LEFT TRIANGULATED CATEGORIES ARISING FROM CONTRAVARIANTLY FINITE SUBCATEGORIES

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Abstract. Let $\text{mod}\,\Lambda$ be the category of finitely generated right $\Lambda$-modules over an artin algebra $\Lambda$, and $F$ be an additive subfunctor of $\text{Ext}_\Lambda^1(\, , )$. Let $\mathcal{P}(F)$ denote the full subcategory of $\mathcal{A}$ with objects the $F$-projective modules. If the functor $F$ has enough $F$-projectives, then we show that the stable category $\text{mod}\,\mathcal{P}(F)\,\Lambda$ has a left triangulated structure. In case $F = \text{Ext}_\Lambda^1(\, , )$, the above statement implies that the stable category $\text{mod}\,\Lambda$ of $\text{mod}\,\Lambda$ has a left triangulated structure. Dual statements for the case of $F$-injective modules are also true.

Introduction

The notion of a triangulated category was introduced almost thirty five years ago by Grothendieck and later by Verdier [13]. It became a very powerful tool first in algebraic geometry and later in many other branches of mathematics. The last decade influenced strongly the theory of representations of artin algebras through the work of Happel [6], [7].

An important field in the theory of representations of artin algebras is the study of the stable category of an artin algebra [1].

In the present paper we relate a triangulated (left or right) structure to stable categories of an artin algebra as follows: Let $\Lambda$ be an artin algebra and $\text{mod}\,\Lambda$ be the category of finitely generated right $\Lambda$-modules.

Main Theorem 3.1.

(i) Any contravariantly finite subcategory $\mathcal{X}$ of $\text{mod}\,\Lambda$ induces on the stable category $\text{mod}\,\mathcal{X}\,\Lambda$ a left triangulated structure.

1991 Mathematics Subject Classification. Primary 18E30, 16G10; Secondary 18G25.

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(ii) Any covariantly finite subcategory $\mathcal{Y}$ of $\text{mod}\Lambda$ induces on the stable category $\text{mod}_\mathcal{Y}\Lambda$ a right triangulated structure.

Let $F$ be an additive subfunctor of $\text{Ext}^1(A, \ )$ and $\mathcal{P}(F)$ (resp. $\mathcal{I}(F)$) be the full subcategory of $\text{mod}\Lambda$ with objects the $F$-projectives (resp. the $F$-injectives). Let $\text{mod}_{\mathcal{P}(F)}\Lambda$ (resp. $\text{mod}_{\mathcal{I}(F)}\Lambda$) denote the corresponding stable category.

From the Theorem above we obtain:

**Theorem 3.2.**

(i) If the functor $F$ has enough $F$-projectives, then $\mathcal{P}(F)$ induces on the stable category $\text{mod}_{\mathcal{P}(F)}\Lambda$ a left triangulated structure.

(ii) If the functor $F$ has enough $F$-injectives, then $\mathcal{I}(F)$ induces on the stable category $\text{mod}_{\mathcal{I}(F)}\Lambda$ a right triangulated structure.

In particular, if $F = \text{Ext}^1(A, \ )$, our theorem implies that the stable category $\text{mod}\Lambda$ of $\text{mod}\Lambda$ modulo the $A$-projectives has a left triangulated structure and similarly the stable category $\text{mod}\Lambda$ modulo the $A$-injectives has a right triangulated structure.

Instead of working in the category of finitely generated right modules of an artin algebra and its stable category, we prefer to work in an arbitrary additive category. Namely, given an additive category $A$ and a contravariantly finite subcategory $\mathcal{X}$, a notion introduced by Auslander and Smalo [z], and assuming that any $X$-epic has a kernel, we show in Theorem 2.12 that the stable category obtained by these ingredients has always a left triangulated structure. Many well known constructions of triangulated categories can be derived using this theorem, by choosing the appropriate additive category and its contravariantly finite subcategory, cf. Remark 2.14.

In our paper we deal also with the dual situation of a covariantly finite subcategory $\mathcal{Y}$ of $A$. In this case the stable category obtained from $A$ and $\mathcal{Y}$ has a right triangulated structure.

### 1. On the Stable Category of an Additive Category by a Homologically Finite Subcategory

In this paper we assume, unless otherwise stated, that all considered categories and their subcategories are additive. Functors between additive categories are assumed to be additive.

Let $A$ be a category and $\mathcal{X}$ a contravariantly finite subcategory of $A$. In the present section we shall construct a covariant endofunctor on the stable category of $A$ determined by $\mathcal{X}$, under the assumption that any right $\mathcal{X}$-approximation has a kernel. A dual construction can be done for a covariantly finite subcategory $\mathcal{Y}$ of $A$, under the assumption that any left $\mathcal{Y}$-approximation has a cokernel. We shall leave this to the reader.

We begin by recalling some definitions and notation concerning contravariantly finite subcategories of an arbitrary category $A$ [2]. Related definitions for covariantly finite, homologically finite or functorially finite subcategories of $A$, can be also found in [2].

Let $\mathcal{X}$ be a subcategory of $A$. A morphism $f_B : X_B \rightarrow B$ of $A$, with $X_B$ an object of $\mathcal{X}$, is said to be a right $\mathcal{X}$-approximation of $B$, if $A(X, f_B) : A(X, X_B) \rightarrow A(X, B)$ is surjective for all objects $X$ of $\mathcal{X}$.
The subcategory \( \mathcal{X} \) is said to be a contravariantly finite subcategory of \( \mathcal{A} \), if any object \( B \) of \( \mathcal{A} \) has a right \( \mathcal{X} \)-approximation.

Given two objects \( A \) and \( B \) of \( \mathcal{A} \), we denote by \( \mathcal{X}(A, B) \) the set of morphisms from \( A \) to \( B \) which factor over some object of \( \mathcal{X} \). It is well known that \( \mathcal{X}(A, B) \) is a subgroup of \( \mathcal{A}(A, B) \), and that the family of these subgroups \( \mathcal{X}(A, B) \) forms an ideal \( \mathcal{J}_\mathcal{X} \) of \( \mathcal{A} \).

Let \( \mathcal{A}_\mathcal{X} \) be the stable category \( \mathcal{A}/\mathcal{J}_\mathcal{X} \). We recall that \( \mathcal{A}_\mathcal{X} \) has the same objects as \( \mathcal{A} \) and that for any two objects \( A \) and \( B \) of \( \mathcal{A}_\mathcal{X} \) the set of morphisms \( \mathcal{A}_\mathcal{X}(A, B) \) is defined to be equal to the factor group \( \mathcal{A}(A, B)/\mathcal{X}(A, B) \). The composition of \( \mathcal{A}_\mathcal{X} \) is induced canonically by the composition of \( \mathcal{A} \).

Let \( \pi_\mathcal{X} : \mathcal{A} \to \mathcal{A}_\mathcal{X} \) be the projection functor induced by \( \text{Id}_\mathcal{A} \). The image \( \pi_\mathcal{X} A \) of any object \( A \) of \( \mathcal{A} \) will be denoted by \( \mathcal{A} \) and the image \( \pi_\mathcal{X} f \) of any morphism \( f \) of \( \mathcal{A} \) will be denoted by \( f \).

Suppose that any right \( \mathcal{X} \)-approximation has a kernel in \( \mathcal{A} \). We associate to any object \( A \) of \( \mathcal{A} \) a sequence of morphisms \( K_A \xrightarrow{\triangle} X_A \xrightarrow{f_A} A \), where \( f_A \) is a right \( \mathcal{X} \)-approximation of \( A \) and \( \triangle_A \) is a kernel of \( f_A \). In this case, we say that an \( \mathcal{X} \)-assignment for \( \mathcal{A} \) has been made.

We shall construct a covariant additive endofunctor of \( \mathcal{A}_\mathcal{X} \), under the assumption that an \( \mathcal{X} \)-assignment for \( \mathcal{A} \) has been made. This construction is inspired from the classical construction of the loop space functor thirty three years ago [9]. In case of a Frobenius category, a similar construction appears in [2].

Given a morphism \( g : A \to B \) of \( \mathcal{A} \), we consider the sequences \( K_A \xrightarrow{\triangle} X_A \xrightarrow{f_A} A \) and \( K_B \xrightarrow{\triangle} X_B \xrightarrow{f_B} B \) determined by the \( \mathcal{X} \)-assignment for \( \mathcal{A} \). Since \( f_B \) is a right \( \mathcal{X} \)-approximation and \( \triangle_B \) its kernel, one obtains the commutative diagram

\[
\begin{array}{ccc}
K_A & \xrightarrow{\triangle} & X_A \\
\downarrow \triangle_A & & \downarrow f_A \\
K_B & \xrightarrow{\triangle} & X_B \\
\end{array}
\]

and sees easily that \( \xi_g \) is independent of the choice of \( x_g \).

