mathcal{A} \leq n + 2 \), by Theorem 11.3 we have \( \text{gl.dim mod-} \mathcal{M} \leq n + 2 \). Then the assertions follow from Theorem 8.23.
(i) $\iff$ (ii) Since $\mathsf{G} \dim \mathcal{A} \leq n + 2$, by Theorem 11.3 it follows that $\mathsf{gl.dim mod-M} \leq n + 2 \leq \mathsf{dom.dim mod-M}$, hence $\mathsf{mod-M}$ is an $n$-Auslander category and therefore, by Theorem 8.23, $\mathcal{M}$ is an $n$-cluster tilting subcategory of $\mathcal{A}$. Then as in Proposition 10.16 we deduce that $\mathcal{A}$ is Frobenius. Conversely if this holds, then $\mathsf{G} \dim \mathcal{A} = 0$ and $\mathsf{dom.dim mod-M} \geq n + 2$ by Theorem 11.3(2)(c). \qed

Keller and Reiten proved that if $\mathcal{X}$ is a 2-cluster tilting subcategory of $\mathcal{A}$, where $\mathcal{A}$ is a Frobenius abelian category and if $\mathcal{A}$ is 2-Calabi–Yau, then $\mathsf{gl.dim mod-M} = 3$, where $\mathcal{M} = \pi^{-1}(\mathcal{X})$, see [46]. The following consequence of Theorems 8.23 and 9.6, and Corollary 9.4, generalizes, among others, the result of Keller and Reiten for any $(n+1)$-cluster tilting subcategory, $n \geq 2$, without assuming any Calabi–Yau condition.

**Corollary 11.5.** Let $\mathcal{A}$ be an abelian category with enough projectives and enough injectives. For a full subcategory $\mathsf{Proj} \mathcal{A} \neq \mathcal{M} \subseteq \mathsf{GProj} \mathcal{A}$, the following are equivalent.

(i) $\mathcal{A}$ is Frobenius and $\mathcal{M}$ is an $(n+1)$-cluster tilting subcategory of $\mathcal{A}$.
(ii) $\mathcal{M}$ is an $n$-cluster tilting subcategory of $\mathcal{A}$.
(iii) $\mathcal{M}$ is an $n$-cluster cotilting subcategory of $\mathcal{A}$.
(iv) $\mathsf{mod-M}$ is an $n$-Auslander category.
(v) $\mathcal{M}$ is $n$-rigid, contravariantly finite in $\mathcal{A}$ and contains the projectives, and $\mathsf{gl.dim mod-M} = n + 2$.
(vi) $\mathcal{M}$ is $n$-rigid, covariantly finite in $\mathcal{A}$ and contains the injectives, and $\mathsf{gl.dim mod-M} = n + 2$.

If (i) holds, and if $\mathcal{M}$ is of finite representation type, then:

$$\mathsf{rep.dim} \mathcal{A} \leq n + 2$$

We continue with some consequences of our previous results in this context. The first one follows from Corollary 7.7 or Corollary 6.5.

**Corollary 11.6.** Let $\mathcal{A}$ be Gorenstein and assume that $\mathsf{GProj} \mathcal{A}$ is $(n+1)$-Calabi–Yau. Let $\mathcal{X}$ be an $(n+1)$-cluster tilting subcategory of $\mathsf{GProj} \mathcal{A}$ and let $\mathcal{M} = \pi^{-1}(\mathcal{X})$. If $\mathsf{Hom}(\mathcal{M}, \Omega^i \mathcal{M}) = 0$, $0 \leq i \leq n - 1$, then $\mathsf{mod-M}$ is $1$-Gorenstein and the stable triangulated category $\mathsf{GProj mod-M}$ is $(n+2)$-Calabi–Yau.

The following result is due to Keller and Reiten, see [46, Theorem 5.4], called relative $(n+2)$-Calabi–Yau duality, proved in [46] in the context of an algebraic $(n+1)$-Calabi–Yau triangulated category over a field. In our setting we give a different proof.

