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LEFT TRIANGULATED CATEGORIES ARISING FROM CONTRAVARIANTLY FINITE SUBCATEGORIES

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ABSTRACT. Let modΛ be the category of finitely generated right Λ-modules over an artin algebra Λ, and F be an additive subfunctor of $\mathrm{Ext}^1_\Lambda(\ ,\ )$. Let $\mathcal{P}(F)$ denote the full sucategory of $\mathcal{A}$ with objects the F-projective modules. If the functor F has enough F- projectives, then we show that the stable category $\underline{\mathrm{mod}}_{\mathcal{P}(F)}\Lambda$ has a left triangulated structure. In case $F = \mathrm{Ext}^1_\Lambda(\ ,\ )$, the above statement implies that the stable category $\underline{\mathrm{mod}}_{\mathcal{P}}\Lambda$ of modΛ has a left triangulated structure. Dual statements for the case of F-injective modules are also true.

INTRODUCTION

The notion of a triangulated category was introduced almost thirty five years ago by Grothendieck and later by Verdier [13]. It became a very powerful tool first in algebraic geometry and later in many other branches of mathematics. The last decade influenced strongly the theory of representations of artin algebras through the work of Happel [6], [7].

An important field in the theory of representations of artin algebras is the study of the stable category of an artin algebra [1].

In the present paper we relate a triangulated (left or right) structure to stable categories of an artin algebra as follows: Let Λ be an artin algebra and modΛ be the category of finitely generated right Λ-modules.

Main Theorem 3.1.

(i) *Any contravariantly finite subcategory $\mathcal{X}$ of modΛ induces on the stable category $\underline{\mathrm{mod}}_{\mathcal{X}}\Lambda$ a left triangulated structure.*

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(ii) *Any covariantly finite subcategory $\mathcal{Y}$ of modΛ induces on the stable category* $\overline{\mathrm{mod}}_{\mathcal{Y}}\Lambda$ *a right triangulated structure.*

Let F be an additive subfunctor of $\mathrm{Ext}^1_\Lambda(\ ,\ )$ and $\mathcal{P}(F)$ (resp. $\mathcal{I}(F)$) be the full subcategory of modΛ with objects the F-projectives (resp. the F-injectives). Let $\underline{\mathrm{mod}}_{\mathcal{P}(F)}\Lambda$ (resp. $\overline{\mathrm{mod}}_{\mathcal{I}(F)}\Lambda$) denote the corresponding stable category.

From the Theorem above we obtain:

Theorem 3.2.

(i) *If the functor F has enough F-projectives, then $\mathcal{P}(F)$ induces on the stable category $\underline{\mathrm{mod}}_{\mathcal{P}(F)}\Lambda$ a left triangulated structure.*

(ii) *If the functor F has enough F-injectives, then $\mathcal{I}(F)$ induces on the stable category $\overline{\mathrm{mod}}_{\mathcal{I}(F)}\Lambda$ a right triangulated structure.*

In particular, if $F = \mathrm{Ext}^1_\Lambda(\ ,\ )$, our theorem implies that the stable category $\underline{\mathrm{mod}}\Lambda$ of modΛ modulo the Λ-projectives has a left triangulated structure and similarly the stable category $\overline{\mathrm{mod}}\Lambda$ modulo the Λ-injectives has a right triangulated structure.

Instead of working in the category of finitely generated right modules of an artin algebra and its stable category, we prefer to work in an arbitrary additive category. Namely, given an additive category $\mathcal{A}$ and a contravariantly finite subcategory $\mathcal{X}$, a notion introduced by Auslander and Smalø [2], and assuming that any $\mathcal{X}$-epic has a kernel, we show in Theorem 2.12 that the stable category obtained by these ingredients has always a left triangulated structure. Many well known constructions of triangulated categories can be derived using this theorem, by choosing the appropriate additive category and its contravariantly finite subcategory, cf. Remark 2.14.

In our paper we deal also with the dual situation of a covariantly finite subcategory $\mathcal{Y}$ of $\mathcal{A}$. In this case the stable category obtained from $\mathcal{A}$ and $\mathcal{Y}$ has a right triangulated structure.

1. On the stable category of an additive category by a homologically finite subcategory

In this paper we assume, unless otherwise stated, that all considered categories and their subcategories are additive. Functors between additive categories are assumed to be additive.

Let $\mathcal{A}$ be a category and $\mathcal{X}$ a contravariantly finite subcategory of $\mathcal{A}$. In the present section we shall construct a covariant endofunctor on the stable category of $\mathcal{A}$ determined by $\mathcal{X}$, under the assumption that any right $\mathcal{X}$-approximation has a kernel. A dual construction can be done for a covariantly finite subcategory $\mathcal{Y}$ of $\mathcal{A}$, under the assumption that any left $\mathcal{Y}$-approximation has a cokernel. We shall leave this to the reader.

We begin by recalling some definitions and notation concerning contravariantly finite subcategories of an arbitrary category $\mathcal{A}$ [2]. Related definitions for covariantly finite, homologically finite or functorially finite subcategories of $\mathcal{A}$, can be also found in [2].

Let $\mathcal{X}$ be a subcategory of $\mathcal{A}$. A morphism $f_B : X_B \to B$ of $\mathcal{A}$, with X_B an object of $\mathcal{X}$, is said to be a right $\mathcal{X}$-approximation of B, if $\mathcal{A}(X, f_B) : \mathcal{A}(X, X_B) \to \mathcal{A}(X, B)$ is surjective for all objects X of $\mathcal{X}$.

The subcategory $\mathcal{X}$ is said to be a contravariantly finite subcategory of $\mathcal{A}$, if any object B of $\mathcal{A}$ has a right $\mathcal{X}$-approximation.

Given two objects A and B of $\mathcal{A}$, we denote by $\mathcal{X}(A, B)$ the set of morphisms from A to B of $\mathcal{A}$ which factor over some object of $\mathcal{X}$. It is well known that $\mathcal{X}(A, B)$ is a subgroup of $\mathcal{A}(A, B)$, and that the family of these subgroups $\mathcal{X}(A, B)$ forms an ideal $\mathcal{J}_{\mathcal{X}}$ of $\mathcal{A}$.

Let $\mathcal{A}_{\mathcal{X}}$ be the stable category $\mathcal{A}/\mathcal{J}_{\mathcal{X}}$. We recall that $\mathcal{A}_{\mathcal{X}}$ has the same objects as $\mathcal{A}$ and that for any two objects A and B of $\mathcal{A}_{\mathcal{X}}$ the set of morphisms $\mathcal{A}_{\mathcal{X}}(A, B)$ is defined to be equal to the factorgroup $\mathcal{A}(A, B)/\mathcal{X}(A, B)$. The composition of $\mathcal{A}_{\mathcal{X}}$ is induced canonically by the composition of $\mathcal{A}$.

Let $\pi_{\mathcal{X}} : \mathcal{A} \to \mathcal{A}_{\mathcal{X}}$ be the projection functor induced by $Id_{\mathcal{A}}$. The image $\pi_{\mathcal{X}} A$ of any object A of $\mathcal{A}$ will be denoted by $\underline{A}$ and the image $\pi_{\mathcal{X}} f$ of any morphism f of $\mathcal{A}$ will be denoted by $\underline{f}$.

Suppose that any right $\mathcal{X}$-approximation has a kernel in $\mathcal{A}$. We associate to any object A of $\mathcal{A}$ a sequence of morphisms $K_A \xrightarrow{\iota_A} X_A \xrightarrow{f_A} A$, where f_A is a right $\mathcal{X}$-approximation of A and ι_A is a kernel of f_A. In this case, we say that an $\mathcal{X}$-*assignment* for $\mathcal{A}$ has been made.

We shall construct a covariant additive endofunctor of $\mathcal{A}_{\mathcal{X}}$, under the assumption that an $\mathcal{X}$-assignment for $\mathcal{A}$ has been made. This construction is inspired from the classical construction of the loop space functor thirty three years ago [9]. In case of a Frobenius category, a similar construction appears in [7].

Given a morphism $g : A \to B$ of $\mathcal{A}$, we consider the sequences $K_A \xrightarrow{\iota_A} X_A \xrightarrow{f_A} A$ and $K_B \xrightarrow{\iota_B} X_B \xrightarrow{f_B} B$ determined by the $\mathcal{X}$-assignment for $\mathcal{A}$. Since f_B is a right $\mathcal{X}$-approximation and ι_B is its kernel, one obtains the commutative diagram

$$\begin{array}{ccccc} K_A & \xrightarrow{\iota_A} & X_A & \xrightarrow{f_A} & A \\ \downarrow{\scriptstyle \kappa_g} & & \downarrow{\scriptstyle x_g} & & \downarrow{\scriptstyle g} \\ K_B & \xrightarrow{\iota_B} & X_B & \xrightarrow{f_B} & B \end{array}$$

and sees easily that $\underline{\kappa_g}$ is independent of the choice of x_g.

Hence, any $\mathcal{X}$-assignment for $\mathcal{A}$ yields a covariant functor $\Omega_{\mathcal{X}} : \mathcal{A}_{\mathcal{X}} \to \mathcal{A}_{\mathcal{X}}$ defined by $\Omega_{\mathcal{X}} \underline{A} = \underline{K_A}$ on the objects $\underline{A}$ of $\mathcal{A}_{\mathcal{X}}$ and by $\Omega_{\mathcal{X}} \underline{g} = \underline{\kappa_g}$ on the morphisms $\underline{g} : \underline{A} \to \underline{B}$ of $\mathcal{A}_{\mathcal{X}}$.

