Mathematisches Forschungsinstitut Oberwolfach

Report No. 10/2011

DOI: 10.4171/OWR/2011/10

Representation Theory of Quivers and Finite Dimensional Algebras

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February 20th – February 26th, 2011

Abstract. Methods and results from the representation theory of quivers and finite dimensional algebras have led to many interactions with other areas of mathematics. Such areas include the theory of Lie algebras and quantum groups, commutative algebra, algebraic geometry and topology, and in particular the new theory of cluster algebras. The aim of this workshop was to further develop such interactions and to stimulate progress in the representation theory of algebras.

Mathematics Subject Classification (2000): 16Exx, 16Gxx, 16L60, 16W35, 18Exx, 18Gxx. Secondary: 13Dxx, 15Axx, 20Cxx, 14Lxx, 16Sxx, 17Bxx.

Introduction by the Organisers

The representation theory of quivers is probably one of the most fruitful parts of modern representation theory because of its various links to other mathematical subjects. This has been the reason for devoting a substantial part of this Oberwolfach meeting to problems that can be formulated and solved involving quivers and their representations. The interaction with neighbouring mathematical subjects like geometry, topology, and combinatorics is one of the traditions of such Oberwolfach meetings; it can be quite challenging for the participants but it certainly continues to be a source of inspiration. There were 27 lectures given at the meeting, and what follows is a quick survey of their main themes.

Representations of quivers. There continues to be rapid development in the theory of representations of quivers, especially in the following interlinked areas:

the geometry of quiver varieties (in the sense of Nakajima, so moduli spaces of representations of the double of a quiver, with relations coming from a moment map, as in the preprojective algebra), moduli spaces of representations of an undoubled quiver, Hall algebras, and Donaldson-Thomas invariants for quivers.

Three of the speakers, T. Hausel, E. Letellier and F. Rodriguez Villegas, spoke on developments arising from their work on arithmetic harmonic analysis on character and quiver varieties. T. Hausel spoke about a series of conjectures relating the cohomology of character varieties with Kac's A-polynomial, which counts the number of absolutely indecomposable representations of a quiver over finite fields of varying sizes. One aim of the conjectures is to find a cohomological proof of Kac's conjecture that the coefficients of the A-polynomial are non-negative. Mostly, the quivers which arise this way are of a specific shape - 'comet-shaped' - but one conjecture extends these ideas to arbitrary quivers. E. Letellier's talk related to the character theory of the general linear group over a finite field. Although the irreducible characters are known since Green's work in 1955, the decomposition of tensor products is not understood. By linking this problem with intersection cohomology of quiver varieties, E. Letellier was able to show in some cases, when the characters are sufficiently generic, that whether or not a given irreducible occurs in a tensor product is determined by a Kac-Moody root system. Rodriguez Villegas explored a refinement of Kac's A-polynomial, and presented an explicit formula for its evaluation at 1 in case the quiver consists of one vertex and several loops.

M. Reineke and S. Mozgovoy both spoke about Donaldson-Thomas invariants associated to symmetric quivers. Reineke described an explicit combinatorial treatment of DT invariants for the quiver with one vertex and several loops. A conjecture of Kontsevich and Soibelman (now proved by Efimov, which we learnt about during the workshop, and with a preprint subsequently posted to the arxiv) implies a positivity property for DT invariants, and S. Mozgovoy showed that this property implies Kac's non-negativity conjecture (mentioned above) for quivers with a loop at every vertex. It is amazing that such different approaches can lead to progress with this conjecture!

Instead of passing to a moduli space of representations of a quiver, one can consider the corresponding group action on the affine space of representations of a fixed dimension vector. Associated to any irreducible closed subset, stable under the group action, there are classes in both equivariant cohomology and K-theory. Such classes have been studied by Buch and others as a way to generalize and unify various notions in Schubert calculus. The classes can be written as linear combinations of products of Schur and Grothendieck polynomials, and in his talk A. Buch described conjectural properties of the coefficients in these linear combinations (known as 'quiver coefficients'), and results obtained in case the quiver is of Dynkin type.

There are many very basic open questions about representations of quivers, and one such is how to construct the indecomposable representations in general. However in his talk, T. Weist showed that for every dimension vector which is an imaginary Schur root, there exists an indecomposable representation given by a tree.