**Proposition 11.7.** Let $\mathcal{A}$ be a Gorenstein abelian Hom-finite $k$-category over a field $k$, and assume that $\mathsf{GProj} \mathcal{A}$ is $(n+1)$-Calabi–Yau. Let $0 \neq \mathcal{X}$ be a $(n+1)$-cluster tilting
subcategory of \( \text{GProj}_\mathcal{A} \), where \( n \geq 1 \). If \( \mathcal{M} = \pi^{-1} \mathcal{X} \), then for any object \( F \in \text{mod-} \mathcal{X} \subseteq \text{mod-} \mathcal{M} \), there is a natural isomorphism:

\[
\text{DHom}_{\text{D}^b(\text{mod-} \mathcal{M})} (F, -) \cong \text{Hom}_{\text{D}^b(\text{mod-} \mathcal{M})} (-, F[n+2])
\]

In particular, for any two objects \( F, G \in \text{mod-} \mathcal{M} \) with \( F \in \text{mod-} \mathcal{X} \):

\[
\text{DExt}_{\text{mod-} \mathcal{M}}^i (F, G) \cong \text{Ext}_{\text{mod-} \mathcal{M}}^{n+2-i} (G, F), \quad i \in \mathbb{Z}
\]

**Proof.** By Theorem 11.3, we have \( \operatorname{gl.dim} \text{mod-} \mathcal{M} < \infty \). Then by [37], [57], the category \( \text{D}^b(\text{mod-} \mathcal{M}) \) admits a Serre functor which is given by \( S = - \otimes^L_M \text{DHom}_\mathcal{A} (-, ?)|_\mathcal{M} \), where \( \text{Hom}_\mathcal{A} (-, ?)|_\mathcal{M} (M) = \text{Hom}_\mathcal{A} (-, M) = \mathbb{H}(M) \). Hence we have a natural isomorphism

\[
\text{DHom}_{\text{D}^b(\text{mod-} \mathcal{M})} (F, -) \cong \text{Hom}_{\text{D}^b(\text{mod-} \mathcal{M})} (-, F \otimes^L_M \text{DHom}_\mathcal{A} (-, ?)|_\mathcal{M})
\]

So it suffices to show that we have an isomorphism \( F[n+2] \xrightarrow{\cong} F \otimes^L_M \text{DHom}_\mathcal{A} (-, ?)|_\mathcal{M} \) in \( \text{D}^b(\text{mod-} \mathcal{M}) \). Applying the triangulated functor \( - \otimes^L_M \text{DHom}_\mathcal{A} (-, ?)|_\mathcal{M} \) to the projective resolution of \( F \) as in (8.5)

\[
\cdots \rightarrow \mathbb{H}(M^n_{AF}) \rightarrow \mathbb{H}(M^{n-1}_{AF}) \rightarrow \cdots \\
\rightarrow \mathbb{H}(M^0_{AF}) \rightarrow \mathbb{H}(M^1) \rightarrow \mathbb{H}(M^0) \rightarrow F \rightarrow 0
\]

with \( F \) deleted, and noting that we may choose \( M^n_{AF} = 0 \) since \( \text{pd} F = n + 2 \), we obtain a complex

\[
0 \rightarrow \text{DHom}_\mathcal{A} (M^n_{AF}, -)|_\mathcal{M} \rightarrow \text{DHom}_\mathcal{A} (M^{n-1}_{AF}, -)|_\mathcal{M} \rightarrow \cdots \\
\rightarrow \text{DHom}_\mathcal{A} (M^0_{AF}, -)|_\mathcal{M} \rightarrow \cdots \\
\rightarrow \text{DHom}_\mathcal{A} (M^1, -)|_\mathcal{M} \rightarrow \text{DHom}_\mathcal{A} (M^0, -)|_\mathcal{M} \rightarrow 0
\]

which is acyclic everywhere except in first position on the left, which corresponds to \( -n - 2 \) degree, where the homology is given by \( \text{DExt}^1 (K_F^{n-1}, -)|_\mathcal{M} \). Since \( \text{GProj}_\mathcal{A} \) is \((n+1)\)-Calabi–Yau, we have natural isomorphisms:

\[
\text{DExt}^1(K_F^{n-1}, -)|_\mathcal{M} \cong \text{DHom}_\mathcal{A}(\Omega K_F^{n-1}, -)|_\mathcal{M} \cong \text{Hom}_\mathcal{A}(-, \Omega^{-n-1} \Omega K_F^{n-1})|_\mathcal{M} = \\
\cong \text{Hom}_\mathcal{A}(-, \Omega^{-n} K_F^{n-1})|_\mathcal{M} \cong \text{Hom}_\mathcal{A}(\Omega^n(-), K_F^{n-1})|_\mathcal{M} \cong \text{Ext}^n_\mathcal{A}(-, K_F^{n-1})|_\mathcal{M} \cong F
\]

where the last isomorphism follows from Lemma 8.6. We infer that in \( \text{D}^b(\text{mod-} \mathcal{M}) \) we have isomorphisms

\[
F[n+2] \xrightarrow{\cong} \text{Ext}^n_\mathcal{A}(-, K_F^{n-1})|_\mathcal{M}[n+2] \xrightarrow{\cong} F \otimes^L_M \text{DHom}_\mathcal{A} (-, ?)|_\mathcal{M} \quad \Box
\]