Definition 1.1. The functor $\Omega_{\mathcal{X}}$ is said to be the *loop space* functor of $\mathcal{A}_{\mathcal{X}}$.

Given two $\mathcal{X}$-assignments for $\mathcal{A}$, we derive easily that the obtained functors $\Omega_{\mathcal{X},1}$ and $\Omega_{\mathcal{X},2}$ are equivalent, because of the following construction:

Let us consider the commutative diagram

$$\begin{array}{ccccc} K_{A,1} & \xrightarrow{\iota_{A,1}} & X_B & \xrightarrow{f_{A,1}} & A \\ \downarrow{\scriptstyle z_A} & & \downarrow{\scriptstyle \alpha} & & \| \\ K_{A,2} & \xrightarrow{\iota_{A,2}} & X_B & \xrightarrow{f_{A,2}} & A \end{array}$$

where the rows correspond to the two different $\mathcal{X}$-assignments for $\mathcal{A}$. The existence of α follows from the fact that $f_{A,2}$ is a right $\mathcal{X}$-approximation of A and the existence of z_A follows from the fact that $\iota_{A,2}$ is the kernel of $f_{A,2}$.

One sees easily that the family $(\underline{z_A})$, where $\underline{A}$ runs over the objects of $\mathcal{A}_{\mathcal{X}}$, determines a natural isomorphism $\underline{z} : \Omega_{\mathcal{X},1} \to \Omega_{\mathcal{X},2}$.

For a covariantly finite subcategory $\mathcal{Y}$ of $\mathcal{A}$, there is the dual procedure. Firstly, we construct the stable category $\mathcal{A}^{\mathcal{Y}}$ of $\mathcal{A}$ by $\mathcal{Y}$ and assuming that any left $\mathcal{Y}$-approximation has a cokernel we define the notion of the *$\mathcal{Y}$-assignment for $\mathcal{A}$*. Any $\mathcal{Y}$-assignment for $\mathcal{A}$, determines a covariant additive endofunctor $\Sigma^{\mathcal{Y}} : \mathcal{A}^{\mathcal{Y}} \to \mathcal{A}^{\mathcal{Y}}$, called *the suspension functor* of $\mathcal{A}^{\mathcal{Y}}$.

We collect the remarks above in the next proposition.

Proposition 1.2. *Let $\mathcal{A}$ be an additive category, $\mathcal{X}$ an additive contravariantly finite subcategory of $\mathcal{A}$ and $\mathcal{Y}$ an additive covariantly finite subcategory of $\mathcal{A}$.*

(i) *If any right $\mathcal{X}$-approximation has a kernel, then any $\mathcal{X}$-assignment for $\mathcal{A}$ yields an endofunctor $\Omega_{\mathcal{X}} : \mathcal{A}_{\mathcal{X}} \to \mathcal{A}_{\mathcal{X}}$ and any two $\mathcal{X}$-assignments yield equivalent functors.*

(ii) *If any left $\mathcal{Y}$-approximation has a cokernel, then any $\mathcal{Y}$-assignment for $\mathcal{A}$ yields an endofunctor $\Sigma^{\mathcal{Y}} : \mathcal{A}^{\mathcal{Y}} \to \mathcal{A}^{\mathcal{Y}}$ and any two $\mathcal{Y}$-assignments yield equivalent functors.*

2. On the left triangulation of an additive category by a contravariantly finite subcategory

In the present section we shall enrich the stable category $\mathcal{A}_{\mathcal{X}}$ with a left triangulated structure, under the assumption that any $\mathcal{X}$-epic, a notion which will be defined in the sequel, has a kernel. We begin by recalling the definition of a left triangulated category [5], [10].

Let $\mathcal{C}$ be a category and $\Omega : \mathcal{C} \to \mathcal{C}$ a covariant endofunctor. Let $\mathcal{LTR}(\mathcal{C}, \Omega)$ denote the category with objects the diagrams of $\mathcal{C}$ of the form $\Omega C \xrightarrow{f} A \xrightarrow{g} B \xrightarrow{h} C$ and with set of morphisms from $\Omega C \xrightarrow{f} A \xrightarrow{g} B \xrightarrow{h} C$ to $\Omega C' \xrightarrow{f'} A' \xrightarrow{g'} B' \xrightarrow{h'} C'$ the triples (α, β, γ) of morphisms of $\mathcal{C}$ from (A, B, C) to (A', B', C'), which make the next diagram commutative:

$$\begin{array}{ccccccc}
\Omega C & \xrightarrow{f} & A & \xrightarrow{g} & B & \xrightarrow{h} & C \\
\downarrow{\scriptstyle \Omega\gamma} & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \gamma} \\
\Omega C' & \xrightarrow{f'} & A' & \xrightarrow{g'} & B' & \xrightarrow{h'} & C'
\end{array}$$

The composition of the morphisms of $\mathcal{LTR}(\mathcal{C}, \Omega)$ is induced in the canonical way by the corresponding composition of the morphisms of $\mathcal{C}$.

Definition 2.1. The category $\mathcal{LTR}(\mathcal{C}, \Omega)$ is said to be the *category of left triangles* of $\mathcal{C}$ and its objects $\Omega C \xrightarrow{f} A \xrightarrow{g} B \xrightarrow{h} C$ of $\mathcal{LTR}(\mathcal{C}, \Omega)$ are said to be the *left triangles* of $\mathcal{C}$.

Whenever it is convenient, the left triangles $\Omega C \xrightarrow{f} A \xrightarrow{g} B \xrightarrow{h} C$ will be denoted by (A, B, C, f, g, h).

Definition 2.2. A full subcategory Δ of $\mathcal{LTR}(\mathcal{C}, \Omega)$ is said to be a *left triangulation* of $(\mathcal{C}, \Omega)$ if it is closed under isomorphisms and satisfies the following four axioms:

(LT1i) For any object A of $\mathcal{C}$, the left triangle $0 \xrightarrow{0} A \xrightarrow{1_A} A \xrightarrow{0} 0$ belongs to Δ.

(LT1ii) For any morphism $h : B \to C$, there is a left triangle in Δ of the form $\Omega C \xrightarrow{f} A \xrightarrow{g} B \xrightarrow{h} C$.

(LT2) For any left triangle $\Omega C \xrightarrow{f} A \xrightarrow{g} B \xrightarrow{h} C$ in Δ, the left triangle $\Omega B \xrightarrow{-\Omega h} \Omega C \xrightarrow{f} A \xrightarrow{g} B$ is also in Δ.

(LT3) For any two left triangles $\Omega C \xrightarrow{f} A \xrightarrow{g} B \xrightarrow{h} C$, $\Omega C' \xrightarrow{f'} A' \xrightarrow{g'} B' \xrightarrow{h'} C'$ in Δ and any two morphisms $\beta : B \to B'$ and $\gamma : C \to C'$ of $\mathcal{C}$ with $\gamma \circ h = h' \circ \beta$, there is a morphism $\alpha : A \to A'$ of $\mathcal{C}$ such that the triple (α, β, γ) is a morphism from the first triangle to the second.

(LT4) For any two left triangles $\Omega C \xrightarrow{f} A \xrightarrow{g} B \xrightarrow{h} C$ and $\Omega D \xrightarrow{i} E \xrightarrow{\ell} C \xrightarrow{k} D$ in Δ, there is a third left triangle $\Omega D \xrightarrow{j} F \xrightarrow{m} B \xrightarrow{k \circ h} D$ in Δ and two morphisms $\alpha : A \to F$ and $\beta : F \to E$ of $\mathcal{C}$, such that the diagram below is fully commutative, where the second column from the left is a left triangle in Δ.

$$\begin{array}{ccccccc}
 & & \Omega E & & & & \\
 & & \downarrow f\circ\Omega\ell & & & & \\
\Omega C & \xrightarrow{f} & A & \xrightarrow{g} & B & \xrightarrow{h} & C \\
\downarrow \Omega k & & \downarrow \alpha & & \downarrow 1_B & & \downarrow k \\
\Omega D & \xrightarrow{j} & F & \xrightarrow{m} & B & \xrightarrow{k\circ h} & D \\
\downarrow 1_{\Omega D} & & \downarrow \beta & & \downarrow h & & \downarrow 1_D \\
\Omega D & \xrightarrow{i} & E & \xrightarrow{\ell} & C & \xrightarrow{k} & D
\end{array}$$

(i.e. the triples $(\alpha, 1_B, k)$ and $(\beta, h, 1_D)$ are morphisms of Δ and $\Omega E \xrightarrow{f\circ\Omega\ell} A \xrightarrow{\alpha} F \xrightarrow{\beta} E$ is in Δ).

Definition 2.3. The triple $(\mathcal{C}, \Omega, \Delta)$ is called a *left triangulated category*

Definition 2.4. A covariant functor $F : \mathcal{C} \to \mathcal{C}'$ is said to be a *triangle equivalence* from the left triangulated category $(\mathcal{C}, \Omega, \Delta)$ to the left triangulated category $(\mathcal{C}', \Omega', \Delta')$, if it is an equivalence and there is a natural isomorphism $\phi : F \circ \Omega \to \Omega' \circ F$, such that if (A, B, C, f, g, h) is a left triangle of Δ, then the left triangle $(FA, FB, FC, Ff \circ \phi_C^{-1}, Fg, Fh)$ is in Δ'.