Hence, any \( \mathcal{X} \)-assignment for \( \mathcal{A} \) yields a covariant functor \( \Omega_\mathcal{X} : \mathcal{A}_\mathcal{X} \to \mathcal{A}_\mathcal{X} \) defined by \( \Omega_\mathcal{X} A = K_A \) on the objects \( A \) of \( \mathcal{A}_\mathcal{X} \) and by \( \Omega_\mathcal{X} g = \xi_g \) on the morphisms \( g : A \to B \) of \( \mathcal{A}_\mathcal{X} \).

**Definition 1.1.** The functor \( \Omega_\mathcal{X} \) is said to be the loop space functor of \( \mathcal{A}_\mathcal{X} \).

Given two \( \mathcal{X} \)-assignments for \( \mathcal{A} \), we derive easily that the obtained functors \( \Omega_{\mathcal{X},1} \) and \( \Omega_{\mathcal{X},2} \) are equivalent, because of the following construction:

Let us consider the commutative diagram

\[
\begin{array}{ccc}
K_{A,1} & \xrightarrow{\triangle_{A,1}} & X_B \\
\downarrow \triangle_A & & \downarrow f_A \\
K_{A,2} & \xrightarrow{\triangle_{A,2}} & X_B \\
\end{array}
\]

where the rows correspond to the two different \( \mathcal{X} \)-assignments for \( \mathcal{A} \). The existence of \( \alpha \) follows from the fact that \( f_{A,2} \) is a right \( \mathcal{X} \)-approximation of \( A \) and the existence of \( \triangle_{A,2} \) follows from the fact that \( \triangle_{A,2} \) is the kernel of \( f_{A,2} \).

One sees easily that the family \( \{ \xi_A \} \), where \( A \) runs over the objects of \( \mathcal{A}_\mathcal{X} \), determines a natural isomorphism \( \xi : \Omega_{\mathcal{X},1} \cong \Omega_{\mathcal{X},2} \).
For a covariantly finite subcategory \( \mathcal{Y} \) of \( \mathcal{A} \), there is the dual procedure. Firstly, we construct the stable category \( \mathcal{A}^\mathcal{Y} \) of \( \mathcal{A} \) by \( \mathcal{Y} \) and assuming that any left \( \mathcal{Y} \)-approximation has a cokernel we define the notion of the \( \mathcal{Y} \)-assignment for \( \mathcal{A} \). Any \( \mathcal{Y} \)-assignment for \( \mathcal{A} \), determines a covariant additive endofunctor \( \Sigma^\mathcal{Y} : \mathcal{A}^\mathcal{Y} \rightarrow \mathcal{A}^\mathcal{Y} \), called the suspension functor of \( \mathcal{A}^\mathcal{Y} \).

We collect the remarks above in the next proposition.

**Proposition 1.2.** Let \( \mathcal{A} \) be an additive category, \( \mathcal{X} \) an additive contravariantly finite subcategory of \( \mathcal{A} \) and \( \mathcal{Y} \) an additive covariantly finite subcategory of \( \mathcal{A} \).

(i) If any right \( \mathcal{X} \)-approximation has a kernel, then any \( \mathcal{X} \)-assignment for \( \mathcal{A} \) yields an endofunctor \( \Omega_\mathcal{X} : \mathcal{A}_\mathcal{X} \rightarrow \mathcal{A}_\mathcal{X} \) and any two \( \mathcal{X} \)-assignments yield equivalent functors.

(ii) If any left \( \mathcal{Y} \)-approximation has a cokernel, then any \( \mathcal{Y} \)-assignment for \( \mathcal{A} \) yields an endofunctor \( \Omega^\mathcal{Y} : \mathcal{A}^\mathcal{Y} \rightarrow \mathcal{A}^\mathcal{Y} \) and any two \( \mathcal{Y} \)-assignments yield equivalent functors.

### 2. On the Left Triangulation of an Additive Category by a Contravariantly Finite Subcategory

In the present section we shall enrich the stable category \( \mathcal{A}_\mathcal{X} \) with a left triangulated structure, under the assumption that any \( \mathcal{X} \)-epic, a notion which will be defined in the sequel, has a kernel. We begin by recalling the definition of a left triangulated category \([5], [10]\).

Let \( \mathcal{C} \) be a category and \( \Omega : \mathcal{C} \rightarrow \mathcal{C} \) a covariant endofunctor. Let \( LTR(\mathcal{C}, \Omega) \) denote the category with objects the diagrams of \( \mathcal{C} \) of the form \( \Omega C \xrightarrow{f} A \xrightarrow{g} B \xrightarrow{h} C \) and with set of morphisms from \( \Omega C \xrightarrow{f} A \xrightarrow{g} B \xrightarrow{h} C \) to \( \Omega C' \xrightarrow{f'} A' \xrightarrow{g'} B' \xrightarrow{h'} C' \) the triples \((\alpha, \beta, \gamma)\) of morphisms of \( \mathcal{C} \) from \((A, B, C)\) to \((A', B', C')\), which make the next diagram commutative:

\[
\begin{array}{ccc}
\Omega C & \xrightarrow{f} & A \\
\downarrow_{\Omega \alpha} & & \downarrow_{\beta} \\
\Omega C' & \xrightarrow{f'} & A'
\end{array}
\quad
\begin{array}{ccc}
A & \xrightarrow{g} & B \\
\downarrow_{\alpha} & & \downarrow_{\gamma} \\
A' & \xrightarrow{g'} & B'
\end{array}
\quad
\begin{array}{ccc}
B & \xrightarrow{h} & C \\
\downarrow_{\beta} & & \downarrow_{\gamma} \\
B' & \xrightarrow{h'} & C'
\end{array}
\]

The composition of the morphisms of \( LTR(\mathcal{C}, \Omega) \) is induced in the canonical way by the corresponding composition of the morphisms of \( \mathcal{C} \).

**Definition 2.1.** The category \( LTR(\mathcal{C}, \Omega) \) is said to be the category of left triangles of \( \mathcal{C} \) and its objects \( \Omega C \xrightarrow{f} A \xrightarrow{g} B \xrightarrow{h} C \) of \( LTR(\mathcal{C}, \Omega) \) are said to be the left triangles of \( \mathcal{C} \).

Whenever it is convenient, the left triangles \( \Omega C \xrightarrow{f} A \xrightarrow{g} B \xrightarrow{h} C \) will be denoted by \((A, B, C, f, g, h)\).

**Definition 2.2.** A full subcategory \( \Delta \) of \( LTR(\mathcal{C}, \Omega) \) is said to be a left triangulation of \( (\mathcal{C}, \Omega) \) if it is closed under isomorphisms and satisfies the following four axioms:

(LT1i) For any object \( A \) of \( \mathcal{C} \), the left triangle \( 0 \xrightarrow{0} A \xrightarrow{1_A} A \xrightarrow{0} 0 \) belongs to \( \Delta \).
(LT1) For any morphism \( h : B \to C \), there is a left triangle in \( \Delta \) of the form
\[
\Omega C \xrightarrow{f} A \xrightarrow{g} B \xrightarrow{h} C.
\]

(LT2) For any left triangle \( \Omega C \xrightarrow{f} A \xrightarrow{g} B \xrightarrow{h} C \) in \( \Delta \), the left triangle
\[
\Omega B \xrightarrow{\\Omega g} \Omega C \xrightarrow{f} A \xrightarrow{g} B \xrightarrow{h} C
\]
is also in \( \Delta \).

(LT3) For any two left triangles \( \Omega C \xrightarrow{f} A \xrightarrow{g} B \xrightarrow{h} C \), \( \Omega C' \xrightarrow{f'} A' \xrightarrow{g'} B' \xrightarrow{h'} C' \) in \( \Delta \), and any morphisms \( \beta : B \to B' \) and \( \gamma : C \to C' \) of \( C \) with \( \gamma \circ h = h' \circ \beta \), there is a morphism \( \alpha : A \to A' \) of \( C \) such that the triple \((\alpha, \beta, \gamma)\) is a morphism from the first triangle to the second.

