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Representation Theory of Finite-Dimensional Algebras

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Abstract. Methods and results from the representation theory of finite dimensional algebras have led to many interactions with other areas of mathematics. The aim of this workshop was, in addition to stimulating progress in the representation theory of algebras, to further develop such interactions with commutative algebra, algebraic geometry, group representation theory, Lie-algebras and quantum groups, but also with the new theory of cluster algebras.

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Introduction by the Organisers

The workshop was well attended with about 50 participants from many continents. The core group was from the representation theory of algebras, with several experts from related areas.

The previous Oberwolfach meeting devoted to representations of finite-dimensional (associative) algebras took place in 2000. After discussions with the then director, we decided to shift the activities to other mathematical centers. We are grateful to the new director who convinced us that all parts of active mathematical research are again welcome at Oberwolfach.

The 2005 Oberwolfach meeting was preceded by a two days conference held at Bielefeld with the title *Perspectives in Mathematics: Algebras and Representations.* The lectures at Bielefeld provided outlines of new developments. They were given by J. F. Carlson (Athens), W. W. Crawley-Boevey (Leeds), S. Iyengar (Lincoln), J. C. Jantzen (Aarhus), M. Reineke (Münster), I. Reiten (Trondheim), R. Rouquier (Paris, New Haven), J. Schröer (Leeds), Jie Xiao (Beijing), A. Zelevinsky (Boston), and E. Zelmanov (San Diego). Since most of the Oberwolfach participants took part in the Bielefeld conference, it was possible to arrange the Oberwolfach schedule in a complementary way.

By now, the usefulness of methods and results from the representation theory of algebras is well-known. In particular, the notion of a quiver, its representations and the corresponding quiver varieties have become quite popular in many parts of mathematics. In this way, there have been a lot of interactions between the representation theory of algebras and other areas. A main focus of this meeting was to promote the interaction with such areas, and most of the talks dealt with topics of general interest.

One of the important connections with other areas is given by the Hall algebras and their connection with quantum groups. Recent developments in this area, partially inspired also by work of Drinfeld, were presented by Hubery and by Keller. The latter report was based on a recent construction of Toën.

The cluster algebras introduced by Fomin–Zelevinsky have had a lot of influence on various parts of algebra, including representation theory of algebras. Some recent investigations on quantum cluster algebras were presented by Zelevinsky, with challenging questions about further connections with finite dimensional algebras. The work on cluster algebras inspired work on what are called cluster categories and cluster tilted algebras, which gives some feedback on the theory of cluster algebras, in particular in the acyclic case. This was discussed in talks by Marsh and Buan. The cluster tilted algebras are of interest for several reasons: they provide a new class of algebras whose representation theory is controlled by a quadratic form, and they shed light on the tilted algebras themselves: any tilted algebra is the factor algebra of a corresponding cluster tilted algebra. Further relationship between cluster algebras and finite dimensional preprojective algebras was discussed by Geiß.

The cluster categories are Calabi-Yau categories of dimension 2 (and related categories give arbitrary dimensions). They contain the stable categories of preprojective algebras of finite type, and such stable categories are Calabi-Yau of dimension 2 for finite dimensional preprojective algebras in general. This has put an emphasis on the study of Calabi-Yau categories and their dimensions for categories related to finite dimensional algebras, in particular for stable categories of selfinjective algebras. Results of this nature were discussed by Erdmann. Iyama's higher analogue of almost split sequences in maximal n-orthogonal subcategories is also related to this, and his talk dealt with complements in Calabi-Yau categories. There are interesting examples in commutative ring theory, as discussed by Yoshino. With many experts on areas where Calabi-Yau categories appear, an evening session was organized to provide a survey on Calabi-Yau phenomena, with contributions by Buchweitz, Geiß, Hille, Lenzing, Neeman and Van den Bergh. A wide range of topics was touched, for example mirror symmetry and reflexive polytopes, A_∞ categories, elliptic curves. Three short abstracts concerning these evening lectures are included at the end of the report.

Derived and triangulated categories were also discussed from other points of view, in connection with coherent sheaves by Burban, infinitesimal deformations by Keller, thick subcategories by Krause, and with homotopy categories of projectives and of injectives by Iyengar. And we have to mention here the various aspects of Koszul duality. The corresponding Koszul algebras are a topic of central interest. Questions concerning Koszul algebras were discussed by Martínez-Villa, Martsinkovsky, Green, and Zacharia.

An important collection of problems in finite dimensional algebra theory are the homological conjectures, including the finitistic dimension conjecture, the (generalized) Nakayama conjecture and the Gorenstein symmetry conjecture. Some of these problems are of interest also in commutative algebra. A survey, along with new ideas for attacking the first conjecture, was given by Xi, and ideas for the Gorenstein symmetry conjecture by Beligiannis. A simple counterexample to a more general conjecture of Auslander was presented by Smalø. Here, a first example was given within commutative algebra. A related homological conjecture is the conjecture of Happel, that the eventual vanishing of Hochschild cohomology implies finite global dimension, where a counterexample was given in the talk by Green. Homological techniques in commutative algebra were discussed by Avramov.

Various aspects of quiver representations were dealt with in talks by Reineke and Buchweitz, and de la Peña discussed problems related to spectral radii. Applications of techniques and results for proving finite, tame or wild representation type were given by Schmidmeier and Farnsteiner.

The workshop presented a vivid picture of the present state of the art. And it provided a clear sight of the many still open problems, and on methods which may be helpful to attack them. We are sure that the interaction between the participants will lead to further progress in the coming years.

We thank Angela Holtmann (Bielefeld) for her careful preparation of this report.

Trondheim and Bielefeld, May 16th, 2005

Idun Reiten, Claus Michael Ringel

Workshop: Representation Theory of Finite-Dimensional Algebras

Table of Contents

Abstracts

The Calabi-Yau dimension of tame symmetric algebras

Karin Erdmann
(joint work with Andrzej Skowroński)

Let K be a field and $\mathcal{T}$ a K-linear triangulated category which has Serre duality, that is there is a triangle functor S such that $D\operatorname{Hom}_{\mathcal{T}}(X,-) \cong \operatorname{Hom}_{\mathcal{T}}(-,SX)$ for each object X in $\mathcal{T}$. By a definition of Kontsevich [8], $\mathcal{T}$ is Calabi-Yau if S is isomorphic to some power of the shift of $\mathcal{T}$, if so, then the CY-dimension is the minimal $d \geq 0$ such that S is isomorphic to $[d]$ (see also [7]).

Let A be a finite-dimensional selfinjective algebra over K and assume that K is algebraically closed. The stable category $\underline{\operatorname{mod}}(A)$ is triangulated, with shift given by Ω^{-1}, which is the inverse of the functor which sends a module M to the kernel $\Omega(M)$ of a minimal projective cover. The stable category $\underline{\operatorname{mod}}(A)$ has Serre duality given by $\Omega\circ\nu$. Here ν is the Nakayama functor $\nu = D\operatorname{Hom}_A(-,A)$. Then $\underline{\operatorname{mod}}(A)$ is Calabi-Yau of CY-dimension d if $d \geq 0$ is minimal such $\Omega\circ\nu$ is isomorphic to Ω^{-d}. When A is symmetric, $\Omega\circ\nu \cong \Omega^{-d}$ if and only if Ω^{d+1} is isomorphic to the identity on $\underline{\operatorname{mod}}(A)$.

Suppose A is symmetric. If $\underline{\operatorname{mod}}(A)$ has finite CY-dimension then it is necessary that all simple A-modules are Ω-periodic. Recently we completed classifying tame symmetric algebras which have only Ω-periodic simple modules [4]. These are precisely the algebras whose connected components are, up to Morita equivalence,

(1) symmetric algebras of Dynkin type;
(2) symmetric algebras of tubular type;
(3) algebras of quaternion type;
(4) socle deformations of algebras in (1) or (2).

The algebras in (1) and (2) are of the form $\hat{B}/(\varphi)$ where $\hat{B}$ is the repetitive algebra of B and φ is an appropriate root of the Nakayama automorphism $\nu_{\hat{B}}$. Here the algebras B are tilted of Dynkin type (in (1)), or of tubular type (in (2)); for details and further references see [3]. An algebra A is of quaternion type if it is connected, tame and symmetric with non-singular Cartan matrix, and such that all indecomposable non-projective A-modules are periodic of period ≤ 4. Any such algebra belongs, up to Morita equivalence, to a small list, explicitly given by quivers and relations [2].

These algebras have been known and studied extensively over the last years. In [4] we show that these are all tame symmetric algebras with only periodic simple modules. Moreover, we have:

Theorem 1. *Assume A is tame and symmetric. Then $\underline{\operatorname{mod}}(A)$ has finite CY-dimension if and only if A is one of the algebras in this list.*

This is proved in [3]. In each case, we determine the Calabi-Yau dimension of $\underline{\operatorname{mod}}(A)$ explicitly. For algebras as in (1) the CY-dimension is given by a formula involving the Coxeter number of the associated Dynkin diagram, and it turns out that all integers can occur. These are the symmetric algebras of finite type. As a contrast, for selfinjective algebras of finite type, the stable category need not have finite CY-dimension. For the algebras in (2) the CY-dimensions are precisely the prime numbers $2, 3, 5, 7$ and 11.

If A is an algebra of quaternion type, then the stable module category has CY-dimension 3. As the main part of the proof, we show, using [6], that all derived equivalence classes of such algebras, except for a few (which are of tubular type), contain an algebra which has a periodic bimodule resolution of period 4. As a consequence we can complete the classification of algebras of quaternion type. Namely it follows that for the algebras in the list given in [2], all indecomposable non-projective modules have Ω-period at most 4.

Furthermore, we study arbitrary selfinjective algebras A such that $\underline{\mathrm{mod}}(A)$ has CY-dimension 2. For this to happen it is necessary that every simple A-module S satisfies $\nu(S) \cong \Omega^{-3}(S)$; and algebras with this property were studied in [1]. The main result is:

Theorem 2 ([1]). *Let A be a connected finite-dimensional selfinjective algebra. Then the following are equivalent:*

(a) *Every simple A-module S satisfies $\nu(S) \cong \Omega^{-3}(S)$;*
(b) *A is either generalized preprojective, i. e. A is Morita equivalent to $P(\Delta)$ with Δ either Dynkin of type ADE, or of type L, or A is Morita equivalent to a certain deformation $P^f(\Delta)$.*

Moreover, any such algebra has a periodic bimodule resolution.

The *preprojective algebra* $P(\Delta)$ for Δ a Dynkin graph has quiver Q_Δ obtained from Δ by replacing each edge by a pair of vertices, one in each direction, denoted by a and $\bar{a}$, setting $\bar{\bar{a}} = a$. Then $P(\Delta) = KQ_\Delta/I$ where I is the ideal of the path algebra generated by all relations of the form

$$\sum_{a, ia=v} a\bar{a} \quad (v \text{ a vertex of } Q_\Delta).$$

The algebra $P(L_n)$, which we call *generalized preprojective*, is defined similarly. Its quiver is obtained from Q_{A_n} by by attaching a loop, ε say, to one of the end vertices. We set $\bar{\varepsilon} = \varepsilon$ and define $P(L_n)$ by the same relations as the preprojective algebras; see also [7]. The algebras $P^f(\Delta)$ are deformations of $P(\Delta)$ where only the relation at the branch vertex (or at the loop) is deformed. The precise definition is given in [1]. This theorem is proved by exploiting subadditive functions, as studied in [5].

The stable category of an algebra $P(\Delta)$ in the Dynkin case is known to have CY-dimension 2; and for $\Delta = L_n$ this also holds. Our theorem implies that the stable categories of the deformed algebras $P^f(\Delta)$ have finite CY-dimension. We do not know at present whether they also have CY-dimension 2.

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On the growth of the Coxeter transformations of derived-hereditary algebras

José Antonio de la Peña

(joint work with Helmut Lenzing)

For a finite dimensional k-algebra A of finite global dimension, the *Coxeter transformation* φ_A is an automorphism of the Grothendieck group $K_0(A)$. Moreover, for any complex $X^\bullet$ in the bounded derived category $D(A) := D^b(\operatorname{mod} A)$ of finite dimensional A-modules, we have $[X^\bullet]\varphi_A = [\tau_{D(A)}X^\bullet]$, where $\tau_{D(A)}$ is the automorphism of $D(A)$ given by the Auslander-Reiten translation. The characteristic polynomial $\chi_A(T)$ of φ_A, called the *Coxeter polynomial* and the corresponding spectral radius $\rho(\varphi_A) = \{\|\lambda\| \colon \lambda \in \operatorname{Spec}\varphi_A\}$ control the growth behavior of φ_A and hence of $\tau_{D(A)}$. Clearly, χ_A and $\rho(\varphi_A)$ are invariant under derived equivalences of the algebra A and provide natural links between the representation theory of finite dimensional algebras and other theories: the theory of Lie algebras, the theory of C^*-algebras, the spectral theory of graphs and the theory of knots and links, among other topics.

In the representation theory of algebras several cases have been extensively studied. For a hereditary algebra $A = k[\vec{\Delta}]$ associated to a finite quiver $\vec{\Delta}$ without oriented cycles, either Δ is Dynkin or affine and $\rho(\varphi_A) = 1$, or A is of wild type and $\rho(\varphi_A)$ is a simple root of the Coxeter polynomial; moreover, if A is wild for any non-preprojective indecomposable module X, the sequence of vectors $([\tau_A^n X])_{n\in\mathbb{N}}$ grows exponentially with ratio $\rho(\varphi_A)$, where τ_A denotes the Auslander-Reiten translation in the module category $\operatorname{mod} A$. Moreover, if A is wild, we have $\rho(\varphi_B) < \rho(\varphi_A)$ for any algebra $B = k[\vec{\Delta}']$ where Δ' is a proper full subgraph of Δ. For a canonical algebra $A = A(\mathbf{p}, \boldsymbol{\lambda})$, associated to a weight sequence $\mathbf{p} = (p_1, \ldots, p_t)$ of positive integers and a parameter sequence $\boldsymbol{\lambda} = (\lambda_3, \ldots, \lambda_t)$ of pairwise distinct non-zero elements from the base field k, the K-theory is well understood. In this case

$\rho(\varphi_A) = 1$, even while A is a one-point extension $B[M]$ of the hereditary star $B = \mathbb{T}_{p_1,\dots,p_t}$

$$\begin{tikzcd}[row sep=small]
(1,1) \arrow[r] & (1,2) \arrow[r] & \cdots \arrow[r] & (1,p_1-1) \arrow[ddr] & \\
(2,1) \arrow[r] & (2,2) \arrow[r] & \cdots \arrow[r] & (2,p_2-1) \arrow[dr] & \\
 & & \vdots & & \bullet \\
(t,1) \arrow[r] & (t,2) \arrow[r] & \cdots \arrow[r] & (t,p_t-1) \arrow[ur] &
\end{tikzcd}$$

which has spectral radius $\rho(\varphi_B)$ arbitrarily large. In case A is wild, that is $\mathbb{T}_{p_1,\dots,p_t}$ is not Dynkin or affine, the growth of τ_A is more complicated than in the hereditary case, since there are indecomposable A-modules X and Y for which $([\tau_A^n X])_n$ and $([\tau_A^{-n} Y])_n$ grow exponentially with $\rho(\varphi_B)$ while $([\tau_A^{-n} X])_n$ and $([\tau_A^n Y])_n$ grow linearly.

We consider the case of an algebra A *derived equivalent to a hereditary algebra* $k[\vec{\Delta}]$. These algebras may be obtained by a finite sequence of tilting processes starting from $k[\vec{\Delta}]$. Moreover, if Δ is of Dynkin or affine type, the construction of A and its Auslander-Reiten quiver is well described. In general, φ_A is conjugate to $\varphi_{k[\vec{\Delta}]}$, hence if Δ is of wild type, $\rho(\varphi_A)$ is a simple root of $\chi_A(T)$. On the other hand, we show simple examples of derived hereditary algebras A and B, with B a full convex subcategory of A and $\rho(\varphi_B) > \rho(\varphi_A)$. We give conditions on a B-module M such that, for the one-point extension $A = B[M]$, the inequality $\rho(\varphi_B) \leq \rho(\varphi_A)$ is satisfied. Namely, we prove that such a module should be *derived-directing*, that is, $M = F(X)$ for X a direct sum of directing complexes in $D(k[\vec{\Delta}])$ and $F\colon D(k[\vec{\Delta}]) \to D(A)$ an equivalence of triangulated categories.

We describe all possible one-point extensions $B[M]$, of certain representation-finite algebras B derived equivalent to wild hereditary algebras, by an indecomposable B-module M. For modules which are not derived-directing, we find algebras $A = B[M]$ which are not derived canonical or derived tame or wild hereditary; nevertheless, the spectral radius $\rho(\varphi_A)$ of the Coxeter polynomial is 1, but not an eigenvalue of φ_A. This new class of algebras will be further studied.

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Quantum affine $\mathfrak{gl}_n$ via Ringel-Hall algebras

Andrew Hubery

Ringel-Hall algebras were introduced in [6] and provide a generalisation of the classical Hall algebra of a discrete valuation ring with finite residue field to an arbitrary finitary ring. It was later shown in [4] that in the case of a hereditary algebra, the (twisted, generic) composition algebra (informally, the subalgebra of the Ringel-Hall algebra generated by the simple modules) realises the quantum group of the same type. In particular, this isomorphism identifies the simple modules with the Chevalley generators.

For an affine Lie algebra, Drinfeld gave a 'new realisation' of the quantised enveloping algebra by quantising the loop-algebra construction of the Lie algebra [2]. An explicit isomorphism between the two presentations was given by Beck [1] in the untwisted case, but the question of understanding the Drinfeld generators in terms of Ringel-Hall algebras remained open.

In the talk, we solved this problem for the affine Lie algebra $\widehat{\mathfrak{sl}}_n$ using the Ringel-Hall algebra of the cyclic quiver with n vertices. In fact, we extended the result to include $\widehat{\mathfrak{gl}}_n$ and thus proved a conjecture of Schiffmann [9].

Let C_n be the cyclic quiver with vertices $1, \dots, n$ and arrows $i \to i+1 \bmod n$. The (generic) composition algebra $\mathcal{C}_v(C_n)$ was originally studied in [7], and then Schiffmann [8] proved that the whole Ringel-Hall algebra $\mathcal{H}_v(C_n)$ consists of the composition algebra together with a central polynomial subalgebra $\mathcal{Z}_n$ on countably many generators. In particular, since the composition algebra is isomorphic to the affine quantum group of type $\widehat{\mathfrak{sl}}_n$, this result showed that the whole Ringel-Hall algebra is isomorphic to the quantum group of type $\widehat{\mathfrak{gl}}_n$.

Explicit generators for this central subalgebra were subsequently given in [5], where it was also shown that this is in fact the whole of the centre of the Ringel-Hall algebra. Furthermore, a Hopf algebra monomorphism was given from Macdonald's ring of symmetric functions to the centre $\Psi_n : \Lambda \to \mathcal{Z}_n$.

Let g_{ir} for $i = 1, \dots, n$ and $r > 0$ be the Heisenberg generators for Drinfeld's new realisation of $\mathcal{U}_v(\widehat{\mathfrak{gl}}_n)$ (see for example [3]). Then the isomorphism between the quantum group and the Ringel-Hall algebra sends $v^{ir} g_{ir}$ to the element $-\pi_{ir} + v^r \pi_{i-1r}$, where $\frac{r}{[r]}\pi_{ir}$ is the image of the r-th power sum function under the composition $\Lambda \xrightarrow{\Psi_i} \mathcal{Z}_i \subset \mathcal{H}_v(C_i) \hookrightarrow \mathcal{H}_v(C_n)$. (Here we have used the natural embedding of the Hall algebras arising from the embedding of the module categories $\mathrm{mod}C_i \hookrightarrow \mathrm{mod}C_n$ which identifies the first $i-1$ simple modules.)

We remark that this Hopf algebra isomorphism restricts to Green's isomorphism $\mathcal{U}_v^+(\widehat{\mathfrak{sl}}_n) \to \mathcal{C}_v(C_n)$ (after using Beck's isomorphism, suitably normalised). Moreover, the natural 'upper left corner' embeddings on the quantum group side, as described in [3], correspond to the natural embeddings of Ringel-Hall algebras mentioned above.

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An introduction to B. Toën's construction of derived Hall algebras

Bernhard Keller

The Ringel-Hall algebra $\mathcal{H}(\mathcal{A})$ of a finitary abelian category $\mathcal{A}$ is the free abelian group on the isomorphism classes of $\mathcal{A}$ endowed with the multiplication whose structure constants are given by the Hall numbers f_{XY}^Z, which count the number of subobjects of Z isomorphic to X and such that Z/X is isomorphic to Y, *cf.* [1]. Thanks to Ringel's famous theorem [6] [7], for each simply laced Dynkin diagram Δ, the *positive part* of the Drinfeld-Jimbo quantum group $U_q(\Delta)$ (*cf. e.g.* [4]) is obtained as the (generic, twisted) Ringel-Hall algebra of the abelian category of finite-dimensional representations of a quiver $\vec{\Delta}$ with underlying graph Δ. Since Ringel's discovery, it has been pointed out by several authors, *cf. e.g.* [3], that an extension of the construction of the Ringel-Hall algebra to the *derived category* of the representations of $\vec{\Delta}$ might yield the *whole* quantum group. However, if one tries to mimic the construction of $\mathcal{H}(\mathcal{A})$ for a triangulated category $\mathcal{T}$ by replacing short exact sequences by triangles, one obtains a multiplication which fails to be associative, *cf.* [2]. A solution to this problem has been proposed by Bertrand Toën in his recent preprint [8]. He obtains an explicit formula[1] for the structure constants ϕ_{XY}^Z of an associative multiplication on the rational vector

[1] not yet included in the first version of [8]

space generated by the isomorphism classes of any triangulated category $\mathcal{T}$ which appears as the perfect derived category of a dg category T over a finite field all of whose Hom-complexes have homology of finite total dimension. The resulting $\mathbb{Q}$-algebra is the *derived Hall algebra*. Toën's formula for the structure constants reads as follows:

$$\phi_{XY}^{Z} = \sum_{f} |\operatorname{Aut}(f/Z)|^{-1} \prod_{i>0} |\operatorname{Ext}^{-i}(X,Z)|^{(-1)^i} |\operatorname{Ext}^{-i}(X,X)|^{(-1)^{i+1}},$$

where f ranges over the set of orbits of the group $\operatorname{Aut}(X)$ in the set of morphisms $f : X \to Z$ whose cone is isomorphic to Y, and $\operatorname{Aut}(f/Z)$ denotes the stabilizer of f under the action of $\operatorname{Aut}(X)$. Toën's proof of associativity is inspired by methods from the study of higher moduli spaces [11] [9] [10] and by the homotopy theoretic approach to K-theory [5]. It remains to be investigated if and how the derived Hall algebra of the category of representations of $\vec{\Delta}$ over a finite field is related to the quantum group $U_q(\Delta)$. In any case, it seems likely that Toën's construction will prove influential in the study of Ringel-Hall algebras.

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The K-theory of triangulated derivators

Amnon Neeman

It has been known for a long time that chain complexes are very useful, and it is a good idea to study categories of chain complexes. If $\mathcal{A}$ is an abelian category then the chain complexes in $\mathcal{A}$ are sequences

$$\dots \xrightarrow{\partial} X^{i-1} \xrightarrow{\partial} X^{i} \xrightarrow{\partial} X^{i+1} \xrightarrow{\partial} \dots$$

with $\partial\partial = 0$. It is an old idea to look at categories whose objects are the chain complexes.

It is not quite so clear what the morphisms in the category should be. There are two traditional choices. In the derived category $D(\mathcal{A})$ the morphisms between two chain complexes X and Y are composites of homotopy equivalence classes of chain maps and of inverses of homology isomorphisms. The derived category $D(\mathcal{A})$ satisfies a short list of properties, which are formulated as the axioms of a *triangulated category.* One can find an extensive treatment of this subject in, for example, [9].

Another very classical construction is to consider the category $C(\mathcal{A})$. The objects are still the chain complexes, but the morphisms are now the chain maps (not homotopy equivalence classes, and nothing formally inverted). It is customary to view $C(\mathcal{A})$ as a model category. There are at least three ways to give an axiomatic description of model categories: Quillen closed model categories [7], Waldhausen model categories [10] and the complicial biWaldhausen categories of Thomason [8]. In all of these we assume we are given a mapping cone functor. We also declare some morphisms to be special. Certain of the morphisms are the so–called *cofibrations,* while some others are declared to be *weak equivalences.* The combined data is assumed to satisfy a fairly long list of axioms. For us the important feature is that the category $C(\mathcal{A})$, with all the added structure that it carries by virtue of being a model category, carries the information needed to construct the derived category $D(\mathcal{A})$. Given any model category $\mathcal{C}$ there is an associated homotopy category $\operatorname{ho}\mathcal{C}$, and $\operatorname{ho} C(\mathcal{A})$ is just $D(\mathcal{A})$.

For various reasons people have, over the last decade, been led to consider constructions intermediate between model categories and triangulated categories. The constructions fall into two broad categories:

(1) dg–categories, or the more general A_∞ categories.
(2) Grothendieck derivators.

I will say almost nothing about (1). The basic idea of a dg–category (or of the more general A_∞ categories) is to consider the morphisms in $C(\mathcal{A})$ not as groups, but as chain complexes of abelian groups. For any two objects of $C(\mathcal{A})$, that is for any two chain complexes X and Y in $\mathcal{A}$, we construct a natural chain complex $\operatorname{Hom}(X, Y)$ of abelian groups, whose 0th homology is the usual group of morphisms up to homotopy. The axioms of dg–categories (or A_∞ categories) encapsulate the properties this construction has. The literature is enormous; for a sample, the reader is referred to [1] and [6].

A completely different way to obtain a construction, intermediate between model categories and triangulated categories, is (2) above; it goes by the name *Grothendieck derivator.* The idea is to consider not just the derived category of $\mathcal{A}$, but the derived categories of all functor categories $\operatorname{Hom}(I^{\mathrm{op}}, \mathcal{A})$.

Suppose I is a small category, and $\mathcal{A}$ is any (fixed) abelian category. Then the category $\mathrm{Hom}(I^{\mathrm{op}}, \mathcal{A})$ is naturally an abelian category. We can form the derived category of $\mathrm{Hom}(I^{\mathrm{op}}, \mathcal{A})$, that is

$$\mathbb{D}(I) = D\big(\mathrm{Hom}(I^{\mathrm{op}}, \mathcal{A})\big).$$

If $F : I \longrightarrow J$ is a functor of small categories, we get an induced functor of triangulated categories

$$\mathbb{D}(F) : \mathbb{D}(J) \longrightarrow \mathbb{D}(I).$$

If F, G are two functors $F, G : I \longrightarrow J$ and $\phi : F \Longrightarrow G$ is a natural transformation, then we deduce a natural transformation

$$\mathbb{D}(\phi) : \mathbb{D}(G) \Longrightarrow \mathbb{D}(F).$$

This data assembles to give a 2–functor from the category $\mathcal{C}at$ of small categories to the category $\mathcal{T}ri$ of triangulated categories. Since this 2–functor is contravariant, we denote it

$$\mathbb{D} : \mathcal{C}at^{\mathrm{op}} \longrightarrow \mathcal{T}ri.$$

The idea of derivators is to encapsulate the extra structure of the 2–functors $\mathbb{D} : \mathcal{C}at^{\mathrm{op}} \longrightarrow \mathcal{T}ri$ which arise as $D\big(\mathrm{Hom}(I^{\mathrm{op}}, \mathcal{A})\big)$. For example, they have the useful property that for any functor $F : I \longrightarrow J$ of small categories, the induced functor $\mathbb{D}(F) : \mathbb{D}(J) \longrightarrow \mathbb{D}(I)$ has both a right and a left adjoint.