We leave it for the reader to define the dual notion of a right triangulated category $(\mathcal{C}, \Sigma, \Delta')$, where $\Sigma : \mathcal{C} \to \mathcal{C}$ is a covariant endofunctor and Δ' is a full subcategory of the category of right triangles $\mathcal{RTR}(\mathcal{C}, \Sigma)$ satisfying the right analogs to LT1-LT4 axioms.

Remark 2.5. If the endofunctor $\Omega : \mathcal{C} \to \mathcal{C}$ is an equivalence, then LT1-LT4 imply that the left triangulated category $(\mathcal{C}, \Omega, \Delta)$ is also a right triangulated category. So, $(\mathcal{C}, \Omega, \Delta)$ is a triangulated category in the classical sense, [13].

Suppose that $\mathcal{X}$ is a contravariantly finite subcategory of $\mathcal{A}$.

Definition 2.6. A morphism $g : A \to B$ of $\mathcal{A}$ is said to be an *$\mathcal{X}$-epic*, if for any object X of $\mathcal{X}$ the induced homomorphism $\mathcal{A}(X, g) : \mathcal{A}(X, A) \to \mathcal{A}(X, B)$ is surjective.

Obviously, any $\mathcal{X}$-approximation is an $\mathcal{X}$-epic. For a covariantly finite subcategory $\mathcal{Y}$ of $\mathcal{A}$, there is the dual notion of a $\mathcal{Y}$-*monic*.

From now on, we assume that given a contravariantly finite subcategory $\mathcal{X}$ of $\mathcal{A}$ an $\mathcal{X}$-assignment for $\mathcal{A}$ has been made and that any $\mathcal{X}$-epic has a kernel. For the rest of the paper, we shall deal only with the contravariantly finite case, leaving for the reader to state and prove dual results for the covariantly finite case. Results concerning covariantly finite subcategories will be stated only in theorems.

Lemma 2.7. *Let $f : C \to B$ and $g : A \to B$ be two morphisms of $\mathcal{A}$. If f is an $\mathcal{X}$-epic, then*

(i) *The morphism $(g, f) : A \oplus C \to B$ is also an $\mathcal{X}$-epic.*
(ii) *The pullback $(h : P \to A, k : P \to C)$ of f and g exists and the morphism h is also an $\mathcal{X}$-epic.*

Proof. (i) Since, $\mathcal{A}(X, f) : \mathcal{A}(X, C) \to \mathcal{A}(X, B)$ is surjective for any X of $\mathcal{X}$, the same is true for $\mathcal{A}(X, (g, f)) : \mathcal{A}(X, A \oplus C) \to \mathcal{A}(X, B)$. (ii) The kernel of (g, f) exists because (g, f) is also an $\mathcal{X}$-epic. So we have

$$P \xrightarrow{\binom{k}{-h}} A \oplus C \xrightarrow{(g,f)} B$$

where the left morphism is the kernel of (g, f). Given a morphism $\ell : L \to A$, there is a morphism $m : L \to C$ with $g \circ \ell = f \circ m$ and because $(h : P \to A, k : P \to C)$ is the pullback of (f, g), there is a morphism $n : L \to P$ satisfying $h \circ n = \ell$. □

Let $\mathcal{LTR}(\mathcal{A}_{\mathcal{X}}, \Omega_{\mathcal{X}})$ denote the category of left triangles of $\mathcal{A}_{\mathcal{X}}$. We shall construct in $\mathcal{LTR}(\mathcal{A}_{\mathcal{X}}, \Omega_{\mathcal{X}})$ two kinds of left triangles, the distinguished and the induced ones.
The distinguished triangles are obtained as follows:
Let $g : A \to B$ be a morphism of $\mathcal{A}$ and $f_B : X_B \to B$ be the right $\mathcal{X}$-approximation of B determined by the $\mathcal{X}$-assignment for $\mathcal{A}$. Because of Lemma 2.7 the morphism $(g, f_B) : A \oplus X_B \to B$ is an $\mathcal{X}$-epic and the pullback of g and f_B exists. This pullback is uniquely determined up to isomorphism by g, because f_B is fixed. So, given a morphism $g : A \to B$, we have the commutative diagram

$$\begin{array}{ccccc} K_B & \xrightarrow{\zeta_g} & \mathrm{Con}(g) & \xrightarrow{\eta_g} & A \\ \downarrow 1_{K_B} & & \downarrow \theta_g & & \downarrow g \\ K_B & \xrightarrow{\iota_B} & X_B & \xrightarrow{f_B} & B \end{array} \tag{2.1}$$

where $(\mathrm{Con}(g), \eta_g, \theta_g)$ is the pullback of f_B and g and where η_g is an $\mathcal{X}$-epic with kernel $\zeta_g : K_B \to \mathrm{Con}(g)$.

Definition 2.8. A left triangle $\Omega_{\mathcal{X}}\underline{G} \xrightarrow{k} \underline{D} \xrightarrow{\ell} \underline{F} \xrightarrow{m} \underline{G}$ of $\mathcal{LTR}(\mathcal{A}_{\mathcal{X}}, \Omega_{\mathcal{X}})$ is said to be $\mathcal{X}$-*distinguished*, if it is isomorphic to $\Omega_{\mathcal{X}}\underline{B} \xrightarrow{\zeta_g} \underline{\mathrm{Con}(g)} \xrightarrow{-\eta_g} \underline{A} \xrightarrow{g} \underline{B}$ for some $g : A \to B$ of $\mathcal{A}$.

We shall call the $\mathcal{X}$-distinguished left triangles, just distinguished left triangles, whenever it is clear from the context which is the contravariantly finite subcategory $\mathcal{X}$ of $\mathcal{A}$.

We leave it for the reader to state the dual definition for the $\mathcal{Y}$-*distinguished* right triangles of the category $\mathcal{RTR}(\mathcal{A}^{\mathcal{Y}}, \Sigma^{\mathcal{Y}})$, for a covariantly finite subcategory $\mathcal{Y}$ of $\mathcal{A}$.

The induced triangles are obtained as follows:

Let $g : A \to B$ be an $\mathcal{X}$-epic of $\mathcal{A}$ and $f_B : X_B \to B$ be the right $\mathcal{X}$-approximation of B determined by the $\mathcal{X}$-assignment for $\mathcal{A}$. We consider the commutative diagram

$$\begin{array}{ccccc} K_B & \xrightarrow{\iota_B} & X_B & \xrightarrow{f_B} & B \\ \downarrow \gamma & & \downarrow \delta & & \downarrow 1_B \\ C & \xrightarrow{h} & A & \xrightarrow{g} & B \end{array} \tag{2.2}$$

where ι_B is the kernel of f_B and h the kernel of g. The morphism δ exists because g is an $\mathcal{X}$-epic and X_B belongs to $\mathcal{X}$. The morphism γ exists because h is the kernel of g. We claim that the morphism $\underline{\gamma}$ of $\mathcal{A}_{\mathcal{X}}$ is uniquely determined by γ. Indeed, if $\delta' : X_B \to A$ satisfies $g \circ \delta' = f_B$ and $\gamma' : K_B \to C$ satisfies $h \circ \gamma' = \delta' \circ \iota_B$, then there is a morphism $m : X_B \to C$ such that $\delta - \delta' = h \circ m$, because $g \circ (\delta - \delta') = 0$. Hence, $h \circ (\gamma - \gamma') = (\delta - \delta') \circ \iota_B = h \circ m \circ \iota_B$ and since h is a monomorphism, we obtain $\gamma - \gamma' = m \circ \iota_B$. So, $\gamma - \gamma'$ factorizes over X_B, i.e. $\underline{\gamma} = \underline{\gamma}'$.

This uniquely determined by the $\mathcal{X}$-epic g, morphism $\underline{\gamma}$ of $\mathcal{A}_{\mathcal{X}}$ is said to be the *characteristic class* of g and it is denoted by $ch(g)$.

Definition 2.9. A left triangle $\Omega_{\mathcal{X}}\underline{G} \xrightarrow{k} \underline{D} \xrightarrow{\ell} \underline{F} \xrightarrow{m} \underline{G}$ of $\mathcal{LTR}(\mathcal{A}_{\mathcal{X}}, \Omega_{\mathcal{X}})$ is said to be $\mathcal{X}$-*induced*, if it is isomorphic to $\Omega_{\mathcal{X}}\underline{B} \xrightarrow{ch(g)} \underline{C} \xrightarrow{h} \underline{A} \xrightarrow{g} \underline{B}$ for some $\mathcal{X}$-epic $g : A \to B$ of $\mathcal{A}$.

In the following we call the $\mathcal{X}$-induced left triangles, just induced left triangles, whenever it is clear from the context which is the contravariantly finite subcategory $\mathcal{X}$ of $\mathcal{A}$.

We leave it for the reader to state the dual definition for the $\mathcal{Y}$-*induced* right triangles of $\mathcal{RTR}(\mathcal{A}^{\mathcal{Y}}, \Sigma^{\mathcal{Y}})$, where $\mathcal{Y}$ is a covariantly finite subcategory of $\mathcal{A}$.

Proposition 2.10. *Any distinguished left triangle is isomorphic to an induced one and any induced left triangle is isomorphic to a distinguished one.*

Proof. Given a morphism $g : A \to B$, we form the $\mathcal{X}$-epic $(g, f_B) : A \oplus X_B \to B$ and the commutative diagram:

$$\begin{array}{ccccc} K_B & \xrightarrow{\iota_B} & X_B & \xrightarrow{f_B} & B \\ \downarrow \zeta_g & & \downarrow \begin{pmatrix} 0 \\ 1_{X_B} \end{pmatrix} & & \downarrow 1_B \\ \mathrm{Con}(g) & \xrightarrow{\begin{pmatrix} -\eta_g \\ \theta_g \end{pmatrix}} & A \oplus X_B & \xrightarrow{(g, f_B)} & B \end{array}$$

This diagram shows that the distinguished left triangle obtained from g is isomorphic the induced left triangle obtained from the $\mathcal{X}$-epic (g, f_B).