Under the assumptions of Proposition 11.7, Lemma 8.6 and Remark 8.11 admit the following consequence:
Corollary 11.8. Let $F \in \modmod{\mathcal{M}}$, $G \in \modmod{\mathcal{M}}$, and $M, M' \in \mathcal{M}$. Then there are isomorphisms:

$$\text{DExt}_{\modmod{\mathcal{M}}}^{n+2}(F, G) \cong \text{Hom}(G, F) \quad \text{and} \quad \text{DExt}_{\modmod{\mathcal{M}}}^{n+2}(F, (\cdot, M')) \cong F(M')$$

$$\text{DExt}_{\modmod{\mathcal{M}}}^{n+2}((-\cdot), (-, M')) \cong \text{DExt}_{\modmod{\mathcal{M}}}^{n+1}(M, M') \cong \text{Hom}(M', M)$$

We close this section with some applications of our previous results to Artin algebras. The first one is a consequence of Theorem 11.3 and Corollary 11.5.

Corollary 11.9. For an Artin algebra $\Lambda$ the following are equivalent.

(i) $\Lambda$ is Gorenstein and $\text{Gproj} \Lambda$ contains a $(n+1)$-cluster tilting object.

(ii) $\modmod{\Lambda}$ contains a generator $M$ such that $\text{add} M = M^\perp_n \cap \text{Gproj} \Lambda$ and $\text{gl.dim} \text{End}_{\Lambda}(M) < \infty$.

If (ii) holds, then $\text{rep.dim} \Lambda \leq n + 2$. Further $M$ is a cogenerator if and only if $\Lambda$ is self-injective.

Our final result illustrates some of the main results of the paper in the context of Gorenstein-projective modules over an Artin algebra $\Lambda$. Note that if $\Lambda$ is Gorenstein, then $\text{Gproj} \Lambda$ admits a Serre functor $\mathcal{S}$ which is given by $\mathcal{S}(A) = \Omega^{-1}X_{\text{DTr}A}$, where $X_{\text{DTr}A}$ is the minimal right $\text{Gproj} \Lambda$-approximation of $\text{DTr}A$. It follows that $\text{Gproj} \Lambda$ is $(n + 1)$-Calabi–Yau if and only if there is a natural isomorphism $X_{\text{DTr}A} \cong \Omega^{-n}A$, see [15].

Theorem 11.10. Let $\Lambda$ be an Artin algebra and let $T$ be an $(n+1)$-cluster tilting object of $\text{Gproj} \Lambda$, $n \geq 1$.

1. $\Lambda$ is Gorenstein if and only if $\text{gl.dim} \text{End}_{\Lambda}(T) < \infty$. If $\Lambda$ is Gorenstein, then:
   (a) $\text{gl.dim} \text{End}_{\Lambda}(T) = \max \{n + 2, \text{id} \Lambda\}$.
   (b) If $\text{id} \Lambda \leq n + 2$, then $\text{gl.dim} \text{End}_{\Lambda}(T) = n + 2$ and $\text{End}_{\Lambda}(T)$ admits an injective resolution
      $$0 \to \text{End}_{\Lambda}(T) \to J^0 \to J^1 \to \cdots \to J^{n+1} \to J^{n+2} \to 0$$
      in $\modmod{\text{End}_{\Lambda}(T)}$, where $\text{pd} J^t \leq \text{id} \Lambda$, $0 \leq t \leq n + 1$. Moreover the following are equivalent:
      (i) $\text{dom.dim} \text{End}_{\Lambda}(T) \geq n + 2$.
      (ii) $\Lambda$ is self-injective.
      (iii) $\text{End}_{\Lambda}(T)$ is an $n$-Auslander algebra.
      (iv) $T$ is an $n$-cluster (co)tilting $\Lambda$-module.
      In this case $\text{rep.dim} \Lambda \leq n + 2$. 
2. Assume that \( \text{Ext}^t_A(T, \text{DTr}T) = 0, \) \( 2 \leq t \leq n - k + 1, \) where \( 0 \leq k \leq n - 1. \)
   (a) If \( k \leq \frac{n+1}{2}, \) then the cluster tilted algebra \( \text{End}_A(T) \) is \( k \)-Gorenstein.
   (b) If \( 0 \leq k \leq 1, \) and \( \text{Gproj} \Lambda \) is \( (n+1) \)-Calabi–Yau, then \( \text{Gproj} \text{End}_A(T) \) is \( (n+2) \)-Calabi–Yau.
   (c) If \( k \leq \frac{n}{2}, \) denote by \( \mathcal{M}_k(T) \) the full subcategory of \( \text{Gproj} \Lambda: \)