The first attempt to describe this was made by Heller [4]. Independently, but a little later, there is Keller's PhD thesis [5], and the manuscript by Grothendieck [3]. Still later there is the work of Franke [2], which cites Heller and Keller. Heller, Keller and Franke should undoubtedly receive recognition for their independent contributions. But in the last few years the name that has become attached to these is "Grothendieck derivators", possibly because the manuscript which Grothendieck wrote was so massive.

In the late 1990s Maltsiniotis took it upon himself to edit Grothendieck's manuscript and publish it. The work is still ongoing, with contributions by Cisinski and Keller. Much more can be found on Maltsiniotis' web page

`http://www.math.jussieu.fr/~maltsin`

In the process of editing the manuscript Maltsiniotis has done a great deal of work. In particular he defined for every derivator $\mathbb{D}$ a K–theory $K(\mathbb{D})$. And he formulated three conjectures about the K–theory of triangulated derivators (see pages 6–8 of the manuscript *La K–théorie d'un dérivateur triangulé* on Maltsiniotis' web page, as above). We wish to report on recent progress regarding Conjecture 3.

Conjecture 3 of Maltsiniotis says that additivity should hold for derivator K–theory. One way to formalise the conjecture is the following: A derivator $\mathbb{D}$ is a functor from small categories to triangulated categories. Given a derivator $\mathbb{D}$ we can define a new derivator $\mathbb{D}'$ by the rule

$$\mathbb{D}'(I) = \mathbb{D}(\mathbf{1} \times I),$$

where $\mathbf{1}$ is the category

$$\mathbf{1} \quad = \quad \cdot \longrightarrow \cdot$$

There are two inclusions of the one–point, terminal category into **1**. These induce two inclusions of I into $\mathbf{1} \times I$, and hence two maps

$$\mathbb{D}(I) \xleftarrow{\pi_0} \mathbb{D}(\mathbf{1} \times I) \xrightarrow{\pi_1} \mathbb{D}(I)$$

As we let I vary, these give two natural transformations, π_0 and π_1, from $\mathbb{D}'$ to $\mathbb{D}$. They induce two maps in K–theory

$$K(\mathbb{D}) \xleftarrow{K(\pi_0)} K(\mathbb{D}') \xrightarrow{K(\pi_1)} K(\mathbb{D}).$$

Conjecture 3 of Maltsiniotis, the "additivity conjecture", asserts that the map

$$K(\mathbb{D}') \xrightarrow{\begin{pmatrix} K(\pi_0) \\ K(\pi_1) \end{pmatrix}} K(\mathbb{D}) \times K(\mathbb{D})$$

is an isomorphism.

Very recently Garkusha proved the conjecture in the special case where the derivator comes from a biWaldhausen complicial model. Garkusha's paper should appear soon in *Mathematische Zeitschrift.* A little later Cisinski, Keller, Maltsiniotis and I found a proof that works for a general derivator.

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Cluster tilting I

Robert J. Marsh
(joint work with Aslak B. Buan, Markus Reineke, Idun Reiten and Gordana Todorov)

1. Tilting Theory

For an introduction to tilting theory, see e.g. [1]. The initial motivation was provided by Gabriel's Theorem [11], which states that the path algebra $H = kQ$ of a quiver Q over an algebraically closed field k has finite representation type if and only if Q is a *Dynkin quiver* of type A, D or E. Bernstein, Gelfand and Ponomarev [3] found an alternative proof employing so-called *reflection functors*, which relate the representation theory of a quiver with that of a second quiver in which all arrows incident with a fixed source or sink of the original quiver have been reversed.

These reflection functors can be realised as Hom-functors; see [2]. If S is a projective noninjective simple module corresponding to the source or sink, then the reflection functor is realised in the form $\mathrm{Hom}(T, -)$, where T is the direct sum of $\tau^{-1}S$ and the indecomposable projective modules not isomorphic to S.

2. Cluster-tilting theory

A key result in the work on cluster-tilted algebras [4, 6] has been the generalisation of APR-tilting theory to arbitrary vertices of Q. Let S be the simple module associated to a vertex i in Q. Then it is shown in [4] that there is an algebra B with simple module S' such that $\frac{\mathrm{mod}\, H}{\mathrm{add}(S)} \simeq \frac{\mathrm{mod}\, B}{\mathrm{add}(S')}$, where $\mathrm{add}(M)$ denotes the additive subcategory generated by a module M. In fact, this result, suitably adapted, holds more generally for a large family of algebras known as the *cluster-tilted algebras.* This theory was inspired by recent development of the theory of *cluster algebras* (see [10]).

We define the *cluster category* $\mathcal{C} = \mathcal{C}_H$ as the quotient of the bounded derived category of its module category by the autoequivalence $F = [1]\tau^{-1}$ (see [6]), where [1] denotes the shift. Keller [12] has shown that $\mathcal{C}$ is naturally triangulated. A combinatorial/geometric definition in type A_n has been given in [8]. Cluster categories are also studied in [7, 9, 5, 13, 14]. We also remark that the cluster category is Calabi-Yau of dimension 2. This category can be regarded as an extension of the usual module category in which any almost complete cluster-tilting object has precisely two complements. An object T in $\mathcal{C}$ is labelled a *(cluster-) tilting object* if $\mathrm{Ext}^1_{\mathcal{C}}(T, T) = 0$ and T has a maximal number of nonisomorphic indecomposable direct summands. Any tilting module over H can be regarded as a tilting object in $\mathcal{C}$.

A *cluster-tilted algebra* is an algebra of the form $\mathrm{End}_{\mathcal{C}}(T)^{\mathrm{op}}$ where T is a tilting object in $\mathcal{C}$; it is easy to see that H itself is cluster-tilted. Suppose that $\overline{T}$ is an almost complete tilting object in $\mathcal{C}$. Then in [6] it is shown that there are precisely two ways in which $\overline{T}$ can be completed to a tilting object, giving rise to tilting objects $T = \overline{T} \oplus M$ and $T' = \overline{T} \oplus M'$. Let $A = \mathrm{End}(T)^{\mathrm{op}}$ and $B = \mathrm{End}(T)^{opp}$. In [4] it is shown that, in this situation,

$$\frac{\mathrm{mod}\, A}{\mathrm{add}(S)} \simeq \frac{\mathrm{mod}\, B}{\mathrm{add}(S')},$$

where S and S' are certain simple modules over A and B respectively. Thus it is natural to define B as an algebra “cluster-tilted” from A at the vertex corresponding to M. APR-tilting is a special case of this construction. For example, the quiver of the algebra cluster-tilted from the path algebra of the quiver in Figure 1(a) at the vertex 2 is shown in Figure 1(b), with relations given by $ab = bc = ca = 0$.

Figure 1. Cluster-tilting in type A_3

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Cluster tilting II

Aslak B. Buan
(joint work with Robert J. Marsh and Idun Reiten)

This talk was part two in a series, where the first part was given by Robert Marsh. This part is mainly based on results of [2], a paper motivated by the interplay between the recent development of the theory of cluster algebras defined by Fomin and Zelevinsky in [4] (see [5] for an introduction) and the subsequent theory of cluster categories and cluster-tilted algebras [3, 1]. Our main results can be considered to be interpretations within cluster categories of the essential concepts in the theory of cluster algebras.

1. Matrix mutation

Given a skew-symmetric integer $n \times n$-matrix $B = (b_{ij})$, and an index $k \in \{1, \ldots, n\}$, let a *mutation in direction* k denote the following operation

$$b'_{ij} = \begin{cases} -b_{ij} & \text{if } k = i \text{ or } k = j, \\ b_{ij} + \frac{|b_{ik}|b_{kj} + b_{ik}|b_{kj}|}{2} & \text{otherwise.} \end{cases}$$

One can associate with B a quiver Q_B with n vertices and with b_{ij} arrows from i to j if $b_{ij} > 0$. It is clear that Q_B will have no loops and no oriented cycles of length two. In fact, the skew-symmetric integer matrices are in one-one correspondence with quivers with these properties. So mutation induces an operation on such quivers.

2. Matrix mutation via quiver representations

Let $H = KQ$ be a hereditary algebra which is the path algebra of a quiver Q for some algebraically closed field K. Given a cluster-tilted algebra $\Gamma = \operatorname{End}_{\mathcal{C}_H}(T)^{\mathrm{op}}$, with $T = T_1 \amalg \cdots \amalg T_n$ a direct sum of n nonisomorphic indecomposable objects T_i in $\mathcal{C}_H$, there is a unique indecomposable object $T_i^* \not\simeq T_i$ in $\mathcal{C}_H$, such that we get a tilting object T' by replacing T_i by T_i^*. Our main result is to obtain a formula for passing from the quiver of $\Gamma = \operatorname{End}_{\mathcal{C}}(T)^{\mathrm{op}}$ to the quiver of $\Gamma' = \operatorname{End}_{\mathcal{C}}(T')^{\mathrm{op}}$, not involving any information on relations. In fact, we show that this formula coincides with the formula for matrix mutation in direction i.

3. Cluster algebras

This has a nice interpretation in the case of cluster algebras. A cluster algebra (without coefficients) is defined via a choice of a free generating set $\underline{x} = \{x_1, \ldots, x_n\}$ in the field $\mathcal{F}$ of rational polynomials over $\mathbb{Q}$ and a skew-symmetrizable integer matrix B indexed by the elements of $\underline{x}$. The pair $(\underline{x}, B)$, called a seed, determines the cluster algebra as a subring of $\mathcal{F}$. More specifically, for each $i = 1, \ldots, n$, a new seed $\mu_i(\underline{x}, B) = (\underline{x'}, B')$ is obtained by replacing x_i in $\underline{x}$ by ${x_i}' \in \mathcal{F}$, where ${x_i}'$ is obtained by a so called *exchange multiplication rule* and B' is obtained from B by applying so called *matrix mutation* at row/column i.

Mutation in any direction is also defined for the new seed, and by iterating this process one obtains a countable number of seeds. For a seed $(\underline{x}, B)$, the set $\underline{x}$ is called a *cluster*, and the elements in $\underline{x}$ are called *cluster variables*. The desired subring of $\mathcal{F}$ is by definition generated by the cluster variables. Given a finite quiver Q with no oriented cycles, one can define on the one hand a cluster algebra $\mathcal{A}$, and on the other hand the cluster category $\mathcal{C}$ of kQ.

It was shown in [3] that in case Q is a Dynkin quiver, the cluster variables of $\mathcal{A}$ correspond to the indecomposable objects of $\mathcal{C}$, and that this correspondence induces a correspondence between the clusters and the tilting objects in $\mathcal{C}$. This was also conjectured to generalize to arbitrary quivers, except that in this case the exceptional objects should correspond to the cluster variables. Combining this with the results of [2], one obtains for finite type a precise interpretation of cluster algebras in terms of tilting theory in cluster categories. In [2] there is also an interpretation beyond finite type.

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The prime ideal spectrum of a tensor triangulated category

Henning Krause
(joint work with Aslak B. Buan and Øyvind Solberg)

Given a triangulated category, it is an interesting challenge to classify all thick subcategories. In my talk, I presented some recent work of Paul Balmer [1]. He defines a prime ideal spectrum for each tensor triangulated category and assigns to each object its support. This idea leads to a complete classification of all thick tensor ideals. The model for such a classification is Thomason's classification of thick tensor ideals for the category of perfect complexes on a scheme [3].

Balmer's classification provides an extremely elegant and conceptual explanation of various existing classifications. This includes the classification of thick tensor ideals for the category of perfect complexes on a scheme [Hopkins, Neeman, Thomason] and a similar classification for the stable category of representations of a finite group [Benson, Carlson, Rickard].

It turns out that Balmer's idea can be extended to obtain classifications of ideals in various settings. This should be relevant in representation theory, in particular when one studies the support varieties of representations.

The general set-up for the classification of ideals in terms of the prime ideal spectrum is the following: We consider an *ideal lattice*, that is, a partially ordered set $L = (L, \leq)$, together with an associative multiplication $L \times L \to L$, such that the following holds.

(L1) The poset L is a *complete lattice*, that is

$$\bigvee_{a \in A} a := \sup A \quad \text{and} \quad \bigwedge_{a \in A} a := \inf A$$

exist in L for every subset $A \subseteq L$.

(L2) The lattice L is *compactly generated*, that is, every element in L is the supremum of compact elements.

(L3) We have for all $a, b, c \in L$

$$a(b \vee c) = ab \vee ac \quad \text{and} \quad (a \vee b)c = ac \vee bc.$$

(L4) The product of two compact elements is again compact.

For example, the thick tensor ideals in a small tensor triangulated category form such an ideal lattice. The compact elements are precisely the finitely generated ideals.

Call $p \in L$ *prime* if $ab \leq p$ implies $a \leq p$ or $b \leq p$ for all $a, b \in L$. An element $q \in L$ is *semi-prime* if $aa \leq q$ implies $a \leq q$ for all $a \in L$. Let $\operatorname{Spec} L$ denote the set of all primes in L. For $a \in L$, let

$$U(a) = \{p \in \operatorname{Spec} L \mid a \leq p\} \quad \text{and} \quad \operatorname{supp}(a) = \{p \in \operatorname{Spec} L \mid a \not\leq p\}.$$

The subsets of $\operatorname{Spec} L$ of the form $U(a)$ for some compact $a \in L$ are closed under forming finite intersections and finite unions; they form the basis of a topology on $\operatorname{Spec} L$.

Theorem. *The assignments*

$$L \ni a \mapsto \operatorname{supp}(a) = \bigcup_{\substack{b \leq a \\ b \text{ compact}}} \operatorname{supp}(b) \quad \textit{and} \quad \operatorname{Spec} L \supseteq Y \mapsto \bigvee_{\substack{\operatorname{supp}(b) \subseteq Y \\ b \text{ compact}}} b$$

induce mutually inverse and inclusion preserving bijections between

(1) *the set of all semi-prime elements in L, and*

(2) *the set of all subsets $Y \subseteq \operatorname{Spec} L$ of the form $Y = \bigcup_{i \in \Omega} Y_i$ with quasi-compact open complement $\operatorname{Spec} L \setminus Y_i$ for all $i \in \Omega$.*

To give an example, take a commutative noetherian ring R and let $L(R)$ denote the lattice of thick tensor ideals of the category of perfect complexes over R. Note that in this case all elements in $L(R)$ are semi-prime. Using the description of $L(R)$ due to Hopkins and Neeman, one can show that $\operatorname{Spec} L(R)$ is homeomorphic to the prime ideal spectrum of R, endowed with the usual Zariski topology.

This example, as well as many more, are beautifully explained in Balmer's work [1]. What seems to be new is the general approach via ideal lattices. It covers for instance equally well the related classification of Serre subcategories of the category of finitely generated modules over a commutative noetherian ring.

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Cohen-Macaulay modules and virtually Gorenstein algebras

Apostolos Beligiannis

Let Λ be an Artin algebra. We denote by $\mathsf{Mod}\text{-}\Lambda$ the category of all right Λ-modules and by $\mathsf{mod}\text{-}\Lambda$ the full subcategory of finitely generated modules. We let $\mathsf{CM}(\mathsf{Mod}\text{-}\Lambda)$ be the category of *Cohen-Macaulay* modules which is defined as the maximal subcategory of $\mathsf{Mod}\text{-}\Lambda$ which contains the projectives as an Ext-injective cogenerator. Following [3] we let $\mathfrak{P}_{\Lambda}^{\prec\infty}$ be the subcategory of modules of *virtually finite projective dimension* which is defined as the right Ext-orthogonal subcategory of $\mathsf{CM}(\mathsf{Mod}\text{-}\Lambda)$. The full subcategories $\mathsf{CoCM}(\mathsf{Mod}\text{-}\Lambda)$ of *CoCohen-Macaulay* modules and $\mathfrak{I}_{\Lambda}^{\prec\infty}$ of modules of *virtually finite injective dimension* are defined dually. Note that $\mathsf{CM}(\mathsf{Mod}\text{-}\Lambda)$, resp. $\mathsf{CoCM}(\mathsf{Mod}\text{-}\Lambda)$, is an exact Frobenius definable subcategory of $\mathsf{Mod}\text{-}\Lambda$ and its stable category modulo projectives, resp. injectives, is a monogenic compactly generated triangulated category. Also the subcategories $\mathfrak{P}_{\Lambda}^{\prec\infty}$ and $\mathfrak{I}_{\Lambda}^{\prec\infty}$ are resolving and coresolving subcategories of $\mathsf{Mod}\text{-}\Lambda$ and there exist cotorsion pairs $\big(\mathsf{CM}(\mathsf{Mod}\text{-}\Lambda), \mathfrak{P}_{\Lambda}^{\prec\infty}\big)$ and $\big(\mathfrak{I}_{\Lambda}^{\prec\infty}, \mathsf{CoCM}(\mathsf{Mod}\text{-}\Lambda)\big)$ in $\mathsf{Mod}\text{-}\Lambda$. Λ is called *virtually Gorenstein* if $\mathfrak{P}_{\Lambda}^{\prec\infty} = \mathfrak{I}_{\Lambda}^{\prec\infty}$, see [3].

In this talk I shall report on some recent results, extracted from [2] and [3], on Cohen-Macaulay modules and virtually Gorenstein algebras. We study the virtual Gorensteinness property by using the above cotorsion pairs in the module category $\mathsf{Mod}\text{-}\Lambda$ and the induced torsion pairs in the stable category of $\mathsf{Mod}\text{-}\Lambda$ modulo projectives or injectives. The class of virtually Gorenstein algebras, which provides a common generalization of Gorenstein algebras and algebras of finite representation or Cohen-Macaulay type, on the one hand is closed under various operations and on the other hand has rich homological structure and satisfies several representation/torsion theoretic finiteness conditions. In this context we characterize the virtually Gorenstein algebras in terms of finitely generated modules by showing, among other equivalent conditions, that Λ is virtually Gorenstein if and only if the class of finitely generated Λ-modules of virtually finite projective dimension (which coincides with the class of finitely generated Λ-modules of virtually finite injective dimension) is contravariantly finite, or equivalently covariantly finite in $\mathsf{mod}\text{-}\Lambda$. Moreover, we show that virtually Gorenstein algebras enjoy the following properties, referring to [2], [3] for more details:

(1) The virtual Gorensteinness property is left-right symmetric.
(2) The class of virtually Gorenstein algebras is closed under derived equivalences and stable equivalences of Morita type.

(3) If the Artin algebra Λ is virtually Gorenstein, then:
  (a) The full subcategories $\mathsf{CM}(\mathsf{Mod}\text{-}\Lambda)$ and $\mathsf{CoCM}(\mathsf{Mod}\text{-}\Lambda)$ are functorially finite in $\mathsf{Mod}\text{-}\Lambda$, and their full subcategories of finitely generated modules are functorially finite in $\mathsf{mod}\text{-}\Lambda$ with free Grothendieck group of finite rank.
  (b) The full subcategory $\mathfrak{P}_\Lambda^{\prec\alpha} = \mathfrak{I}_\Lambda^{\prec\alpha}$ is thick, definable and functorially finite in $\mathsf{Mod}\text{-}\Lambda$, and its full subcategory of finitely generated modules is thick and functorially finite in $\mathsf{mod}\text{-}\Lambda$, hence it has Auslander-Reiten sequences, with free Grothendieck group of finite rank.
  (c) The full subcategories $\mathsf{CM}(\mathsf{Mod}\text{-}\Lambda)$, $\mathsf{CoCM}(\mathsf{Mod}\text{-}\Lambda)$, $\mathfrak{P}_\Lambda^{\prec\alpha}$ and $\mathfrak{I}_\Lambda^{\prec\alpha}$ are completely determined by their intersection with the finitely generated modules (as their closure under filtered colimits).
  (d) The subcategory of compact objects of the compactly generated triangulated category of Cohen-Macaulay modules modulo projectives admits a Serre functor and therefore has Auslander-Reiten triangles.

In addition, virtual Gorensteinness provides a useful tool for the study of the Gorenstein Symmetry Conjecture and modified versions of the Telescope Conjecture for module or stable categories. Recall that the former asserts that Λ is Gorenstein provided it has finite right or left self-injective dimension [1], and a generalized version of the latter asserts that any torsion pair of finite type, in the sense of [3], in a suitable "homotopy" category $\mathcal{C}$ is generated in a certain sense by compact objects induced from $\mathcal{C}$, see [4], [5], [6]. For instance, we show that Λ is virtually Gorenstein if and only if the monogenic compactly generated triangulated category of Cohen-Macaulay modules modulo projectives is smashing if and only if all of its compact objects are induced from finitely generated modules. Moreover, in the context of the above conjectures we show (in particular) the following:

- The Gorenstein Symmetry Conjecture holds for any virtually Gorenstein algebra.
- The Telescope Conjecture holds in the stable category modulo projectives for the torsion pair induced by the Cohen-Macaulay modules over a virtually Gorenstein algebra.

In particular, both conjectures hold for any algebra lying in the derived equivalence class or the stable equivalence class (of Morita type) of an algebra of finite representation or Cohen-Macaulay type.

As it is clear from the above that the class of virtually Gorenstein algebras is rather large, since it contains on the one hand algebras or finite global dimension and self-injective algebras or, more generally, Gorenstein algebras, and on the other hand algebras of finite representation or Cohen-Macaulay type. This gives the motivation for the following:

Problem: Find an Artin algebra which is not virtually Gorenstein.

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Free divisors in representation varieties of quivers

Ragnar-Olaf Buchweitz

1. Let K be an algebraically closed field of characteristic zero. A formal power series $f \in S = K[\![z_1, ..., z_N]\!]$ is a *free divisor* if there exists a *discriminant matrix* A for f, that is, a $N \times N$–matrix over S that satisfies:

$$\det A = f \quad \text{and} \quad \left(\frac{\partial f}{\partial z_1}, ..., \frac{\partial f}{\partial z_N}\right) \cdot A \equiv 0 \bmod f\,.$$

If f is *square free*, an equivalent description can be obtained through *logarithmic vector fields* along the hypersurface $f = 0$. These are those derivations D on S with $D(\log f) = D(f)/f \in S$. By *Saito's criterion*, f is a free divisor, iff the S–module $\Theta_S(-\log f)$ of all such is *free of rank N* (and that explains the name...) Equivalently, the*Jacobian ideal* describing the singular locus of $f = 0$ is a maximal Cohen–Macaulay module on that hypersurface. All these characterizations imply that f is "highly singular". Indeed, it is already rare for a polynomial or power series to be representable as a determinant in a nontrivial way.

2. This concept of free divisors was first isolated by K. Saito [6] in the context of discriminants in versal deformations of isolated hypersurface singularities and includes the classical discriminants of polynomials in one variable. It was shown by Teissier (isolated complete intersection singularities; see [4]), van Straten (some classes of curve singularities [9]), Damon et al. (bifurcation sets of versal unfoldings; see [1] for a survey), Buchweitz–Ebeling (Gorenstein surface singularities in 5–space; *in preparation*), Buchweitz (Hilbert scheme of a smooth surface; *in preparation*) and various others that discriminants in versal deformations are often (but not always) free divisors. Another rich source of free divisors are *free* (hyperplane) *arrangements*, and the survey [5] is a good starting point.

3. Here we identify the discriminants in representation varieties of certain real Schur roots of quivers as yet another source of such free divisors that are even *linear*, that is, the discriminant matrix has only linear entries. (Together with D. Mond we are currently investigating further such examples and variations thereof.) The key point is to interpret representation varieties as versal deformations of quiver representations.

4. To be explicit, fix a finite connected quiver $Q = (Q_0, Q_1)$ without oriented cycles, $\mathbf{d} \in (\mathbb{N}_{>0})^{Q_0}$ a *sincere real Schur root* of Q. With $t\alpha \in Q_0$ the *tail* of an arrow $\alpha \in Q_1$ and $h\alpha \in Q_0$ its *head*, the *representation variety* $\mathrm{Rep}(Q, \mathbf{d})$ is the affine K–space of families of matrices $\prod_{\alpha\in Q_1} \mathrm{Hom}_K\left(K^{\mathbf{d}(t\alpha)}, K^{\mathbf{d}(h\alpha)}\right) \cong \mathbb{A}_K^{\sum_{\alpha\in Q_1}\mathbf{d}(h\alpha)\cdot\mathbf{d}(t\alpha)}$. Its dimension is $\delta := \sum_{\alpha\in Q_1} \mathbf{d}(h\alpha)\mathbf{d}(t\alpha) = -1 + \sum_{i\in Q_0} \mathbf{d}(i)^2$. The ring of polynomial functions on $\mathrm{Rep}(Q, \mathbf{d})$ is written $R = K[X(\alpha), \alpha \in Q_1]$, where each $X(\alpha) = (x_{rs}(\alpha))_{r=1,\ldots,h\alpha}^{s=1,\ldots,t\alpha}$ is a matrix of indeterminates. The representation variety carries the *universal* $\mathbf{d}$*–dimensional representation* M of Q, a (right) module over the path algebra RQ on Q with coefficients in R, and $\mathrm{Rep}(Q, \mathbf{d})$ constitutes a *versal deformation* of the Q–representation $M(p)$ at each of its points p.

5. The group $\mathrm{GL}(\mathbf{d}) = \prod_{i\in Q_0} \mathrm{GL}(\mathbf{d}(i))$ acts on $\mathrm{Rep}(Q, \mathbf{d})$ through $(A_i)_{i\in Q_0} * (X_\alpha)_{\alpha\in Q_1} := (A_{h\alpha}X_\alpha A_{t\alpha}^{-1})_{\alpha\in Q_1}$, and the orbits correspond to the isomorphism classes of $\mathbf{d}$–dimensional representations of Q over K. The action factors through the projective linear group $\mathrm{PGL}(\mathbf{d}) = \mathrm{GL}(\mathbf{d})/K^*$. As $\mathbf{d}$ is a real Schur root, there is a (unique) open dense orbit and, by [3], its complement is the union of $n-1$ distinct components of codimension one, where $n = |Q_0|$ is the number of vertices of Q. The polynomials $f_1, ..., f_{n-1} \in R$ defining these components are algebraically independent and span the ring of *semi-invariants* $R^{\mathrm{SL}(\mathbf{d})}$, with $\mathrm{SL}(\mathbf{d}) = \{(A_i)_{i\in Q_0} \in \mathrm{GL}(\mathbf{d}) \mid \prod_i \det A_i = 1\}$ the corresponding special linear group. They can be obtained explicitly as shown by Schofield [7], Schofield–Van den Bergh [8], and Derksen–Weyman [2].

6. With $\Theta_R = \mathrm{Der}_K(R) \cong R\otimes_K \mathrm{Rep}(Q, \mathbf{d}) \cong \prod_{\alpha\in Q_1} R\otimes_K \mathrm{Hom}_K\left(K^{\mathbf{d}(t\alpha)}, K^{\mathbf{d}(h\alpha)}\right)$ the module of derivations of R or vector fields on $\mathrm{Rep}(Q, \mathbf{d})$, the group action yields a homomorphism of K–Lie algebras $\mathfrak{pgl}(\mathbf{d}) \to \Theta_R, [(A_i)_{i\in Q_0}] \mapsto (A_{h\alpha}X(\alpha) - X(\alpha)A_{t\alpha})_{\alpha\in Q_1}$, that extends to an R–linear map $\varphi\colon R\otimes_K \mathfrak{pgl}(\mathbf{d}) \to \Theta_R$, with cokernel equal to $\mathrm{Ext}^1_{RQ}(M, M)$. The projection $\Theta_R \to \mathrm{Ext}^1_{RQ}(M, M)$ is the *Kodaira–Spencer map* for $\mathrm{Rep}(Q, \mathbf{d})$ as a versal deformation, and the support of $\mathrm{Ext}^1_{RQ}(M, M)$ is its *discriminant*, the locus of non-rigid representations.