Conversely, we shall prove that given an $\mathcal{X}$-epic $g : A \to B$, the corresponding induced left triangle $\Omega_{\mathcal{X}}\underline{B} \xrightarrow{ch(g)} \underline{C} \xrightarrow{\underline{h}} \underline{A} \xrightarrow{\underline{g}} \underline{B}$ is isomorphic to the distinguished left triangle obtained from g.

Let $(\mathrm{Con}(g), \eta_g, \theta_g)$ be the pullback of f_B and $g$and let $\delta : X_B \to A$, be a morphism as in diagram (2.2) satisfying $g\circ\delta = f_B$. This pullback and the morphisms δ and 1_{X_B}, yield a morphism $\lambda_g : X_B \to \mathrm{Con}(g)$, such that $\delta = \eta_g\circ\lambda_g$ and $1_{X_B} = \theta_g\circ\lambda_g$.

Since g is an $\mathcal{X}$-epic, the morphism θ_g is also an $\mathcal{X}$-epic because of 2.7, and $(\mathrm{Con}(g), \eta_g, \theta_g)$ yields the commutative diagram

$$\begin{array}{ccccc} C & \xrightarrow{m} & \mathrm{Con}(g) & \xrightarrow{\theta_g} & X_B \\ \downarrow 1_C & & \downarrow \eta_g & & \downarrow f_B \\ C & \xrightarrow{h} & A & \xrightarrow{g} & B \end{array}$$

where h is the kernel of g and m is the kernel of θ_g. Let us consider the commutative diagram:

$$\begin{array}{ccccc} C & \xrightarrow{h} & A & \xrightarrow{g} & B \\ \Big\downarrow m & & \Big\downarrow \begin{pmatrix} 1_A \\ 0 \end{pmatrix} & & \Big\downarrow 1_B \\ \mathrm{Con}(g) & \xrightarrow{\begin{pmatrix} \eta_g \\ -\theta_g \end{pmatrix}} & A \oplus X_B & \xrightarrow{(g, f_B)} & B \\ \Big\downarrow t & & \Big\downarrow (1_A, \delta) & & \Big\downarrow 1_B \\ C & \xrightarrow{h} & A & \xrightarrow{g} & B \end{array}$$

The morphism $t : \mathrm{Con}(g) \to C$ exists and makes the left bottom square commutative, because h is the kernel of g. Because of the same reason, $t\circ m$ has to be equal to 1_C.

The commutativity of the left bottom square implies $h\circ t\circ\zeta_g = \eta_g\circ\zeta_g - \delta\circ\theta_g\circ\zeta_g = -\delta\circ\iota_B$. This shows that in the construction of the induced left triangle of g as in diagram (2.2) we may take $\gamma = -t\circ\zeta_g$, i.e. $ch(g) = -\underline{t}\circ\underline{\zeta}_g$.

Let us consider the morphism $1_{Con(g)} - \lambda_g\circ\theta_g : \mathrm{Con}(g) \to \mathrm{Con}(g)$. Since, $\theta_g\circ(1_{Con(g)} - \lambda_g\circ\theta_g) = 0$ and m is the kernel of θ_g, there is a morphism $\xi : \mathrm{Con}(g) \to C$ with $m\circ\xi = 1_{Con(g)} - \lambda_g\circ\theta_g$. So, $t\circ m\circ\xi = t\circ(1_{Con(g)} - \lambda_g\circ\theta_g)$ and so $\xi = t - t\circ\lambda_g\circ\theta_g$. This implies that in $\mathcal{A}_{\mathcal{X}}$ we have $\underline{\xi} = \underline{t}$, $\underline{m}\circ\underline{t} = \underline{1}_{Con(g)}$ and $\underline{\zeta}_g = \underline{m}\circ\underline{t}\circ\underline{\zeta}_g = -\underline{m}\circ ch(g)$

Hence, in the diagram below all squares commute and the vertical morphisms are isomorphisms

$$\begin{array}{ccccccc} \Omega_{\mathcal{X}}\underline{B} & \xrightarrow{ch(g)} & \underline{C} & \xrightarrow{\underline{h}} & \underline{A} & \xrightarrow{\underline{g}} & \underline{B} \\ \| & & \downarrow -\underline{m} & & \| & & \| \\ \Omega_{\mathcal{X}}\underline{B} & \xrightarrow{\underline{\zeta}_g} & \underline{\mathrm{Con}(g)} & \xrightarrow{-\underline{\eta}_g} & \underline{A} & \xrightarrow{\underline{g}} & \underline{B} \end{array}$$

This finishes the proof of the second part of our claim. $\square$

Let $\mathcal{LTR}(\mathcal{A}_{\mathcal{X}}, \Omega_{\mathcal{X}})$ be the category of left triangles of $(\mathcal{A}_{\mathcal{X}}, \Omega_{\mathcal{X}})$ and $\Delta_{\mathcal{X}}$ be the full subcategory of $\mathcal{LTR}(\mathcal{A}_{\mathcal{X}}, \Omega_{\mathcal{X}})$ with objects the distinguished left triangles of

$\mathcal{LTR}(\mathcal{A}_{\mathcal{X}},\Omega_{\mathcal{X}})$. We shall show that the triple $(\mathcal{A}_{\mathcal{X}},\Omega_{\mathcal{X}},\Delta_{\mathcal{X}})$ is a left triangulated category

Lemma 2.11. *If $h : B \to C$, $k : C \to D$ are two $\mathcal{X}$-epics of $\mathcal{A}$ with kernels $g : A \to B$ and $\ell : E \to C$ respectively, then there is a fully commutative diagram*

$$\begin{array}{ccccc} A & \xrightarrow{g} & B & \xrightarrow{h} & C \\ \downarrow\alpha & & \downarrow 1_B & & \downarrow k \\ F & \xrightarrow{m} & B & \xrightarrow{k\circ h} & D \\ \downarrow\beta & & \downarrow h & & \downarrow 1_D \\ E & \xrightarrow{\ell} & C & \xrightarrow{k} & D \end{array}$$

where $k\circ h$ and β are $\mathcal{X}$-epics with kernels m and α respectively.

Proof. The kernel $m : F \to B$ exists, because $k\circ h$ is an $\mathcal{X}$-epic. The morphisms α and β exist because m and ℓ are kernels. We claim that β is also an $\mathcal{X}$-epic. Let us consider in the category of the abelian groups, the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc} 0 & \to & \mathcal{A}(\mathcal{X},F) & \xrightarrow{m_*} & \mathcal{A}(\mathcal{X},B) & \xrightarrow{(k\circ h)_*} & \mathcal{A}(\mathcal{X},D) & \to & 0 \\ & & \downarrow\beta_* & & \downarrow h_* & & \| & & \\ 0 & \to & \mathcal{A}(\mathcal{X},E) & \xrightarrow{\ell_*} & \mathcal{A}(\mathcal{X},C) & \xrightarrow{k_*} & \mathcal{A}(\mathcal{X},D) & \to & 0 \end{array}$$

Because h_* is an epimorphism, β_* is also an epimorphism. So, β is $\mathcal{X}$-epic. Since $\ell\circ\beta\circ\alpha = h\circ g = 0$ and ℓ is a monomorphism, we obtain $\beta\circ\alpha = 0$. Let $n : H \to F$ be a morphism with $\beta\circ n = 0$. We have $\ell\circ\beta\circ n = h\circ m\circ n = 0$. Hence, there is a morphism $p : H \to A$ with $g\circ p = m\circ n$ and so $m\circ\alpha\circ p = m\circ n$. Since m is a monomorphism we obtain $\alpha\circ p = n$. This implies that α is the kernel of β. □

Theorem 2.12. *Let $\mathcal{A}$ be an additive category, $\mathcal{X}$ a contravariantly finite subcategory and $\mathcal{Y}$ a covariantly finite subcategory of $\mathcal{A}$.*

(i) *If any $\mathcal{X}$-epic has a kernel and an $\mathcal{X}$-assignment for $\mathcal{A}$ has been made, then the full subcategory $\Delta_{\mathcal{X}}$ of the distinguished left triangles of $\mathcal{LTR}(\mathcal{A}_{\mathcal{X}},\Omega_{\mathcal{X}})$ is a left triangulation of $(\mathcal{A}_{\mathcal{X}},\Omega_{\mathcal{X}})$.*

(ii) *If any $\mathcal{Y}$-monic has a cokernel and a $\mathcal{Y}$-assignment for $\mathcal{A}$ has been made, then the full subcategory $\Delta^{\mathcal{Y}}$ of the distinguished right triangles of $\mathcal{RTR}(\mathcal{A}^{\mathcal{Y}},\Sigma^{\mathcal{Y}})$ is a right triangulation of $(\mathcal{A}^{\mathcal{Y}},\Sigma^{\mathcal{Y}})$.*

Proof. (i) We shall verify all axioms for a left triangulated category one by one.

LT1 i. The induced left triangle of the $\mathcal{X}$-epic $0 : A \to 0$, is equal to $0 \xrightarrow{0} A \xrightarrow{1_A} A \xrightarrow{0} 0$.