   \[
   \mathcal{M}_k(T) := \{ \text{add } T \circ \text{add } \Omega^{-1}T \circ \cdots \circ \text{add } \Omega^{-k}T \}
   \]

   \[
   \bigcap \{ A \in \text{Gproj} \Lambda \mid \text{Ext}^t_A(A, \text{DTr}T) = 0, \ 2 \leq t \leq k + 1 \}
   \]

   Then \( \text{gl.dim } \text{End}_A(T) < \infty \) if and only if the functor

   \[
   \text{Hom}_A(T, -) : \mathcal{M}_k(T) \xrightarrow{\cong} \text{mod-End}_A(T)
   \]

   is an equivalence. In this case \( \text{gl.dim } \text{End}_A(T) = k \) with equality if \( \text{Ext}^{n-k+2}(T, \text{DTr}T) \neq 0. \)

3. Assume that \( \text{Ext}^t_A(T, \text{DTr}T) = 0, \) \( 2 \leq t \leq n. \)
   (a) If \( \text{End}_A(T) \) is of finite CM-type, then \( \text{rep.dim } \text{End}_A(T) \leq 3. \)
   (b) If \( \text{Gproj} \Lambda \) is \( (n+1) \)-Calabi–Yau and \( \text{End}_A(T) \) has finite global dimension, then \( \text{End}_A(T) \) is hereditary and there is a triangle equivalence:

   \[
   \text{Gproj} \Lambda \xrightarrow{\cong} \mathcal{C}^{n+1}_{\text{End}_A(T)}
   \]

   sending \( T \) to \( \pi(\text{End}_A(T)), \) where \( \mathcal{C}^{n+1}_{\text{End}_A(T)} \) is the \( (n+1) \)-cluster category associated to \( \text{End}_A(T). \)

4. If \( \text{Gproj} \Lambda \) is \( (n+1) \)-Calabi–Yau, then for any \( F \in \text{mod-End}_A(T) \) and any \( G \in \text{mod-End}_A(T): \)

   \[
   \text{DExt}^i_{\text{End}_A(T)}(F, G) \xrightarrow{\cong} \text{Ext}^{n-i+2}_{\text{End}_A(T)}(G, F), \ \forall i \in \mathbb{Z}
   \]

   \[
   \text{Ext}^{n+2}_{\text{End}_A(T)}(\text{End}_A(T), \text{End}_A(T)) \xrightarrow{\cong} \text{DEnd}_A(T)
   \]

**Proof.** Part 1. follows from Theorem 11.3 and Corollary 11.4. Part 2. follows from Theorems 6.4, 6.10, and 7.6, by observing that \( T \) is \( (n-k) \)-corigid, i.e. \( \text{Hom}_A(T, \Omega^iT) = 0, \) \( 1 \leq t \leq n-k \), if and only if \( \text{Ext}^t(T, \text{DTr}T) = 0, \ 2 \leq t \leq n-k+1, \) where \( 0 \leq k \leq n-1, \) if and only if \( T \in \mathcal{M}_k(T), \) as follows from Auslander–Reiten duality:

   \[
   \text{Ext}^{i+1}_A(T, \text{DTr}T) \cong \text{Ext}^1_A(\Omega^iT, \text{DTr}T) \cong \text{DHom}_A(T, \Omega^iT)
   \]

Since the triangulated category \( \text{Gproj} \Lambda \) is algebraic, part 3. follows from 2.(a) and Theorems 6.9 and 6.15. Finally part 4. follows from Proposition 11.7 and Corollary 11.8. \( \square \)
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