7. As $\mathbf{d}$ is a real Schur root, φ is injective, with source and target of φ free R–modules of the same rank. Moreover, the image of φ is a sub K–Lie algebra of $\Theta_R(-\log f)$, the Lie algebra of logarithmic vector fields along the discriminant $f = f_1 \cdots f_{n-1}$, that is, the complement of the open orbit. In case of equality, (the Taylor series of) f is a free divisor (at each point) and its discriminant matrix, the matrix of φ, has only linear entries. This provides for families of examples of such linear free divisors and exhibits at the same time yet another special property of the hypersurfaces bounding the open orbit. We may summarize the results thus.

Theorem. *With notation as explained above, one has*

(1) (Basic facts) *The determinant of* φ *is a homogeneous semi-invariant polynomial of degree* δ *and of weight* $\prod_{i\in Q_0}(\det \mathrm{GL}(d_i))^{\mathrm{in}(i)-\mathrm{out}(i)}$*, where* $\mathrm{in}(i)$ *is the in–degree and* $\mathrm{out}(i)$ *is the out–degree of the vertex* i*. It has the form* $\det\varphi = (\text{a unit})\cdot f_1^{a_1}\cdots f_{n-1}^{a_{n-1}}$*, with* a_ν *the dimension of the space of self extensions of the generic module over the component* $f_\nu = 0$*.*

(2) (Openness of Versality *or* Voigt's Lemma) *A component $f_\nu = 0$ contains itself an open dense orbit, if, and only if, $a_\nu = 1$.*
(3) (Saito's Criterion) *If each $a_\nu = 1$, then f is a* linear free divisor.

Condition (3) *is, in particular, satisfied if* $\mathrm{Rep}(Q, \mathbf{d})$ *is union of only finitely many orbits, for example, if the quiver Q is of finite representation type.*

A simple example may suffice to illustrate the result.

Example. *Let Δ_ν, for $\nu = 1, \ldots, n-1$, be the maximal minors of a generic $(n-2) \times (n-1)$–matrix. The product $f = \Delta_1 \cdots \Delta_{n-1}$ is then such a free divisor that arises as a discriminant in a suitable representation variety of a quiver with n vertices.*

Exercise: Write down the discriminant matrix and verify that f is a linear free divisor! What are possible quivers Q and dimension vectors $\mathbf{d}$ for this example? Hint: For n = 4, the Dynkin quiver $Q = D_4$, in any orientation, and the dimension vector $\mathbf{d} = \begin{pmatrix} & 1 & \\ 1 & 2 & 1 \end{pmatrix}$, will do.

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Rigid Cohen-Macaulay modules over a three dimensional Gorenstein ring

Yuji Yoshino

1. Main theorem

Let k be an algebraically closed field of characteristic zero, and let $S = k[[x, y, z]]$ be a formal power series ring in three variables x, y and z. The cyclic group $G = \mathbb{Z}/3\mathbb{Z}$ of order 3 acts linearly on S in such a way that

$$x^\sigma = \zeta x, \quad y^\sigma = \zeta y, \quad z^\sigma = \zeta z,$$

where σ is a generator of G and $\zeta \in k$ is a primitive cubic root of unity. We denote by R the invariant subring of S by this action of G. It is easy to see that

$$R = k[[\{\text{monomials of degree three in } x, y, z\}]],$$

which is often called (the completion of) the Veronese subring of degree three. It is known and is easy to prove that R is a Gorenstein complete local normal domain that has an isolated singularity.

The action of G gives a G-graded structure on S such as

$$S = S_0 \oplus S_1 \oplus S_2,$$

where each S_j is the R-module of semi-invariants that is defined as

$$S_j = \{f \in S \mid f^\sigma = \zeta^j f\}.$$

Note that $S_0 = R$. It is known that S_j $(0 \leqq j \leqq 2)$ are maximal Cohen-Macaulay modules over R, and in particular they are reflexive R-modules of rank one, whose classes form the divisor class group of R;

$$\mathrm{Cl}(R) = \{[S_0], [S_1], [S_2]\}.$$

In particular, any maximal Cohen-Macaulay module of rank one over R is isomorphic to one of S_j $(0 \leqq j \leqq 2)$.

It is not difficult to see that the category $\mathrm{CM}(R)$ of maximal Cohen-Macaulay modules over R is of wild representation type. Actually, one can construct a family of nonisomorphic classes of indecomposable maximal Cohen-Macaulay modules over R in relation with the representations of the following quiver.

$$Q = \left(\bullet \quad \begin{smallmatrix}\longrightarrow \\ \longrightarrow \\ \longrightarrow\end{smallmatrix} \quad \bullet \right)$$

In this talk I am interested in rigid maximal Cohen-Macaulay modules that are defined as follows:

Definition. An R-module M is called *rigid* if $\mathrm{Ext}^1_R(M, M) = 0$. And we denote the full subcategory of $\mathrm{mod}\, R$ consisting of all rigid maximal Cohen-Macaulay modules by $\mathcal{C}$.

By computation, the modules S_j $(0 \leqq j \leqq 2)$ and any of their syzygies and any of their cosyzygies are rigid (and indecomposable). Our main theorem is the following:

Theorem. *Let $\mathcal{S}$ be a sequence of indecomposable rigid maximal Cohen-Macaulay modules defined as follows:*

$$\mathcal{S} = (\cdots, \Omega^{-2}S_1,\ \Omega^{-2}S_2,\ \Omega^{-1}S_1,\ \Omega^{-1}S_2,\ S_1,\ S_2,\ \Omega^1 S_1,\ \Omega^1 S_2,\ \Omega^2 S_1,\ \Omega^2 S_2, \cdots).$$

Then any object in $\mathcal{C}$ is isomorphic to a module of the following form:

$$P^a \oplus Q^b \oplus R^c,$$

where a, b, c are nonnegative integers and $\{P, Q\}$ is a pair of two adjacent modules in the sequence $\mathcal{S}$.

2. Outline of Proof

The proof of the theorem is divided into the following four steps.

2.1. First Step (Approximation).

Let $\mathcal{E}$ be the full subcategory of $\operatorname{mod} R$ consisting of modules M which can be embedded in an exact sequence of the following type:

$$(*) \qquad 0 \longrightarrow S_1^n \longrightarrow S_2^m \oplus R^\ell \longrightarrow M \longrightarrow 0$$

If $M \in \mathcal{E}$, then the sequence $(*)$ gives a right $\operatorname{add}_R S$-approximation of M that is, of course, right minimal.

Claim 1. *Let M be an indecomposable object in $\mathcal{C}$. Suppose that M is isomorphic neither to S_1 nor $\Omega^{-1}S_2$. Then M belongs to $\mathcal{E}$.*

The claim means that $\operatorname{Ind}(\mathcal{C}) = \operatorname{Ind}(\mathcal{C} \cap \mathcal{E}) \cup \{S_1, \Omega^{-1}S_2\}$.

2.2. Second Step (Rigidity).

Claim 2. *Let M and M' be objects in $\mathcal{C} \cap \mathcal{E}$. Suppose there are exact sequences:*

$$0 \longrightarrow S_1^n \xrightarrow{\ f\ } S_2^m \oplus R^\ell \longrightarrow M \longrightarrow 0,$$
$$0 \longrightarrow S_1^{n'} \xrightarrow{\ f'\ } S_2^{m'} \oplus R^{\ell'} \longrightarrow M' \longrightarrow 0.$$

If $n = n'$ and $m = m'$, then M and M' are stably isomorphic to each other.

2.3. Third Step (Tate-Vogel cohomology).

The Tate-Vogel cohomology for maximal Cohen-Macaulay modules is defined as follows:

$$\check{\operatorname{Ext}}^i_R(M, N) = \underline{\operatorname{Hom}}_R(\Omega^i M, N),$$

for any $i \in \mathbb{Z}$ and $M, N \in \operatorname{CM}(R)$. We define

$$e^i_j(M) = \dim_k\ \check{\operatorname{Ext}}^i_R(S_j, M)$$

for any $i \in \mathbb{Z}$, $j \in G$ and $M \in \operatorname{CM}(R)$.

Claim 3. *Let M be an object in $\mathcal{C} \cap \mathcal{E}$, and suppose there is an exact sequence:*

$$0 \longrightarrow S_1^n \longrightarrow S_2^m \oplus R^\ell \longrightarrow M \longrightarrow 0.$$

Then $n = e_1^1(M)$ and $m = e_2^0(M)$.

Now we define a mapping e from the isomorphism classes of modules in $\mathcal{C} \cap \mathcal{E}$ to nonnegative integral vectors $\mathbb{Z}^2_{\geqq 0}$ by

$$e(M) = (e_1^1(M),\ e_2^0(M)).$$

Note that it follows from Claim 2 that the mapping

$$e : \mathcal{C} \cap \mathcal{E}/\cong \ \longrightarrow \ \mathbb{Z}^2_{\geqq 0}$$

is an injection. Hence, to classify the objects in $\mathcal{C} \cap \mathcal{E}$, it is enough to determine the image of the mapping e.

Remark. *Note that the Auslander-Reiten-Serre duality says that*

$$\mathrm{Ext}^3_R(\check{\mathrm{E}}\mathrm{xt}^i_R(M,N), R) \cong \check{\mathrm{E}}\mathrm{xt}^{2-i}_R(N,M),$$

for any $i \in \mathbb{Z}$ and $M, N \in \mathrm{CM}(R)$. Therefore, the triangulated category $\underline{\mathrm{CM}}(R)$ is 2-Calabi-Yau.

2.4. **Fourth Step (Root system).**

Let H be the set of nonnegative integral vectors (x, y) with $x^2 - 3xy + y^2 \geqq 1$:

$$H = \{(x,y) \in \mathbb{Z}^2_{\geqq 0} \mid x^2 - 3xy + y^2 \geqq 1\}$$

It is easy to see that $H = H_+ \cup H_-$ where

$$H_+ = \{(x,y) \in \mathbb{Z}^2_{\geqq 0} \mid 2x - (3+\sqrt{5})y \geqq 0\},$$
$$H_- = \{(x,y) \in \mathbb{Z}^2_{\geqq 0} \mid 2x - (3-\sqrt{5})y \leqq 0\},$$

each of which is a semigroup. We can prove the following claim.

Claim 4. *The image of the mapping $e : \mathcal{C} \cap \mathcal{E} \to \mathbb{Z}^2_{\geqq 0}$ is exactly H.*

The main theorem follows from this claim with a little observation.

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Modules with injective cohomology

David Benson
(joint work with John Greenlees)

Let G be a finite group, and let k be an algebraically closed field of characteristic p. Then the cohomology ring $H^*(G,k) = \operatorname{Ext}^*_{kG}(k,k)$ is a Noetherian graded commutative k-algebra, so we can form the maximal ideal spectrum $V_G = \max\operatorname{spec} H^*(G,k)$. This is a closed homogeneous affine variety, and was studied extensively by Quillen [6, 7]. If M is a finitely generated kG-module then there is a ring homomorphism

$$H^*(G,k) \xrightarrow{M\otimes_k -} \operatorname{Ext}^*_{kG}(M,M),$$

and the *support variety* $V_G(M)$ is defined to be the subvariety of V_G determined by the kernel of this homomorphism. Support varieties have been investigated extensively by Carlson and others.

If $\mathfrak{p}$ is a homogeneous prime ideal in $H^*(G,k)$ corresponding to a closed homogeneous irreducible subvariety V of V_G, then there is a *kappa module* $\kappa_{\mathfrak{p}} = \kappa_V$, introduced by Benson, Carlson and Rickard [1], with the following properties:

(i) $V \subseteq V_G(M) \iff \kappa_V \otimes_k M$ is not projective,
(ii) κ_V is idempotent, in the sense that $\kappa_{\mathcal{V}} \otimes_k \kappa_{\mathcal{V}} \cong \kappa_V \oplus$ (projective), and
(iii) $\kappa_{\mathcal{V}}$ is usually not finite dimensional.

The modules κ_V were used by Benson, Carlson and Rickard in [1] to develop a theory of varieties for infinitely generated kG-modules. Instead of associating a single variety to M, we associate a collection of subvarieties of V_G:

$$\mathcal{V}_G(M) = \{V \subseteq V_G \mid \kappa_V \otimes_k M \text{ is not projective}\}.$$

For example, $\mathcal{V}_G(\kappa_V) = \{V\}$. One of the most important properties of this variety theory is the tensor product formula

$$\mathcal{V}_G(M \otimes_k N) = \mathcal{V}_G(M) \cap \mathcal{V}_G(N).$$

This, together with the statement that $\mathcal{V}_G(M) = \varnothing$ if and only if M is projective, are what make the variety theory useful.

The purpose of the joint work with Greenlees was to determine the cohomology of these modules κ_V. It turns out that it is more sensible to ask about Tate cohomology. The answer, together with some consequences, is given by the following theorem.

Theorem (Benson and Greenlees [2])**.** (i) *The Tate cohomology of the kappa modules is given by*

$$\hat{H}^*(G,\kappa_V) \cong I_{\mathfrak{p}}[d].$$

Here, $I_{\mathfrak{p}}$ denotes the injective hull of $H^*(G,k)/\mathfrak{p}$ in the category of graded modules over $H^*(G,k)$, and d is the dimension of the variety V (i.e., the Krull dimension of $H^*(G,k)/\mathfrak{p}$).

(ii) *The kappa modules are the representing objects for the Matlis dual of Tate cohomology:*

$$\underline{\mathrm{Hom}}_{kG}(M, \kappa_V) \cong \mathrm{Hom}_{H^*(G,k)}(\hat{H}^*(G, M), I_{\mathfrak{p}}[d]);$$

these representing objects were investigated in [3].

(iii) *The modules κ_V are pure injective—there are no phantom maps into them.*

(iv) $\widehat{\mathrm{Ext}}^*_{kG}(\kappa_V, \kappa_V) \cong H^*(G, k)^\wedge_{\mathfrak{p}} = \varprojlim_n H^*(G, k)/\mathfrak{p}^n$.

The extraordinary thing about the theorem is that its proof involves translating to the context of modules over E_∞ ring spectra and solving the problem there. The context is as follows. Let BG be the classifying space of G, so that $\Omega BG \simeq G$. The Rothenberg–Steenrod construction gives for any space X a quasiisomorphism between the differential graded algebras $\mathbb{R}\,\mathrm{End}_{C_*(\Omega X)}(k)$ and $C^*(X; k)$. In particular, for a finite group G this gives $\mathbb{R}\,\mathrm{End}_{kG}(k) \simeq C^*(BG; k)$. Writing $\mathcal{R}$ for $\mathbb{R}\,\mathrm{End}_{kG}(k)$ and $\mathcal{C}$ for $C^*(BG; k)$, the following diagram of categories and functors explains the route we took:

$$\begin{tikzcd}
\mathsf{Mod}(kG) \arrow[r] \arrow[dr] & D(kG) \arrow[d] \arrow[r, shift right, "\mathbb{R}\,\mathrm{Hom}_{kG}(k,-)"'] & D(\mathcal{R}^{\mathrm{op}}) \arrow[l, shift right, "-\otimes^{\mathbb{L}}_{\mathcal{R}} k"'] \arrow[r, "\simeq"] & D(\mathcal{C}) \\
 & \mathsf{StMod}(kG) & &
\end{tikzcd}$$

Here, $D(kG)$ stands for the derived category of *all* chain complexes of kG-modules. Similarly, $D(\mathcal{R}^{\mathrm{op}})$ is the derived category obtained from the homotopy category of differential graded *right* $\mathcal{R}$-modules by inverting quasi-isomorphisms. Since $\mathcal{R} \simeq \mathcal{R}^{\mathrm{op}}$, this is equivalent to the derived category formed from the differential graded *left* $\mathcal{R}$-modules. We regard $\mathcal{C}$ (or rather, the Eilenberg–Mac Lane spectrum of $\mathcal{C}$) as an E_∞ ring spectrum; here, E_∞ means "commutative and associative up to all higher homotopies." This allows us, for example, to take two objects A and B in $D(\mathcal{C})$ and regard $A \otimes^{\mathbb{L}}_{\mathcal{C}} B$ as another object in $D(\mathcal{C})$, just as we can regard the tensor product of two modules over a commutative ring as another module over the same ring. For this purpose, it is essential to be working in a category of spectra in which the smash product is commutative and associative up to coherent natural isomorphism, and not just up to all higher homotopies; there are nowadays a number a candidates for such a category, and we chose to work in the framework of Elmendorf, Kříž, Mandell and May [5].

Another construction requiring the E_∞ structure is localization at a prime ideal in the homotopy. Since $\pi_*\mathcal{C} = H^{-*}(G, k)$, we can form the localization $\mathcal{C}_{\mathfrak{p}}$, and then use tensor products to apply a stable Koszul type construction with respect to a homogeneous system of parameters in $\mathfrak{p}$. This construction gives the image in $D(\mathcal{C})$ of a suitable lift to $D(kG)$ of the kappa module $\kappa_{\mathfrak{p}}$ in $\mathsf{StMod}(kG)$. This construction can therefore be regarded as a sort of local cohomology object in $D(\mathcal{C})$ for the prime $\mathfrak{p}$. The statement that its cohomology is injective is a sort of Gorenstein duality for $\mathcal{C}_{\mathfrak{p}}$.

The statement that $\mathcal{C}$ is Gorenstein in the appropriate sense appeared in the work of Dwyer, Greenlees and Iyengar [4]. The usual proof that localization at a prime ideal of a Gorenstein ring gives a Gorenstein ring no longer works in this context, because it relies on the characterization of Gorenstein via finite injective dimension, which doesn't make much sense in this context. So proving that $\mathcal{C}_{\mathfrak{p}}$ is Gorenstein went via a different route. We applied Grothendieck duality with respect to a normalization coming from an embedding of G in $SU(n)$, and proved the corresponding dual statement.

To summarize, the proof involves translating the original problem from modular representation theory into the language of modules over an E_∞ ring spectrum from algebraic topology, and then using methods from commutative algebra to solve the problem there. The level of machinery involved is formidable, but the hope is that other problems in modular representation theory will succumb to a similar route.

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Quantum cluster algebras

Andrei Zelevinsky

Cluster algebras were introduced and studied by S. Fomin and A. Zelevinsky in [3, 5, 1]. This is a family of commutative rings designed to serve as an algebraic framework for the theory of total positivity and canonical bases in semisimple groups and their quantum analogs. Here we report on a joint work with A. Berenstein [2], where we introduce and study quantum deformations of cluster algebras.

We start by recalling the definition of cluster algebras (of geometric type). Let m and n be two positive integers with $m \geq n$. Let $\mathcal{F}$ be the field of rational functions over $\mathbb{Q}$ in m independent (commuting) variables.

Definition 1. A *seed* in $\mathcal{F}$ is a pair $(\tilde{\mathbf{x}}, \tilde{B})$, where

- $\tilde{\mathbf{x}} = \{x_1, \ldots, x_m\}$ is a free (i.e., algebraically independent) generating set for $\mathcal{F}$.

- $\tilde{B}$ is an $m \times n$ integer matrix with rows labeled by $[1, m] = \{1, \ldots, m\}$ and columns labeled by an n-element subset $\mathbf{ex} \subset [1, m]$, such that, for some positive integers d_j ($j \in \mathbf{ex}$), we have $d_i b_{ij} = -d_j b_{ji}$ for all $i, j \in \mathbf{ex}$.

The subset $\mathbf{x} = \{x_j : j \in \mathbf{ex}\} \subset \tilde{\mathbf{x}}$ (resp. $\mathbf{c} = \tilde{\mathbf{x}} - \mathbf{x}$) is called the *cluster* (resp. the *coefficient set*) of a seed $(\tilde{\mathbf{x}}, \tilde{B})$. The seeds are defined up to a relabeling of elements of $\tilde{\mathbf{x}}$ together with the corresponding relabeling of rows and columns of $\tilde{B}$.

Definition 2. Let $(\tilde{\mathbf{x}}, \tilde{B})$ be a seed in $\mathcal{F}$. For any $k \in \mathbf{ex}$, the *seed mutation* in direction k transforms $(\tilde{\mathbf{x}}, \tilde{B})$ into a seed $(\tilde{\mathbf{x}}', \tilde{B}')$ given by:

- $\tilde{\mathbf{x}}' = \tilde{\mathbf{x}} - \{x_k\} \cup \{x'_k\}$, where $x'_k \in \mathcal{F}$ is determined by the *exchange relation*

$$x'_k = x_k^{-1} \Big(\prod_{\substack{i \in [1,m] \\ b_{ik} > 0}} x_i^{b_{ik}} + \prod_{\substack{i \in [1,m] \\ b_{ik} < 0}} x_i^{-b_{ik}} \Big) \ . \tag{1}$$

- The entries of $\tilde{B}'$ are given by

$$b'_{ij} = \begin{cases} -b_{ij} & \text{if } i = k \text{ or } j = k; \\ b_{ij} + \dfrac{|b_{ik}| b_{kj} + b_{ik} |b_{kj}|}{2} & \text{otherwise.} \end{cases} \tag{2}$$

The seed mutations generate an equivalence relation: we say that two seeds $(\tilde{\mathbf{x}}, \tilde{B})$ and $(\tilde{\mathbf{x}}', \tilde{B}')$ are *mutation-equivalent* if $(\tilde{\mathbf{x}}', \tilde{B}')$ can be obtained from $(\tilde{\mathbf{x}}, \tilde{B})$ by a sequence of seed mutations.

Fix a mutation-equivalence class $\mathcal{S}$ of seeds. Let $\mathcal{X} \subset \mathcal{F}$ denote the union of clusters, and $\mathbf{c}$ the common coefficient set of all seeds from $\mathcal{S}$. The *cluster algebra* $\mathcal{A}(\mathcal{S})$ associated with $\mathcal{S}$ is the $\mathbb{Z}[\mathbf{c}^{\pm 1}]$-subalgebra of $\mathcal{F}$ generated by $\mathcal{X}$.

We now define a family of q-deformations of $\mathcal{A}(\mathcal{S})$. The following setup is a simplified version of that in [2]. The main idea is to deform each extended cluster $\tilde{\mathbf{x}}$ to a quasi-commuting family $\tilde{\mathbf{X}} = \{X_1, \ldots, X_m\}$ satisfying

$$X_i X_j = q^{\lambda_{ij}} X_j X_i \tag{3}$$

for some skew-symmetric integer $m \times m$ matrix $\Lambda = (\lambda_{ij})$. Let $\mathcal{F}_q$ denote the skew-field of fractions of the ring $\mathbb{Z}[q^{\pm 1/2}, X_1, \ldots, X_m]$, where $X_1, \ldots, X_m$ are algebraically independent variables satisfying (3). For any $a = (a_1, \ldots, a_m) \in \mathbb{Z}^m$, we set

$$X^a = q^{\frac{1}{2} \sum_{i>j} \lambda_{ij} a_i a_j} X_1^{a_1} \cdots X_m^{a_m} \ . \tag{4}$$

Definition 3. A *free generating set* for $\mathcal{F}_q$ is a subset $\{Y_1, \ldots, Y_m\} \subset \mathcal{F}_q$ of the following form: $Y_j = \varphi(X^{c_j})$, where φ is a $\mathbb{Q}(q^{\pm 1/2})$-linear automorphism of $\mathcal{F}_q$, and $\{c_1, \ldots, c_m\}$ is a basis of the lattice $\mathbb{Z}^m$.

Note that the subset $\{Y_1, \ldots, Y_m\}$ can be used instead of $\{X_1, \ldots, X_m\}$ in the definition of the ambient field $\mathcal{F}_q$, with the matrix Λ replaced by $C^T \Lambda C$, where C is the matrix with columns $c_1, \ldots, c_m$.

Definition 4. A *quantum seed* in $\mathcal{F}_q$ is a pair $(\tilde{\mathbf{X}}, \tilde{B})$, where

- $\tilde{\mathbf{X}} = \{X_1, \ldots, X_m\}$ is a free generating set for $\mathcal{F}_q$.
- $\tilde{B}$ is a $m \times n$ integer matrix with rows labeled by $[1, m]$ and columns labeled by an n-element subset $\mathbf{ex} \subset [1, m]$, which is *compatible* with the matrix Λ given by (3), in the following sense: for some positive integers d_j $(j \in \mathbf{ex})$, we have

$$\sum_{k=1}^{m} b_{kj}\lambda_{ki} = \delta_{ij} d_j \quad (j \in \mathbf{ex},\ i \in [1, m]) \ . \tag{5}$$

As in Definition 1, the quantum seeds are defined up to a relabeling of elements of $\tilde{\mathbf{X}}$ together with the corresponding relabeling of rows and columns of $\tilde{B}$.

Note that (5) implies that $d_i b_{ij} = -d_j b_{ji}$ for all $i, j \in \mathbf{ex}$, i.e., $\tilde{B}$ is as in Definition 1.

Example. *Let $m = 2n$, $\mathbf{ex} = [1, n]$, and let $\tilde{B}$ be of the form*

$$\tilde{B} = \begin{pmatrix} B \\ I \end{pmatrix} ,$$

where I is the identity $n \times n$ matrix. Here B is an arbitrary integer $n \times n$ matrix satisfying $d_i b_{ij} = -d_j b_{ji}$ for some positive integers $d_1, \ldots, d_n$: in other words, B is skew-symmetrizable*, that is, DB is skew-symmetric, where D is the diagonal matrix with diagonal entries $d_1, \ldots, d_n$. An easy calculation shows that the skew-symmetric matrices Λ compatible with $\tilde{B}$ in the sense of (5) are those of the form*

$$\Lambda = \begin{pmatrix} \Lambda_0 & -D - \Lambda_0 B \\ D - B^T \Lambda_0 & -DB + B^T \Lambda_0 B \end{pmatrix} , \tag{6}$$

where Λ_0 is an arbitrary skew-symmetric integer $n \times n$ matrix.

Definition 5. Let $(\tilde{\mathbf{X}}, \tilde{B})$ be a quantum seed in $\mathcal{F}_q$. For any $k \in \mathbf{ex}$, the *quantum seed mutation* in direction k transforms $(\tilde{\mathbf{X}}, \tilde{B})$ into a quantum seed $(\tilde{\mathbf{X}}', \tilde{B}')$ given by:

- $\tilde{\mathbf{X}}' = \tilde{\mathbf{X}} - \{X_k\} \cup \{X'_k\}$, where $X'_k \in \mathcal{F}_q$ is given by

$$X'_k = X^{-e_k + \sum_{b_{ik}>0} b_{ik} e_i} \ + \ X^{-e_k - \sum_{b_{ik}<0} b_{ik} e_i} \ , \tag{7}$$

  where the terms on the right are defined via (4), and $\{e_1, \ldots, e_m\}$ is the standard basis in $\mathbb{Z}^m$.
- The matrix entries of $\tilde{B}'$ are given by (2).

The fact that $(\tilde{\mathbf{X}}', \tilde{B}')$ is a quantum seed is not automatic: for the proof see [2, Proposition 4.7].

Based on definitions (4) and (5), one defines the *quantum cluster algebra* associated with a mutation-equivalence class of quantum seeds, in exactly the same way as the ordinary cluster algebra. It is shown in [2] that practically all the structural results on cluster algebras obtained in [3, 5, 1] extend to the quantum setting. This includes the Laurent phenomenon obtained in [3, 4, 1] and the classification of cluster algebras of finite type given in [5].