(LT4) For any two left triangles \( \Omega C \xrightarrow{f} A \xrightarrow{g} B \xrightarrow{h} C \) and \( \Omega D \xrightarrow{f'} A \xrightarrow{g'} B \xrightarrow{h'} D \) in \( \Delta \), there is a third left triangle \( \Omega D \xrightarrow{f''} F \xrightarrow{m} B \xrightarrow{k \circ h'} D \) in \( \Delta \) and two morphisms \( \alpha : A \to F \) and \( \beta : F \to E \) of \( C \), such that the diagram below is fully commutative, where the second column from the left is a left triangle in \( \Delta \).

\[
\begin{array}{c}
\Xi E \\
\Omega C \xrightarrow{f} A \xrightarrow{g} B \xrightarrow{h} C \\
\downarrow \Omega k & \downarrow \alpha & \downarrow \beta & \downarrow k \\
\Omega D \xrightarrow{f'} A \xrightarrow{g'} B \xrightarrow{h'} D \\
\downarrow \Omega m & \downarrow \beta & \downarrow h & \downarrow \Omega D \\
\Omega D \xrightarrow{f''} F \xrightarrow{m} B \xrightarrow{k \circ h'} D
\end{array}
\]

(i.e. the triples \((\alpha, 1_B, k)\) and \((\beta, h, 1_D)\) are morphisms of \( \Delta \) and \( \Omega E \xrightarrow{f''} A \xrightarrow{\alpha} F \xrightarrow{\beta} E \) is in \( \Delta \)).

**Definition 2.3.** The triple \((C, \Omega, \Delta)\) is called a left triangulated category.

**Definition 2.4.** A covariant functor \( F : C \to C' \) is said to be a triangle equivalence from the left triangulated category \((C, \Omega, \Delta)\) to the left triangulated category \((C', \Omega', \Delta')\), if it is an equivalence and there is a natural isomorphism \( \phi : F \circ \Omega \to \Omega' \circ F \), such that if \((A, B, C, f, g, h)\) is a left triangle of \( \Delta \), then the left triangle \((FA, FB, FC, Ff, Fg, Fh)\) is in \( \Delta' \).

We leave it for the reader to define the dual notion of a right triangulated category \((C, \Sigma, \Delta')\), where \( \Sigma : C \to C \) is a covariant endofunctor and \( \Delta' \) is a full subcategory of the category of right triangles \( RTR(C, \Sigma) \) satisfying the right analogs to LT1-LT4 axioms.

**Remark 2.5.** If the endofunctor \( \Omega : C \to C \) is an equivalence, then LT1-LT4 imply that the left triangulated category \((C, \Omega, \Delta)\) is also a right triangulated category. So, \((C, \Omega, \Delta)\) is a triangulated category in the classical sense, [13].

Suppose that \( X \) is a contravariantly finite subcategory of \( A \).

**Definition 2.6.** A morphism \( g : A \to B \) of \( A \) is said to be an \( X \)-epic, if for any object \( X \) of \( X \) the induced homomorphism \( A(X, g) : A(X, A) \to A(X, B) \) is surjective.
Obviously, any $X$-approximation is an $X$-epic. For a covariantly finite subcategory $\mathcal{Y}$ of $\mathcal{A}$, there is the dual notion of a $\mathcal{Y}$-monic.

From now on, we assume that given a contravariantly finite subcategory $X$ of $\mathcal{A}$ an $X$-assignment for $\mathcal{A}$ has been made and that any $X$-epic has a kernel. For the rest of the paper, we shall deal only with the contravariantly finite case, leaving for the reader to state and prove dual results for the covariantly finite case. Results concerning covariantly finite subcategories will be stated only in theorems.

Lemma 2.7. Let $f : C \to B$ and $g : A \to B$ be two morphisms of $\mathcal{A}$. If $f$ is an $X$-epic, then

(i) The morphism $(g, f) : A \oplus C \to B$ is also an $X$-epic.

(ii) The pullback $(h : P \to A, k : P \to C)$ of $f$ and $g$ exists and the morphism $h$ is also an $X$-epic.

Proof. (i) Since, $\mathcal{A}(X, f) : \mathcal{A}(X, C) \to \mathcal{A}(X, B)$ is surjective for any $X$ of $\mathcal{X}$, the same is true for $\mathcal{A}(X, (g, f)) : \mathcal{A}(X, A \oplus C) \to \mathcal{A}(X, B)$. (ii) The kernel of $(g, f)$ exists because $(g, f)$ is also an $X$-epic. So we have

$$P \xrightarrow{\begin{bmatrix} k \\ h \end{bmatrix}} A \oplus C \xrightarrow{(g,f)} B$$

where the left morphism is the kernel of $(g, f)$. Given a morphism $\ell : L \to A$, there is a morphism $m : L \to C$ with $g \ell = f \circ m$ and because $(h : P \to A, k : P \to C)$ is the pullback of $(f, g)$, there is a morphism $n : L \to P$ satisfying $h \circ n = \ell$. \qed

Let $\mathcal{LTR}(\mathcal{A}_X, \Omega_X)$ denote the category of left triangles of $\mathcal{A}_X$. We shall construct in $\mathcal{LTR}(\mathcal{A}_X, \Omega_X)$ two kinds of left triangles, the distinguished and the induced ones.

The distinguished triangles are obtained as follows:

Let $g : A \to B$ be a morphism of $\mathcal{A}$ and $f_B : X_B \to B$ be the right $X$-approximation of $B$ determined by the $X$-assignment for $\mathcal{A}$. Because of Lemma 2.7 the morphism $(g, f_B) : A \oplus X_B \to B$ is an $X$-epic and the pullback of $g$ and $f_B$ exists. This pullback is uniquely determined up to isomorphism by $g$, because $f_B$ is fixed. So, given a morphism $g : A \to B$, we have the commutative diagram

$$
\begin{array}{ccc}
K_B & \xrightarrow{\zeta_g} & \text{Con}(g) \\
\downarrow \text{id} & & \downarrow \text{id} \\
K_B & \xrightarrow{\eta_g} & X_B \xrightarrow{f_B} B
\end{array}
$$

where $(\text{Con}(g), \eta_g, \theta_g)$ is the pullback of $f_B$ and $g$ and where $\eta_g$ is an $X$-epic with kernel $\zeta_g : K_B \to \text{Con}(g)$.

Definition 2.8. A left triangle $\Omega_X G \xrightarrow{h} \mathcal{D} \xrightarrow{f} \mathcal{E} \xrightarrow{m} \mathcal{G}$ of $\mathcal{LTR}(\mathcal{A}_X, \Omega_X)$ is said to be $X$-distinguished, if it is isomorphic to $\Omega_X B \xrightarrow{\zeta_g} \text{Con}(g) \xrightarrow{-\eta_g} A \xrightarrow{\theta_g} B$ for some $g : A \to B$ of $\mathcal{A}$.

We shall call the $X$-distinguished left triangles, just distinguished left triangles, whenever it is clear from the context which is the contravariantly finite subcategory $X$ of $\mathcal{A}$.
We leave it for the reader to state the dual definition for the $\mathcal{Y}$-distinguished right triangles of the category $\mathcal{RTR}(A^Y, \Sigma^Y)$, for a covariantly finite subcategory $\mathcal{Y}$ of $A$.

The induced triangles are obtained as follows:

Let $g : A \to B$ be an $\mathcal{X}$-epic of $A$ and $f_B : X_B \to B$ be the right $\mathcal{X}$-approximation of $B$ determined by the $\mathcal{X}$-assignment for $A$. We consider the commutative diagram

\[
\begin{array}{ccccccccc}
K_B & \xrightarrow{\iota_B} & X_B & \xrightarrow{f_B} & B \\
\downarrow \gamma & & \downarrow \varepsilon & & \downarrow 1_B \\
C & \xrightarrow{h} & A & \xrightarrow{g} & B
\end{array}
\]

(2.2)

where $\iota_B$ is the kernel of $f_B$ and $h$ the kernel of $g$. The morphism $\varepsilon$ exists because $g$ is an $\mathcal{X}$-epic and $X_B$ belongs to $\mathcal{X}$. The morphism $\gamma$ exists because $h$ is the kernel of $g$. We claim that the morphism $\gamma$ of $A_X$ is uniquely determined by $\gamma$. Indeed, if $\gamma' : X_B \to A$ satisfies $g \circ \gamma' = f_B$ and $\gamma' : K_B \to C$ satisfies $h \circ \gamma' = \varepsilon \circ \iota_B$, then there is a morphism $m : X_B \to C$ such that $\delta - \delta' = h \circ m$, because $g \circ (\delta - \delta') = 0$.