LT1 ii. Any morphism $g : A \to B$, embeds to the distinguished left triangle $\Omega_{\mathcal{X}}\underline{B} \xrightarrow{\underline{\zeta}_g} \underline{\mathrm{Con}}(g) \xrightarrow{-\underline{\eta}_g} \underline{A} \xrightarrow{\underline{g}} \underline{B}$

LT2. It is enough to prove that for any morphism $g : A \to B$ the left triangle $\Omega_{\mathcal{X}}\underline{A} \xrightarrow{-\Omega_{\mathcal{X}}\underline{g}} \Omega_{\mathcal{X}}\underline{B} \xrightarrow{\underline{\zeta}_g} \underline{\mathrm{Con}}(g) \xrightarrow{-\underline{\eta}_g} \underline{A}$ is in $\Delta_{\mathcal{X}}$, since if a left triangle $(\underline{D},\underline{F},\underline{G},\underline{k},\underline{\ell},\underline{h})$ is isomorphic to some $(\underline{\mathrm{Con}}(g),\underline{A},\underline{B},\underline{\zeta}_g,-\underline{\eta}_g,\underline{g})$,then $(\Omega_{\mathcal{X}}\underline{G},\underline{D},\underline{F},-\Omega_{\mathcal{X}}(\underline{h}),\underline{k},\underline{\ell})$ is isomorphic to $(\Omega_{\mathcal{X}}\underline{B},\underline{\mathrm{Con}}(g),\underline{A},-\Omega_{\mathcal{X}}(\underline{g}),\underline{\zeta}_g,-\underline{\eta}_g)$.

Given the morphism $g : A \to B$, we consider the commutative diagram

$$(2.3)\qquad \begin{tikzcd} K_A \arrow[r, "\iota_A"] \arrow[d, "\gamma"] & X_A \arrow[r, "f_A"] \arrow[d, "\delta"] & A \arrow[d, "1_A"] \\ K_B \arrow[r, "\zeta_g"] \arrow[d, "1_{K_B}"] & \mathrm{Con}(g) \arrow[r, "\eta_g"] \arrow[d, "\theta_g"] & A \arrow[d, "g"] \\ K_B \arrow[r, "\iota_B"] & X_B \arrow[r, "f_B"] & B \end{tikzcd}$$

and the commutative diagram

$$(2.4)\qquad \begin{tikzcd} K_A \arrow[r, "\iota_A"] \arrow[d, "\gamma"] & X_A \arrow[r, "f_A"] \arrow[d, "\theta_g\circ\delta"] & A \arrow[d, "g"] \\ K_B \arrow[r, "\iota_B"] & X_B \arrow[r, "f_B"] & B \end{tikzcd}$$

which is obtained from (2.3) by composition. Because of (2.4) we have that $\Omega_{\mathcal{X}}\underline{g} = \underline{\gamma} = ch(g)$. Hence, the induced left triangle obtained from the $\mathcal{X}$-epic η_g equals $\Omega_{\mathcal{X}}\underline{A} \xrightarrow{\Omega_{\mathcal{X}}\underline{g}} \Omega_{\mathcal{X}}\underline{B} \xrightarrow{\underline{\zeta}_g} \underline{\mathrm{Con}}(g) \xrightarrow{\underline{\eta}_g} \underline{A}$. The triple $(1_{\Omega_{\mathcal{X}}\underline{B}}, 1_{\underline{\mathrm{Con}}(g)}, -1_{\underline{A}})$ is an isomorphism from $\Omega_{\mathcal{X}}\underline{A} \xrightarrow{\Omega_{\mathcal{X}}\underline{g}} \Omega_{\mathcal{X}}\underline{B} \xrightarrow{\underline{\zeta}_g} \underline{\mathrm{Con}}(g) \xrightarrow{\underline{\eta}_g} \underline{A}$ to $\Omega_{\mathcal{X}}\underline{A} \xrightarrow{-\Omega_{\mathcal{X}}\underline{g}} \Omega_{\mathcal{X}}\underline{B} \xrightarrow{\underline{\zeta}_g} \underline{\mathrm{Con}}(g) \xrightarrow{-\underline{\eta}_g} \underline{A}$. Hence, this last triangle belongs to $\Delta_{\mathcal{X}}$.

LT3. We assume without loss of generality, that our triangles are left distinguished, since a comparison of left triangles via isomorphisms in $\mathcal{LTR}(\mathcal{C},\Omega)$, gives the general case, compare [7] Th.2.6, for the special case of a Frobenius category.

Let $T_g = (\underline{\mathrm{Con}}(g), \underline{A}, \underline{B}, \underline{\zeta}_g, -\underline{\eta}_g, \underline{g})$ and $T_h = (\underline{\mathrm{Con}}(h), \underline{C}, \underline{D}, \underline{\zeta}_h, -\underline{\eta}_h, \underline{h})$ be two distinguished left triangles and let $\underline{k} : \underline{A} \to \underline{C}$, $\underline{\ell} : \underline{B} \to \underline{D}$ be two morphisms of $\mathcal{A}_{\mathcal{X}}$ satisfying $\underline{\ell}\circ\underline{g} = \underline{h}\circ\underline{k}$. The triangle T_g is determined by the commutative diagram (2.1). The commutative diagram below determines T_h:

$$(2.5)\qquad \begin{tikzcd} K_D \arrow[r, "\zeta_h"] \arrow[d, "1_{K_D}"] & \mathrm{Con}(h) \arrow[r, "\eta_h"] \arrow[d, "\theta_h"] & C \arrow[d, "h"] \\ K_B \arrow[r, "\iota_D"] & X_D \arrow[r, "f_D"] & D \end{tikzcd}$$

We consider also the commutative diagram

$$(2.6)\qquad \begin{tikzcd} K_B \arrow[r, "\iota_B"] \arrow[d, "w"] & X_B \arrow[r, "f_B"] \arrow[d, "x_\ell"] & B \arrow[d, "\ell"] \\ K_D \arrow[r, "\iota_D"] & X_D \arrow[r, "f_D"] & D \end{tikzcd}$$

which gives the image $\Omega_{\mathcal{X}}\underline{\ell} = \underline{w}$ of $\underline{\ell}$.

Firstly, we construct a morphism $\underline{s} : \underline{\mathrm{Con}}(g) \to \underline{\mathrm{Con}}(h)$. Since $\underline{\ell}\circ\underline{g} = \underline{h}\circ\underline{k}$, the morphism $h\circ k - \ell\circ g : A \to D$ of $\mathcal{A}$ factorizes over some object of $\mathcal{X}$, and so there a morphism $r : A \to X_D$ with $h\circ k = f_D\circ r + \ell\circ g$. Composing the last equality from the right with η_g and using $\ell\circ g\circ\eta_g = \ell\circ f_B\circ\theta_g = f_D\circ x_\ell\circ\theta_g$, which holds because the diagrams (2.1) and (2.6) are commutative, we have $h\circ k\circ\eta_g = f_D\circ(r\circ\eta_g + x_\ell\circ\theta_g)$. Since $(\mathrm{Con}(h), \eta_h, \theta_h)$ is the pullback of h and f_D, there is a morphism $s : \mathrm{Con}(g) \to \mathrm{Con}(h)$ with $\eta_h\circ s = k\circ\eta_g$ and $\theta_h\circ s = r\circ\eta_g + x_\ell\circ\theta_g$.

Now, we shall show that $\underline{s}\circ\underline{\zeta}_g = \underline{\zeta}_h\circ\Omega_{\mathcal{X}}\underline{\ell}$. We construct firstly a representative v of $\Omega_{\mathcal{X}}\underline{\ell}$ in $\mathcal{A}$. Since, $\eta_h\circ s\circ\zeta_g = k\circ\eta_g\circ\zeta_g = 0$, there is a morphism $v : K_B \to K_D$ with $\zeta_h\circ v = s\circ\zeta_g$. We claim that the image $\underline{v}$ of v in $\mathcal{A}_{\mathcal{X}}$ equals $\Omega_{\mathcal{X}}\underline{\ell}$. For this it is enough to verify that setting v in place of w, the left square of (2.6) becomes commutative. Indeed, $\iota_D\circ v = \theta_h\circ\zeta_h\circ v = (\theta_h\circ s)\circ\zeta_g = r\circ\eta_g\circ\zeta_g + x_\ell\circ\theta_g\circ\zeta_g = x_\ell\circ\iota_B$.

So, we proved that given the morphisms $\underline{\ell}$ and $\underline{k}$ satisfying $\underline{\ell}\circ\underline{g} = \underline{h}\circ\underline{k}$, there is a morphism $\underline{s} : \underline{\mathrm{Con}}(g) \to \underline{\mathrm{Con}}(h)$ making the first two squares from the left of the diagram below commutative

$$\begin{array}{ccccccc}
\Omega_{\mathcal{X}}\underline{B} & \xrightarrow{\zeta_g} & \underline{\mathrm{Con}}(g) & \xrightarrow{-\eta_g} & \underline{A} & \xrightarrow{\underline{g}} & \underline{B} \\
\downarrow{\scriptstyle \Omega_{\mathcal{X}}\ell} & & \downarrow{\scriptstyle \underline{s}} & & \downarrow{\scriptstyle \underline{k}} & & \downarrow{\scriptstyle \underline{\ell}} \\
\Omega_{\mathcal{X}}\underline{D} & \xrightarrow{\zeta_h} & \underline{\mathrm{Con}}(h) & \xrightarrow{-\eta_h} & \underline{C} & \xrightarrow{\underline{h}} & \underline{D}
\end{array}$$

i.e. the triple $(\underline{s}, \underline{k}, \underline{\ell})$ is a morphism from T_g to T_h.