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Maximal orthogonal subcategories of triangulated categories satisfying Serre duality

Osamu Iyama

1. Motivation

The classical Auslander correspondence gives a bijection between the set of Morita-equivalence classes of representation-finite finite-dimensional algebras Λ and that of finite-dimensional algebras Γ with $\operatorname{gl.dim}\Gamma \leq 2$ and $\operatorname{dom.dim}\Gamma \geq 2$. Our motivation comes from a higher dimensional generalization [5] of the Auslander correspondence in Theorem 1.2.

Definition 1.1. Let $\mathbf{T}$ be a triangulated category (resp. a full subcategory of abelian category) and $n \geq 0$. For a functorially finite full subcategory $\mathbf{C}$ of $\mathbf{T}$, put

$$\begin{aligned} \mathbf{C}^{\perp_n} &:= \{X \in \mathbf{T} \mid \operatorname{Ext}^i(\mathbf{C}, X) = 0 \text{ for any } i \ (0 < i \leq n)\}, \\ {}^{\perp_n}\mathbf{C} &:= \{X \in \mathbf{T} \mid \operatorname{Ext}^i(X, \mathbf{C}) = 0 \text{ for any } i \ (0 < i \leq n)\}. \end{aligned}$$

We call $\mathbf{C}$ a *maximal n-orthogonal subcategory* of $\mathbf{T}$ if $\mathbf{C} = \mathbf{C}^{\perp_n} = {}^{\perp_n}\mathbf{C}$ holds [4].

By definition, $\mathbf{T}$ is a unique maximal 0-orthogonal subcategory of $\mathbf{T}$.

Theorem 1.2. *For any $n \geq 1$, there exists a bijection between the set of equivalence classes of maximal $(n-1)$-orthogonal subcategories $\mathbf{C}$ of $\operatorname{mod}\Lambda$ with additive generators M and finite-dimensional algebras Λ, and the set of Morita-equivalence classes of finite-dimensional algebras Γ with $\operatorname{gl.dim}\Gamma \leq n+1$ and $\operatorname{dom.dim}\Gamma \geq n+1$. It is given by $\mathbf{C} \mapsto \Gamma := \operatorname{End}_\Lambda(M)$.*

Important examples of maximal orthogonal subcategories appear in the work of Buan-Marsh-Reineke-Reiten-Todorov on cluster categories [1], that of Geiß-Leclerc-Schröer on preprojective algebras [3], and in considerations of invariant subrings of finite subgroups G of $\mathrm{GL}_d(k)$ (see [4]). Let us find some kind of higher dimensional analogy of Auslander-Reiten theory by considering maximal orthogonal subcategories.

2. Triangulated categories

In this section, let $\mathbf{T}$ be a triangulated category with a Serre functor S, and $\mathbf{C}$ a maximal $(n-1)$-orthogonal subcategory of $\mathbf{T}$.

Theorem 2.1 ([6]). (1) *$S_n := S \circ [-n]$ gives an autoequivalence of* $\mathbf{C}$.
(2) $\mathbf{C}$ *has "Auslander-Reiten $(n+2)$-angles", i.e. any $X \in \mathbf{C}$ has a complex*

$$S_n X \xrightarrow{f_n} C_{n-1} \xrightarrow{f_{n-1}} \cdots \xrightarrow{f_1} C_0 \xrightarrow{f_0} X$$

which is obtained by glueing triangles $X_{i+1} \longrightarrow C_i \xrightarrow{f_i} X_i \longrightarrow X_{i+1}[1]$, $0 \leq i < n$, with $X_0 = X$, $X_n = S_n X$, $C_i \in \mathbf{C}$ and the following sequences are exact.

$$\mathbf{C}(-, S_n X) \xrightarrow{\cdot f_n} \mathbf{C}(-, C_{n-1}) \xrightarrow{\cdot f_{n-1}} \cdots \xrightarrow{\cdot f_1} \mathbf{C}(-, C_0) \xrightarrow{\cdot f_0} J_{\mathbf{C}}(-, X) \longrightarrow 0$$

$$\mathbf{C}(X, -) \xrightarrow{f_0 \cdot} \mathbf{C}(C_0, -) \xrightarrow{f_1 \cdot} \cdots \xrightarrow{f_{n-1} \cdot} \mathbf{C}(C_{n-1}, -) \xrightarrow{f_n \cdot} J_{\mathbf{C}}(S_n X, -) \longrightarrow 0$$

It is quite interesting to study the relationship among all maximal $(n-1)$-orthogonal subcategories of $\mathbf{T}$. In the rest of this section, assume that $\mathbf{T}$ is n-*Calabi-Yau*, i.e. $S_n = 1$. For example, if Λ is a d-dimensional symmetric order, then $\underline{\mathrm{CM}}\Lambda$ is $(d-1)$-Calabi-Yau.

Definition 2.2. Assume that $\mathbf{C}$ *satisfies the strict no-loop condition*, i.e. for any $X \in \operatorname{ind} \mathbf{C}$, $X \notin \operatorname{add} \bigoplus_{i=1}^{n-1} C_i$ holds in Theorem 2.1, (2). Define a full subcategory $\mu_{X,i}(\mathbf{C})$ of $\mathbf{T}$ by

$$\operatorname{ind} \mu_{X,i}(\mathbf{C}) := (\operatorname{ind} \mathbf{C} \backslash \{X\}) \cup \{X_i\} \qquad (X \in \operatorname{ind} \mathbf{C},\ i \in \mathbf{Z}/n\mathbf{Z})$$

where X_i is the term of the triangle in Theorem 2.1, (2). This can be regarded as a higher dimensional generalization of the Fomin-Zelevinsky mutation in [1] and [3].

Theorem 2.3 ([6]). *Assume that $\mathbf{C}$ satisfies the strict no-loop condition. For any $X \in \operatorname{ind} \mathbf{C}$, $\{\mu_{X,i}(\mathbf{C}) \mid i \in \mathbf{Z}/n\mathbf{Z}\}$ is the set of all maximal $(n-1)$-orthogonal subcategories of $\mathbf{T}$ containing $\operatorname{ind} \mathbf{C} \backslash \{X\}$.*

2.4. It is an interesting question when transitivity holds in $\mathbf{T}$, i.e. the set of all maximal $(n-1)$-orthogonal subcategories of $\mathbf{T}$ is transitive under the action of mutations defined in Definition 2.2. It is known that transitivity holds for cluster categories $\mathbf{T}$ [1], and $\mathbf{T} = \underline{\mathrm{CM}}\,\Lambda$ for the Veronese subring Λ of degree 3 of $k[[x,y,z]]$ (see [8]).

3. Derived equivalence

It is suggestive to relate our question in 2.4 to Van den Bergh's generalization [7] of the Bondal-Orlov conjecture [2] in algebraic geometry, which asserts that *all (commutative or non-commutative) crepant resolutions of a normal Gorenstein domain have the same derived category.* Let us generalize the concept of Van den Bergh's non-commutative crepant resolutions [7] of commutative normal Gorenstein domains to our situation.

3.1. Let Λ be an R-order which is an isolated singularity. We call $M \in \mathrm{CM}\,\Lambda$ a *NCC resolution* of Λ if $\Lambda \oplus \mathrm{Hom}_R(\Lambda, R) \in \mathrm{add}\, M$ and $\Gamma := \mathrm{End}_\Lambda(M)$ is an R-order with $\mathrm{gl.dim}\,\Gamma = d$. We have the remarkable relationship below between NCC resolutions and maximal $(d-2)$-orthogonal subcategories [5].

Proposition. *Let $d \geq 2$. Then $M \in \mathrm{CM}\,\Lambda$ is a NCC resolution of Λ if and only if $\mathrm{add}\, M$ is a maximal $(d-2)$-orthogonal subcategory of $\mathrm{CM}\,\Lambda$.*

3.2. We conjecture that *the endomorphism rings $\mathrm{End}_\Lambda(M)$ are derived equivalent for all maximal $(n-1)$-orthogonal subcategories $\mathrm{add}\, M$ of $\mathrm{CM}\,\Lambda$.* This is an analogy of the Bondal-Orlov and Van den Bergh conjecture by 3.1, and true for $n = 2$.

Theorem ([5]). *Let $\mathbf{C}_i = \mathrm{add}\, M_i$ be a maximal 1-orthogonal subcategory of $\mathrm{CM}\,\Lambda$ and $\Gamma_i := \mathrm{End}_\Lambda(M_i)$, $i = 1, 2$. Then Γ_1 and Γ_2 are derived equivalent.*

Corollary 3.3 ([5, 6]). *All NCC resolutions of Λ are derived equivalent if*

(1) *$d \leq 3$, or*
(2) *Λ is a symmetric order and transitivity holds in $\underline{\mathrm{CM}}\Lambda$ (2.4).*

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A construction of maximal 1-orthogonal modules for preprojective algebras

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(joint work with Bernard Leclerc and Jan Schröer)

For a Dynkin quiver $Q = (Q_0, Q_1, t, h)$ we consider its double $\bar{Q}$, which is obtained from Q by adding an extra arrow $a^*\colon h(a) \to t(a)$ for each arrow $a\colon t(a) \to h(a)$ in Q_1. The preprojective algebra $\Lambda = \mathrm{k}\bar{Q}/\langle \sum_{a \in Q_1} [a, a^*] \rangle$ is in this situation a finite dimensional, selfinjective algebra.

Let $F\colon \widetilde{\Lambda} \to \Lambda$ be the universal covering of Λ. Consider moreover an embedding $J\colon \Gamma_Q \to \widetilde{\Lambda}$ where Γ_Q is the Auslander algebra of $\mathrm{k}Q$. To be precise, we should replace here our algebras by locally bounded categories, and consider contravariant functors instead of right modules.

Consider now $I'_Q = \mathrm{Hom}_{\mathrm{k}}(\Gamma_Q, \mathrm{k})$ as a right Γ_Q-module and define the Λ-module $I_Q := F_\lambda J^{\cdot}(I_Q)$. Here, $F_\lambda\colon \text{mod-}\widetilde{\Lambda} \to \text{mod-}\Lambda$ is the usual push down functor associated to F and $J^{\cdot}\colon \text{mod-}\Gamma_Q \to \text{mod-}\widetilde{\Lambda}$ is the "extension by 0" associated to J. Clearly, I_Q is a direct sum of $|\Pi_Q|$ pairwise non-isomorphic indecomposable summands, where Π_Q is the set of positive roots associated to Q.

We can describe $\mathcal{E}_Q = \mathrm{End}_\Lambda(I_Q)$ as a quiver with relations: The quiver $\bar{A}_Q$ of $\mathrm{End}_\Lambda(I_Q)$ is obtained from the Auslander-Reiten quiver A_Q of $\mathrm{k}Q$ by inserting an additional arrow $\rho_x\colon x \to \tau x$ for each non-projective vertex x of A_Q. The relations are the usual mesh relations for A_Q and, moreover, for each arrow $\beta\colon x \to y$ with y not a projective vertex there is a relation $\tau(\beta)\rho_x - \rho_y\beta$ (interpret this as $\rho_y\beta$ if x is projective). In other words, precisely for each arrow $\alpha\colon u \to v$ in $\bar{A}_Q$ with not both u and v injective there is a homogeneous relation of length 2 from v to u.

Now, a slightly tricky calculation shows that

$$\dim_{\mathrm{k}} \mathcal{E}_Q Q = q_Q(\underline{\dim}\, I_Q), \tag{1}$$

where q_Q is the quadratic form associated to Q.

Remarks. (1) *For $X \in \text{mod-}\Lambda$ let $\mathbf{v} = \underline{\dim}\, X$. If we denote by x the corresponding point in the preprojective variety $\Lambda_{\mathbf{v}}$, one has*

$$\dim \mathrm{Ext}^1_\Lambda(X, X) = 2\, \mathrm{codim}_{\Lambda_{\mathbf{v}}}(\mathrm{Gl}_{\mathbf{v}} \cdot x) = 2(\dim \mathrm{End}_\Lambda(X) - q_Q(\mathbf{v})).$$

We conclude from (1) that I_Q is rigid, i.e. $\mathrm{Ext}^1_\Lambda(I_Q, I_Q) = 0$.

(2) *In [4] it was shown that $|\Pi_Q|$ is an upper bound for the number of pairwise non-isomorphic direct summands of a rigid module. As we have seen for I_Q, this upper bound is reached. We call such modules* complete rigid.

(3) *In [3] we show that if there exists a complete rigid module T such that the quiver of* $\mathrm{End}_\Lambda(T)$ *has no loops, then each complete rigid module is even maximal 1-orthogonal in the sense of Iyama [5].*

(4) *Let now $\mathrm{k} = \mathbb{C}$ and N be a maximal unipotent subgroup of a simple (complex) Lie group of type $|Q|$. It follows from [1] that the coordinate ring $\mathbb{C}[N]$ has the structure of an (upper) cluster algebra. The exchange matrix for the initial seed constructed there can be codified in a quiver which coincides with the quiver of our $\mathcal{E}_Q$ (for a proper reduced word for the longest element in the corresponding Weyl group).*

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Resolutions over Koszul algebras and a question of D. Happel

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(joint work with Ragnar-Olaf Buchweitz, Dag Madsen and Øyvind Solberg)

Suppose that Λ is a finite dimensional K-algebra where K is a field. We denote the n-th Hochschild cohomology group of Λ by $\mathrm{HH}^n(\Lambda)$. In [5], Dieter Happel asked: If $\mathrm{HH}^n(\Lambda) = 0$ for sufficiently large n, then is the global dimension of Λ finite? Using the quantum exterior algebra in two variables as an example, we give a negative answer to the question [2]. It should be noted that L. Avramov and S. Iyengar [1] show that if Λ is a commutative finite dimensional K-algebra, then Happel's question has an affirmative answer.

We now give the example. Let $\Lambda_q = K\langle x, y\rangle/(x^2, y^2, xy + qyx)$ where $K\langle x, y\rangle$ is the free associative algebra in two variables and $q \in K$. We have the following facts. For all $q \in K$, the dimension of Λ_q is 4 and the global dimension of Λ_q is infinite. If $q \neq 0$ then Λ_q is a self-injective Koszul algebra with Koszul dual being the quantum affine plane $K\langle x, y\rangle/(yx - qxy)$. We note that a minimal projective resolution of K, as a right Λ_q-module, is

$$\cdots \to \Lambda_q^4 \xrightarrow{\begin{pmatrix} x & y & 0 & 0 \\ 0 & qx & y & 0 \\ 0 & 0 & q^2x & 0 \end{pmatrix}} \Lambda_q^3 \xrightarrow{\begin{pmatrix} x & y & 0 \\ 0 & qx & y \end{pmatrix}} \Lambda_q^2 \xrightarrow{\begin{pmatrix} x & y \end{pmatrix}} \Lambda_q \to K \to 0.$$

Using this resolution, we are able to find a minimal projective Λ_q^e-resolution of Λ_q, where Λ_q^e is the enveloping algebra $\Lambda_q^{op} \otimes_K \Lambda_q$. We use this resolution to calculate the Hochschild cohomology groups.

We prove the following result.

Theorem. *Let $\Lambda_q = K\langle x, y\rangle/(x^2, y^2, xy + qyx)$ such that q is not a root of unity in K. Then the Hochschild cohomology ring, $\mathrm{HH}^*(\Lambda_q)$ is isomorphic to $K[z]/(z^2) \times_K \wedge^*(u_0, u_1)$ as graded algebras where z has degree one, u_0 and u_1 are of degree 1 and $\wedge^*(u_0, u_1)$ denotes the exterior algebra in two variables.*

From this theorem, we see that $\dim_K(\mathrm{HH}^0(\Lambda_q)) = 2 = \dim_K(\mathrm{HH}^1(\Lambda_q))$, $\dim_K(\mathrm{HH}^2(\Lambda_q)) = 1$, and $\mathrm{HH}^n(\Lambda_q) = 0$ for $n \geq 3$. Thus, each Λ_q, q not a root of unity in K, provides a counterexample to Happel's question.

If q is a root of unity in K, then the Hochschild cohomology ring for Λ_q, in all characteristics, is also completely described in [2]. In particular, it follows that in this case, $\mathrm{HH}^n(\Lambda_q) \neq 0$ for an infinite number of n's. On the other hand, by appropriate choices of the root of unity q, it is shown that there can be arbitrarily large gaps where the Hochschild cohomology vanishes.

The story of the Hochschild homology groups of the Λ_q is different. Y. Han [4] shows that for any q, $\mathrm{HH}_n(\Lambda_q) \neq 0$ for all n.

The main technique to describe the Hochschild cohomology rings is to find a minimal projective Λ_q^e-resolution of Λ_q. It turns out that the basic requirement in finding such resolutions is that Λ_q is a Koszul algebra for $q \neq 0$. The techniques we employ generalize to arbitrary Koszul algebras and can be found in [3].

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Local Ext-limitations do not exist

Sverre O. Smalø

In this talk it was shown that for k a field and the four dimensional algebra $\Lambda = k\langle x, y\rangle/\langle x^2, y^2, xy+qyx\rangle$ when $q^n \neq 1, 0$ for all n, there exist a two dimensional module M and a family of two dimensional modules M_i, $i = 1, 2, \dots$, such that $\dim_k \operatorname{Ext}^i_\Lambda(M, M_j) = 1$ for $i = 0, j$ and $j+1$ and zero otherwise. This is probably the easiest example giving a negative answer to a question raised by Maurice Auslander.

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Double Poisson algebras

Michel Van den Bergh

Let k be a field. If Q is a finite quiver and $\bar{Q}$ is its associated double quiver, then $k\bar{Q}/[k\bar{Q}, k\bar{Q}]$ is equipped with a natural Lie bracket $\{-,-\}$, the so-called necklace bracket [1, 5, 6].

The necklace bracket has a connection with representation spaces as follows. Let α be a dimension vector. Then $\operatorname{Rep}(\bar{Q}, \alpha)$ is the cotangent bundle of $\operatorname{Rep}(Q, \alpha)$, and as such it comes equipped with a classical Poisson bracket. The trace map

$$\operatorname{Tr} : k\bar{Q}/[k\bar{Q}, k\bar{Q}] \to \mathcal{O}(\operatorname{Rep}(Q, \alpha))^{Gl(\alpha)}$$

is a Lie algebra homomorphism.

This theory is somewhat unsatisfying since

- As $k\bar{Q}/[k\bar{Q}, k\bar{Q}]$ has no algebra structure, it cannot itself be regarded as a kind of (non-commutative) Poisson algebra.
- The above trace map only explains the Poisson bracket between *invariant* functions on $\operatorname{Rep}(Q, \alpha)$.

To solve these problems we introduce the notion of a *double Poisson* structure on a non-commutative algebra A [7]. This is by definition a bilinear map

$$\{\!\{-,-\}\!\} : A \otimes A \to A \otimes A$$

satisfying suitable analogues of the axioms of a Poisson algebra. If A is a double Poisson algebra then $A/[A,A]$ carries an induced Lie bracket $\{-,-\}$ and furthermore all representation spaces of A carry an induced Poisson bracket.

We show that $k\bar{Q}$ has a natural double Poisson structure whose associated Lie bracket is the necklace bracket and which induces the standard Poisson structure on $\operatorname{Rep}(\bar{Q},\alpha)$.

The algebra DA of double poly-vector fields associated to A is defined as $T_A \operatorname{Der}(A, A\otimes A)$ [2]. This definition can be motivated by showing that the elements of DA induce poly-vector fields on all representation spaces. We show that DA has a natural (super) double Poisson structure which induces the Schouten bracket on all representation spaces. If A is quasi-free, then a double Poisson bracket on A can be described as an element P of $(DA/[DA,DA])_2$ such that $\{P,P\}=0$.

For more information on non-commutative Poisson geometry, and in particular an application to the multiplicative preprojective algebras recently introduced by Crawley-Boevey and Shaw [4], see [7]. For a related approach see [3].

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Microscopy of simple representations

Markus Reineke

Let $Q=(Q_0,Q_1)$ be a finite quiver, and let $d \in \mathbf{N}Q_0$ be a dimension vector. A year ago I proved:

Theorem 1. *There exists a polynomial $a_d^Q(t) \in \mathbf{Z}[t]$ such that, for any finite field k, the evaluation $a_d^Q(|k|)$ equals the number of isomorphism classes of absolutely simple representations S of kQ of dimension vector d (i.e. $\overline{k} \otimes_k S$ is a simple representation of $\overline{k}Q$).*

Computer experiments show that the nature of these polynomials is rather mysterious. However, a special value has a simple interpretation:

Theorem 2. *If* $\dim d > 1$*, the polynomial* $a_d^Q(t)$ *has a zero at* $t = 1$*, and* $\left.\frac{a_d^Q(t)}{t-1}\right|_{t=1}$ *equals the number of cyclic equivalence classes of primitive cycles in* Q *of weight* d*.*

A cycle ω in Q is of weight d if it passes d_i times through each vertex $i \in Q_0$. It is called primitive if it is not a proper power of another cycle. The equivalence relation is cyclic rotation of paths.

The proof works as follows:

Step 1: Let $R_d(Q) = \bigoplus_{(\alpha:i\to j)\in Q_1} \mathrm{Hom}(\mathbf{C}^{d_i}, \mathbf{C}^{d_j})$ be the variety of complex representations of Q of dimension vector d, on which the algebraic group $G_d := \prod_{i\in Q_0} \mathrm{GL}_{d_i}(\mathbf{C})$ acts by base change. The projective space $\mathbf{P}R_d(Q)$ contains an open subset U corresponding to the simple representations, which admits a geometric quotient $\mathbf{P}\mathcal{M}_d(Q) := U/G_d$, a smooth, but non-projective complex variety. By Theorem 1 and some properties of ℓ-adic cohomology, the value $\left.\frac{a_d^Q(t)}{t-1}\right|_{t=1}$ equals the Euler characteristic in cohomology with compact support $\chi_c(\mathbf{P}\mathcal{M}_d(Q))$. This reduces the theorem to a topological statement.

Step 2: The Borel localization formula in equivariant cohomology gives the following: given a torus action on a complex variety, the Euler characteristic χ_c is preserved under passage to the fixed point set. Here we have an action of the torus $T_Q = (\mathbf{C}^*)^{Q_1}$ on $R_d(Q)$ by rescaling of arrows, which passes to an action on $\mathbf{P}\mathcal{M}_d(Q)$. It thus suffices to compute (the Euler characteristic of) $\mathbf{P}\mathcal{M}_d(Q)^{T_q}$.

Step 3: Given an indivisible vector $\lambda \in \mathbf{N}Q_1$, define a quiver Q_λ (an *almost universal abelian covering* of Q) with set of vertices $Q_0 \times \mathbf{Z}Q_1/\mathbf{Z}\lambda$ and arrows $(\alpha, \mu) : (i, \mu) \to (j, \mu + e_\alpha)$ for all $(\alpha : i \to j) \in Q_1$ and all $\mu \in \mathbf{Z}Q_1/\mathbf{Z}\lambda$. Given $d \in \mathbf{N}Q_0$, consider dimension vectors $\widetilde{d} \in \mathbf{N}(Q_\lambda)_0$ such that $\sum_\mu \widetilde{d}_{i,\mu} = d_i$ for all $i \in Q_0$.

Proposition. *The fixed point set* $\mathbf{P}\mathcal{M}_d(Q)^{T_Q}$ *is isomorphic to the disjoint union* $\bigcup_{\lambda,\widetilde{d}} \mathbf{P}\mathcal{M}_{\widetilde{d}}(Q_\lambda)$ *running over all* λ *and* $\widetilde{d}$ *as above.*

By additivity of Euler characteristic and Step 2, $\chi_c(\mathbf{P}\mathcal{M}_d(Q))$ equals the sum $\sum_{\lambda,\widetilde{d}} \chi_c(\mathbf{P}\mathcal{M}_{\widetilde{d}}(Q_\lambda))$.

Step 4: The theorem can now be proved by induction on $|Q_0|$, assuming in each step w.l.o.g. that $\mathrm{supp}(d) = Q$ and that Q is connected. The reduction process ends with quivers $\overline{Q}$ such that either $\mathbf{P}\mathcal{M}_d(\overline{Q}) = \emptyset$, or $\overline{Q}$ is an $\widetilde{A}_n$-quiver with cyclic orientation, and $d_i = 1$ for all $i \in I$, in which case $\mathbf{P}\mathcal{M}_d(Q)$ is a single point, thus of Euler characteristic 1. To count how many times this quiver is produced in the reduction process, its arrows have to be labelled (up to cyclic permutation) by arrows of the original quiver Q which form a primitive cycle. This proves Theorem 2.

This principle of proof may be called microscopy for two reasons: on the one hand, the iterated application of localization "zooms" into the moduli space $\mathbf{P}\mathcal{M}_d(Q)$ of simple representations. On the other hand, simples belonging to the fixed point set $\mathbf{P}\mathcal{M}_d(Q)^{T_Q}$ possess an inner structure (they lift to a simple representation of some Q_λ), so the proof also "looks at simples under a microscope".

Auslander-Reiten sequences, locally free sheaves and Chebysheff polynomials

Dan Zacharia

Let R be the exterior algebra in $n+1$ variables, and let S denote the symmetric algebra in $n+1$ variables. It is well known that R is a selfinjective Koszul algebra and S is its Koszul dual. By $\mathcal{K}_R$ and $\mathcal{K}_S$ we denote the categories of linear R-modules (S-modules respectively) where the morphisms are the degree zero homomorphisms. The Koszul duality can be then used to obtain mutually inverse dualities between the category of linear R-modules and that of the linear S-modules:

$$\begin{tikzcd} \mathcal{K}_R \arrow[r, bend left, "\mathcal{E}"] & \mathcal{K}_S \arrow[l, bend left, "\mathcal{F}"] \end{tikzcd}$$

By $\mathrm{coh}(\mathbf{P}^n)$ we denote the category of coherent sheaves over the projective n-space, and if M is a finitely generated graded S-module, we set $\widetilde{M} \in \mathrm{coh}(\mathbf{P}^n)$ to be its sheafification. A theorem of Avramov and Eisenbud [1] states that for every finitely generated graded S-module M, there exists an integer k such that the shifted truncation $M_{\geq k}[-k] = M_k \oplus M_{k+1} \oplus \ldots [-k]$ is a linear S-module. This means that every indecomposable coherent sheaf $\mathcal{F}$, is the sheafification of the graded shift of some linear R-module, and using the Koszul duality we can write $\mathcal{F} \cong \widetilde{\mathcal{E}(M)}[i]$ for some linear R-module M and some integer i.

The main ingredient is the following theorem from [2]:

Theorem 1. *Let M be an indecomposable linear nonfree R-module. Then there exists an exact sequence $0 \to A \to B \to M \to 0$ that is an Auslander-Reiten sequence in $\mathcal{K}_R$. Moreover the Loewy length of A is precisely 2.*

It turns out that if we denote by J the radical of R, then the module M/J^2M is indecomposable, and the induced sequence $0 \to A \to B/J^2B \to M/J^2M \to 0$ is an Auslander-Reiten sequence in the category $\mathrm{gr}_0\, R/J^2$ of graded R/J^2-modules generated in degree zero. Since the algebra R/J^2 is stably equivalent to the generalized Kronecker algebra we can use this algebra to describe the Auslander-Reiten quiver of $\mathcal{K}_R$:

Theorem 2. *Let R denote the exterior algebra in $n+1$ variables, where $n > 1$. The A-R quiver of $\mathcal{K}_R$ has a connected component that coincides with the preinjective component of $\mathrm{gr}_0\, R/J^2$, a component consisting only of the module R, and all the remaining connected components are quivers of the type $\mathbf{N}^- A_\infty$.*

We can use now the Koszul duality and Serre's theorem to show that certain subcategories of the category of coherent sheaves over the projective n-space have one sided Auslander-Reiten sequences, and describe the shapes of their Auslander-Reiten quivers. First, for each integer i, denote by $\mathcal{K}_S[i]$ the graded shifts of the category of linear S-modules and by $\widetilde{\mathcal{K}_S[i]}$ their sheafifications. We have the following:

Theorem 3. *For each integer i, the category $\widetilde{\mathcal{K}_S[i]}$ has left Auslander-Reiten sequences, that is, given an indecomposable coherent sheaf $\mathcal{F}$ in $\widetilde{\mathcal{K}_S[i]}$, there exist an exact sequence*

$$0 \to \mathcal{F} \to \mathcal{B} \to \mathcal{C} \to 0$$

that is almost split in $\widetilde{\mathcal{K}_S[i]}$. Moreover, the Auslander-Reiten quiver of the subcategory $\widetilde{\mathcal{K}_S[i]}$ of $\mathrm{coh}(\mathbf{P}^n)$, where $n > 1$, has one preprojective component, and the remaining components are all quivers of the type $\mathbf{N}A_\infty$.