Cluster algebras and cluster categories. Cluster algebras are certain commutative algebras whose generators and relations are constructed recursively. They were invented by S. Fomin and A. Zelevinsky in the year 2000 to serve as a combinatorial framework for the study of Kashiwara/Lusztig's canonical bases in quantum groups and of the closely related notion of total positivity in algebraic groups. More than a decade after Fomin-Zelevinsky's invention, the precise connection between cluster algebras and canonical bases remains a mystery. The best results confirming that such a link exists are certainly those due to Geiss-Leclerc-Schröer, who have shown that in the cluster algebras arising as rings of coordinates on unipotent cells in Kac–Moody groups, all cluster monomials belong to Lusztig's dual semi-canonical basis. In his talk, C. Geiss presented an interpretation of the dual semi-canonical basis as the 'generic basis', which allows its conjectural generalization to arbitrary cluster algebras. These remarkable results were complemented in P.-G. Plamondon's talk, devoted to a mutation-invariant parametrization of the elements of the conjectural generic basis in an arbitrary cluster algebra. This parametrization yields an important connection between Geiss-Leclerc-Schröer's conjecture and Fock-Goncharov series of duality conjectures motivated by their higher Teichmüller theory. It is expected that a quantum version of the generic basis will yield a generalization of Kashiwara/Lusztig's canonical basis to an arbitrary cluster algebra. The very first results in the direction of this long-term goal are due to P. Lampe, who, in his talk, explained how in type A, the quantum cluster algebra identifies with a quantum coordinate algebra in such a way that the quantum cluster variables correspond to certain canonical basis vectors.

In their 2008 preprint 'Stability structures, Donaldson-Thomas invariants and cluster transformations', Kontsevich-Soibelman have interpreted individual cluster transformations as wall-crossing formulas for DT-invariants of certain 3-Cala-bi-Yau categories. In remarkable work, K. Nagao has extended their idea to compositions of cluster transformations and combined it with D. Joyce's results to prove a series of conjectures formulated by Fomin-Zelevinsky in 2006 (and recently proved using different methods first by Derksen-Weyman-Zelevinsky and then by Plamondon). Nagao's talk combined a streamlined introduction to Donaldson-Thomas theory with a beautiful presentation of the key ideas of his approach.

Cluster categories are triangulated 2-Calabi-Yau categories used to 'categorify' cluster algebras (as in the talks by Geiss and Plamondon). The study of particular classes of such categories reveals intricate combinatorial structures. For finite cluster type and for tubes, these were explored in the talks by T. Holm and by K. Baur. For cluster categories associated with 'ciliated surfaces', R. Marsh analyzed the combinatorics of the mutation of rigid objects in terms of coloured quivers. Generalized cluster categories of higher Calabi-Yau dimension were at the center of O. Iyama's talk. After a beautiful introduction to this circle of ideas he sketched an extension of his 'higher Auslander-Reiten theory' to the new class of 'n-representation-controlled algebras' (joint work with S. Oppermann and M. Herschend).

Derived categories and tilting theory. Much of the recent progress in representation theory of algebras is formulated in terms of derived categories. In fact, the derived category of an algebra captures a wealth of homological information and is an interesting invariant in its own right. The study of the existence and properties of (cluster) tilting objects provides one of the challenges in this subject. The talks of A. Beligiannis and L. Hille were devoted to this aspect. M. Van den Bergh talked about autoequivalences of derived categories for singular elliptic curves and pointed out the connection with mirror symmetry. Another method to approach a derived category is the use of stratifications. In Koenig's talk, the stratification of the derived category of an algebra was discussed in terms of recollements. A completely different way of stratifying the stable module category of a finite group was explained in S. Iyengar's talk. He used group cohomology and presented the connection with specific properties of the Bousfield lattice. An interesting numerical invariant of a derived category is its dimension as a triangulated category. It is a somewhat surprising result presented by S. Oppermann that this dimension is finite for the derived category of finite dimensional modules, while any proper triangulated subcategory containing the projectives is of infinite dimension.

Auslander-Reiten theory. One classical invariant in the representation theory of artin algebras is the Auslander-Reiten quiver of the category of the finitely generated (left or right) modules. The quiver records the isomorphism classes of the indecomposable modules and their relative position with respect to the radical of the category. In fact it records the category of finitely generated modules modulo the infinite radical, $\operatorname{rad}^\omega = \cap_{i\geqslant 0}\operatorname{rad}^i$. The talk of C. M. Ringel discussed how to describe the module category of finitely generated modules modulo $\operatorname{rad}^{\omega 2}$ for suitable (1-domestic) special biserial algebras. This is obtained through the socalled Auslander-Reiten quilt of the algebra, which is the Auslander-Reiten quiver with additional vertices inserted for indecomposable algebraically compact infinite dimensional modules and a convergence relation. The talk illustrated this construction through a careful study of one example.