Hence, $h \circ (\gamma - \gamma') = (\delta - \delta') \circ \iota_B = h \circ m \circ \iota_B$ and since $h$ is a monomorphism, we obtain $\gamma - \gamma' = m \circ \iota_B$. So, $\gamma - \gamma'$ factorizes over $X_B$, i.e. \( \gamma = \gamma' \).

This uniquely determined by the $\mathcal{X}$-epic $g$, morphism $\gamma$ of $A_X$ is said to be the characteristic class of $g$ and it is denoted by $\text{ch}(g)$.

**Definition 2.9.** A left triangle $\Omega_X G \xrightarrow{\beta} D \xrightarrow{f} E \xrightarrow{m} G$ of $\mathcal{LTR}(A_X, \Omega_X)$ is said to be $\mathcal{X}$-induced, if it is isomorphic to $\Omega_X B \xrightarrow{\text{ch}(g)} C \xrightarrow{h} A \xrightarrow{g} B$ for some $\mathcal{X}$-epic $g : A \to B$ of $A$.

In the following we call the $\mathcal{X}$-induced left triangles, just induced left triangles, whenever it is clear from the context which is the contravariantly finite subcategory $\mathcal{X}$ of $A$.

We leave it for the reader to state the dual definition for the $\mathcal{Y}$-induced right triangles of $\mathcal{RTR}(A^Y, \Sigma^Y)$, where $\mathcal{Y}$ is a covariantly finite subcategory of $A$.

**Proposition 2.10.** Any distinguished left triangle is isomorphic to an induced one and any induced left triangle is isomorphic to a distinguished one.

**Proof.** Given a morphism $g : A \to B$, we form the $\mathcal{X}$-epic $(g, f_B) : A \oplus X_B \to B$ and the commutative diagram:

\[
\begin{array}{ccccccccc}
K_B & \xrightarrow{\iota_B} & X_B & \xrightarrow{f_B} & B \\
\downarrow \zeta & & \downarrow \left( \begin{smallmatrix} 0 \\ 1_X \end{smallmatrix} \right) & & \downarrow 1_B \\
\text{Con}(g) & \xrightarrow{\left( \begin{smallmatrix} -\gamma \\ \eta \end{smallmatrix} \right)} & A \oplus X_B & \xrightarrow{(g, f_B)} & B
\end{array}
\]

This diagram shows that the distinguished left triangle obtained from $g$ is isomorphic the induced left triangle obtained from the $\mathcal{X}$-epic $(g, f_B)$. 

\[\]
Conversely, we shall prove that given an $X$-epic $g : A \to B$, the corresponding induced left triangle $\Omega_X B \xrightarrow{\lambda(g)} C \xrightarrow{h} A \xrightarrow{g} B$ is isomorphic to the distinguished left triangle obtained from $g$.

Let $(\text{Con}(g), \eta_g, \theta_g)$ be the pullback of $f_B$ and $g$ and let $\delta : X_B \to A$, be a morphism as in diagram (2.2) satisfying $g \circ \delta = f_B$. This pullback and the morphisms $\delta$ and $1_{X_B}$, yield a morphism $\lambda_g : X_B \to \text{Con}(g)$, such that $\delta = \eta_g \circ \lambda_g$ and $1_{X_B} = \theta_g \circ \lambda_g$.

Since $g$ is an $X$-epic, the morphism $\theta_g$ is also an $X$-epic because of 2.7, and $(\text{Con}(g), \eta_g, \theta_g)$ yields the commutative diagram

$$
\begin{array}{ccc}
C & \xrightarrow{m} & \text{Con}(g) & \xrightarrow{\theta_g} & X_B \\
\downarrow h & & \downarrow \eta_g & & \downarrow f_B \\
C & \xrightarrow{h} & A & \xrightarrow{g} & B
\end{array}
$$

where $h$ is the kernel of $g$ and $m$ is the kernel of $\theta_g$. Let us consider the commutative diagram:

$$
\begin{array}{ccc}
C & \xrightarrow{h} & A & \xrightarrow{g} & B \\
\downarrow m & & \downarrow \begin{pmatrix} 1_A \\ 0 \end{pmatrix} & & \downarrow 1_B \\
\text{Con}(g) & \xrightarrow{\begin{pmatrix} \eta_g \\ \theta_g \end{pmatrix}} & A \oplus X_B & \xrightarrow{(\eta, f_B)} & B \\
\downarrow t & & \downarrow \begin{pmatrix} 1_A, \delta \end{pmatrix} & & \downarrow 1_B \\
C & \xrightarrow{h} & A & \xrightarrow{g} & B
\end{array}
$$

The morphism $t : \text{Con}(g) \to C$ exists and makes the left bottom square commutative, because $h$ is the kernel of $g$. Because of the same reason, $tm$ has to be equal to $1_C$.

The commutativity of the left bottom square implies $h \circ t \circ \xi_g = \eta_g \circ \xi_g - \delta \circ \theta_g \circ \xi_g = -\delta \circ t \circ \lambda_B$. This shows that in the construction of the induced left triangle of $g$ as in diagram (2.2) we may take $\gamma = -t \circ \xi_g$, i.e. $\lambda(g) = -t \circ \xi_g$.

Let us consider the morphism $1_{\text{Con}(g)} - \lambda_g : \text{Con}(g) \to \text{Con}(g)$. Since, $\theta_g \circ (1_{\text{Con}(g)} - \lambda_g) = 0$ and $m$ is the kernel of $\theta_g$, there is a morphism $\xi : \text{Con}(g) \to C$ with $m \circ \xi = 1_{\text{Con}(g)} - \lambda_g$. So, $tm \circ \xi = t \circ (1_{\text{Con}(g)} - \lambda_g)$ and so $\xi = t - t \circ \lambda_B \circ \theta_g$. This implies that in $A_X$ we have $\xi = t; m \circ \xi = 1_{\text{Con}(g)}$ and $\xi_g = m \circ t \circ \xi_g = -m \circ \lambda(g)$.

Hence, in the diagram below all squares commute and the vertical morphisms are isomorphisms

$$
\begin{array}{ccc}
\Omega_X B & \xrightarrow{\lambda(g)} & C & \xrightarrow{h} & A & \xrightarrow{g} & B \\
\downarrow m & & \downarrow 1 & & \downarrow 1 & & \downarrow 1 \\
\Omega_X B & \xrightarrow{\lambda(g)} & \text{Con}(g) & \xrightarrow{-\eta_g} & A & \xrightarrow{-g} & B
\end{array}
$$

This finishes the proof of the second part of our claim. $\Box$

Let $\text{LTR}(A_X, \Omega_X)$ be the category of left triangles of $(A_X, \Omega_X)$ and $\Delta_X$ be the full subcategory of $\text{LTR}(A_X, \Omega_X)$ with objects the distinguished left triangles of
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\textbf{Theorem 2.11.} If \( h : B \twoheadrightarrow C, k : C \twoheadrightarrow D \) are two \( \mathcal{X} \)-epics of \( \mathcal{A} \) with kernels \( g : A \to B \) and \( \ell : E \to C \) respectively, then there is a fully commutative diagram

\[
\begin{array}{ccc}
A & \xrightarrow{\alpha} & B & \xrightarrow{h} & C \\
\downarrow \alpha & & \downarrow \beta & & \downarrow \kappa \\
F & \xrightarrow{m} & B & \xrightarrow{k \circ h} & D \\
\downarrow \beta & & \downarrow \kappa & & \downarrow \ell_D \\
E & \xrightarrow{\ell} & C & \xrightarrow{\kappa} & D
\end{array}
\]

where \( k \circ h \) and \( \beta \) are \( \mathcal{X} \)-epics with kernels \( m \) and \( \alpha \) respectively.