We can use the fact that every coherent sheaf is the sheafification of a linear S-module, to compute the rank of a locally free sheaf. Namely, if $\mathcal{F}$ is locally free, then we can write $\mathcal{F} = \widetilde{\mathcal{E}(M)}$ for some linear R-module M, and then we have $\operatorname{rk} F = \sum_{i=0}^{p}(-1)^i \dim M_i$ where the M_i are the graded pieces of M. Using the Koszul duality we can prove that if a component of the Auslander-Reiten quiver of some $\widetilde{\mathcal{K}_S[i]}$ contains a locally free sheaf, then all the sheaves in that component are locally free. Then it is easy to show that each sheaf lying in the preprojective component of one of the $\widetilde{\mathcal{K}_S[i]}$ is locally free. We have the following:

Proposition. *Let $\mathcal{F}_0, \mathcal{F}_1, \mathcal{F}_2 \ldots$ be the locally free sheaves lying in the preprojective component of the subcategory $\widetilde{\mathcal{K}_S[i]}$ of $\mathrm{coh}(\mathbf{P}^n)$. Denote by*

$$T_k(x) = \sum_{m=0}^{[k/2]} (-1)^m \binom{k-m}{m} (2x)^{k-2m}$$

the Chebysheff polynomials of the second kind. Then for each $k \geq 1$, we have

$$\operatorname{rk} \mathcal{F}_k = T_k(\frac{n+1}{2}) - T_{k-1}(\frac{n+1}{2}).$$

In addition, if $n > 1$, then for each k, $\operatorname{rk} \mathcal{F}_{k+1} > \operatorname{rk} \mathcal{F}_k$.

It is a long standing problem to determine whether there are indecomposable locally free sheaves of small ranks over $\mathbf{P^n}$. In this regard we have the following theorem.

Theorem 4. *Let $n > 1$. For each integer i, each A-R component of $\widetilde{\mathcal{K}_S[i]}$ contains at most one locally free sheaf of rank less than n. Moreover, in each component the ranks increase exponentially.*

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On the finitistic dimension conjecture

Changchang Xi

1. Introduction

Given an artin algebra A, the finitistic dimension of A is defined to be the supremum of the projective dimensions of the finitely generated left A-modules of finite projective dimension. The famous **finitistic dimension conjecture** says that for any artin algebra A the finitistic dimension of A is finite. This conjecture was proposed 45 years ago and still remains open, and has been related to at least five other homological conjectures (see the last 6 conjectures of the total 13 conjectures in the book [4]):

Strong Nakayama conjecture [7]: If M is a non-zero module over an artin algebra A, then there is an integer $n \geq 0$ such that $\mathrm{Ext}^n_A(M, A) \neq 0$.

Generalized Nakayama conjecture [2]: If $0 \to {}_AA \to I_0 \to I_1 \to \dots$ is a minimal injective resolution of an artin algebra A, then any indecomposable injective is a direct summand of some I_j. Equivalently, if M is a finitely generated A-module such that $\mathrm{add}(A) \subseteq \mathrm{add}(M)$ and $\mathrm{Ext}^i_A(M, M) = 0$ for all $i \geq 1$, then M is projective.

Nakayama conjecture [15]: If all I_j in a minimal injective resolution of an artin algebra A, say $0 \to {}_AA \to I_0 \to I_1 \to \dots$, are projective, then A is self-injective.

Gorenstein symmetry conjecture: Let A be an artin algebra. If the injective dimension of ${}_AA$ is finite, then the injective dimension of A_A is finite.

In general, all the above conjectures are still open. They have the following well-known relationship: The finitistic dimension conjecture $\Longrightarrow$ the strong Nakayama conjecture $\Longrightarrow$ the generalized Nakayama conjecture $\Longrightarrow$ the Nakayama conjecture. And, the finitistic dimension conjecture $\Longrightarrow$ the Gorenstein symmetry conjecture.

In this talk I shall report on some new developments attacking the finitistic dimension conjecture. Our idea to approach the conjecture is to use a chain of subalgebras with certain radical conditions. Let us introduce the following notion:

Definition. (1) Given an artin algebra A, we say that the *left representation distance* of A, denoted by $\operatorname{lrep.dis}(A)$, is the minimum of the lengths of chains of subalgebras $A = A_0 \subseteq A_1 \subseteq \cdots \subseteq A_s$ such that $\operatorname{rad}(A_i)$ is a left ideal in A_{i+1} for all i and that A_s is representation-finite. Here we have denoted the Jacobson radical of A by $\operatorname{rad}(A)$.

(2) A homomorphism $f : B \longrightarrow A$ between two algebras A and B is said to be *radical-full* if $\operatorname{rad}(A) = \operatorname{rad}({}_BA)$.

Note that the left representation distance of an artin algebra is always finite by [19] and invariant under Morita equivalences. Every surjective homomorphism is radical-full. Note that if B is a subalgebra of an artin algebra A, the inclusion map being radical-full does not imply that $\operatorname{rad}(B)$ is a left ideal in A.

2. Main results

In this section we shall summarize some new results in the recent papers [19, 20]. For some known results on finitistic dimension conjecture we refer to [3, 9, 10, 8, 11, 16] and many other papers. (I apologize that I could not display all literature here.)

Our main results are the following.

Theorem 1. *Let A be an artin algebra.*

(1) *If* $\operatorname{lrep.dis}(A) \leq 2$, *then the finitistic dimension conjecture is true for A.*
(2) *Let B be a subalgebra of A such that* $\operatorname{rad}(B)$ *is a left ideal of A and that the inclusion map is radical-full. If the global dimension of A is at most 4, then the finitistic dimension conjecture is true for B.*

We may also use a chain of factor algebras to bound the finitistic dimension. In this direction, we have the following result.

Theorem 2. *Let A be an artin algebra, and let I and J be two ideals in A with $IJ\operatorname{rad}(A) = 0$. If A/I and A/J are representation-finite, then the finitistic dimension conjecture is true for A.*

The proofs of Theorem 1 and Theorem 2 are based on the following lemmas.

Lemma 1. *Suppose B is a subalgebra of A such that* $\operatorname{rad}(B)$ *is a left ideal in A. Then, for any B-module X and integer $i \geq 2$, there is a projective A-module Q and an A-module Z such that $\Omega^i_B(X) \simeq \Omega_A(Z) \oplus Q$ as A-modules, where Ω_B stands for the first syzygy over the algebra B.*

Lemma 2 ([11]). *For any artin algebra A there is a function Ψ from the finitely generated A-modules to the non-negative integers such that*

(1) $\Psi(M) = \operatorname{proj.dim}(M)$ *if M has finite projective dimension.*
(2) *For any natural number n, $\Psi(\bigoplus_{j=1}^{n} M) = \Psi(M)$.*
(3) *For any A-modules X and Y, $\Psi(X) \leq \Psi(X \oplus Y)$.*
(4) *If $0 \to X \to Y \to Z \to 0$ is an exact sequence in A-mod such that the projective dimension of Z is finite, then $\Psi(Z) \leq \Psi(X \oplus Y) + 1$.*

Based on the above results, there are many elementary questions, for example, if the left representation distance of A is 3, could we prove the finitistic dimension conjecture for A? For more information and the details of the proofs of the above main results we refer to the papers [19, 20]. Preprints can be downloaded from http:/math.bnu.edu.cn/∼ccxi/.

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Stable cohomology over local rings

Luchezar L. Avramov

(joint work with Oana Veliche)

In the mid-1980s Pierre Vogel introduced a cohomology theory that associates to each pair (M, N) of modules over an associative ring A groups $\widehat{\mathrm{Ext}}^n_A(M,N)$ defined for every $n \in \mathbb{Z}$, which vanish when either M or N has finite projective dimension. The first published account is in [5], and different constructions were independently found by Benson and Carlson [2] and by Mislin [8]. Kropholler's survey [6, §4] contains background and details. Known as *stable cohomology*, this theory contains as a special case Tate's cohomology theory for modules over a finite group G (namely, $\widehat{\mathrm{Ext}}^n_{\mathbb{Z}G}(\mathbb{Z}, N) = \widehat{\mathrm{H}}(G, N)$, where $\mathbb{Z}G$ is the group ring), as well as its extension by Buchweitz [3] to two-sided noetherian Gorenstein rings.

Little is known about the meaning or the properties of stable cohomology outside of the original context of group representations. One reason for that may be the fact that the stable groups, and the multiplicative structures they support, are not readily amenable to computations through classical techniques.

We develop new approaches to their computation and present applications to commutative algebra. For the rest of this text, R denotes a commutative local ring with residue field k. Historical precedents indicate that considerable ring theoretic information on R is reflected in the homological behavior of k, so we focus on the stable cohomology of that module.

The classical Auslander-Buchsbaum-Serre theorem characterizes regular local rings as the local rings of finite global dimension. In particular, when R is regular all functors $\widehat{\mathrm{Ext}}^n_R(-,-)$ are trivial. We prove a strong converse:

1. If $\widehat{\mathrm{Ext}}^n_R(k,k) = 0$ for a single $n \in \mathbb{Z}$, then R is regular.

When R is Gorenstein and M is finitely generated, $\widehat{\mathrm{Ext}}^n_R(M,N)$ can be computed from a complete resolution of M, which is a complex of finite free R-modules. It follows that if N is finitely generated as well, then so is $\widehat{\mathrm{Ext}}^n_R(M,N)$ for each $n \in \mathbb{Z}$. No characterization of Gorenstein rings is known in terms of the numbers $\operatorname{rank}_k \mathrm{Ext}^n_R(k,k)$, so the next result comes as a surprise:

2. If $\operatorname{rank}_k \widehat{\mathrm{Ext}}^n_R(k,k) < \infty$ for a single $n \in \mathbb{Z}$, then R is Gorenstein.

The statements above concern R-module structures, but their proofs use the fact that $\mathcal{E} = \mathrm{Ext}_R(k,k)$ and $\mathcal{S} = \widehat{\mathrm{Ext}}_R(k,k)$ are graded k-algebras, linked by a canonical homomorphism $\iota\colon \mathcal{E} \to \mathcal{S}$. The structure of $\mathcal{E}$ has been the subject of numerous investigations. The structure of $\mathcal{S}$ is a major topic of the talk.

When R is regular, **1.** yields $\mathcal{S} = 0$. Martsinkovsky [7] proved that for singular rings ι is injective. We reprove this as part of the next result, where Σ denotes the translation functor and $\mathcal{E}$ acts canonically on $\mathcal{I} = \operatorname{Hom}_k(\mathcal{E}, k)$. This theorem leads to an effective procedure for checking the finiteness condition in **2.**

3. If R is singular, then there is an exact sequence

$$0 \longrightarrow \mathcal{E} \xrightarrow{\ \iota\ } \mathcal{S} \longrightarrow \coprod_{i=d-1}^{\infty} (\Sigma^{-i}\mathcal{I})^{\mu^{i+1}} \longrightarrow 0$$

of graded left $\mathcal{E}$-modules, where $d = \operatorname{depth} R$ and $\mu^i = \operatorname{rank}_k \operatorname{Ext}^i_R(k, R)$.

One measure of the singularity of R is provided by a non-negative number, $\operatorname{codepth} R = \operatorname{edim} R - \operatorname{depth} R$, where $\operatorname{edim} R$ denotes the minimal number of generators of $\mathfrak{m}$ and $\operatorname{depth} R$ the depth of the ring. One has $\operatorname{codepth} R = 0$ precisely when R is regular. The condition $\operatorname{codepth} R \leq 1$ characterizes hypersurface rings. Their stable cohomology algebra, determined by Buchweitz [3], satisfies:

4. When R is a hypersurface, $\mathcal{S} = \mathcal{E}[\vartheta^{-1}]$, where $\vartheta \in \mathcal{E}^2$ is a central non-zero-divisor and $\mathcal{E}/(\vartheta)$ is an exterior algebra on $\operatorname{edim} R$ generators of degree 1.

Except for the special case of group algebras of finite abelian groups, little is known about the structure of $\mathcal{S}$ for local rings R having $\operatorname{codepth} R \geq 2$.

Our results on the subject involve the number

$$\operatorname{depth} \mathcal{E} = \inf\{n \in \mathbb{Z} \mid \operatorname{Ext}^n_{\mathcal{E}}(k, \mathcal{E}) \neq 0\}.$$

Clearly, one always has $\operatorname{depth} \mathcal{E} \geq 0$. When R is regular, the k-algebra $\mathcal{E}$ is finite dimensional, so $\operatorname{depth} \mathcal{E} = 0$. The converse also holds, but this time for a non-trivial reason. Indeed, a fundamental structure theorem, due to Milnor and Moore, André, and Sjödin, shows that $\mathcal{E}$ is the universal enveloping algebra of a graded Lie algebra π_R. If R is singular, then $\pi^2_R \neq 0$, so the Poincaré-Birkhoff-Witt theorem implies $\operatorname{depth} \mathcal{E} \geq 1$; see [1] for details on π_R. Félix *et al.* [4] pioneered the use of $\operatorname{depth} \mathcal{E}$ in the study of the structure of $\mathcal{E}$. We show that this invariant provides also a lot of information on the structure of the k-algebra $\mathcal{S}$.

To describe the structure of $\mathcal{S}$ we use the subset

$$\mathcal{N} = \{\tau \in \mathcal{S} \mid \mathcal{E}^{\geqslant i}\tau = 0 \text{ for some } i \geq 0\}\,.$$

For instance, if $\operatorname{codim} R = 1$, then **4.** shows that $\operatorname{depth} \mathcal{E} = 1$ and $\mathcal{N} = 0$. From the next result a completely different picture emerges 'in general'.

5. If R is a Gorenstein ring and one of the following conditions holds:

(a) $\operatorname{depth} \mathcal{E} \geq 2$; or

(b) $\operatorname{codepth} R \geq 2$, and $\mathcal{E}^{\geqslant 1}$ contains a central non-zero-divisor,

then $\mathcal{N}$ is a two-sided ideal of $\mathcal{S}$, such that

$$\mathcal{S} = \iota(\mathcal{E}) \oplus \mathcal{N} \quad \text{and} \quad \mathcal{N}^2 = 0\,.$$

The theorem applies in many cases. For example, we prove that (a) holds when R is Gorenstein and $\operatorname{codim} R = 3$; when R has minimal multiplicity; when R is a localization of a graded Gorenstein Koszul algebra; or when R is a tensor product of singular Gorenstein algebras over a field. Condition (b) is known to apply to all complete intersection rings R with $\operatorname{codepth} R \geq 2$.

However, there exist examples of Gorenstein rings for which $\operatorname{depth} \mathcal{E} = 1$ and $\mathcal{E}^{\geqslant 1}$ does not contain non-zero central elements. The structure of their stable cohomology algebra is not known at present.

Our results on the structure of the stable cohomology algebra $\mathcal{S} = \widehat{\operatorname{Ext}}_R(k,k)$ for a Gorenstein ring R are similar to—and partly motivated by—results of Benson and Carlson [2] on the structure of the Tate cohomology algebra $\widehat{\mathrm{H}}(G,k)$ for a finite group G. The similarity is rather unexpected, as the cohomology algebra $\mathrm{H}(G,k)$ is always noetherian, while the absolute cohomology algebra $\mathcal{E} = \operatorname{Ext}_R(k,k)$ is noetherian precisely when R is complete intersection.

The structure of the algebra $\mathcal{S}$ when R is not Gorenstein is the subject of work in progress. We have found out that in some cases $\mathcal{S}$ can be described in terms of $\iota(\mathcal{E})$ and $\mathcal{N}$, as in **5.**, but that fundamentally new phenomena also occur.

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An equivalence between the homotopy categories of projectives and of injectives

Srikanth Iyengar

(joint work with Henning Krause)

Let R be a commutative noetherian ring with a dualizing complex D; thus D is a bounded complex of injective R-modules, with $H(D)$ finitely generated, and the natural morphism $R \to \operatorname{Hom}_R(D,D)$ is a homology isomorphism. The starting point of the work described in this talk was the realization that $\mathbf{K}(\operatorname{Proj} R)$ and

$\mathbf{K}(\operatorname{Inj} R)$, the homotopy categories of complexes of projective and injective R-modules, respectively, are equivalent. This equivalence comes about as follows: D consists of injective modules and, R being noetherian, direct sums of injectives are injective, so $D \otimes_R -$ defines a functor from $\mathbf{K}(\operatorname{Proj} R)$ to $\mathbf{K}(\operatorname{Inj} R)$. This functor factors through $\mathbf{K}(\operatorname{Flat} R)$, the homotopy category of flat R-modules, and provides the lower row in the following diagram:

$$\begin{tikzcd} \mathbf{K}(\operatorname{Proj} R) \arrow[r, shift right, "\pi=\mathrm{inc}"'] & \mathbf{K}(\operatorname{Flat} R) \arrow[l, shift right, "\pi_r"'] \arrow[r, shift right, "D\otimes_R -"'] & \mathbf{K}(\operatorname{Inj} R) \arrow[l, shift right, "\operatorname{Hom}_R(D,-)"'] \end{tikzcd}$$

The triangulated structures on the homotopy categories are preserved by π and $D \otimes_R -$. The functors in the upper row of the diagram are the corresponding right adjoints; π_r exists because π preserves coproducts and $\mathbf{K}(\operatorname{Proj} R)$ is compactly generated; the latter property was discovered by Jørgensen [3]. Then one has:

Theorem 1. *The functor* $D \otimes_R - \colon \mathbf{K}(\operatorname{Proj} R) \to \mathbf{K}(\operatorname{Inj} R)$ *is an equivalence of triangulated categories, with quasi-inverse* $\pi_r \circ \operatorname{Hom}_R(D, -)$.

This equivalence is closely related to, and may be viewed as an extension of, Grothendieck's duality theorem for $\mathbf{D}^f(R)$, the derived category of complexes whose homology is bounded and finitely generated. To see this connection one has to consider the commutative diagram of functors:

$$\begin{tikzcd} \mathbf{K}^c(\operatorname{Proj} R) \arrow[r, "D\otimes_R -"] \arrow[d, "\mathsf{P}"', "\simeq"] & \mathbf{K}^c(\operatorname{Inj} R) \arrow[d, "\simeq"', "\mathsf{Q}"] \\ \mathbf{D}^f(R) \arrow[r, "\mathbf{R}\operatorname{Hom}_R(-,D)"] & \mathbf{D}^f(R) \end{tikzcd}$$

where the top row consists of the compact objects in $\mathbf{K}(\operatorname{Proj} R)$ and $\mathbf{K}(\operatorname{Inj} R)$, respectively. The functor P is the composition of $\operatorname{Hom}_R(-, R) \colon \mathbf{K}(\operatorname{Proj} R) \to \mathbf{K}(R)$ with the canonical functor $\mathbf{K}(R) \to \mathbf{D}(R)$; it is a theorem of Jørgensen [3] that P is an equivalence of categories. The functor Q is induced by $\mathbf{K}(R) \to \mathbf{D}(R)$, and Krause [4] proves that it is an equivalence. Given these descriptions it is not hard to verify that $D \otimes_R -$ preserves compactness; this explains the top row of the diagram. Now, Theorem 1 implies that the $D \otimes_R -$ restricts to an equivalence between compact objects, so the diagram above implies $\mathbf{R}\operatorname{Hom}_R(-, D)$ is an equivalence; this is one form of the duality theorem; *cf.* Hartshorne [2]. Conversely, given that $\mathbf{R}\operatorname{Hom}_R(-, D)$ is an equivalence, it follows that the top row of the diagram is an equivalence; this is the crux of the proof of Theorem 1.

We develop Theorem 1 in two directions. The first one deals with the difference between $\mathbf{K}_{\mathrm{ac}}(\operatorname{Proj} R)$, the category of acyclic complexes in $\mathbf{K}(\operatorname{Proj} R)$, and $\mathbf{K}_{\mathrm{tac}}(\operatorname{Proj} R)$, its subcategory of totally acyclic complexes. We consider also the injective counterparts. The main new result in this context is summarized in:

Theorem 2. *The quotient triangulated categories* $\mathbf{K}_{\mathrm{ac}}(\operatorname{Proj} R)/\mathbf{K}_{\mathrm{tac}}(\operatorname{Proj} R)$ *and* $\mathbf{K}_{\mathrm{ac}}(\operatorname{Inj} R)/\mathbf{K}_{\mathrm{tac}}(\operatorname{Inj} R)$ *are compactly generated. The compact objects in each of these categories are equivalent to* $\operatorname{Thick}(R, D)/\operatorname{Thick}(R)$, *up to direct factors.*

The quotient $\operatorname{Thick}(R,D)/\operatorname{Thick}(R)$ is a subcategory of $\mathbf{D}^f(R)/\operatorname{Thick}(R)$, the stable category of R. Since D has finite projective dimension if and only if R is Gorenstein, we deduce: R is Gorenstein if and only if every acyclic complex of projectives is totally acyclic, if and only if every acyclic complex of injectives is totally acyclic. An interesting feature of Theorem 2 is, that it draws our attention to the (monogenic) category $\operatorname{Thick}(R,D)/\operatorname{Thick}(R)$ as a measure of the failure of a ring R from being Gorenstein. Its role is thus analogous to that of the full stable category with regards to regularity: $\mathbf{D}^f(R)/\operatorname{Thick}(R)$ is trivial if and only if R is regular. This observation, and others of this ilk, suggest that $\operatorname{Thick}(R,D)/\operatorname{Thick}(R)$ is an object worth investigating.

Next we turn to the functors induced on $\mathbf{D}(R)$ by the ones in Theorem 1. This involves two different realizations of the derived category as a subcategory of $\mathbf{K}(R)$, both obtained from the localization functor $\mathbf{K}(R) \to \mathbf{D}(R)$: one by restricting it to K-proj(R) the subcategory of K-projective complexes, and the other by restricting it to K-inj(R), the subcategory of K-injective complexes. The inclusion K-proj$(R) \subseteq \mathbf{K}(\operatorname{Proj} R)$ admits a right adjoint $\mathbf{p}$, the inclusion K-inj$(R) \subseteq \mathbf{K}(\operatorname{Proj} R)$ admits a left adjoint $\mathbf{i}$, and one obtains a diagram

$$\begin{tikzcd}
\mathbf{K}(\operatorname{Proj} R) \arrow[rrr, shift left, leftarrow, "\pi_r \circ \operatorname{Hom}_R(D,-)"] \arrow[rrr, shift right, "D\otimes_R -"'] \arrow[rrr, phantom, "\simeq"] \arrow[d, shift left, "\mathbf{p}"] & & & \mathbf{K}(\operatorname{Inj} R) \arrow[d, shift right, "\mathbf{i}"'] \\
\text{K-proj}(R) \arrow[u, shift left] \arrow[rrr, shift left, leftarrow, "\mathsf{F}"] \arrow[rrr, shift right, "\mathsf{G}"'] & & & \text{K-inj}(R) \arrow[u, shift right]
\end{tikzcd}$$

where G is $\mathbf{i} \circ (D \otimes_R -)$ restricted to K-proj(R), and F is $\mathbf{p} \circ \pi_r \circ \operatorname{Hom}_R(D,-)$ restricted to K-inj(R). It follows that (G,F) is an adjoint pair of functors. However, the equivalence in the upper row of the diagram does not imply an equivalence in the lower one. Indeed, using Theorem 1, we prove:

The natural morphism $X \to \mathsf{FG}(X)$ is an isomorphism if and only if the mapping cone of the morphism $(D \otimes_R X) \to \mathbf{i}(D \otimes_R X)$ is totally acyclic.

The point being that the mapping cones of resolutions are, in general, only acyclic. Complexes in K-inj(R) for which the morphism $\mathsf{GF}(Y) \to Y$ is an isomorphism can be characterized in a similar fashion. This is the key observation that allows us to describe the subcategories of K-proj(R) and K-inj(R) where the functors G and F restrict to equivalences. A further extension of these results, when translated to the derived category, reads:

Theorem 3. *A complex X of R-modules has finite G-projective dimension if and only if the morphism $X \to \mathbf{R}\operatorname{Hom}_R(D, D \otimes^{\mathbf{L}}_R X)$ in $\mathbf{D}(R)$ is an isomorphism and $H(D \otimes^{\mathbf{L}}_R X)$ is bounded on the left.*

This theorem, together with its counterpart for G-injective dimensions, recovers recent results of Christensen, Frankild, and Holm [1], who arrived at them from another route. In the talk I focused on commutative rings. However, the results carry over, with suitable modifications in the statements and with nearly identical proofs, to non-commutative rings that possess dualizing complexes. The details are given in our article, which we intend to post on the Math arXiv shortly; I am writing this on 26th February, 2005.

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Algebras derived equivalent to self-injective algebras

Jeremy Rickard

This talk describes some work from the recent PhD thesis of my student Salah Al-Nofayee. [1]

Recall that two algebras A and B over a field k are said to be *derived equivalent* if the derived categories $D(A - \mathrm{Mod})$ and $D(B - \mathrm{Mod})$ of the module categories of A and B are equivalent as triangulated categories.

Many properties are preserved under derived equivalence. Here is one example that we proved some time ago.

Theorem 1 ([2], Corollary 5.3). *A finite-dimensional algebra derived equivalent to a symmetric algebra is itself symmetric.*

In fact, there is a rather more satisfactory proof than the one that appears there, as symmetric algebras can be characterized by properties of their derived categories.

Theorem 2 ([3], Corollary 3.2). *A finite dimensional algebra A is symmetric if and only if the vector spaces $\mathrm{Hom}(P, M)$ and $\mathrm{Hom}(M, P)$ are naturally dual whenever M and P are objects of $D(A - \mathrm{Mod})$ such that M is isomorphic to a bounded complex of finitely generated modules and P is perfect (i.e., isomorphic to a bounded complex of finitely generated projective modules).*

For some time, the corresponding statement for self-injective algebras has been open. Recently, in his PhD thesis, it was proved by Salah Al-Nofayee.

Theorem 3 (Al-Nofayee, [1]). *A finite-dimensional algebra derived equivalent to a self-injective algebra is itself self-injective.*

The proof uses a result of Saorín and Zimmermann-Huisgen on rigidity of tilting complexes [4], stating that for a given finite sequence $\{P_i, i \in \mathbb{Z}\}$ of finitely generated projective modules for a finite dimensional algebra, there are, up to isomorphism, only a finite number of tilting complexes of the form

$$\cdots \to P_1 \to P_0 \to P_{-1} \to \ldots.$$

In fact, Saorín and Zimmermann-Huisgen prove this for algebras over an algebraically closed field, but it is easy to deduce from this the statement for general fields.