LT4. We assume without loss of generality, compare the argumentation in LT3, that the two left triangles in the assumptions of LT4 are induced, i.e. we assume that there are two $\mathcal{X}$-epics $h : B \to C$ and $k : C \to D$ of $\mathcal{A}$ with kernels $g : A \to B$ and $\ell : E \to C$ respectively, inducing the left triangles $\Omega_{\mathcal{X}}\underline{C} \xrightarrow{ch(h)} \underline{A} \xrightarrow{\underline{g}} \underline{B} \xrightarrow{\underline{h}} \underline{C}$ and $\Omega_{\mathcal{X}}\underline{D} \xrightarrow{ch(k)} \underline{E} \xrightarrow{\underline{\ell}} \underline{C} \xrightarrow{\underline{k}} \underline{D}$. Because of Lemma 2.11, we obtain the diagram

$$\begin{array}{ccccccc}
 & & \Omega_{\mathcal{X}}\underline{E} & & & & \\
 & & \downarrow{\scriptstyle ch(\beta)} & & & & \\
\Omega_{\mathcal{X}}\underline{C} & \xrightarrow{ch(h)} & \underline{A} & \xrightarrow{\underline{g}} & \underline{B} & \xrightarrow{\underline{h}} & \underline{C} \\
\downarrow{\scriptstyle \Omega_{\mathcal{X}}\underline{k}} & & \downarrow{\scriptstyle \underline{\alpha}} & & \| & & \downarrow{\scriptstyle \underline{k}} \\
\Omega_{\mathcal{X}}\underline{D} & \xrightarrow{ch(k\circ h)} & \underline{F} & \xrightarrow{\underline{m}} & \underline{B} & \xrightarrow{\underline{k\circ h}} & \underline{D} \\
\| & & \downarrow{\scriptstyle \underline{\beta}} & & \downarrow{\scriptstyle \underline{h}} & & \| \\
\Omega_{\mathcal{X}}\underline{D} & \xrightarrow{ch(k)} & \underline{E} & \xrightarrow{\underline{\ell}} & \underline{C} & \xrightarrow{\underline{k}} & \underline{D}
\end{array}$$

where all squares , with possible exception of the two left hand ones, commute.

In order to prove LT4, we must show that $ch(k) = \underline{\beta}\circ ch(k\circ h)$, $\underline{\alpha}\circ ch(h) = ch(k\circ h)\circ\Omega_{\mathcal{X}}\underline{k}$ and that $ch(\beta) = ch(h)\circ\Omega_{\mathcal{X}}\underline{\ell}$.

Let us consider the commutative diagrams used for the construction of the characteristic classes of h, k, $k\circ h$ and β respectively:

$$\begin{array}{ccccc}
K_C & \xrightarrow{\iota_C} & X_C & \xrightarrow{f_C} & C \\
\downarrow{\scriptstyle \gamma_1} & & \downarrow{\scriptstyle \delta_1} & & \| \\
A & \xrightarrow{g} & B & \xrightarrow{h} & C
\end{array}
\qquad
\begin{array}{ccccc}
K_D & \xrightarrow{\iota_D} & X_D & \xrightarrow{f_D} & D \\
\downarrow{\scriptstyle \gamma_2} & & \downarrow{\scriptstyle \delta_2} & & \| \\
E & \xrightarrow{\ell} & C & \xrightarrow{k} & D
\end{array}$$

$$\begin{array}{ccccc}
K_D & \xrightarrow{\iota_D} & X_D & \xrightarrow{f_D} & D \\
\downarrow{\scriptstyle \gamma_3} & & \downarrow{\scriptstyle \delta_3} & & \| \\
F & \xrightarrow{m} & B & \xrightarrow{k\circ h} & D
\end{array}
\qquad
\begin{array}{ccccc}
K_E & \xrightarrow{\iota_E} & X_E & \xrightarrow{f_E} & E \\
\downarrow{\scriptstyle \gamma_4} & & \downarrow{\scriptstyle \delta_4} & & \| \\
A & \xrightarrow{\alpha} & F & \xrightarrow{\beta} & E
\end{array}$$

Since $k\circ(h\circ\delta_3) = f_D$, we may choose $\delta_2 = h\circ\delta_3$ and then since $(\ell\circ\beta)\circ\gamma_3 = h\circ(m\circ\gamma_3) = (h\circ\delta_3)\circ\iota_D$ we may choose $\gamma_2 = \beta\circ\gamma_3$. Hence, $ch(k) = \underline{\beta}\circ ch(k\circ h)$.

Let us consider the two commutative diagrams used for the construction of $\Omega_{\mathcal{X}}\underline{k}$ and $\Omega_{\mathcal{X}}\underline{\ell}$ respectively:

$$\begin{array}{ccccc}
K_C & \xrightarrow{\iota_C} & X_C & \xrightarrow{f_C} & C \\
\downarrow{\scriptstyle \kappa_k} & & \downarrow{\scriptstyle x_k} & & \downarrow{\scriptstyle k} \\
K_D & \xrightarrow{\iota_D} & X_D & \xrightarrow{f_D} & D
\end{array}
\qquad
\begin{array}{ccccc}
K_E & \xrightarrow{\iota_E} & X_E & \xrightarrow{f_E} & E \\
\downarrow{\scriptstyle \kappa_\ell} & & \downarrow{\scriptstyle x_\ell} & & \downarrow{\scriptstyle \ell} \\
K_C & \xrightarrow{\iota_C} & X_C & \xrightarrow{f_C} & C
\end{array}$$

We have $(k \circ h) \circ \delta_1 = k \circ f_C = f_D \circ x_k = (k \circ h) \circ \delta_3 \circ x_k$. Since m is the kernel of $k \circ h$, there is a morphism $\rho : X_C \to F$ with $m \circ \rho = \delta_1 - \delta_3 \circ x_k$. Hence, $m \circ \rho \circ \iota_C = \delta_1 \iota_C - \delta_3 \circ (x_k \circ \iota_C) = g \circ \gamma_1 - (\delta_3 \circ \iota_D) \circ \kappa_k = m \circ \alpha \circ \gamma_1 - m \circ \gamma_3 \circ \kappa_k$. Since m is a monomorphism we obtain $\rho \circ \iota_C = \alpha \gamma_1 - \gamma_3 \circ \kappa_k$. Passing to the stable category $\mathcal{A}_\mathcal{X}$ we have $\underline{\alpha} \circ ch(h) = ch(k \circ h) \circ \Omega_\mathcal{X} \underline{k}$.

Since $(h \circ m) \circ \delta_4 = \ell \circ (\beta \circ \delta_4) = \ell \circ f_E = f_C \circ x_\ell = (h \circ \delta_1) \circ x_\ell$ and since g is the kernel of h, there is a morphism $t : X_E \to A$ with $g \circ t = m \circ \delta_4 - \delta_1 \circ x_\ell$. Because $\delta_1 \circ (x_\ell \circ \iota_E) = (\delta_1 \circ \iota_C) \circ \kappa_\ell = g \circ \gamma_1 \circ \kappa_\ell$, we obtain $(g \circ t) \circ \iota_E = m \circ \delta_4 \circ \iota_E - g \circ \gamma_1 \circ \kappa_\ell$. Hence, $g \circ (t \circ \iota_E + \gamma_1 \circ \kappa_\ell) = m \circ \delta_4 \circ \iota_E$. Because m is a monomorphism and $g = m \circ \alpha$, we obtain $\alpha \circ (t \circ \iota_E + \gamma_1 \circ \kappa_\ell) = \delta_4 \circ \iota_E$. This implies that we may choose for γ_4 the morphism $(t \circ \iota_E + \gamma_1 \circ \kappa_\ell)$ and so $ch(\beta) = \underline{\gamma}_4 = \underline{\gamma}_1 \circ \underline{\kappa}_\ell = ch(h) \circ \Omega_\mathcal{X} \underline{\ell}$.

(ii) Left for the reader. □

We close the present section showing that given a contravariantly finite subcategory $\mathcal{X}$ of $\mathcal{A}$, the choice of some $\mathcal{X}$-assignment is inessential, in the sense that any two $\mathcal{X}$-assignments yield equivalent left triangulated structures.

Let us consider two $\mathcal{X}$-assignments of $\mathcal{A}$ and let $\Omega_{\mathcal{X},1}$, $\Omega_{\mathcal{X},2}$ be the corresponding loop-space functors of $\mathcal{A}_\mathcal{X}$. Let $(\mathcal{A}_\mathcal{X}, \Omega_{\mathcal{X},1}, \Delta_1)$ and $(\mathcal{A}_\mathcal{X}, \Omega_{\mathcal{X},2}, \Delta_2)$ be the two left triangulated categories induced by the two $\mathcal{X}$-assignments.

In Proposition 1.2 we mentioned already, that there is a natural equivalence $(\underline{z}) : \Omega_{\mathcal{X},1} \to \Omega_{\mathcal{X},2}$. Using $(\underline{z})$ we shall show in the next theorem, that the identity functor of $\mathcal{A}_\mathcal{X}$ is a triangle equivalence.