\textbf{Proof.} The kernel \( m : F \to B \) exists, because \( k \circ h \) is an \( \mathcal{X} \)-epic. The morphisms \( \alpha \) and \( \beta \) exist because \( m \) and \( \ell \) are kernels. We claim that \( \beta \) is also an \( \mathcal{X} \)-epic. Let us consider in the category of the abelian groups, the following commutative diagram with exact rows:

\[
\begin{array}{ccc}
0 & \to & \mathcal{A}(X, F) & \xrightarrow{m} & \mathcal{A}(X, B) & \xrightarrow{(k \circ h)} & \mathcal{A}(X, D) & \to & 0 \\
& & \downarrow h_\ast & & \downarrow h \circ \beta & & \downarrow \kappa \\
0 & \to & \mathcal{A}(X, E) & \xrightarrow{\ell} & \mathcal{A}(X, C) & \xrightarrow{k \circ \beta} & \mathcal{A}(X, D) & \to & 0
\end{array}
\]

Because \( h_\ast \) is an epimorphism, \( \beta \) is also an epimorphism. So, \( \beta \) is \( \mathcal{X} \)-epic. Since \( \ell \circ \beta \circ \alpha = h \circ g = 0 \) and \( \ell \) is a monomorphism, we obtain \( \beta \circ \alpha = 0 \). Let \( n : H \to F \) be a morphism with \( \beta \circ n = 0 \). We have \( \ell \circ \beta \circ n = h \circ m \circ n = 0 \). Hence, there is a morphism \( p : H \to A \) with \( g \circ p = m \circ n \) and \( m \circ \alpha \circ p = m \circ n \). Since \( m \) is a monomorphism we obtain \( \alpha \circ p = n \). This implies that \( \alpha \) is the kernel of \( \beta \). \( \blacksquare \)

\textbf{Theorem 2.12.} Let \( \mathcal{A} \) be an additive category, \( \mathcal{X} \) a contravariantly finite subcategory and \( \mathcal{Y} \) a covariantly finite subcategory of \( \mathcal{A} \).

(i) If any \( \mathcal{X} \)-epic has a kernel and \( \mathcal{X} \)-assignment for \( \mathcal{A} \) has been made, then the full subcategory \( \mathcal{D}_X \) of the distinguished left triangles of \( \mathcal{LTR}(\mathcal{A}_X, \Omega_X) \) is a left triangulation of \( (\mathcal{A}_X, \Omega_X) \).

(ii) If any \( \mathcal{Y} \)-monic has a cokernel and \( \mathcal{Y} \)-assignment for \( \mathcal{A} \) has been made, then the full subcategory \( \mathcal{D}_Y \) of the distinguished right triangles of \( \mathcal{RTR}(\mathcal{A}_Y, \Omega_Y) \) is a right triangulation of \( (\mathcal{A}_Y, \Omega_Y) \).

\textbf{Proof.} (i) We shall verify all axioms for a left triangulated category one by one.

\textbf{LT1.1.} The induced left triangle of the \( \mathcal{X} \)-epic \( 0 : A \to 0 \), is equal to \( 0 \xrightarrow{0} A \xrightarrow{1} A \xrightarrow{0} 0 \).

\textbf{LT1.ii.} Any morphism \( g : A \to B \), embeds to the distinguished left triangle \( \Omega_X B \xrightarrow{\zeta_g} \mathcal{Con}(g) \xrightarrow{-\eta_{\mathcal{X} \cdot g}} A \xrightarrow{g} B \).

\textbf{LT2.} It is enough to prove that for any morphism \( g : A \to B \) the left triangle \( \Omega_X A \xrightarrow{-\eta_{\mathcal{X} \cdot g}} \Omega_X B \xrightarrow{\zeta_g} \mathcal{Con}(g) \xrightarrow{-\eta_g} A \) is in \( \mathcal{D}_X \), since if a left triangle \( (D, F, G, k, \ell, h) \) is isomorphic to some \( (\mathcal{Con}(g), A, B, \zeta_g, -\eta_{\mathcal{X} \cdot g}, -\eta_g) \), then \( (\Omega_X G, D, F, -\Omega_X (h), k, \ell) \) is isomorphic to \( (\Omega_X B, \mathcal{Con}(g), \mathcal{D}, -\Omega_X (g), \zeta_g, -\eta_g) \).
Given the morphism \( g : A \to B \), we consider the commutative diagram

\[
\begin{array}{ccc}
K_A & \xrightarrow{\eta} & X_A & \xrightarrow{\epsilon} & A \\
\downarrow{\gamma} & & \downarrow{\epsilon} & & \downarrow{\lambda_A} \\
K_B & \xrightarrow{\eta} & \text{Con}(g) & \xrightarrow{\epsilon} & A \\
\downarrow{\gamma_B} & & \downarrow{\epsilon_g} & & \downarrow{\eta_A} \\
K_B & \xrightarrow{\eta} & X_B & \xrightarrow{\epsilon} & B
\end{array}
\]  
(2.3)

and the commutative diagram

\[
\begin{array}{ccc}
K_A & \xrightarrow{\eta} & X_A & \xrightarrow{\epsilon} & A \\
\downarrow{\gamma} & & \downarrow{\epsilon_g} & & \downarrow{\eta_A} \\
K_B & \xrightarrow{\eta} & X_B & \xrightarrow{\epsilon} & B
\end{array}
\]  
(2.4)

which is obtained from (2.3) by composition. Because of (2.4) we have that \( \Omega_X \eta_g = \gamma = \text{ch}(g) \). Hence, the induced left triangle obtained from (2.3) gives the general case, compare [7] Th.2.6, for the special case of a Frobenius category.

LT3. We assume without loss of generality, that our triangles are left distinguished, since a comparison of left triangles via isomorphisms in \( \text{CTR}(\mathcal{C}, \Omega) \), gives the general case, compare [7] Th.2.6, for the special case of a Frobenius category.

Let \( T_g = (\text{Con}(g), A, B, \xi, -\eta_g, -\gamma) \) and \( T_h = (\text{Con}(h), C, D, \zeta, -\eta_h, -\eta_h, -\gamma_h) \) be two distinguished left triangles and let \( \xi : A \to C, \xi : B \to D \) be two morphisms of \( \mathcal{A}_X \) satisfying \( \xi g = h \). The triangle \( T_g \) is determined by the commutative diagram (2.1). The commutative diagram below determines \( T_h \):

\[
\begin{array}{ccc}
K_D & \xrightarrow{\xi} & \text{Con}(h) & \xrightarrow{\eta} & C \\
\downarrow{\gamma_D} & & \downarrow{\eta_h} & & \downarrow{\lambda_C} \\
K_B & \xrightarrow{\eta} & X_D & \xrightarrow{\epsilon} & D
\end{array}
\]  
(2.5)

We consider also the commutative diagram

\[
\begin{array}{ccc}
K_B & \xrightarrow{\eta} & X_B & \xrightarrow{\epsilon} & B \\
\downarrow{\gamma_B} & & \downarrow{\epsilon_g} & & \downarrow{\eta_A} \\
K_D & \xrightarrow{\eta} & X_D & \xrightarrow{\epsilon} & D
\end{array}
\]  
(2.6)

which gives the image \( \Omega_X \xi = w \) of \( \xi \).

Firstly, we construct a morphism \( s : \text{Con}(g) \to \text{Con}(h) \). Since \( \xi g = h \), the morphism \( \eta h = \xi g : A \to D \) of \( \mathcal{A} \) factorizes over some object of \( \mathcal{X} \), and there is a morphism \( r : A \to X_D \) with \( h g k = f_D r + \xi g \). Composing the last equality from the right with \( \eta_g \) and using \( \xi g \eta_g = \xi g f_D r + \xi g \theta_g = f_D x \xi g \theta_g \), which holds because the diagrams (2.1) and (2.6) are commutative, we have \( h g k \eta_g = f_D r h \eta_g + x \xi g \theta_g \).

Since \( \text{Con}(h), \eta_h, \theta_h \) is the pullback of \( h \) and \( f_D \), there is a morphism \( s : \text{Con}(g) \to \text{Con}(h) \) with \( \eta h s = k \eta_g \) and \( \theta h s = r \eta_g + x \xi g \theta_g \).