Using this result, one can show that if T is a tilting complex for a self-injective algebra A, then there is some power ν_A^t of the Nakayama functor $\nu_A = ? \otimes_A DA$ for which

$$\nu_A^t(T) \cong T.$$

This implies that if B is derived equivalent to A, and therefore isomorphic to the endomorphism algebra of some tilting complex for A, then some power $(\mathbf{L}\nu_B)^t$ of the left derived functor of the Nakayama functor takes projective modules to projective modules. From this one can prove by reverse induction on t that this is true for *all* powers of $\mathbf{L}\nu_B$, and in particular, an injective cogenerator $\nu_B(B) = DB$ is projective, and so B is self-injective.

Unfortunately there seems to be no simple property of the derived category that characterizes the self-injective algebras, as there is in the case of symmetric algebras.

With the help of this theorem, Al-Nofayee also generalized a theorem [3, Theorem 5.1] that characterizes the sets of objects in the derived category of a symmetric algebra that correspond to the simple modules under some derived equivalence. For a self-injective algebra, his necessary and sufficient conditions for such a set $\{X_1, \ldots, X_n\}$ of objects of $D^b(A - \mathrm{mod})$ are:

- $\mathrm{Hom}(X_i, X_j[t]) = 0$ for all $1 \leq i, j \leq n$ and $t < 0$.
- $\mathrm{Hom}(X_i, X_j) = 0$ for $i \neq j$, and $\mathrm{End}(X_i)$ is a division ring for every i.
- $X_1, \ldots, X_n$ generate $D^b(A - \mathrm{mod})$ as a triangulated category.
- The set $\{X_1, \ldots, X_n\}$ is closed (up to isomorphism) under the Nakayama functor ν_A.

The last condition is automatic for symmetric algebras, since then the Nakayama functor is isomorphic to the identity functor, but for non-symmetric algebras there are simple examples that show that the first three conditions are not sufficient.

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The Gabriel-Serre category, the Tate-Vogel category, and Koszul duality

Alex Martsinkovsky
(joint work with Roberto Martínez-Villa)

The BGG correspondence [1] establishes an equivalence

$$\underline{\mathrm{gr}}(\wedge^{\bullet} V^{n+1}) \simeq D^b(\mathrm{coh}(P^n))$$

between the stable category of finitely generated graded modules over the exterior algebra on $n+1$ letters and the bounded derived category of coherent sheaves on the n-dimensional projective space. Using Koszul duality, this result can be significantly generalized [2]. In this paper, we provide a more transparent and less technically involved proof of that result. Our approach is based on Koszul duality and universal constructions.

Let Λ be a finite-dimensional graded algebra such that Λ_1 is a semisimple Λ_0-module. The Yoneda algebra of Λ will be denoted Γ. It is naturally graded by the cohomological degree. Let $\mathrm{gr}(\Lambda)$ be the category of finitely generated graded Λ-modules. Our first goal is to produce, for an arbitrary $M \in \mathrm{gr}(\Lambda)$, a complex of finitely generated projective graded Γ-modules. This was done in [2]. We shall now review that construction. Starting with the multiplication map $\Lambda_1 \otimes_{\Lambda_0} M_n \to M_{n+1}$ and applying the functor $D(-) := \mathrm{Hom}_{\Lambda_0}(-, \Lambda_0)$, we have a homomorphism of Γ_0-modules $D(M_{n+1}) \to D(M_n) \otimes_{\Gamma_0} \Gamma_1$. To this map, we can apply the functors $- \otimes_{\Gamma_0} \Gamma_l$, which results, for each n, in a degree zero homomorphism of graded Γ-modules

$$d_{n+1} : D(M)_{-n-1} \otimes_{\Gamma_0} \Gamma \to D(M)_{-n} \otimes_{\Gamma_0} \Gamma[1].$$

Lemma. $d^2 = 0$.

Thus the above construction yields a (linear) complex of projectives in $\mathrm{gr}(\Gamma)$. In fact, the construction is functorial and we have a contravariant functor

$$\lambda : \mathrm{gr}(\Lambda) \to \mathcal{LCP}^b(\mathrm{gr}(\Gamma)),$$

where the target is the category of bounded linear complexes of finitely generated projective graded Γ-modules.

Proposition. *If Λ is quadratic, then λ is a duality. If λ is Koszul, then M is (a shift of) a Koszul module if and only if $\lambda(M)$ is exact at the non-minimal degrees (i. e., $\lambda(M)$ is a (shifted) projective graded resolution of the Γ-module Koszul-dual to M).*

Composing λ with the tautological functor $\mathcal{LCP}^b(\mathrm{gr}(\Gamma)) \to D^b(\mathrm{gr}(\Gamma))$, we have a functor

$$\gamma : \mathrm{gr}(\Lambda) \to D^b(\mathrm{gr}(\Gamma)).$$

Taking the Verdier quotient of the target category by the subcategory of all complexes isomorphic to finite complexes of graded modules of finite length and identifying the result with the bounded derived category $D^b(\mathrm{Qgr}\,(\Gamma))$ of finitely generated graded Γ-modules modulo modules of finite length, we have a composite functor

$$\pi : D^b(\mathrm{gr}(\Gamma)) \to D^b(\mathrm{Qgr}\,(\Gamma)).$$

If $M \in \mathrm{gr}(\Lambda)$ is projective, then $\lambda(M)$ is semisimple and $\pi\gamma(M)$ is a zero object in $D^b(\mathrm{Qgr}\,(\Gamma))$. Therefore the composition $\pi\gamma$ factors through the stable category $\underline{\mathrm{gr}}(\Lambda)$. On that category, the syzygy operation Ω becomes an endofunctor. Applying $\pi\gamma$ to the short exact sequence $0 \to \Omega M \to P \to M \to 0$, where P is projective, we see that $\pi\gamma(\Omega M) \simeq \pi\gamma(M)[-1]$. In other words, $\pi\gamma$ "inverts" Ω and therefore factors, in a unique way, through the Tate-Vogel category $v(\mathrm{gr}(\Lambda))$.[1] The above observations are codified in the following commutative diagram:

$$\begin{tikzcd}
\mathrm{gr}(\Lambda) \arrow[r, "\gamma"] \arrow[d] & D^b(\mathrm{gr}(\Gamma)) \arrow[dd, "\pi"'] \arrow[dr] & \\
\underline{\mathrm{gr}}(\Lambda) \arrow[d] \arrow[dr, dashed] & & D^b(\mathrm{gr}(\Gamma))/\mathrm{ffl}(\mathrm{gr}(\Gamma)) \arrow[dl] \\
v(\mathrm{gr}(\Lambda)) \arrow[r, dashed, "\theta"] & D^b(\mathrm{Qgr}\,(\Gamma)) &
\end{tikzcd}$$

We can now state our main result.

Theorem. *Suppose Λ is a finite-dimensional Koszul algebra such that Λ_1 is a semisimple Λ_0-module and the Yoneda algebra Γ is noetherian. Then the functor θ is a (contravariant) equivalence of triangulated categories.*

If Λ is an exterior algebra, then we recover the BGG correspondence.

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[1]This category is a universal solution to the problem of inverting an endofunctor, in our case, Ω.

Graded and Koszul categories

Roberto Martínez-Villa
(joint work with Øyvind Solberg)

1. Graded categories

Let K be a field, and let $\mathcal{C}$ be an additive K-category. We say $\mathcal{C}$ is *graded* if for each pair of objects, C and D we have a decomposition $\operatorname{Hom}_{\mathcal{C}}(C,D) = \bigoplus_{i\in\mathbb{Z}} \operatorname{Hom}_{\mathcal{C}}(C,D)_i$ as $\mathbb{Z}$-graded K-vector spaces and if f is in $\operatorname{Hom}_{\mathcal{C}}(C,C')_i$ and g is in $\operatorname{Hom}_{\mathcal{C}}(C',D)_j$, then gf is in $\operatorname{Hom}_{\mathcal{C}}(C,D)_{i+j}$. In particular, the identity maps are in degree zero.

Examples. (1) *Let $\Lambda = \bigoplus_{i\geq 0} \Lambda_i$ be a positively graded K-algebra. Denote by $\operatorname{Gr}(\Lambda)_0$ the category of graded modules and degree zero maps, and by $\operatorname{Gr}(\Lambda)$ the category of graded modules and maps $\operatorname{Hom}_{\operatorname{Gr}(\Lambda)}(M,N) = \bigoplus_{i\in\mathbb{Z}} \operatorname{Hom}_{\operatorname{Gr}(\Lambda)_0}(M,N[i])$. Then $\operatorname{Gr}(\Lambda)$ is a graded*

(2) *Let $\mathcal{C}$ be an additive K-category and denote by $\operatorname{rad}\mathcal{C}$ the radical of $\mathcal{C}$. Then the associated graded category $\mathcal{A}_{\mathrm{gr}}(\mathcal{C})$ has the same objects as $\mathcal{C}$ and maps $\operatorname{Hom}_{\mathcal{A}_{\mathrm{gr}}(\mathcal{C})}(C,D) = \bigoplus_{i\geq 0} \operatorname{rad}^i(C,D)/\operatorname{rad}^{i+1}(C,D)$.*

(3) *Let $\mathcal{C}$ be an abelian K-category with enough projective (injective) objects. The Yoneda or Ext-category $E(\mathcal{C})$ has the same objects as $\mathcal{C}$ and maps $\operatorname{Hom}_{E(\mathcal{C})}(A,B) = \bigoplus_{k\geq 0} \operatorname{Ext}^k_{\mathcal{C}}(A,B)$.*

2. Functors between graded K-categories

Let $\mathcal{C}$ and $\mathcal{D}$ be two graded K-categories. A contravariant functor $F\colon \mathcal{C} \to \mathcal{D}$ is a functor of graded categories if it is a functor such that it induces a degree zero homomorphism of K-vector spaces $F\colon \operatorname{Hom}_{\mathcal{C}}(C,D) \to \operatorname{Hom}_{\mathcal{D}}(F(C),F(D))$.

Example. *Let $\mathcal{C}$ be a graded K-category. For an object C in $\mathcal{C}$ the representable functors $\operatorname{Hom}_{\mathcal{C}}(C,-)\colon \mathcal{C} \to \operatorname{Gr}(K)$ and $\operatorname{Hom}_{\mathcal{C}}(-,C)\colon \mathcal{C}^{\mathrm{op}} \to \operatorname{Gr}(K)$ are functors of graded categories.*

Denote by $\operatorname{Gr}(\mathcal{C})_0$ the category with objects the functors of graded categories $F\colon \mathcal{C}^{\mathrm{op}} \to \operatorname{Gr}(K)$ and morphisms the natural transformations $\eta\colon F \to G$ with each $\eta_C\colon F(C) \to G(C)$ a degree zero morphism. This is an abelian category.

Let $\operatorname{Gr}(\mathcal{C})$ be the category with the same objects as $\operatorname{Gr}(\mathcal{C})_0$ and maps $\operatorname{Hom}_{\operatorname{Gr}(\mathcal{C})}(F,G) = \bigoplus_{i\in\mathbb{Z}} \operatorname{Hom}_{\operatorname{Gr}(\mathcal{C})_0}(F,G[i])$. The category $\operatorname{Gr}(\mathcal{C})$ is a graded K-category.

3. Weakly Koszul and Koszul categories

Let $\mathcal{C}$ be a graded K-category. We say that $\mathcal{C}$ is *generated in degree zero and one*, if it is positively graded, that is: $\operatorname{Hom}_{\mathcal{C}}(C,D) = \bigoplus_{i\geq 0} \operatorname{Hom}_{\mathcal{C}}(C,D)_i$ and for any triple of objects A, B and C and for $i,j \geq 0$ the maps $\operatorname{Hom}_{\mathcal{C}}(A,C)_i \times \operatorname{Hom}_{\mathcal{C}}(C,B)_j \to \operatorname{Hom}_{\mathcal{C}}(A,B)_{i+j}$ given by $(f,g) \mapsto gf$ are onto.

Definition 1. A functor F in $\mathrm{Gr}(\mathcal{C})_0$ is a *Koszul functor* if there exists an exact sequence of graded functors and degree zero maps

$$\cdots \to \mathrm{Hom}_{\mathcal{C}}(-,C_k)[-k] \to \cdots \to \mathrm{Hom}_{\mathcal{C}}(-,C_2)[-2] \\ \to \mathrm{Hom}_{\mathcal{C}}(-,C_1)[-1] \to \mathrm{Hom}_{\mathcal{C}}(-,C_0) \to F \to 0$$

Definition 2. Let $\mathcal{C}$ be a Krull-Schmidt category, then the simple functors $\mathcal{C}^{\mathrm{op}} \to \mathrm{Mod}\, K$ are of the form $S_C = \mathrm{Hom}_{\mathcal{C}}(-,C)/\operatorname{rad}(-,C)$ with C indecomposable.

Assume that $\mathcal{C}$ is graded and generated in degrees zero and one, then it is *Koszul* if every graded simple object $S_C\colon \mathcal{C}^{\mathrm{op}} \to \mathrm{Gr}(K)$ is Koszul.

Definition 3. Let $\mathcal{C}$ be a Krull-Schmidt K-category (not necessarily graded).

(1) A functor $F\colon \mathcal{C}^{\mathrm{op}} \to \mathrm{Mod}\, K$ is *weakly Koszul* if it has a minimal projective resolution $\cdots \to P_k \to P_{k-1} \to \cdots \to P_1 \to P_0 \to F \to 0$ with P_i finitely generated and $\operatorname{rad}^{i+1}(P_j) \cap \Omega^{j+1}(G) = \operatorname{rad}^i(\Omega^{j+1}(G))$ for $j \geq 0$ and $i \geq 1$.
(2) If every simple functor in $\mathrm{mod}(\mathcal{C})$ is weakly Koszul, then $\mathcal{C}$ is *weakly Koszul.*

The results for weakly Koszul algebras obtained in [4, 5] extend to weakly Koszul categories.

4. Applications of Koszul categories to the representation theory of finite dimensional algebras

Let Λ be a finite dimensional K-algebra, and denote by $\operatorname{ind}\Lambda$ the category of indecomposable finitely generated modules. The category $\mathcal{A}_{\mathrm{gr}}(\operatorname{ind}\Lambda)$ has the same objects as $\operatorname{ind}\Lambda$ and maps $\mathrm{Hom}_{\mathcal{A}_{\mathrm{gr}}(\operatorname{ind}\Lambda)}(X,Y) = \bigoplus_{i\geq 0} \operatorname{rad}^i(X,Y)/\operatorname{rad}^{i+1}(X,Y)$.

The objects in $\operatorname{ind}\Lambda$ decompose as a disjoint union $\cup_{\sigma\in\Sigma}\mathcal{C}_\sigma$, where $\mathcal{C}_\sigma$ are Auslander-Reiten components. The category $\mathcal{A}_{\mathrm{gr}}(\operatorname{ind}\Lambda)$ is a disjoint union $\cup_{\sigma\in\Sigma}\mathcal{A}_{\mathrm{gr}}(\mathcal{C}_\sigma)$ of categories. Hence, $\mathrm{Gr}(\mathcal{A}_{\mathrm{gr}}(\operatorname{ind}\Lambda)) = \prod_{\sigma\in\Sigma}\mathrm{Gr}(\mathcal{A}_{\mathrm{gr}}(\mathcal{C}_\sigma))$. The categories $\mathrm{Gr}(\mathcal{A}_{\mathrm{gr}}(\operatorname{ind}\Lambda))$ and each $\mathrm{Gr}(\mathcal{A}_{\mathrm{gr}}(\mathcal{C}_\sigma))$ have global dimension 2.

We obtain generalizations of results given by the first author and results related to the hereditary categories with Serre duality studied by D. Happel, H. Lenzing, I. Reiten and M. Van den Bergh.

Theorem 1. (a) *The category* $\operatorname{ind}\Lambda$ *is weakly Koszul.*

(b) *The category* $\mathcal{A}_{\mathrm{gr}}(\operatorname{ind}\Lambda)$ *is Koszul, in particular each* $\mathcal{A}_{\mathrm{gr}}(\mathcal{C}_\sigma)$ *is Koszul.*
(c) *Denote by* $\mathrm{Fin}(\mathcal{A}_{\mathrm{gr}}(\mathcal{C}_\sigma))$ *the full subcategory of* $\mathrm{Gr}(\mathcal{A}_{\mathrm{gr}}(\mathcal{C}_\sigma))$ *of all functors whose minimal projective resolutions consist of finitely generated projective functors. Then for each F in* $\mathrm{Fin}(\mathcal{A}_{\mathrm{gr}}(\mathcal{C}_\sigma))$ *there exists a subfunctor G of F such that some shift $G[i]$ is Koszul and F/G is of finite length.*
(d) *Any simple S_C with C indecomposable non-projective satisfies the Gorenstein condition, that is;*
   (i) $\mathrm{Hom}(S_C, \mathrm{Hom}_{\mathcal{A}_{\mathrm{gr}}(\mathcal{C}_\sigma)}(-,X[n])) = 0$ *for all X and n.*
   (ii) $\mathrm{Ext}^1_{\mathrm{Gr}(\mathcal{A}_{\mathrm{gr}}(\mathcal{C}_\sigma))}(S_C, \mathrm{Hom}_{\mathcal{A}_{\mathrm{gr}}(\mathcal{C}_\sigma)}(-,X[n])) = 0$ *for all X and n.*
   (iii) $\mathrm{Ext}^2_{\mathrm{Gr}(\mathcal{A}_{\mathrm{gr}}(\mathcal{C}_\sigma))}(S_C, \mathrm{Hom}_{\mathcal{A}_{\mathrm{gr}}(\mathcal{C}_\sigma)}(-,X[n])) = \overline{S}_{\tau C}[n+2](X)$, *where* $\overline{S}_{\tau C} = \mathrm{Hom}_{\mathcal{A}_{\mathrm{gr}}(\mathcal{C}_\sigma)}(\tau C,-)/\operatorname{rad}(\tau C,-)$ *and τC is the Auslander-Reiten translation of C.*

Theorem 2. *Let $\mathcal{C}$ be a regular Auslander-Reiten component of a finite dimensional algebra Λ and $E(S(\mathcal{C}))$ the associated* Ext*-category. Then the following statements are true.*

(a) *$E(S(\mathcal{C}))$ is a Frobenius category of radical cube zero.*

(b) *The categories $E(S(\mathcal{C}))/\operatorname{soc} E(S(\mathcal{C}))$ and $\mathcal{C}^{\mathrm{op}}/\operatorname{rad}^2$ are equivalent and $\operatorname{Gr}(\mathcal{C}^{\mathrm{op}}/\operatorname{rad}^2)$ is stably equivalent to $\operatorname{Gr}(\mathcal{S})$, where $\operatorname{Gr}(\mathcal{S})$ decomposes as a product of sections $\operatorname{Gr}(\mathcal{S}) = \prod_j \operatorname{Gr}(\mathcal{S}_j) \times \operatorname{Gr}(\mathcal{S}_j^{\mathrm{op}})$ and each $\mathcal{S}_j$ is a hereditary category such that $\mathcal{S}_j$ and $\mathcal{S}_i$ have the same quiver Q but $\mathcal{S}_j$ and $\mathcal{S}_j^{\mathrm{op}}$ have opposite quivers.*

(c) *If the quiver Q of $\mathcal{S}_j$ is finite, then $\mathcal{S}_j$ is of infinite representation type.*

Theorem 3. *Let $\mathcal{C}$ be a regular Auslander-Reiten component of a finite dimensional algebra Λ. Assume the quiver Q of the sections $\mathcal{S}_j$ of $E(S(\mathcal{C}))$ is infinite and is not of type $\mathbb{A}_\infty$, $\mathbb{D}_\infty$, or $\mathbb{A}_\infty^\infty$.*

(a) *Then any finitely presented functor F in $\operatorname{gr}(\mathcal{A}_{\mathrm{gr}}(\mathcal{C}))$ is either of finite length or it has infinite Gelfand-Krillov dimension.*

(b) *The category of finitely presented functors $\operatorname{gr}(\mathcal{A}_{\mathrm{gr}}(\mathcal{C}))$ is not noetherian.*

(c) *If $E(S(\mathcal{C}))$ has sections of type $\mathbb{A}_\infty$, $\mathbb{D}_\infty$ or $\mathbb{A}_\infty^\infty$, then it is noetherian of Gelfand-Krillov dimension 2.*

Our last theorem is the following.

Theorem 4. *Let $\mathcal{C}$ be a regular Auslander-Reiten component of a finite dimensional algebra Λ. Assume that $E(S(\mathcal{C}))$ has sections of type $\mathbb{A}_\infty$, $\mathbb{D}_\infty$ or $\mathbb{A}_\infty^\infty$. Then the quotient category of the finitely presented functors modulo the functors of finite length, $\operatorname{Qgr}(\mathcal{A}_{\mathrm{gr}}(\mathcal{C}))$, is hereditary and noetherian with Serre duality.*

If the sections of $E(S(\mathcal{C}))$ are not infinite of type $\mathbb{A}_\infty$, $\mathbb{D}_\infty$ or $\mathbb{A}_\infty^\infty$, then $\operatorname{Qgr}(\mathcal{A}_{\mathrm{gr}}(\mathcal{C}))$ is not noetherian.

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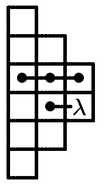
A remark by M. C. R. Butler on subgroup embeddings

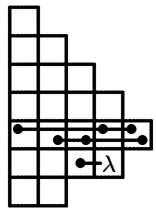
Markus Schmidmeier

An object in the submodule category $\mathcal{S}(\Lambda)$ is a pair $M = (M_0; M_1)$ which consists of a finitely generated Λ-module M_0 together with a Λ-submodule M_1 of M_0. A morphism $f : M \to N$ in $\mathcal{S}(\Lambda)$ is given by a Λ-linear map $f : M_0 \to N_0$ which preserves the submodules, that is, $f(M_1) \subseteq N_1$ holds. In this abstract, Λ usually will be a commutative local uniserial ring; we will call Λ uniserial for short. The radical factor field will be denoted by k and t will be a radical generator (thus $\Lambda/\langle t\rangle = k$).

We have two special cases in mind: In the first case, Λ is the ring $\mathbb{Z}/\langle p^n\rangle$ where p is a prime. Then we are dealing with the category of all possible embeddings of a subgroup in a p^n-bounded finite abelian group. The classification problem for the objects in $\mathcal{S}(\mathbb{Z}/\langle p^n\rangle)$ was raised by Birkhoff [1] in 1934. In the second case, Λ is the factor ring $k[T]/\langle T^n\rangle$ of the polynomial ring one variable T over the field k. Then we consider the possible invariant subspaces of a nilpotent operator: The objects in $\mathcal{S}(k[T]/\langle T^n\rangle)$ may be written as triples (V, ϕ, U), where V is a k-space, $\phi : V \to V$ is a k-linear transformation with $\phi^n = 0$ and U is a subspace of V with $\phi(U) \subseteq U$.

The type $t(B)$ of a finite length Λ-module B is the partition $\mu = (\mu_1, \ldots, \mu_t)$ such that $B \cong \bigoplus_{i=1}^t \Lambda/\langle t^{\mu_i}\rangle$. Thus the pair $(t(B); t(A))$ is an isomorphism invariant of a submodule embedding $(B; A)$. Birkhoff showed that the number of isomorphism classes of subgroup embeddings $(B; A) \in \mathcal{S}(\mathbb{Z}/\langle p^6\rangle)$ with $t(B) = (6, 4, 2)$ and $t(A) = (4, 2)$ tends to infinity with p; namely, for each value $0 < \lambda < p$, the following embeddings are pairwise nonisomorphic. The group B is generated by elements x, y, z of order p^6, p^4, p^2, respectively, and the subgroup A_λ is given by the generators $p^2x + py + z$ and $p^2y + p\lambda z$ of order p^4 and p^2, as pictured below.

$(A_\lambda \subseteq B)$: 

$(C_\lambda \subseteq D)$: 

Is this family $(B; A_\lambda)$ of pairwise nonisomorphic subgroup embeddings the first family which occurs? We know from [2] that the category $\mathcal{S}(\mathbb{Z}/\langle p^5\rangle)$ has finite type, and hence in Birkhoff's example the exponent of the big group, which is p^6, is minimal. However, the exponent of the subgroup, which is p^4, is not minimal, as the above examples of embeddings $(D; C_\lambda)$ in $\mathcal{S}(\mathbb{Z}/\langle p^7\rangle)$ shows, where this exponent is p^3.

For Λ a uniserial ring of length n, and $m \leq n$, let $\mathcal{S}_m(\Lambda)$ be the full subcategory of $\mathcal{S}(\Lambda)$ of all pairs $(B; A)$ where $t^m A = 0$. For each pair (n, m) where $m \leq n$, the representation type of the category $\mathcal{S}_m(k[T]/\langle T^n\rangle)$ has been determined in [5], see also [4] for several finite cases. Recall that the categories $\mathcal{S}_3(k[T]/\langle T^6\rangle)$ and $\mathcal{S}_2(k[T]/\langle T^7\rangle)$ have finite type, in fact, all categories of type $\mathcal{S}_2(k[T]/\langle T^n\rangle)$ are representation finite. It follows that in case $\Lambda = k[T]/\langle T^n\rangle$, the above two families are minimal in the following sense: If we fix the exponent of the submodule (or the big module) then the exponent of the big module (the submodule, respectively,) is as small as possible.

In the classical case where $\Lambda = \mathbb{Z}/\langle p^n\rangle$, the results in [5] per se do not answer the question whether or not the above two families are minimal. This is the point of Butler's remark. In fact, it is not surprising that the special case $\Lambda = k[T]/\langle T^n\rangle$ is better understood, since in this case many powerful techniques are available (in particular covering theory). In the following we describe two example classes where the representation theory is independent of the underlying commutative uniserial ring Λ. Our last theorem can be used to answer the question in the positive.

Controlled wildness

Let $\mathcal{A}$ be an additive category and $\mathcal{C}$ a class of objects (or a full subcategory) in $\mathcal{A}$. Given objects A, A' in $\mathcal{A}$, we will write $\operatorname{Hom}(A, A')_{\mathcal{C}}$ for the set of maps $A \to A'$ which factor through a (finite) direct sum of objects in $\mathcal{C}$. Here we attach to $\mathcal{C}$ the ideal $\langle\mathcal{C}\rangle$ in $\mathcal{A}$ generated by the identity morphisms of the objects in $\mathcal{C}$. The same convention will apply to a single object C in $\mathcal{A}$: We denote by $\operatorname{Hom}(A, A')_C$ the set of maps $A \to A'$ which factor through a (finite) direct sum of copies of C. Given an ideal $\mathcal{I}$ of $\mathcal{A}$, we write $\mathcal{A}/\mathcal{I}$ for the corresponding factor category, as usual. It has the same objects as $\mathcal{A}$ and for any two objects A, A' of $\mathcal{A}$, the group $\operatorname{Hom}_{\mathcal{A}/\mathcal{I}}(A, A')$ is defined as $\operatorname{Hom}_{\mathcal{A}}(A, A')/\mathcal{I}(A, A')$. In particular, the category $\mathcal{A}/\langle\mathcal{C}\rangle$ has the same objects as $\mathcal{A}$ and $\operatorname{Hom}_{\mathcal{A}/\langle\mathcal{C}\rangle}(A, A') = \operatorname{Hom}_{\mathcal{A}}(A, A')/\operatorname{Hom}(A, A')_{\mathcal{C}}$.