Theorem 2.13. *The identity functor $Id_{\mathcal{A}_\mathcal{X}}$ of $\mathcal{A}_\mathcal{X}$ is a triangle equivalence from $(\mathcal{A}_\mathcal{X}, \Omega_{\mathcal{X},1}, \Delta_1)$ to $(\mathcal{A}_\mathcal{X}, \Omega_{\mathcal{X},2}, \Delta_2)$*

Proof. Since $(\underline{z})$ is a natural isomorphism from $Id_{\mathcal{A}_\mathcal{X}} \circ \Omega_{\mathcal{X},1} = \Omega_{\mathcal{X},1}$ to $\Omega_{\mathcal{X},2} \circ Id_{\mathcal{A}_\mathcal{X}} = \Omega_{\mathcal{X},2}$, it is remains to show that given the induced left triangle $\Omega_{\mathcal{X},1}\underline{B} \xrightarrow{ch(g)_1} \underline{C} \xrightarrow{\underline{h}} \underline{A} \xrightarrow{\underline{g}} \underline{B}$ of Δ_1, then $\Omega_{\mathcal{X},2}\underline{B} \xrightarrow{ch(g)_1 \circ \underline{z}_B^{-1}} \underline{C} \xrightarrow{\underline{h}} \underline{A} \xrightarrow{\underline{g}} \underline{B}$ is a left triangle of Δ_2. Let us consider the commutative diagram

$$\begin{array}{ccccc}
K_{B,1} & \xrightarrow{\iota_{B,1}} & X_{B,1} & \xrightarrow{f_{B,1}} & B \\
\downarrow z_B & & \downarrow \alpha & & \| \\
K_{B,2} & \xrightarrow{\iota_{B,2}} & X_{B,2} & \xrightarrow{f_{B,2}} & B \\
\downarrow \gamma & & \downarrow \delta & & \| \\
C & \xrightarrow{h} & A & \xrightarrow{g} & B
\end{array}$$

where the first (resp. second) row from the top corresponds to the first (resp. second) $\mathcal{X}$-assignment and where g is an $\mathcal{X}$-epic with kernel h. Since $h \circ \gamma \circ z_B = \delta \circ \alpha \circ \iota_{B,1}$ and $\underline{\gamma} = ch(g)_2$, we obtain $ch(g)_1 = ch(g)_2 \circ \underline{z}_B$. Hence, $\Omega_{\mathcal{X},2}\underline{B} \xrightarrow{ch(g)_1 \circ \underline{z}_B^{-1}} \underline{C} \xrightarrow{\underline{h}} \underline{A} \xrightarrow{\underline{g}} \underline{B}$ coincides with the induced left triangle $\Omega_{\mathcal{X},2}\underline{B} \xrightarrow{ch(g)_2} \underline{C} \xrightarrow{\underline{h}} \underline{A} \xrightarrow{\underline{g}} \underline{B}$ of Δ_2, which is obtained from the $\mathcal{X}$-epic h. This finishes the proof. □

Remark 2.14. Our Theorem 2.12 covers many of the well known constructions of triangulated categories. In case $\mathcal{A}$ is an abelian category with enough projectives (resp. injectives) relative to a projective (resp. injective) class $\mathcal{E}$ (resp. $\mathcal{M}$) of morphisms of $\mathcal{A}$, then the full subcategory $\mathcal{P}$ (resp. $\mathcal{I}$) of the relative projectives (resp. injectives) is a contravariantly finite (resp. covariantly finite) subcategory of $\mathcal{A}$ and so our Theorem can be applied directly. For example if $\mathcal{B}$ is an exact category, [11], [10] with enough injectives, then these injectives form a covariantly finite subcategory of $\mathcal{B}$ and so $\mathcal{B}$ has a right triangulated structure, compare [10]. In case $\mathcal{B}$ is Frobenius, then the corresponding suspension functor is an equivalence and so $\mathcal{B}$ has a triangulated structure, [6], [7]. Well known special case of this situation is the category of complexes, where the injective class $\mathcal{E}$ of morphisms consists of the monomorphisms which split if we forget the differentials and consider any complex as a graded object,compare [13], [6], [7].

3. On stable categories obtained from subfunctors of $\mathrm{Ext}^1_\Lambda(\ , \ )$.

Let Λ be an artin algebra and modΛ be the category of the finitely generated right Λ-modules.

We write $\mathcal{Z} \subseteq \mathcal{W}$ in case $\mathcal{Z}$ and $\mathcal{W}$ are subcategories of modΛ with $\mathcal{Z}$ a subcategory of $\mathcal{W}$.

Let $\mathcal{X}$ (resp. $\mathcal{Y}$) be a contravariantly finite (resp. covariantly finite) subcategory of modΛ, and $\underline{\mathrm{mod}}_{\mathcal{X}}\Lambda$ (resp. $\overline{\mathrm{mod}}_{\mathcal{Y}}\Lambda$) be the corresponding stable category. As direct application of Theorem 2.12 we obtain the main theorem of our paper:

Main Theorem 3.1.

(i) *The triple* $(\underline{\mathrm{mod}}_{\mathcal{X}}\Lambda, \Omega_{\mathcal{X}}, \Delta_{\mathcal{X}})$ *is a left triangulated category.*

(ii) *The triple* $(\overline{\mathrm{mod}}_{\mathcal{Y}}\Lambda, \Sigma^{\mathcal{Y}}, \Delta^{\mathcal{Y}})$ *is a right triangulated category.*

Proof. According to Theorem 2.12, one has to check that the $\mathcal{X}$-epics (resp. $\mathcal{Y}$-monics) have kernels (resp. cokernels). This is obvious, since modΛ is abelian category. ☐

In the remaining part of this section all considered subcategories $\mathcal{Z}$ of modΛ are assumed to be full, strict, additive, and closed under direct summands.

Let F be an additive subfunctor of $\mathrm{Ext}^1_\Lambda(\ , \ )$ and $\mathcal{P}(F)$ (resp. $\mathcal{I}(F)$ be the full subcategory of modΛ consisting of the F-projective (resp. F-injective) Λ-modules, cf. [3].

Let $\underline{\mathrm{mod}}_{\mathcal{P}(F)}\Lambda$ (resp. $\overline{\mathrm{mod}}_{\mathcal{I}(F)}\Lambda$) be the stable category $\mathrm{mod}\Lambda/\mathcal{J}_{\mathcal{P}(F)}$ (resp. $\mathrm{mod}\Lambda/\mathcal{J}_{\mathcal{I}(F)}$), where $\mathcal{J}_{\mathcal{P}(F)}$ (resp. $\mathcal{J}_{\mathcal{I}(F)}$) is the ideal of morphisms of modΛ, which factor over the F-projectives (resp. F-injectives) and let $\Omega_{\mathcal{P}(F)} : \underline{\mathrm{mod}}_{\mathcal{P}(F)}\Lambda \to \underline{\mathrm{mod}}_{\mathcal{P}(F)}\Lambda$ (resp. $\Sigma^{\mathcal{I}(F)} : \overline{\mathrm{mod}}_{\mathcal{I}(F)}\Lambda \to \overline{\mathrm{mod}}_{\mathcal{I}(F)}\Lambda$) be the corresponding loop (rep. suspension) functor.

In case F has enough projectives (resp. injectives), then $\mathcal{P}(F)$ (resp. $\mathcal{I}(F)$) is a contravariantly finite (resp. covariantly finite) subcategory of modΛ, our main Theorem 3.1 implies:

Theorem 3.2.

(i) *The triple* $(\underline{\mathrm{mod}}_{\mathcal{P}(F)}\Lambda, \Omega_{\mathcal{P}(F)}, \ \Delta_{\mathcal{P}(F)})$ *is a left triangulated category.*

(ii) *The triple* $(\overline{\mathrm{mod}}_{\mathcal{I}(F)}\Lambda, \Sigma^{\mathcal{I}(F)}, \ \Delta^{\mathcal{I}(F)})$ *is a right triangulated category.*

Our aim is to study, when the the triple $(\underline{\mathrm{mod}}_{\mathcal{P}(F)}\Lambda, \Omega_{\mathcal{P}(F)}, \ \Delta_{\mathcal{P}(F)})$ or the triple $(\overline{\mathrm{mod}}_{\mathcal{I}(F)}\Lambda, \Sigma^{\mathcal{I}(F)}, \Delta^{\mathcal{I}(F)})$ is a triangulated category, i.e. when $\Omega_{\mathcal{P}(F)} : \underline{\mathrm{mod}}_{\mathcal{P}(F)}\Lambda \to \underline{\mathrm{mod}}_{\mathcal{P}(F)}\Lambda$ or $\Sigma^{\mathcal{I}(F)} : \overline{\mathrm{mod}}_{\mathcal{I}(F)}\Lambda \to \overline{\mathrm{mod}}_{\mathcal{I}(F)}\Lambda$ is an equivalence.

Let F be an additive subfunctor of $\mathrm{Ext}^1_\Lambda(\ , \ )$ with enough projectives and injectives.

Definition 3.3. The algebra Λ is said to be *F-Frobenius* if $\mathcal{P}(F) = \mathcal{I}(F)$.