Now, we shall show that \( s \xi g \eta_h = \zeta_h \eta h \xi g \). We construct firstly a representative \( v \) of \( \Omega_X \xi \) in \( A \). Since, \( \eta h s \xi g \zeta_h = k \eta_g \xi g \zeta_h = 0 \), there is a morphism \( v : K_B \to K_D \) with \( \zeta_h s \xi g = s \xi g \eta_h \zeta_h = 0 \). We claim that the image \( v \) of \( v \) in \( \mathcal{A}_X \) equals \( \Omega_X \xi \). For this it is enough to verify that setting \( v \) in place of \( w \), the left square of (2.6) becomes commutative. Indeed, \( \xi D \xi g = \theta h \xi g s \xi g = (\theta h \eta s) \xi g = r \eta g \xi g + x \xi g \theta g \xi g = x \xi g \xi g \).
So, we proved that given the morphisms $f$ and $k$ satisfying $f \circ k = h$, there is a morphism $g : \text{Con}(g) \rightarrow \text{Con}(h)$ making the first two squares from the left of the diagram below commutative

\[
\begin{array}{cccc}
\Omega X B & \xrightarrow{\delta g} & \text{Con}(g) & \xrightarrow{\eta g}  \\
\downarrow_{\Omega X \ell} & & \downarrow_{g} & \\
\Omega X D & \xrightarrow{\delta h} & \text{Con}(h) & \xrightarrow{\eta h} \\
\end{array}
\]

i.e. the triple $(g, k, \ell)$ is a morphism from $T_B$ to $T_h$.

LT4. We assume without loss of generality, compare the argumentation in LT3, that the two left triangles in the assumptions of LT4 are induced, i.e. we assume that there are two $X$-epics $h : B \rightarrow C$ and $k : C \rightarrow D$ of $A$ with kernels $g : A \rightarrow B$ and $\ell : E \rightarrow C$ respectively, inducing the left triangles $\Omega X C \xrightarrow{ch(h)} A \xrightarrow{g} B \xrightarrow{h} C$ and $\Omega X D \xrightarrow{ch(k)} E \xrightarrow{\ell} C \xrightarrow{k} D$. Because of Lemma 2.11, we obtain the diagram

\[
\begin{array}{cccc}
\Omega X E & \xrightarrow{ch(\beta)} & A & \xrightarrow{g} B \xrightarrow{h} C  \\
\downarrow_{\Omega X k} & & \downarrow_{\eta g} & \\
\Omega X D & \xrightarrow{ch(k\circ h)} & F & \xrightarrow{m} B \xrightarrow{k \circ h} D  \\
\downarrow_{\Omega X \ell} & & \downarrow_{\eta h} & \\
\Omega X D & \xrightarrow{ch(\ell)} & E & \xrightarrow{\ell} C \xrightarrow{k} D  \\
\end{array}
\]

where all squares, with possible exception of the two left hand ones, commute.

In order to prove LT4, we must show that $ch(k) = \beta \circ ch(k \circ h)$, $\alpha \circ ch(h) = ch(k \circ h) \circ \Omega X k$ and that $ch(\beta) = ch(h) \circ \Omega X \ell$.

Let us consider the commutative diagrams used for the construction of the characteristic classes of $h$, $k$, $k \circ h$ and $\beta$ respectively:

\[
\begin{array}{cccc}
K_C & \xrightarrow{\Delta} & X_C & \xrightarrow{\Delta} C  \\
\downarrow_{\gamma_1} & & \downarrow_{\ell_1} & \\
A & \xrightarrow{g} & B & \xrightarrow{h} C  \\
\end{array}
\]

\[
\begin{array}{cccc}
K_D & \xrightarrow{\Delta} & X_D & \xrightarrow{\Delta} D  \\
\downarrow_{\gamma_3} & & \downarrow_{\ell_3} & \\
F & \xrightarrow{m} & B & \xrightarrow{k \circ h} D  \\
\end{array}
\]

Since $k \circ (h \circ \delta_3) = f_D$, we may choose $\delta_2 = h \circ \delta_3$ and then since $(\ell \circ \beta) \circ \gamma_3 = h \circ (m \circ \gamma_3) = (h \circ \delta_3) \circ \ell_D$ we may choose $\gamma_2 = \beta \circ \gamma_3$. Hence, $ch(k) = \beta \circ ch(k \circ h)$.

Let us consider the two commutative diagrams used for the construction of $\Omega X k$ and $\Omega X \ell$ respectively:

\[
\begin{array}{cccc}
K_C & \xrightarrow{\Delta} & X_C & \xrightarrow{\Delta} C  \\
\downarrow_{\gamma_1} & & \downarrow_{\ell_1} & \\
K_E & \xrightarrow{\Delta} & X_E & \xrightarrow{\Delta} E  \\
\downarrow_{\gamma_3} & & \downarrow_{\ell_3} & \\
K_C & \xrightarrow{\Delta} & X_C & \xrightarrow{\Delta} C  \\
\downarrow_{\gamma_4} & & \downarrow_{\ell_4} & \\
K_E & \xrightarrow{\Delta} & X_E & \xrightarrow{\Delta} E  \\
\end{array}
\]
We have \((k,h)\delta_1 = k_2 f_C = f_D \circ x_k = (k,h)\delta_3 \circ x_k\). Since \(m\) is the kernel of \(k,h\), there is a morphism \(\rho : X_C \to F\) with \(m \circ \rho = \delta_1 - \delta_3 \circ x_k\). Hence, \(m \circ \delta_3 \circ x_k = \rho \circ \delta_1 - (\delta_3 \circ x_k) = m \circ \alpha \circ \gamma_1 - m \circ \gamma_3 \circ x_k\). Since \(m\) is a monomorphism we obtain \(\rho \circ \delta_3 \circ x_k = \alpha \gamma_1 - \gamma_3 \circ x_k\). Passing to the stable category \(\mathcal{A}_X\) we have \(\alpha \gamma_1 - \gamma_3 \circ x_k = \gamma_4\).

Since \(m\) is the kernel of \(h\), there is a morphism \(p : X_{c} \to \mathcal{F}\) with \(m \circ p = 61 - b_3 \circ x_k\). Hence, \(m \circ p \circ \mathcal{L}_k = b_1 \circ \mathcal{L}_k - 6_3 \circ (x_k \circ \mathcal{L}_k) = \gamma_4 \circ \mathcal{L}_k - \gamma_3 \circ x_k\). Since \(7n\) is a monomorphism we obtain \(p \circ \mathcal{L}_k = m \circ \gamma_1 - \gamma_3 \circ x_k\).

Passing to the stable category \(\mathcal{A}_X\) we have \(\gamma_4 \circ \mathcal{L}_k - \gamma_3 \circ x_k = \gamma_4\).

(ii) Left for the reader. □

We close the present section showing that given a contravariantly finite subcategory \(\mathcal{X}\) of \(\mathcal{A}\), the choice of some \(\mathcal{X}\)-assignment is inessential, in the sense that any two \(\mathcal{X}\)-assignments yield equivalent left triangulated structures.

Let us consider two \(\mathcal{X}\)-assignments of \(\mathcal{A}\) and let \(\Omega_{\mathcal{X}_1}\), \(\Omega_{\mathcal{X}_2}\) be the corresponding loop-space functors of \(\mathcal{A}_X\). Let \((\mathcal{A}_X, \Omega_{\mathcal{X}_1}, \Delta_1)\) and \((\mathcal{A}_X, \Omega_{\mathcal{X}_2}, \Delta_2)\) be the two left triangulated categories induced by the two \(\mathcal{X}\)-assignments.

In Proposition 1.2 we mentioned already, that there is a natural equivalence \((\mathcal{X}) : \Omega_{\mathcal{X}_1} \to \Omega_{\mathcal{X}_2}\). Using \((\mathcal{X})\) we shall show in the next theorem, that the identity functor of \(\mathcal{A}_X\) is a triangle equivalence.

**Theorem 2.13.** The identity functor \(\text{Id}_{\mathcal{A}_X}\) of \(\mathcal{A}_X\) is a triangle equivalence from \((\mathcal{A}_X, \Omega_{\mathcal{X}_1}, \Delta_1)\) to \((\mathcal{A}_X, \Omega_{\mathcal{X}_2}, \Delta_2)\).