Definition. We say that $\mathcal{A}$ is *controlled k-wild* provided there are full subcategories $\mathcal{C} \subseteq \mathcal{B} \subseteq \mathcal{A}$ such that $\mathcal{B}/\langle\mathcal{C}\rangle$ is equivalent to $\operatorname{mod} k\langle X, Y\rangle$ where $k\langle X, Y\rangle$ is the free k-algebra in two generators. We will call $\mathcal{C}$ the *control class*, and in case $\mathcal{C}$ is given by a single object C then this object C will be the *control object.*

Theorem 1. ([3, Theorem 2]) *Let Λ be a uniserial ring of length $n \geq 7$ and let k be its radical factor. Then the category $\mathcal{S}_4(\Lambda)$ is controlled k-wild.*

Auslander-Reiten quivers in the representation finite case

For $\mathcal{P}$ a (finite) poset, let $\operatorname{sub}_\Lambda \mathcal{P}$ denote the category of Λ-linear subspace representations of $\mathcal{P}$. For example, if $\mathcal{P}$ is the one point poset then $\operatorname{sub}_\Lambda \mathcal{P} = \mathcal{S}(\Lambda)$. We construct Auslander-Reiten sequences which are not split exact in each component.

Notation. For X a Λ-module and S a subset of $\mathcal{P}$ denote by X^S the representation which has the space X in each component labelled by a point in S and which is zero otherwise.

Suppose that $0 \to X \to Y \to Z \to 0$ is an Auslander-Reiten sequence in $\operatorname{mod} \Lambda$ and $i \in \mathcal{P}$ and let $X \to E$ be the injective envelope for X. Then there is an Auslander-Reiten sequence in the category $\operatorname{sub}_\Lambda \mathcal{P}$ of the following type.

$$(*) \qquad 0 \to X^{\leq i} \oplus E^{\not\leq i} \to Y^{=i} \oplus X^{<i} \oplus E^{\not\leq i} \oplus Z^{>i} \to Z^{\geq i} \to 0.$$

This sequence is split exact in each component different from the i-th.

Lemma. *Each other Auslander-Reiten sequence $0 \to A \to B \to C \to 0$ in $\operatorname{sub}_\Lambda \mathcal{P}$ is split exact in each component, that is, for each $i \in \mathcal{P}$, $B_i = A_i \oplus C_i$ holds.*

Corollary. *If Λ is a uniserial ring then the type detects:*

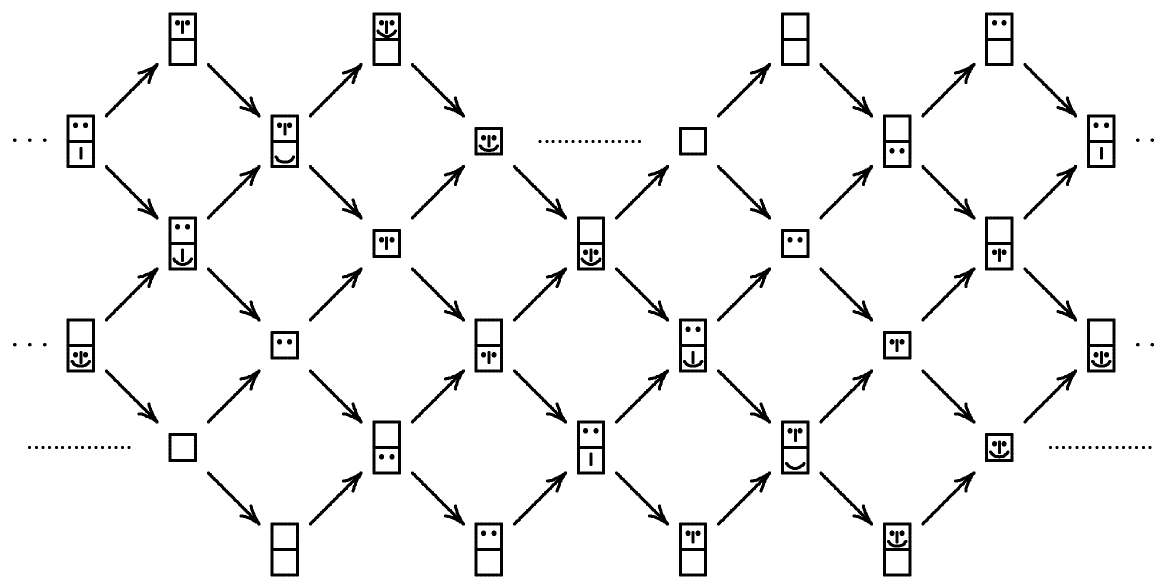
- *projective modules in $\operatorname{sub}_\Lambda \mathcal{P}$ and their radicals,*
- *injective modules in $\operatorname{sub}_\Lambda \mathcal{P}$ and the end term of their source maps, and*
- *starting terms and end terms of AR-sequences in $\operatorname{sub}_\Lambda \mathcal{P}$ of type $(*)$.*

Theorem 2. *Suppose Λ, Δ are commutative uniserial rings of the same length, $\mathcal{C}$ and $\mathcal{D}$ are connected components of their Auslander-Reiten quiver and $\mathcal{K}$ and $\mathcal{L}$ are slices in $\mathcal{C}$ and $\mathcal{D}$, respectively. Suppose*

(1) *$\mathcal{K}$ and $\mathcal{L}$ are isomorphic as graphs,*
(2) *their points correspond to objects of the same type and*
(3) *certain objects in $\mathcal{K}$ are determined uniquely by their type.*

Then the connected components $\mathcal{C}$ and $\mathcal{D}$ are isomorphic as graphs and condition (2) holds for all points.

Example. *Let $\mathcal{P}$ be the chain of three points and Λ any uniserial ring of length 2. We obtain the following Auslander-Reiten quiver. In fact, the AR-sequences starting at a module with simple total space are exactly the sequences of type $(*)$; these sequences form a slice.*



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On the derived category of coherent sheaves on an irreducible projective curve of arithmetic genus one

Igor Burban

(joint work with Bernd Kreußler)

In my talk based on a joint work with B. Kreußler I am going to discuss various properties of the bounded derived category of coherent sheaves on a singular Weierstrass cubic curve.

Singular Weierstrass curves are irreducible one-dimensional Calabi-Yau manifolds and the study of their derived category is important from the point of view of the homological mirror symmetry [6], applications to F-theory [5] and to the theory of the Yang-Baxter equation [7].

A classification of the indecomposable objects of the derived category of coherent sheaves on a smooth elliptic curve follows from the classification of vector bundles of Atiyah [1]. The main difference in the case of a singular Weierstrass curve is that the homological dimension of the category of coherent sheaves is infinite, and hence there are indecomposable complexes with an arbitrary number of non-zero cohomologies, see [3]. In the singular case there are indecomposable vector bundles and torsion free sheaves which are not semi-stable and there are indecomposable sheaves which are neither torsion sheaves nor torsion free sheaves. The indecomposable objects of the derived category of a nodal cubic curve were described in [3], in [4] the Fourier-Mukai transform on Weierstrass curves was studied. One of the goals of my talk is to compare common features and to point out main differences between the derived category of a smooth and a singular Weierstrass cubic curve.

Let $\mathbf{E}$ be a a Calabi-Yau curve, i.e. a curve with trivial canonical bundle $\omega_{\mathbf{E}} = \mathcal{O}$ and let $\mathcal{E}$ be a spherical object of $D^b(\mathrm{Coh}_{\mathbf{E}})$, i.e. a perfect complex such that

$$\mathrm{Hom}(\mathcal{E}, \mathcal{E}[i]) = \begin{cases} \mathbf{k} & \text{if } \ i = 0, 1 \\ 0 & \text{otherwise.} \end{cases}$$

It was shown by Polishchuk [7] that one can associate to a pair $(\mathbf{E}, \mathcal{E})$ a solution of the classical Yang-Baxter equation over the Lie algebra $\mathfrak{g} = \mathfrak{sl}(n)$. Motivated by this application the following two conjectures were posed [7]:

1. Let $\mathbf{E}$ be a Calabi-Yau curve, $\mathcal{E}$ be a spherical object, then there exists $\mathbb{F} \in \mathrm{Aut}(D^b(\mathrm{Coh}_{\mathbf{E}}))$ such that $\mathcal{E} = \mathbb{F}(\mathcal{O}_{\mathbf{E}})$;
2. The group $\mathrm{Aut}(D^b(\mathrm{Coh}_{\mathbf{E}}))$ is generated by $\mathrm{Aut}(\mathbf{E})$, $\mathrm{Pic}^0(\mathbf{E})$ and tubular mutations.

I use the technique of Harder-Narasimhan filtrations in triangulated categories [2] to prove these conjectures in the case of Weierstrass cubic curves.

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Parabolic group actions and tilting modules

Lutz Hille

1. Parabolic group actions

Let k be an algebraically closed field and $V_0 = \{0\} \subset V_1 \subset V_2 \subset \ldots V_{t-1} \subset V_t$ be a flag of finite dimensional vector spaces with $d_i := \dim V_i - \dim V_{i-1}$. The stabiliser of this flag is a parabolic subgroup in $\mathrm{GL}(V_t)$ denoted by $P(d)$. It is also the group of invertible elements in $\mathrm{End}_{k\mathbb{A}_t}(\bigoplus P(i)^{d_i})$, where the modules $P(i)$ are the indecomposable projective modules over the path algebra $k\mathbb{A}_t$ of a directed quiver of type $\mathbb{A}_t$. Let further I be a subset in $\{(i,j) \mid 1 \leq i < j \leq t\}$ so that for all (i,j) in I also $(i,j+1)$, for $j < t$, and $(i-1,j)$, for $i > 1$, are both in I. The set I can be seen as a root ideal in the positive roots of the root system of type $\mathbb{A}_t$ or simply as a subset closed under taking right and upper neighbours. To such a subset we associate a function $h : \{1,\ldots,t\} \longrightarrow \{0,\ldots,t-1\}$ defined by $h(j) := \max\{i \mid (i,j) \in I\}$ if such an i exists and $h(j) = 0$ otherwise. Using this notation we can define the group $P(d)$ and several Lie algebras in $P(d)$.

$$\begin{aligned} P(d) &:= \{f \in \mathrm{Aut}(V_t) \mid f(V_i) \subseteq V_i\} \\ \mathfrak{p}_u(d) &:= \{f \in \mathrm{End}(V_t) \mid f(V_i) \subseteq V_{i-1}\} \\ \mathfrak{n}(I,d) &:= \{f \in \mathrm{End}(V_t) \mid f(V_i) \subseteq V_{h(i)}\} \\ \mathfrak{p}_u(d)^{(l)} &:= \{f \in \mathrm{End}(V_t) \mid f(V_i) \subseteq V_{i-1-l}\} \end{aligned}$$

Main Question: When does $P(d)$ act with a dense orbit on $\mathfrak{n}(I,d)$ and $\mathfrak{p}_u(d)^{(l)}$?

Example 1. a) *By a classical result of Richardson, it is well-known that $P(d)$ acts always with a dense orbit on $\mathfrak{p}_u(d)$.*

b) *Let I be the set generated by $\{(1,2),(3,4),(5,6)\}$ and $d = (1,1,1,1,1,1)$, where $t = 6$, then $P(d)$ does not act wit a dense orbit on $\mathfrak{n}(I,d)$.*

c) *Let $l = 1$, $t = 9$ and $d = (2,1,2,2,1,2,2,1,2)$, then $P(d) \subset \mathrm{GL}_{15}$ does not act with a dense orbit on $\mathfrak{p}_u(d)^{(1)}$.*

The aim of the talk is to present an equivalent problem to the questions above for the existence of certain modules without self extensions over certain subalgebras of the Auslander algebra of $k[T]/T^t$. Moreover, we present several partial results concerning the question above.

2. Subalgebras of the Auslander algebra of $k[T]/T^t$

To the ideals defined above one can associate certain subquotient algebras of the Auslander algebra of $k[T]/T^t$. We denote by $\mathcal{A}_t$ the algebra $\mathrm{End}(\bigoplus_{i=1}^t k[T]/T^i)$ (it is the Auslander algebra of $k[T]/T^t$). The quiver of this algebra consists of t vertices and arrows α_i (corresponding to the inclusion of $k[T]/T^i$ into $k[T]/T^{i+1}$) and of arrows β_i (corresponding to the projection of $k[T]/T^{i+1}$ onto $k[T]/T^i$. The subquotient algebras $\mathcal{A}(I)$ and $\mathcal{A}_{t,l}$ corresponding to the ideals $\mathfrak{n}(I,d)$ and $\mathfrak{p}_u(d)^{(l)}$ have arrows α_i and γ_j consisting of certain compositions of the arrows β_i: the algebra $\mathcal{A}_{t,l}$ has arrows α_i (starting in i and ending in $i+1$) for $i = 1, \ldots, t-1$ and arrows γ_j (starting in $j+l+1$ and ending in i) for $j = 1, \ldots, t-1-l$ defined as $\gamma_j := \beta_j\beta_{j+1}\ldots\beta_{j+l}$. The algebra $\mathcal{A}(I)$ has arrows α_i (starting in i and ending in $i+1$) for $i = 1, \ldots, t-1$ and arrows γ_j (starting in $h(j)$ and ending in j) for all pairs $(h(j), j)$ in I with $(h(j)+1, j+1) \in I$) defined as $\gamma_j := \beta_j\beta_{j+1}\ldots\beta_{h(j)}$ (see [1] for details).

Note that the constructions coincide in the special cases when $I = \{(i,j) \mid 1 \leq i < j - l \leq t - l\}$.

Theorem 2.1. *The algebras $\mathcal{A}(I)$ are quasi-hereditary and the category of Δ–good modules coincides with the set of all modules M satisfying one of the following equivalent conditions*

a) *the maps $M(\alpha_i)$ are injective,*
b) *the projective dimension of M is at most 1, and*
c) *the restriction of M to the subalgebra generated by α_i (it is the path algebra of a directed quiver of type $\mathbb{A}_t$) is projective.*

It is proven in [6, 2] that the orbits for the $P(d)$ action are in bijection with the isomorphism classes of modules over the corresponding algebra $\mathcal{A}(I)$.

3. Richardson's result and tilting modules
(joint work with T. Brüstle, C. M. Ringel, and G. Röhrle)

Richardson's result (see Example 1,a), [7]) implies that for each dimension vector e with $e_i - e_{i-1} \geq 0$ there exists precisely one good module $M(e)$ of dimension vector e without self extensions (it is not indecomposable in general). We construct this module explicitly (so we also construct all indecomposable modules without self extensions explicitly). Moreover, we can use an analogous construction to get so called *standard* modules over the algebra $\mathcal{A}_{t,l}$ and also over the algebra $\mathcal{A}(I)$, which also do not have self extensions. For $l = 1$ and $t \geq 6$ there exist modules without self extensions which are not standard (see Example 2).

Let P be the largest indecomposable finite dimensional projective $\mathcal{A}_t$–module (it is the projective cover of the simple module $S(t)$). Note that P has all finite dimensional indecomposable projective modules $P(i)$ as a submodule, they are generated by one element say $p(i)$ in P. Let $A = \{a_1, \ldots, a_r\}$ for $1 \leq a_1 < a_2 < \ldots < a_r \leq t$ be an ordered set of natural numbers. For each such set A we define a unique submodule $\Delta(A)$ of P which is Δ–good as the module generated by the elements $\alpha^{a_i - i}(p(i))$ (where $\alpha^{a_i - i}$ denotes the composition of $a_i - i$ arrows α_i, so that $\alpha^{a_i-i}(p(i))$ makes sense).

Theorem 3.1. *An indecomposable Δ–good module without self extensions is isomorphic to $\Delta(A)$ for some subset A. Each basic tilting module (note that a module of projective dimension at most one is already Δ–filtered) is isomorphic to the direct sum $\bigoplus_{i=1}^{t} \Delta(\sigma(1), \ldots, \sigma(i))$ for some element σ in the symmetric group S_t.*

4. A reduction theorem

The classification of all pairs (I, d), so that $P(d)$ acts with a dense orbit on $\mathfrak{p}_u(I, d)$ seems to be more difficult than the classification of all pairs (t, l), so that $P(d)$ acts with a dense orbit on $\mathfrak{p}_u(d)^{(l)}$. In this part we claim, that both classifications are equivalent. One direction is obvious, so we concentrate on the non-obvious one. We show two results. First, if we allow d to have entries 0, then one can show, that $\mathfrak{p}_u(I, d)$ is isomorphic (together with the group action) to some $\mathfrak{p}_u(\overline{d})^{(l)}$ (where $\overline{d}$ is a certain dimension vector obtained from d by filling in some zeros) by [5], Theorem 1.4.2. Using [5], Theorem 1.4.1 we can even replace $\overline{d}$ by a dimension vector $\underline{d}$ without an entry zero, so that $P(\underline{d})$ acts with a dense orbit on $\mathfrak{p}_u(\underline{d})^{(l)}$ precisely when $P(\overline{d})$ acts with a dense orbit on $\mathfrak{p}_u(\overline{d})^{(l)}$.

5. Actions of the Borel subgroup
(joint work with S. Goodwin)

In this section we consider the special case, when $d_i \leq 1$ (for simplicity we allow $d_i = 0$, instead of working with the subset I, however, both approaches are equivalent by Section 4). If we consider an ideal $\mathfrak{b}_u^{(1)} \subseteq \mathfrak{n} \subseteq \mathfrak{b}_u$, then we can ask whether B acts with a dense orbit (it corresponds to a good module over $\mathcal{A}_{t,1}$ without self extensions and dimension vector e with $e_i - e_{i-1} \leq 1$) and whether this orbit is indecomposable (it corresponds to an indecomposable good module). A dimension vector d as above consists of certain strings of entries 1 of some length, say $a_0, \ldots, a_r$ with $a_i > 0$. The strings not in the beginning and the end are called *intermediate*, so the length of the intermediate strings is $a_1, \ldots, a_{r-1}$.

Theorem 5.1. *Let d be a dimension vector with $d_i \leq 1$. Then B acts with a dense orbit on $\mathfrak{p}_u(d)^{(1)}$ precisely when one of the following conditions is satisfied:*

a) *if $d_i = 1$ for some i, then $d_{i-1} = 0$ and $d_{i+1} = 0$ (d is* standard*) or*
b) *there is at most one intermediate string of entries 1 of even length.*

The orbit above is indecomposable precisely when d is either standard or there exists precisely one intermediate string of even length.

Example 2.

a) *The dimension vector in Example 1,b) is* $(1,0,1,1,0,1,1,0,1)$*, so it is the minimal dimension vector d, so that B does not act with a dense orbit on* $\mathfrak{p}_u(d)^{(1)}$.

b) *The minimal non-standard dimension vector, so that B acts with a dense orbit and the orbit is indecomposable is* $(1,0,1,1,0,1)$.

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Block representation type for groups and Lie algebras

Rolf Farnsteiner

(joint work with Andrzej Skowroński and Detlef Voigt)

Group Algebras. Let k be an algebraically closed field of characteristic $p > 0$. Throughout, all algebras and modules are assumed to be finite dimensional. An associative k-algebra Λ decomposes into a direct sum $\Lambda = \mathcal{B}_1 \oplus \mathcal{B}_2 \oplus \cdots \oplus \mathcal{B}_s$ of two-sided ideals, that are indecomposable associative k-algebras. The relevance of this *block decomposition* for representation theory was first observed by Brauer and Nesbitt in their study of non-semisimple group algebras of finite groups.

Because of these historical origins, results on group algebras have often served as a paradigm for other classes of algebras, such as reduced enveloping algebras of restricted Lie algebras or distribution algebras of infinitesimal group schemes. In my talk, I will compare the representation theories of finite groups and restricted Lie algebras, focusing on the notion of representation type. In retrospect, most phenomena characteristic of infinitesimal group schemes already occur at the level of restricted Lie algebras [2, 3, 4].

Let me begin by collecting some of the methods and results from the modular representation theory of finite groups. We fix a finite group G, and recall that the unique block $\mathcal{B}_0(G) \subset k[G]$ containing the trivial $k[G]$-module k is the *principal block.*

Mackey Decomposition. If $H \subset G$ is a subgroup and M is an H-module, then

$$k[G] \otimes_{k[H]} M|_H \cong \bigoplus_{HgH} k[H] \otimes_{k[H \cap gHg^{-1}]} M^g.$$

In particular, M is always a direct summand of the restriction of the induced module. Mackey's result leads to the important notion of the defect: Each block $\mathcal{B} \subset k[G]$ gives rise to a p-subgroup $D_{\mathcal{B}} \subset G$ that measures the complexity of $\mathcal{B}$. Since the defect group of $\mathcal{B}_0(G)$ is a Sylow-p-subgroup, it is the most complicated block of $k[G]$.

The aforementioned facts together with Brauer correspondence imply that representation type behaves well under passage from the principal block to other blocks, or from a group to a subgroup.

Reduced Enveloping Algebras. Let $(\mathfrak{g}, [\,,\,])$ be a Lie algebra, $B \subset \mathfrak{g}$ a basis. If for every element $x \in B$ the p-th power of the inner derivation $\operatorname{ad} x : \mathfrak{g} \longrightarrow \mathfrak{g} \ ; \ y \mapsto [x, y]$ is again inner, then a theorem by Jacobson ensures the existence of a map $[p] : \mathfrak{g} \longrightarrow \mathfrak{g} \ ; \ x \mapsto x^{[p]}$ that enjoys the basic properties of the p-power operator of an associative algebra. In particular, we have

$$(\operatorname{ad} x)^p = \operatorname{ad} x^{[p]} \qquad \forall\, x \in \mathfrak{g}.$$

The pair $(\mathfrak{g}, [p])$ is then referred to as a *restricted Lie algebra*.

In the 1970's Kac and Weisfeiler noticed that much of the representation theory of $\mathfrak{g}$, or equivalently that of its universal enveloping algebra $U(\mathfrak{g})$, is captured by an algebraic family of $(U_\chi(\mathfrak{g}))_{\chi \in \mathfrak{g}^*}$ of associative algebras of dimension $p^{\dim \mathfrak{g}}$. The study of this family has since been one of the focal points in the representation theory of modular Lie algebras. By definition, we have $U_\chi(\mathfrak{g}) := U(\mathfrak{g})/I_\chi$, where $I_\chi := (\{x^p - x^{[p]} - \chi(x)^p 1 \ ; \ x \in \mathfrak{g}\})$. The algebra $U_\chi(\mathfrak{g})$ is a Frobenius algebra, though in general not symmetric. Contrary to finite groups, the Cartan matrix of $U_\chi(\mathfrak{g})$ may be singular. The example of the Steinberg module shows that one cannot expect to have good control of the composition of induction and restriction in the sense of Mackey. By analogy with finite groups, special attention is given to the principal block $\mathcal{B}_0(\mathfrak{g}) \subset U_0(\mathfrak{g})$. In a similar vein, the algebra $U_0(\mathfrak{g})$, being located at the generic point of the family, is thought of as the most complicated member of the family.

To this date, the most promising replacement of a defect appears to be given by Carlson's concepts of support varieties and rank varieties, that were transferred to our context by Friedlander-Parshall [7]. Let $\mathcal{V}_\mathfrak{g} := \{x \in \mathfrak{g} \ ; \ x^{[p]} = 0\}$ be the *nullcone* of $\mathfrak{g}$. Given a $U_\chi(\mathfrak{g})$-module M the *rank variety* $\mathcal{V}_\mathfrak{g}(M)$ is defined via

$$\mathcal{V}_\mathfrak{g}(M) := \{x \in \mathcal{V}_\mathfrak{g} \ ; \ M|_{U_{\chi|_{kx}}(kx)} \text{ is not free}\} \cup \{0\}.$$

If $\mathcal{B} \subset U_\chi(\mathfrak{g})$ is a block with simple modules $S_1, \ldots, S_n$, then we put

$$\mathcal{V}_\mathcal{B} := \bigcup_{i=1}^n \mathcal{V}_\mathfrak{g}(S_i) \subset \mathcal{V}_\mathfrak{g}.$$

This is our replacement of a defect. Again, $\mathcal{V}_{\mathcal{B}} \subset \mathcal{V}_{\mathcal{B}_0(\mathfrak{g})} = \mathcal{V}_{\mathfrak{g}}$, so that $\mathcal{B}_0(\mathfrak{g})$ has the largest defect.

Facts. Let $\mathcal{B} \subset U_\chi(\mathfrak{g})$ be a block.

(1) $\mathcal{B}$ is representation-finite if and only if $\dim \mathcal{V}_{\mathcal{B}} \leq 1$.

(2) If $\mathcal{B}$ is tame (and representation-infinite), then $\dim \mathcal{V}_{\mathcal{B}} = 2$.

From now on we assume that $p \geq 3$. In the early eighties, Drozd, Rudakov and Fischer independently showed that $\mathcal{B}_0(\mathfrak{sl}(2))$ is Morita equivalent to the trivial extension of the Kronecker algebra. It turns out that for Lie algebras $\mathfrak{g} = \mathrm{Lie}(G)$ of algebraic groups, all tame blocks of $U_0(\mathfrak{g})$ are of this type [1].

Examples. We consider the Lie algebra $\mathfrak{g} := \mathfrak{sl}(2) \oplus kz$, where $[z, \mathfrak{sl}(2)] = (0)$. Using the standard basis $\{e, h, f\} \subset \mathfrak{sl}(2)$, we introduce two p-maps on $\mathfrak{g}$:

(1) The algebra $\mathfrak{sl}(2)_n$ is defined via $e^{[p]} = 0$; $h^{[p]} = h$; $f^{[p]} = z$; $z^{[p]} = 0$.

(2) The algebra $\mathfrak{sl}(2)_s$ is defined via $e^{[p]} = 0$; $h^{[p]} = h + z$; $f^{[p]} = 0$; $z^{[p]} = 0$.

Let $C(\mathfrak{g}) := \{x \in \mathfrak{g} \ ; \ [x, \mathfrak{g}] = (0)\}$ be the *center* of $\mathfrak{g}$. By general theory, we have a "Fitting decomposition"

$$(*) \qquad C(\mathfrak{g}) = \mathfrak{t} \oplus \mathfrak{u}$$

of $C(\mathfrak{g})$ into its toral and unipotent parts. Here is a recognition criterion for tameness:

Theorem ([2]). *Let $\mathfrak{g}$ be a restricted Lie algebra.*

(1) *Then $\mathcal{B}_0(\mathfrak{g})$ is tame if and only if $\mathfrak{g}/C(\mathfrak{g})^{[p]} \cong \mathfrak{sl}(2)$, $\mathfrak{sl}(2)_s$.*

(2) *If $\mathcal{B}_0(\mathfrak{g})$ is tame and $C(\mathfrak{g})$ is unipotent or toral, then $U_0(\mathfrak{g})$ is tame.*

In particular, the block $\mathcal{B}_0(\mathfrak{sl}(2)_n)$ is wild, while the algebra $U_0(\mathfrak{sl}(2)_s)$ is tame. Moreover, $\mathfrak{h} := ke \oplus kz$ is a p-subalgebra of $\subset \mathfrak{sl}(2)_s$ with $U_0(\mathfrak{h}) \cong k[X, Y]/(X^p, Y^p)$. Thus, $U_0(\mathfrak{h}) \subset U_0(\mathfrak{sl}(2)_s)$ is wild, while $U_0(\mathfrak{sl}(2)_s)$ is tame.

Using rank varieties and schemes of tori one first shows that $\mathfrak{g}/C(\mathfrak{g}) \cong \mathfrak{sl}(2)$, with $\mathfrak{u} \subset C(\mathfrak{g})$ being generated by one element [5, 6]. Let P be a principal indecomposable $U_0(\mathfrak{g})$-module, $\mathcal{B} \subset U_0(\mathfrak{g})$ the block belonging to P, and set $H_P := \mathrm{Rad}(P)/\mathrm{Rad}^3(P)$.