Theorem 3.4. *Let F be an additive subfunctor of $\mathrm{Ext}^1_\Lambda(\ ,\ )$ having enough projectives and injectives. The following are equivalent:*

(i) *The algebra Λ is F-Frobenius.*
(ii) *The triple $(\underline{\mathrm{mod}}_{\mathcal{P}(F)}\Lambda, \Omega_{\mathcal{P}(F)}, \Delta_{\mathcal{P}(F)})$ is a triangulated category.*
(iii) *The triple $(\overline{\mathrm{mod}}_{\mathcal{I}(F)}\Lambda, \Sigma^{\mathcal{I}(F)}, \Delta^{\mathcal{I}(F)})$ is a triangulated category.*

Proof. (i) $\to$ (ii) Let A be a Λ-module and

$$0 \to K \xrightarrow{g} P_A \xrightarrow{f} A \to 0 \tag{3.1}$$

be an F-projective presentation of A. Given an F-injective module I, we obtain the short exact sequence

$$0 \to \mathrm{Hom}_\Lambda(A, I) \to \mathrm{Hom}_\Lambda(P_A, I) \to \mathrm{Hom}_\Lambda(K, I) \to 0. \tag{3.2}$$

Since P_A is also F-injective, the exactness of (3.2) implies that (3.1) is an F-injective copresentation of K. Now, the construction of the functors $\Omega_{\mathcal{P}(F)}$ and $\Sigma^{\mathcal{I}(F)}$ in the first section of our paper, implies that $\Sigma^{\mathcal{I}(F)}\circ\Omega_{\mathcal{P}(F)}$ is natural equivalent to the identity functor of $\underline{\mathrm{mod}}_{\mathcal{P}(F)}\Lambda$. Similarly, $\Omega_{\mathcal{P}(F)}\circ\Sigma^{\mathcal{I}(F)}$ is natural equivalent to the identity functor of $\overline{\mathrm{mod}}_{\mathcal{I}(F)}\Lambda$. The proof is complete.

(ii) $\to$ (i) Let I be an F-injective module. Because $\Omega_{\mathcal{P}(F)}$ is an equivalence, there is a Λ-module A such that $\Omega_{\mathcal{P}(F)}\underline{A} = \underline{I}$. Hence, there is an F-exact sequence $0 \to I \oplus L \xrightarrow{(\alpha,\beta)} P \to A \oplus N \to 0$ with L, N and P in $\mathcal{P}(F)$. Let $(1_I, 0) : I \oplus P \to I$ be the canonical projection to I. Since (α, β) is a $\mathcal{P}(F)$-monic and I is in $\mathcal{I}(F)$, there is a morphism $\gamma : P \to I$ satisfying $\gamma\circ(\alpha, \beta) = (1_I, 0)$. So, $\gamma : I \to P$ is a split monomorphism and I is an F-projective. Hence, $\mathcal{I}(F) \subseteq \mathcal{P}(F)$.

Let P be an F-projective. Since $\mathcal{I}(F)$ is a covariantly finite cogenerator of $\mathrm{mod}\Lambda$, there is an F-exact sequence

$$0 \to P \xrightarrow{h} I \xrightarrow{g} A \to 0 \tag{3.3}$$

where I is F-injective. Since g is a $\mathcal{P}(F)$-epic, we obtain in $\underline{\mathrm{mod}}_{\mathcal{P}(F)}\Lambda$, the $\mathcal{P}(F)$-induced left triangle $\Omega_{\mathcal{P}(F)}\underline{A} \xrightarrow{ch(g)} \underline{P} \xrightarrow{\underline{h}} \underline{I} \xrightarrow{\underline{g}} \underline{A}$. Because I and P are in $\mathcal{P}(F)$, this left triangle equals $\Omega_{\mathcal{P}(F)}\underline{A} \to 0 \to 0 \to \underline{A}$. Because of LT2, the left triangle $0 \to \Omega_{\mathcal{P}(F)}\underline{A} \to 0 \to 0$ belongs also to $\Delta_{\mathcal{P}(F)}$ implying $\Omega_{\mathcal{P}(F)}\underline{A} = 0$. Since $\Omega_{\mathcal{P}(F)}$ is an equivalence, A has to be in $\mathcal{P}(F)$. Now the sequence (3.3) has to be a splitting one, and so we obtain that P is in $\mathcal{I}(F)$. So $\mathcal{P}(F) \subseteq \mathcal{I}(F)$. This finishes the proof.

(ii)$\leftrightarrow$(iii). Left for the reader. $\square$

Remark 3.5. We recall from [4], that an artin algebra Λ is said to be *F-selfinjective*, if $\mathcal{P}(F) = \mathcal{I}(F)$ and $\mathcal{P}(F)$ is a subcategory of finite type. Since any subcategory of finite type is functorially finite, the functor F has enough projectives and injectives and so an F-selfinjective algebra Λ is also F-Frobenius. Hence, if Λ is an F-selfinjective algebra, then $\underline{\mathrm{mod}}_{\mathcal{P}(F)}\Lambda$ is a triangulated category, because of the previous Theorem.

Suppose now that $\mathcal{X}$ is a contravariantly finite subcategory of modΛ and $\mathcal{Y}$ a covariantly finite subcategory of modΛ. We consider the stable categories $\underline{\mathrm{mod}}_{\mathcal{X}}\Lambda$ and $\underline{\mathrm{mod}}_{\mathcal{X}}\Lambda$.

Definition 3.6.

(i) The artin algebra Λ is said to be *contravariantly $\mathcal{X}$-Frobenius* if the triple $(\underline{\mathrm{mod}}_{\mathcal{X}}\Lambda, \Omega_{\mathcal{X}}, \Delta_{\mathcal{X}})$ is a triangulated category.

(ii) The artin algebra Λ is said to be *covariantly $\mathcal{Y}$-Frobenius* if the triple $(\underline{\mathrm{mod}}_{\mathcal{X}}\Lambda, \Sigma^{\mathcal{Y}}, \Delta^{\mathcal{Y}})$ is a triangulated category.

Let $\mathcal{P}(\Lambda)$ (resp. $\mathcal{I}(\Lambda)$) be the subcategory of the projective (resp. injective) Λ-modules.

Proposition 3.7. *Let $\mathcal{X}$ be a functorially finite generator of modΛ and $\mathcal{Y}$ a functorially finite cogenerator of modΛ.*

(i) *Λ is contravariantly $\mathcal{X}$-Frobenius if and only if $\mathcal{X} = \mathcal{I}(\Lambda)\cup\mathrm{DTr}\mathcal{X}$.*

(ii) *Λ is covariantly $\mathcal{Y}$-Frobenius if and only if $\mathcal{Y} = \mathcal{P}(\Lambda)\cup\mathrm{TrD}\mathcal{Y}$.*

Proof. Let us consider the additive subfunctor $F = F_{\mathcal{X}}$ of $\mathrm{Ext}^1_\Lambda(\ ,\ )$. Since $\mathcal{X}$ is a contravariantly finite generator, we have by [2], Th. 1.14(c), that $\mathcal{P}(F) = \mathcal{X}$ and also that F has enough projectives and injectives. Moreover we have by [2], Cor. 1.16(a), that $\mathcal{I}(F) = \mathcal{I}(\Lambda)\cup\mathrm{DTr}(\mathcal{P}(F)$. Hence, $\mathcal{I}(F) = \mathcal{I}(\Lambda)\cup\mathrm{DTr}\mathcal{X}$.

(i) Because $\underline{\mathrm{mod}}_{\mathcal{P}(F)}\Lambda = \underline{\mathrm{mod}}_{\mathcal{X}}\Lambda$, the triple $(\underline{\mathrm{mod}}_{\mathcal{X}}\Lambda, \Omega_{\mathcal{X}}, \Delta_{\mathcal{X}})$ is a triangulated category, and from Theorem 3.4 it follows that Λ is F-Frobenius. Hence $\mathcal{P}(F) = \mathcal{I}(F)$ and so $\mathcal{X} = \mathcal{I}(\Lambda)\cup\mathrm{DTr}\mathcal{X}$.

Conversely, since $\mathcal{X} = \mathcal{I}(\Lambda)\cup\mathrm{DTr}\mathcal{X}$ it follows that $\mathcal{I}(F) = \mathcal{P}(F) = \mathcal{X}$. Hence, Λ is F-Frobenius, which implies by Theorem 3.4, that $(\underline{\mathrm{mod}}_{\mathcal{X}}\Lambda\ , \Omega_{\mathcal{X}}, \Delta_{\mathcal{X}})$ is a triangulated category and hence Λ is contravariantly $\mathcal{X}$-Frobenius.

(ii) Left for the reader. □

Example 3.8. An artin algebra Λ is $\mathrm{Ext}^1_\Lambda(\ ,\ )$-Frobenius if and only if $\mathcal{P}(\Lambda) = \mathcal{I}(\Lambda)$, i.e. if and only if it is selfinjective. Because of Theorem 3.4, an algebra Λ is selfinjective if and only if $\underline{\mathrm{mod}}_{\mathcal{P}(\Lambda)}$ is a triangulated category. A well-known result.

Example 3.9. Let Λ be an artin algebra of finite representation type. Let $\mathcal{J}$ be the subcategory of modΛ with objects the indecomposable modules of the form $\mathrm{DTr}^i I_j$, $i \in \mathbb{N}\cup 0$, where $\{I_j\}$ is a complete set of non-isomorphic indecomposable Λ-injective modules. Let us consider the subcategory $\mathcal{W} = \mathrm{add}\mathcal{J}$. Since $\mathcal{W}$ is of finite type, it is functorially finite and also a generator of modΛ. Moreover it satisfies the equality $\mathcal{W} = \mathcal{I}(\Lambda)\cup\mathrm{DTr}\mathcal{W}$. Hence, any artin algebra Λ of finite representation type is contravariantly $\mathcal{W}$-Frobenius and so the triple $(\underline{\mathrm{mod}}_{\mathcal{W}}\Lambda, \Omega_{\mathcal{W}}, \Delta_{\mathcal{W}})$ is a triangulated category. The Auslander-Reiten quiver of $\underline{\mathrm{mod}}_{\mathcal{W}}\Lambda$ coincides with the stable quiver $\mathrm{A}_s(\Lambda)$ of the algebra Λ, introduced in [12] and later in [8].

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