**Proof.** Since \((\mathcal{X})\) is a natural isomorphism from \(\text{Id}_{\mathcal{A}_X} \circ \Omega_{\mathcal{X}_1} = \Omega_{\mathcal{X}_1}\) to \(\Omega_{\mathcal{X}_2} \circ \text{Id}_{\mathcal{A}_X} = \Omega_{\mathcal{X}_2}\), it remains to show that given the induced left triangle \(\Omega_{\mathcal{X}_1} B \xrightarrow{\mathcal{L}(\mathcal{X})^{-1}} C \xrightarrow{h} A \xrightarrow{g} B\), then \(\Omega_{\mathcal{X}_2} B \xrightarrow{\mathcal{L}(\mathcal{X})^{-1}} C \xrightarrow{h} A \xrightarrow{g} B\) is a left triangle of \(\Delta_2\). Let us consider the commutative diagram

\[
\begin{array}{ccc}
K_{B_1} & \xrightarrow{\mathcal{L}(\mathcal{X})} & X_{B_1} \\
\downarrow & & \downarrow \quad \circ \\
C & \xrightarrow{h} & A \\
\end{array}
\]

where the (first resp. second) row from the top corresponds to the first (resp. second) \(\mathcal{X}\)-assignment and where \(g\) is an \(\mathcal{X}\)-epic with kernel \(h\). Since \(h \circ \gamma_2 = \delta_4 \circ x_k\) and \(2 = \mathcal{L}(\mathcal{X})^{-1}\), we obtain \(\mathcal{L}(\mathcal{X})^{-1} = \mathcal{L}(\mathcal{X})^{-1} \circ x_k\). Hence, \(\Omega_{\mathcal{X}_2} B \xrightarrow{\mathcal{L}(\mathcal{X})^{-1}} C \xrightarrow{h} A \xrightarrow{g} B\) coincides with the induced left triangle \(\Omega_{\mathcal{X}_2} B \xrightarrow{\mathcal{L}(\mathcal{X})^{-1}} C \xrightarrow{h} A \xrightarrow{g} B\) of \(\Delta_2\), which is obtained from the \(\mathcal{X}\)-epic \(h\). This finishes the proof. □

**Remark 2.14.** Our Theorem 2.12 covers many of the well known constructions of triangulated categories. In case \(\mathcal{A}\) is an abelian category with enough projectives (resp. injectives) relative to a projective (resp. injective) class \(E\) of morphisms of \(\mathcal{A}\), then the full subcategory \(\mathcal{P}\) (resp. \(\mathcal{I}\) of the relative projectives (resp. injectives) is a contravariantly finite (resp. covariantly finite) subcategory.
of $A$ and so our Theorem can be applied directly. For example if $B$ is an exact category, $[11], [10]$ with enough injectives, then these injectives form a covariantly finite subcategory of $B$ and so $B$ has a right triangulated structure, compare $[10]$. In case $B$ is Frobenius, then the corresponding suspension functor is an equivalence and so $B$ has a triangulated structure, $[6], [7]$. Well known special case of this situation is the category of complexes, where the injective class $E$ of morphisms consists of the monomorphisms which split if we forget the differentials and consider any complex as a graded object, compare $[13], [6], [7]$.

3. ON STABLE CATEGORIES OBTAINED FROM SUBFUNCTORS OF $\text{EXT}_A^1(\cdot, \cdot)$.

Let $A$ be an artin algebra and $\text{mod}A$ be the category of the finitely generated right $A$-modules.

We write $Z \subseteq W$ in case $Z$ and $W$ are subcategories of $\text{mod}A$ with $Z$ a subcategory of $W$.

Let $X$ (resp. $Y$) be a contravariantly finite (resp. covariantly finite) subcategory of $\text{mod}A$, and $\text{mod}_X\Lambda$ (resp. $\text{mod}_Y\Lambda$) be the corresponding stable category. As direct application of Theorem 2.12 we obtain the main theorem of our paper:

Main Theorem 3.1.

(i) The triple $(\text{mod}_X\Lambda, \Omega_X, \Delta_X)$ is a left triangulated category.
(ii) The triple $(\text{mod}_Y\Lambda, \Sigma_Y, \Delta_Y)$ is a right triangulated category.

Proof. According to Theorem 2.12, one has to check that the $X$-epics (resp. $Y$-monics) have kernels (resp. cokernels). This is obvious, since $\text{mod}A$ is abelian category. $\square$

In the remaining part of this section all considered subcategories $Z$ of $\text{mod}A$ are assumed to be full, strict, additive, and closed under direct summands.

Let $F$ be an additive subfunctor of $\text{Ext}_A^1(\cdot, \cdot)$ and $\mathcal{P}(F)$ (resp. $\mathcal{I}(F)$) be the full subcategory of $\text{mod}A$ consisting of the $F$-projective (resp. $F$-injective) $A$-modules, cf. $[3]$.

Let $\text{mod}_{\mathcal{P}(F)}\Lambda$ (resp. $\text{mod}_{\mathcal{I}(F)}\Lambda$) be the stable category $\text{mod}A/\mathcal{P}(F)$ (resp. $\text{mod}A/\mathcal{I}(F)$), where $\mathcal{P}(F)$ (resp. $\mathcal{I}(F)$) is the ideal of morphisms of $\text{mod}A$, which factor over the $F$-projectives (resp. $F$-injectives) and let $\Omega_{\mathcal{P}(F)} : \text{mod}_{\mathcal{P}(F)}\Lambda \rightarrow \text{mod}_{\mathcal{P}(F)}\Lambda$ (resp. $\Sigma_{\mathcal{I}(F)} : \text{mod}_{\mathcal{I}(F)}\Lambda \rightarrow \text{mod}_{\mathcal{I}(F)}\Lambda$) be the corresponding loop (resp. suspension) functor.

In case $F$ has enough projectives (resp. injectives), then $\mathcal{P}(F)$ (resp. $\mathcal{I}(F)$) is a contravariantly finite (resp. covariantly finite) subcategory of $\text{mod}A$, our main Theorem 3.1 implies:

Theorem 3.2.

(i) The triple $(\text{mod}_{\mathcal{P}(F)}\Lambda, \Omega_{\mathcal{P}(F)}, \Delta_{\mathcal{P}(F)})$ is a left triangulated category.
(ii) The triple $(\text{mod}_{\mathcal{I}(F)}\Lambda, \Sigma_{\mathcal{I}(F)}, \Delta_{\mathcal{I}(F)})$ is a right triangulated category.

Our aim is to study, when the the triple $(\text{mod}_{\mathcal{P}(F)}\Lambda, \Omega_{\mathcal{P}(F)}, \Delta_{\mathcal{P}(F)})$ or the triple $(\text{mod}_{\mathcal{I}(F)}\Lambda, \Sigma_{\mathcal{I}(F)}, \Delta_{\mathcal{I}(F)})$ is a triangulated category, i.e. when $\Omega_{\mathcal{P}(F)} : \text{mod}_{\mathcal{P}(F)}\Lambda \rightarrow \text{mod}_{\mathcal{P}(F)}\Lambda$ or $\Sigma_{\mathcal{I}(F)} : \text{mod}_{\mathcal{I}(F)}\Lambda \rightarrow \text{mod}_{\mathcal{I}(F)}\Lambda$ is an equivalence.

Let $F$ be an additive subfunctor of $\text{Ext}_A^1(\cdot, \cdot)$ with enough projectives and injectives.
Definition 3.3. The algebra $A$ is said to be $F$-Frobenius if $\mathcal{P}(F) = \mathcal{I}(F)$.

Theorem 3.4. Let $F$ be an additive subfunctor of $\text{Ext}^1_A(\ ,\ )$ having enough projectives and injectives. The following are equivalent:

(i) The algebra $A$ is $F$-Frobenius.

(ii) The triple $(\text{mod}_{\mathcal{P}(F)}A, \Omega_{\mathcal{P}(F)}, \Delta_{\mathcal{P}(F)})$ is a triangulated category.

(iii) The triple $(\text{mod}_{\mathcal{I}(F)}A, \Sigma^{\mathcal{I}(F)}, \Delta^{\mathcal{I}(F)})$ is a triangulated category.

Proof. (i) $\rightarrow$ (ii) Let $A$ be a $\Lambda$-module and

$$0 \rightarrow K \xrightarrow{\delta} P_A \xrightarrow{f} A \rightarrow 0$$

be an $F$-projective presentation of $A$. Given an $F$-injective module $I$, we obtain the short exact sequence

$$0 \rightarrow \text{Hom}_A(A, I) \rightarrow \text{Hom}_A(P_A, I) \rightarrow \text{Hom}_A(K, I) \rightarrow 0.$$

Since $P_A$ is also $F$-injective, the exactness of (3.2) implies that (3.1) is an $F$-injective copresentation of $K$. Now, the construction of the functors $\Omega_{\mathcal{P}(F)}$ and $\Sigma^{\mathcal{I}(F)}$ in the first section of our paper, implies that $\Sigma^{\mathcal{I}(F)} \circ \Omega_{\mathcal{P}(F)}$ is natural equivalent to the identity functor of $\text{mod}_{\mathcal{P}(F)}A$. Similarly, $\Omega_{\mathcal{P}(F)} \circ \Sigma^{\mathcal{I}(F)}$ is natural equivalent to the identity functor of $\text{mod}_{\mathcal{I}(F)}A$. The proof is complete.