Proposition 1 ([2]). *The block $\mathcal{B}$ is tame if and only if H_P is decomposable.*

Filtrations by Verma modules and Auslander-Reiten Theory then yield the list of decomposable hearts. Let me illustrate one technical aspect. By general theory, the central extension $\mathfrak{g}$ is given by a p-semilinear map $\psi : \mathfrak{sl}(2) \longrightarrow C(\mathfrak{g})$. The decomposition $(*)$ of $C(\mathfrak{g})$ provides a p-semilinear map $\psi_{\mathfrak{t}} : \mathfrak{sl}(2) \longrightarrow \mathfrak{t}$. One then has

$$U_0(\mathfrak{g}) \cong \bigoplus_{\gamma \in X(\mathfrak{t})} U_{\chi_\gamma}(\mathfrak{g}/\mathfrak{t}),$$

where $X(\mathfrak{t})$ is the character group of $\mathfrak{t}$, and $\chi_\gamma(x+u)^p = \gamma(\psi_{\mathfrak{t}}(x)) \quad \forall\, x \in \mathfrak{sl}(2)$, $u \in \mathfrak{u}$.

The map ψ also gives rise to a p-semilinear form $\hat{\psi} : \mathfrak{sl}(2) \longrightarrow C(\mathfrak{g})/C(\mathfrak{g})^{[p]} \subset k$. For $\chi \in \mathfrak{sl}(2)^* \subset \mathfrak{g}^*$, we define

$$d(\psi, \chi) := \dim \mathcal{V}_{\mathfrak{sl}(2)} \cap \ker \hat{\psi} \cap \ker \chi.$$

A linear form $\chi \in \mathfrak{sl}(2)^*$ is *nilpotent* if it corresponds via the Cartan-Killing form to a nonzero nilpotent element of $\mathfrak{sl}(2)$.

Proposition 2 ([2]). *Let $\mathfrak{g}$ be a central extension of $\mathfrak{sl}(2)$ with $\hat{\psi} \neq 0$.*

(1) *If $C(\mathfrak{g})$ is unipotent, χ is nilpotent, and $d(\psi, \chi) \neq 0$, then $U_\chi(\mathfrak{g})$ is wild.*

(2) *If $d(\psi, \chi_\gamma) \neq 0$ for some nilpotent χ_γ, then $U_0(\mathfrak{g})$ possesses a wild block.*

Examples. (1) Let $\chi \in \mathfrak{sl}(2)_s^*$ be defined via $\chi(e) = 0 = \chi(h)$; $\chi(f) = 1$; $\chi(z) = 0$. Then $U_0(\mathfrak{sl}(2)_s)$ is tame, while $U_\chi(\mathfrak{sl}(2)_s)$ is wild.

(2) Let $\mathfrak{g} := \mathfrak{sl}(2) \oplus kz \oplus kt$, $[kz \oplus kt, \mathfrak{g}] = (0)$, $e^{[p]} = 0$; $h^{[p]} = h + z$; $f^{[p]} = t$; $z^{[p]} = 0$; $t^{[p]} = t$. Then $\mathcal{B}_0(\mathfrak{g})$ is tame, while $U_0(\mathfrak{g})$ is wild.

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Infinitesimal deformations of derived categories

Bernhard Keller

(joint work with Christof Geiß)

According to Kontsevich-Soibelman [3, section 2.1], *cf.* also [1], the shifted Hochschild complex $C(A, A)[1]$ of a differential graded algebra A over a field of characteristic 0 is the 'moduli space of A_∞-categories'. We propose to interpret this statement to the effect that the differential graded Lie algebra $C(A, A)[1]$ should control the deformations of the derived Morita class [8] [2] [9] of A, or, in more sloppy terms, the deformations of the derived category $\mathcal{D}A$. In particular, one expects a canonical bijection between the second Hochschild cohomology $\mathrm{HH}^2(A, A)$ and the equivalence classes of infinitesimal deformations of $\mathcal{D}A$. We show that such a bijection does indeed exist in many cases, notably if A itself has right bounded homology. In the general case, we obtain a bijection between the equivalence classes of Morita deformations of A and the 2-cocycles which act nilpotently in the graded endomorphism ring of each perfect object over A. Our proof starts from the observation that a Hochschild 2-cocycle c naturally gives rise to a deformation $A_c[\varepsilon]$ of A in the category of *curved A_∞-algebras* and that the (flat) derived category of $A_c[\varepsilon]$ admits a compact generator: the lift to $A_c[\varepsilon]$ of the cone over the graded endomorphism of the free module A induced by c. The links of these results with Lowen-Van den Bergh's deformation theory for abelian categories [6] [7] [5] [4] remain to be elucidated.

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Reducing cohomology by split pairs

Steffen König

(joint work with Luca Diracca)

In order to compare cohomology in two abelian categories, and in particular to show non-vanishing of certain cohomology, the following situation is studied:

Let $\mathcal{A}$ and $\mathcal{B}$ be two additive categories. A pair (F,G) of additive functors $F\colon \mathcal{A} \to \mathcal{B}$ and $G\colon \mathcal{B} \to \mathcal{A}$ is a split pair of functors *(between $\mathcal{A}$ and $\mathcal{B}$) if the composition $F \circ G$ is an autoequivalence of the category $\mathcal{B}$. If the categories are equipped with exact structures, and if the two functors are exact with respect to these exact structures, the split pair is called an* exact split pair of functors *(between $\mathcal{A}$ and $\mathcal{B}$).*

An exact split pair on abelian level induces a split pair on derived level; hence cohomology can be compared.

Easy examples of exact split pairs (A,B) are:

Split quotients: B is a *split quotient* of A, if B is a subring of A (via an embedding ε sending the unit of B to that of A) and there exists a surjective homomorphism $\pi\colon A \to B$, such that the composition $\pi \circ \varepsilon$ is the identity on B.

Morita equivalences.

Corner rings eAe, provided Ae is projective over eAe.

A more general class of examples is the following:

Let A be a ring, e an idempotent, and B a split quotient of eAe (viewed as a subring of eAe). Then we call B a corner split quotient *if there is a left A- and right eAe-module S, which is projective as a right B-module (via the embedding of B into eAe) and which satisfies $eS \simeq B$ as left B-modules.*

Up to composition with certain Morita equivalences, every exact split pair between module categories is a corner split quotient.

Applications include a proof of some cases of the strong no loops conjecture, and results relating Brauer algebras with various symmetric groups in the context of [2].

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Special evening session: Calabi-Yau phenomena

On 10 February, 2005, a special session on Calabi-Yau phenomena was organised.

Besides the talks of Helmut Lenzing on weighted projective spaces of Calabi-Yau type, of Ragnar-Olaf Buchweitz presenting a theorem by Bogomolov–Tian–Todorov and a simplification of the proof by Z. Ran, and of Christof Geiß on a generalisation of triangulated categories (so called 4-angulated categories) and their Calabi-Yau dimensions, the following three talks were given:

Calabi-Yau varieties and reflexive polytopes

Lutz Hille

1. Calabi-Yau varieties in $\mathbb{P}^n$

Let k be the field of complex numbers and $\mathbb{P}^n$ the projective n-space over k. The anticanonical sheaf ω^{-1} is isomorphic to $\mathcal{O}(n+1)$, and we can identify a global section in ω^{-1} with a homogeneous polynomial of degree $n+1$ in the $n+1$ variables $x_0, \dots, x_n$.

a) Let $n = 2$, then a generic polynomial f of degree 3 defines an elliptic curve E in $\mathbb{P}^2$.

b) Let $n = 3$, then a generic polynomial of degree 4 defines a K3-surface in $\mathbb{P}^4$.

c) Let $n = 4$, then a generic polynomial of degree 5 defines a 3–dimensional Calabi-Yau variety in $\mathbb{P}^5$.

All these varieties X are Calabi-Yau varieties (see definition below), in particular, $\omega_X \simeq \mathcal{O}$ (the canonical sheaf is trivial) and the Serre duality is of the form $\mathrm{Ext}^l(\mathcal{F},\mathcal{G}) \simeq \mathrm{Ext}^{n-l}(\mathcal{G},\mathcal{F})^*$.

We can also (using the action of the torus k^n on $\mathbb{P}^n$) identify the space of polynomials of degree n with all formal linear combinations of elements in a lattice polytope $\Delta(n)$. (The elements in $\Delta(n)$ correspond to a torus invariant basis of the space of homogeneous polynomials of degree $n+1$ in $n+1$ variables, for the torus action $(\lambda_1,\dots,\lambda_n)(x_0,x_1,\dots,x_n) := (x_0,\lambda_1 x_1,\dots,\lambda_n x_n)$ this basis consists just of the monomials.) So we get

$$\Delta(n) := \{a \in \mathbb{Z}^{n+1} \mid \sum_{i=1} a_i = n+1, a_i \geq 0\},$$

a simplex in the lattice $\mathbb{Z}^{n+1}$. This is a polytope which has precisely one inner lattice point (a lattice point not on the boundary of $\Delta(n)$), it is $(1,1,\dots,1)$.

On the projective n–space, there exists a sequence of line bundles $\mathcal{O},\mathcal{O}(1),\dots,$ $\mathcal{O}(n)$ without any self extensions ($\mathrm{Ext}^l_{\mathbb{P}^n}(\mathcal{O}(i),\mathcal{O}(j)) = 0$ for all $0 \leq i,j \leq n$ and all l) generating the derived category of coherent sheaves on $\mathbb{P}^n$. Classical results on the derived category of coherent sheaves on $\mathbb{P}^n$ allow us to describe it using derived categories of modules over the endomorphism ring of $\bigoplus_{i=0}^n \mathcal{O}(i)$.

2. Calabi-Yau varieties

Definition. A *Calabi-Yau variety* X is a smooth projective variety satisfying

(1) $\omega_X \simeq \mathcal{O}$ (the canonical sheaf is trivial), and
(2) $\mathrm{H}^l(X;\mathcal{O}_X) = 0$ for all $1 \leq l \leq \dim X - 1$.

The definition above can be generalised, sometimes one only wants X to be complete, and in dimension greater or equal to 4, one often allows some mild singularities. Calabi-Yau varieties can be constructed in Fano varieties, we explain the construction in more detail below.

Example.

a) *Let $X \subset \mathbb{P}^n$ be a hyper surface defined by a generic homogeneous polynomial of degree $n+1$ (as in section 1), then X is a Calabi-Yau variety.*

b) *Let F be a smooth Fano variety satisfying $\mathrm{H}^l(F;\mathcal{O}_F) = 0$ for all $1 \leq l \leq \dim X$. Take a generic element f in $\mathrm{H}^0(F;\omega_F^{-1})$, then the hyper surface X defined by f is a Calabi-Yau variety. Condition* 1) *follows from the adjunction formula and condition* 2) *from the long exact cohomology sequence applied to*

$$0 \longrightarrow \omega_F \longrightarrow \mathcal{O}_F \longrightarrow \mathcal{O}_X \longrightarrow 0.$$

To find Calabi-Yau varieties, we need to find Fano varieties F satisfying the condition $\mathrm{H}^l(F,\mathcal{O}_F) = 0$ for all $1 \leq l \leq \dim X$. The conditions on F can be chosen weaker at several places. E.g., it is sufficient that F has only isolated singularities (a generic section does not meet these singularities), and one can also take partial resolutions $\widetilde{F}$ of singular Fano varieties F satisfying $\omega_{\widetilde{F}} \simeq \mathcal{O}_{\widetilde{F}}$.

There exists a large class of those varieties that can be constructed using so-called reflexive polytopes, the class of toric (possibly singular) Fano varieties (see [5]).

3. Reflexive polytopes

Definition. Let M be a lattice in $M_{\mathbb{R}} \simeq \mathbb{R}^n$. A *lattice polytope* Δ in $M_{\mathbb{R}}$ is the convex hull in $M_{\mathbb{R}}$ of a finite number of lattice points (that is points in M). We assume $\dim \Delta = n$ and 0 be an interior lattice point of Δ. The polytope Δ is *reflexive* if its dual polytope

$$\Delta^\circ := \{n \in M_{\mathbb{R}}^* \mid n(m) \geq -1 \quad \forall m \in \Delta\}$$

is also a lattice polytope. A lattice polytope is *smooth* if for each vertex v the cone spanned by $\Delta - v$ (we shift the polytope so that v becomes the zero point and consider the cone with apex in 0 generated by the shifted elements in Δ) is generated by a $\mathbb{Z}$–basis of $M_{\mathbb{R}}$.

To each lattice polytope Δ one can associate a toric variety F_Δ. If Δ is smooth, then F_Δ is smooth, and if Δ is reflexive, then F_Δ is a Fano variety. Conversely, each toric Fano variety also comes from a reflexive polytope, the sections in ω_F^{-1} form a reflexive polytope (similar to the example in section 1).

Let Δ be a lattice polytope. We define a cone $C(\Delta)$ as the cone with apex in 0 generated by $\Delta \times \{1\} \subset M_{\mathbb{R}} \times \mathbb{R}$. The lattice points $C(\Delta)_{\mathbb{Z}}$ in $C(\Delta)$ form a semi-group, and we consider the semi-group ring $S(\Delta)$ of $C(\Delta)_{\mathbb{Z}}$. It is a graded ring, the degree comes from the additional element, so $\deg(x, a) := a$ for $x \in a\Delta$. Then we define the projective algebraic variety F_Δ as $\mathrm{Proj}(S(\Delta))$. This variety is of dimension n, and it comes with an action of an n–dimensional algebraic torus $T \simeq k^n$, the torus acts with a dense orbit. If we consider the T–action on $\mathrm{H}^0(F_\Delta, \mathcal{O}_{F_\Delta}(1))$ (where $\mathcal{O}_{F_\Delta}(1)$ is taken with respect to the given embedding in $\mathbb{P}^N$, where N is the number of lattice points in Δ), then the T–invariant points form the lattice points of the n–dimensional lattice polytope Δ.

We conclude this section with an overview of the classification of reflexive polytopes.

- $n = 1$: There exists precisely one reflexive simplex, it is the convex hull of -1 and 1 in $\mathbb{R}$.
- $n = 2$: It is an exercise to classify them, there exist precisely 16 reflexive polytopes and 5 of them are smooth. These five smooth ones correspond to the five toric del Pezzo surfaces: $\mathbb{P}^2$, $\mathbb{P}^1 \times \mathbb{P}^1$, and the blow up of $\mathbb{P}^2$ in one, two or three points (the three points must not lie on a common line).
- $n = 3$: A classification of the smooth reflexive polytopes can be found in [10], there exist 18. They can be classified using certain double weighted triangulations of the plane. The classification of all reflexive polytopes is done by a computer, the algorithm can be found in [8], there exist $4,319$ of them (see [11]).
- $n = 4$: The classification of 4–dimensional smooth reflexive polytopes was done by Batyrev in [5], there exist 124 of them. The classification of all reflexive

polytopes is mainly a problem on hard disc space (as one of the authors told me), there exist $473,800,776$ of them (see [9, 11]).

For reflexive simplices the classification is much simpler and consists essentially of the classification of so-called weight systems. These weight systems also appear for weighted projective spaces in the sense of Baer, Geigle and Lenzing ([3]).

4. Quivers and reflexive polytopes

Surprisingly, one can construct some reflexive polytopes using quivers, however the class of these polytopes is not very large (see [1, 6]). On the other hand, a smooth reflexive polytope constructed from a quiver comes always with a sequence of line bundles without any self extension (see [1]). There exist also several other approaches to construct exceptional sequences of line bundles on toric varieties. It is an open problem (see [2, 7]) whether there exists on any smooth toric variety a full strong exceptional sequence of line bundles, (similar to the one on $\mathbb{P}^n$). This problem is even open for toric surfaces.

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Introduction to super potentials

Michel Van den Bergh

Boundary conditions for open strings (branes) form a triangulated category. In the B-model, this triangulated category is the derived category $\mathcal{A}$ of coherent sheaves over a Calabi-Yau manifold [6].

It is often useful to consider triangulated subcategories $\mathcal{B} \subset \mathcal{A}$ which are derived equivalent to $D^b(\text{f.l.}A)$ where A is the (completed) path algebra of a quiver with relations. These are the so-called quiver gauge theories (see e.g. [4]).

A standard example is given by the derived category of the canonical bundle on $\mathbb{P}^2$. This is a non-compact Calabi-Yau. The derived category of sheaves supported on the zero section is equivalent to $D^b(k[[x,y,z]] * (\mathbb{Z}/3\mathbb{Z}))$ (see [5]).

It seems therefore interesting to be able to construct A such that $D^b(\text{f.l.}A)$ is Calabi-Yau. Physicists have a construction of such A in terms of so-called super potentials. It is not clear exactly when this construction works, but if it works, then the resulting algebra is Calabi-Yau of dimension 3.

For notational simplicity ,we will explain the construction in the case that A has only one simple module. The general case is entirely similar.

Put $F = k\langle\langle x_1, \ldots, x_n \rangle\rangle$. For a general monomial $a \in F$, we define the *circular derivative* of a with respect to x_i as

$$\frac{{}^\circ\partial a}{\partial x_i} = \sum_{a = u x_i v} vu.$$

The circular derivative extends to a linear map

$$\frac{{}^\circ\partial}{\partial x_i} : F/[F,F] \to F.$$

The ordinary partial derivative of a with respect to x_i is defined as

$$\frac{\partial a}{\partial x_i} = \sum_{a = u x_i v} u \otimes v.$$

This extends to a linear map

$$\frac{\partial}{\partial x_i} : F \to F \otimes F.$$

It is convenient to write

$$\frac{\partial^2 a}{\partial x_i \partial x_j} = \frac{\partial}{\partial x_j} \frac{{}^\circ\partial a}{\partial x_i},$$

and it is easy to check that

$$\frac{\partial^2 a}{\partial x_i \partial x_j} = \tau \frac{\partial^2 a}{\partial x_j \partial x_i},$$

where $\tau(p \otimes q) = q \otimes p$.

A *super potential* is an element $w \in F/[F,F]$ containing only monomials of degree ≥ 3. Put

$$A = F/I,$$

where I is the twosided ideal topologically generated by

$$r_i = \frac{{}^\circ\partial w}{\partial x_i}.$$

Put $dx_i = x_i \otimes 1 - 1 \otimes x_i$. We consider the following complex of A-bimodules.

$$0 \to A^e \xrightarrow{(\cdot dx_i \cdot)_i} (A^e)^n \xrightarrow{\left(\cdot \frac{\partial^2 w}{\partial x_j \partial x_i} \cdot\right)_{ij}} (A^e)^n \xrightarrow{(\cdot dx_j \cdot)_j} A^e \to A \to 0$$

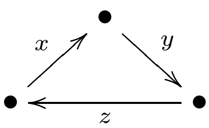
Here A^e is $A\hat{\otimes}A$ equipped with its outer bimodule structure. If this complex is exact then it represents a resolution of A as an A-bimodule.

Furthermore, from the fact that the resolution is self-dual, one may deduce, using standard homological algebra, that $D^b(\text{f.l.}A)$ is indeed Calabi-Yau.

Remark. *I haven't checked the details, but it seems not unlikely that the above construction is reversible and that a* 3*-dimensional Calabi-Yau algebra A is always given by a super potential.*

Remark. *Super potentials form an (infinite dimensional) affine space. This is reminiscent of the smoothness of the moduli-spaces of compact Calabi-Yau manifolds (Tian, Bogomolov, Ran, and Kawamata). See the talk by Ragnar Buchweitz during this evening seminar.*

Unfortunately, the construction does not always work (take the zero super potential). For a generic super potential, one would expect that the construction works if there are enough variables (or arrows in the quiver case), but this is entirely speculative. The following non-example was communicated to me by Berenstein.



$w = xyz$

In this case, it is easy to see that the resulting algebra is self-injective, but not Calabi-Yau.

Cases that are completely understood are, when there are either three or two variables and the degree of w is 3 or 4 respectively. This follows from the classification of 3-dimensional Artin-Schelter regular algebras [1, 2, 3].

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Abelian varieties

Amnon Neeman

In representation theory one is interested in Calabi–Yau triangulated categories. These few lectures were an attempt to survey the classical analogue in algebraic geometry and complex analysis. In this abstract I treat the case of abelian varieties. Much more detail on everything I say may be found in [2, 3].

For the sake of definiteness, we begin with the definitions.

Definition 1. A *Calabi–Yau manifold* is a connected, compact, complex manifold with trivial sheaf of top differential forms.

In other words a connected, compact, complex manifold X of dimension g will be Calabi–Yau if the sheaf Ω_X^g has a nowhere vanishing holomorphic section. We recall

Theorem 1 (Serre Duality). *Let X be a connected, compact, complex manifold of dimension g. If $\mathcal{D}$ is the bounded derived category of chain complexes of coherent analytic sheaves on X, then there is a natural isomorphism*

$$\mathrm{Hom}_{\mathcal{D}}(A,B)^* \;\simeq\; \mathrm{Hom}_{\mathcal{D}}\left(B, A\otimes\Omega_X^g[g]\right).$$

In the language of [1] the category $\mathcal{D}$ has a Serre functor S, given by the formula

$$S(-) \;=\; (-)\otimes\Omega_X^g[g].$$

$\mathcal{D}$ is a Calabi–Yau triangulated category if and only if $\Omega_X^g \simeq \mathcal{O}_X$, that is if and only if the manifold X is Calabi–Yau. The dimension of the Calabi–Yau triangulated category $\mathcal{D}$ agrees with the complex dimension of the manifold X. One very classical case of this is complex tori. We recall the definition

Definition 2. A *complex torus* is a connected, compact, complex Lie group.

We note that every complex torus is automatically Calabi–Yau. The point is that the line bundle Ω_X^g has a unique trivialisation by a left invariant g–form. Take any non–vanishing g–form at the identity, and extend it (uniquely) to a left invariant g–form on all of X.

Let us say a little more about connected, compact, complex Lie groups. We observe

Theorem 2. *Any connected, compact, complex Lie group is commutative.*

Proof. The result is well–known but we include a proof. Let X be a connected, compact, complex Lie group. Consider the map

$$f : X\times X \longrightarrow X$$

given by

$$f(x,y) = xyx^{-1}y^{-1}.$$

If $e\in X$ is the identity, then $f(e,y)=e$ for all $y\in X$. Now let $V\subset X$ be a small ball around e. Then $f^{-1}V$ must contain an open set of the form $U\times X$, with U an open neighbourhood of the identity $e\in X$. For any $u\in U$, the map f induces a holomorphic map from the compact manifold $\{u\} \times X$ to the ball V, and any such map is constant. But then

$$f(u, y) = f(u, e) = e;$$

that is, f sends all of $U \times X$ to the singleton e. Now analytic continuation tells us that f collapses all of $X \times X$ to e. $\square$

It follows that the Lie algebra of X is commutative; it is just the trivial Lie algebra $\mathbb{C}^g$. Furthermore, the exponential map $\mathbb{C}^g \longrightarrow X$ is a group homomorphism, which is locally a diffeomorphism. The image is an open subgroup of the connected group X, and hence the exponential map is surjective. This means that X is isomorphic to a quotient group $\mathbb{C}^g/\Lambda$, where Λ is a discrete closed subgroup of $\mathbb{C}^g$. Since X is compact, Λ must be a lattice. That is the natural map

$$\mathbb{R} \otimes_{\mathbb{Z}} \Lambda \longrightarrow \mathbb{C}^g$$

is an isomorphism. We summarise:

Theorem 3. *Any connected, compact, complex Lie group is $\mathbb{C}^g/\Lambda$, where $\Lambda \subset \mathbb{C}^g$ is a lattice.*

Remark 1. *Theorem 3 justifies the terminology of Definition 2. By Theorem 3 a connected, compact, complex Lie group is $\mathbb{C}^g/\Lambda$, which is nothing other than a $2g$–dimensional real torus with a complex structure. Hence, we call these complex tori.*

Now we come to the question of how many different complex tori are there. The answer is clear. Two complex tori $\mathbb{C}^g/\Lambda$ and $\mathbb{C}^g/\Lambda'$ will agree if there is a linear transformation in $GL(g, \mathbb{C})$ taking Λ to Λ'. If we choose a basis for Λ, we can always, up to a linear transformation in $GL(g, \mathbb{C})$, assume that g elements of this basis are the standard basis vectors for $\mathbb{C}^g$. Our freedom in varying Λ amounts to the freedom in selecting the other g basis vectors. The space of choices is an open subset of $\{\mathbb{C}^g\}^g = \mathbb{C}^{g^2}$. There are g^2 "degrees of freedom" in choosing a g–dimensional complex torus.

Definition 3. A complex torus is called an *abelian variety* if it can be given the structure of an algebraic variety. Equivalently, this means it can be embedded as a complex analytic submanifold of projective space.

How many complex tori are abelian varieties? One classical way to answer the problem is using Theta functions. We briefly explain.

If X admits an embedding into projective space, then it must have a line bundle on it, with plenty of sections. Pulling back the line bundle by the exponential map $\mathbb{C}^g \longrightarrow X$, we get a holomorphic line bundle on $\mathbb{C}^g$, but all such bundles are trivial. The sections of the line bundle on X pull back to sections of the trivial bundle (that is, functions) on $\mathbb{C}^g$, with certain periodicity properties. These functions have been studied classically as Theta functions.

Without giving much detail, Theta functions are constructed as infinite sums. If $z \in \mathbb{C}^g$ and Ω is a symmetric $g \times g$ matrix over $\mathbb{C}$ with a positive definite imaginary part, we can form the sum

$$\Theta(\Omega, z) \;=\; \sum_{n \in \mathbb{Z}^g} \exp \pi i \big({}^t n \Omega n + 2\, {}^t n z \big)$$

If we fix Ω and view this as a function in z, we get one of our sections of holomorphic line bundles on $\mathbb{C}^g$. The point we want to make is that, as we vary the parameter Ω over the symmetric $g \times g$ matrices, the dimension of the parameter space is only $g(g+1)/2$. There is only a $g(g+1)/2$–dimensional space of g–dimensional abelian varieties. Therefore, most complex tori do not admit the structure of algebraic varieties.

The physics literature is divided on whether abelian varieties should be admitted as Calabi–Yau manifolds. From the point of representation of quivers, some of the most interesting examples come from elliptic curves, which are 1–dimensional abelian varieties. Undoubtedly, the quiver theoretic statements one can make about the categories of sheaves over elliptic curves (equivariant with respect to the action of suitable automorphisms) all generalise to higher dimensional abelian varieties.

An elliptic curve admits an involution, which is nothing other than the map taking $x \in X$ to $-x \in X$. Much has been made of the quiver representations giving the category of equivariant sheaves on X. There is no reason why this should not generalise to higher dimension.

If $\sigma : X \longrightarrow X$ is the involution taking $x \in X$ to $-x \in X$, one can study the variety X/σ. If X is a curve, then X/σ is nothing other than $\mathbb{P}^1$, in particular X/σ is smooth. In higher dimensions X/σ is singular. But the singularities of X/σ are not too bad and are well understood. For example, if X is a surface (that is, 2–dimensional), then X/σ has exactly 16 singular points. A minimal resolution of these 16 points gives an Enriques surface. It is not quite Calabi–Yau, but almost. The sheaf Ω^g_X is not trivial, but $\{\Omega^g_X\}^2 = \Omega^g_X \otimes \Omega^g_X$ is. That is, there is an isomorphism $\{\Omega^g_X\}^2 \simeq \mathcal{O}_X$. In other words, the Serre functor

$$S(-) \;=\; (-) \otimes \Omega^g_X[g].$$

is not a shift, but

$$S^2(-) \;=\; (-) \otimes \{\Omega^g_X\}^2[2g]$$

is a shift.

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