(ii) $\rightarrow$ (i) Let $I$ be an $F$-injective module. Because $\Omega_{\mathcal{P}(F)}$ is an equivalence, there is a $\Lambda$-module $A$ such that $\Omega_{\mathcal{P}(F)}A = I$. Hence, there is an $F$-exact sequence

$$0 \rightarrow I \oplus L \xrightarrow{(\alpha, \beta)} P \rightarrow A \oplus N \rightarrow 0$$

with $L$, $N$ and $P$ in $\mathcal{P}(F)$. Let $(1_I, 0) : I \oplus P \rightarrow I$ be the canonical projection to $I$. Since $(\alpha, \beta)$ is a $\mathcal{P}(F)$-monic and $I$ is in $\mathcal{I}(F)$, there is a morphism $\gamma : P \rightarrow I$ satisfying $\gamma_0(\alpha, \beta) = (1_I, 0)$. So, $\gamma : I \rightarrow P$ is a split monomorphism and $I$ is an $F$-projective. Hence, $I(F) \subseteq \mathcal{P}(F)$.

Let $P$ be an $F$-projective. Since $\mathcal{I}(F)$ is a covariantly finite cogenerator of $\text{mod}\Lambda$, there is an $F$-exact sequence

$$0 \rightarrow P \xrightarrow{h} I \xrightarrow{\alpha} A \rightarrow 0$$

where $I$ is $F$-injective. Since $g$ is a $\mathcal{P}(F)$-epic, we obtain in $\text{mod}_{\mathcal{P}(F)}A$, the $\mathcal{P}(F)$-induced left triangle $\Omega_{\mathcal{P}(F)}A \xrightarrow{\text{ch}(g)} P \xrightarrow{h} I \xrightarrow{\alpha} A$. Because $I$ and $P$ are in $\mathcal{P}(F)$, this left triangle equals $\Omega_{\mathcal{P}(F)}A \rightarrow 0 \rightarrow 0 \rightarrow A$. Because of LT2, the left triangle $0 \rightarrow \Omega_{\mathcal{P}(F)}A \rightarrow 0 \rightarrow 0$ belongs also to $\Delta_{\mathcal{P}(F)}$ implying $\Omega_{\mathcal{P}(F)}A = 0$. Since $\Omega_{\mathcal{P}(F)}A$ is an equivalence, $A$ has to be in $\mathcal{P}(F)$. Now the sequence (3.3) has to be a splitting one, and so we obtain that $P$ is in $\mathcal{I}(F)$. So $\mathcal{P}(F) \subseteq \mathcal{I}(F)$. This finishes the proof.

(ii) $\rightarrow$ (iii). Left for the reader. \(\square\)

Remark 3.5. We recall from [4], that an artin algebra $\Lambda$ is said to be $F$-selfinjective, if $\mathcal{P}(F) = \mathcal{I}(F)$ and $\mathcal{P}(F)$ is a subcategory of finite type. Since any subcategory of finite type is functorially finite, the functor $F$ has enough projectives and injectives and so an $F$-selfinjective algebra $\Lambda$ is also $F$-Frobenius. Hence, if $\Lambda$ is an $F$-selfinjective algebra, then $\text{mod}_{\mathcal{P}(F)}A$ is a triangulated category, because of the previous Theorem.
Suppose now that $\mathcal{X}$ is a contravariantly finite subcategory of $\text{mod}A$ and $\mathcal{Y}$ a covariantly finite subcategory of $\text{mod}A$. We consider the stable categories $\text{mod}_\mathcal{X}A$ and $\text{mod}_\mathcal{Y}A$.

**Definition 3.6.**

(i) The artin algebra $A$ is said to be **contravariantly $\mathcal{X}$-Frobenius** if the triple $(\text{mod}_\mathcal{X}A, \Omega_{\mathcal{X}}, \Delta_{\mathcal{X}})$ is a triangulated category.

(ii) The artin algebra $A$ is said to be **covariantly $\mathcal{Y}$-Frobenius** if the triple $(\text{mod}_\mathcal{Y}A, \Sigma_{\mathcal{Y}}, \Delta_{\mathcal{Y}})$ is a triangulated category.

Let $\mathcal{P}(\Lambda)$ (resp. $\mathcal{I}(\Lambda)$) be the subcategory of the projective (resp. injective) $\Lambda$-modules.

**Proposition 3.7.** Let $\mathcal{X}$ be a functorially finite generator of $\text{mod}A$ and $\mathcal{Y}$ a functorially finite cogenerator of $\text{mod}A$.

(i) $A$ is contravariantly $\mathcal{X}$-Frobenius if and only if $\mathcal{X} = \mathcal{I}(\Lambda) \cup \text{DTr} \mathcal{X}$.

(ii) $A$ is covariantly $\mathcal{Y}$-Frobenius if and only if $\mathcal{Y} = \mathcal{P}(\Lambda) \cup \text{DTr} \mathcal{Y}$.

**Proof.** Let us consider the additive subfunctor $F = F_{\mathcal{X}}$ of $\text{Ext}_A^1(\cdot, \cdot)$. Since $\mathcal{X}$ is a contravariantly finite generator, we have by [2], Th. 1.14(c), that $\mathcal{P}(F) = \mathcal{X}$ and also that $F$ has enough projectives and injectives. Moreover we have by [2], Cor. 1.16(a), that $\mathcal{I}(F) = \mathcal{I}(\Lambda) \cup \text{DTr} F$. Hence, $\mathcal{I}(F) = \mathcal{I}(\Lambda) \cup \text{DTr} \mathcal{X}$.

(i) Because $\text{mod}_{\mathcal{P}(F)} A = \text{mod}_\mathcal{X}A$, the triple $(\text{mod}_\mathcal{X}A, \Omega_{\mathcal{X}}, \Delta_{\mathcal{X}})$ is a triangulated category, and from Theorem 3.4 it follows that $A$ is $F$-Frobenius. Hence $\mathcal{P}(F) = \mathcal{I}(F)$ and so $\mathcal{X} = \mathcal{I}(\Lambda) \cup \text{DTr} \mathcal{X}$.

Conversely, since $\mathcal{X} = \mathcal{I}(\Lambda) \cup \text{DTr} \mathcal{X}$ it follows that $\mathcal{I}(F) = \mathcal{P}(F) = \mathcal{X}$. Hence, $\Lambda$ is $F$-Frobenius, which implies by Theorem 3.4, that $(\text{mod}_\mathcal{X}A, \Omega_{\mathcal{X}}, \Delta_{\mathcal{X}})$ is a triangulated category and hence $A$ is contravariantly $\mathcal{X}$-Frobenius.

(ii) Left for the reader. □

**Example 3.8.** An artin algebra $A$ is $\text{Ext}_A^1(\cdot, \cdot)$-Frobenius if and only if $\mathcal{P}(\Lambda) = \mathcal{I}(\Lambda)$, i.e. if and only if it is selfinjective. Because of Theorem 3.4, an algebra $A$ is selfinjective if and only if $\text{mod}_{\mathcal{P}(\Lambda)}$ is a triangulated category. A well-known result.

**Example 3.9.** Let $A$ be an artin algebra of finite representation type. Let $\mathcal{J}$ be the subcategory of $\text{mod}A$ with objects the indecomposable modules of the form $\text{DTr} I_j$, $i \in \mathbb{N} \cup 0$, where $\{I_j\}$ is a complete set of non-isomorphic indecomposable $A$-injective modules. Let us consider the subcategory $\mathcal{W} = \text{add} \mathcal{J}$. Since $\mathcal{W}$ is of finite type, it is functorially finite and also a generator of $\text{mod}A$. Moreover it satisfies the equality $\mathcal{W} = \mathcal{I}(\Lambda) \cup \text{DTr} \mathcal{W}$. Hence, any artin algebra $A$ of finite representation type is contravariantly $\mathcal{W}$-Frobenius and so the triple $(\text{mod}_\mathcal{W}A, \Omega_{\mathcal{W}}, \Delta_{\mathcal{W}})$ is a triangulated category. The Auslander-Reiten quiver of $\text{mod}_\mathcal{W}A$ coincides with the stable quiver $A_+^*(\Lambda)$ of the algebra $\Lambda$, introduced in [12] and later in [8].

**References**


Received: September 1993