The Auslander-Reiten quiver gives rise to different classes of modules and invariants of the algebra. One important notion is that of a module lying on a short chain: Given an almost split sequence $0 \to A \to B \to C \to 0$, the end terms are connected via the Auslander-Reiten translate τ, that is, $A \simeq \tau(C)$. Then an indecomposable module M is *on a short chain* if there are non-zero homomorphisms $X \to M \to \tau(X)$ for some indecomposable module X. An interesting fact about indecomposable modules not in the middle of a short chain is that they are determined up to isomorphism by their composition factors. An omnipresent class of algebras in the representation theory of artin algebras is the *tilted* algebras, that is, endomorphism rings of a tilting module over a hereditary algebra. In the talk of A. Skowronski tilted algebras were characterized by the existence of a sincere module which is not in the middle of a short chain.

Some links to commutative algebra. Eisenbud pointed out early on a link between complete intersections and group rings by showing that ideas from homological algebra for complete intersections transferred to group rings by studying projective resolutions and in addition introducing matrix factorizations. This approach was extended in the talk of R.O. Buchweitz to construct complete resolutions and showing that maximal Cohen-Macaulay modules over complete intersections are also determined by matrix factorizations, not over the original ring, but over a naturally associated larger ring.

In the mid 1980's there was a strong influence from representation theory on commutative algebra, through Auslander-Reiten theory for maximal Cohen-Macaulay modules over isolated singularities and a theory of homologically finite subcategories. The talk of J. Weyman extended this interplay as it dealt with a new connection between generic free resolutions in commutative algebra and Kac-Moody Lie algebras. A free complex

$$0 \to F_n \to F_{n-1} \to \cdots F_1 \to F_0 \to 0,$$

has format $(r_n, r_{n-1}, \ldots, r_1)$ over a commutative ring if the rank of the i-th differential d_i equals to r_i for all i. Then an acyclic complex F_{gen} over a given ring R_{gen} is *generic* if for every complex G of a given format $(r_n, r_{n-1}, \ldots, r_1)$ over a Noetherian ring S there exists a homomorphism $f\colon R_{gen} \to S$ such that $G \simeq F_{\mathrm{gen}} \otimes_{R_{\mathrm{gen}}} S$. For $n = 3$ associate to the format (r_3, r_2, r_1) a graph $T_{p,q,r}$ with three arms of length $p = r_3$, $q = r_2 - 2$ and $r = r_1$. Then Weyman showed, among other things, that there exists a Noetherian generic ring for this format if and only if the graph $T_{p,q,r}$ is Dynkin. For the case $n = 2$ the problem was solved by Hochster and Huneke.

Further links to Lie theory and algebraic groups. M. Brion spoke on a representation-theoretic approach to the study of homogeneous bundles on abelian varieties. This allowed him not only to recover a classical structure theorem of Miyanishi and Mukai (in characteristic zero) but also to obtain new results on projective bundles, which he linked to standard representations of Heisenberg groups.

C. Stroppel presented ongoing joint work with E. Frenkel and J. Sussan motivated by the problem of categorifying Turaev-Viro invariants of 3-manifolds. A key step is the categorification of tensor products of representations of quantum groups and of the intertwiners between tensor products. She focused on the case of sl_2, where she showed that a categorification with excellent properties is provided by a certain category of Harish-Chandra bimodules. By work of Futorny-Mazorchuk-König, this category is properly stratified, which provides a beautiful link to the representation theory of finite-dimensional algebras.

The format of the workshop has been a combination of introductory survey lectures and more specialized talks on recent progress. In addition there was plenty of time for informal discussions. Thus the workshop provided an ideal atmosphere

for fruitful interaction and exchange of ideas. It is a pleasure to thank the administration and the staff of the Oberwolfach Institute for their efficient support and hospitality.

Workshop: Representation Theory of Quivers and Finite Dimensional Algebras

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Abstracts

Geometric realization of orbit categories

Karin Baur
(joint work with Robert Marsh)

The first part of the talk contained the presentation of various geometric models for cluster categories. We write $\mathcal{C}_Q$ to refer to a cluster category of type Q.

- In type A_n, the cluster categories can be modelled via diagonals in a disc with $n+3$ marked points on the boundary, cf. [CCS]: diagonals give rise to indecomposable objects of $\mathcal{C}_{A_n}$ and simple rotations about endpoints correspond to irreducible maps.
- For cluster categories of type D_n, one uses (tagged) arcs in a disc with n marked points on the boundary to describe indecomposable objects, see [S].
- m-cluster categories in types A_n and D_n can be obtained from m-diagonals in a disc with marked points on the boundary resp. from m-arcs in a punctured disc with marked points on the boundary ([BM1], [BM2]).
- To construct cluster categories of type A_∞, [HJ] uses diagonals in an ∞-gon.

More recently, [BZ] have associated a cluster category $\mathcal{C}(S,M)$ for any Riemann surface S with a set M of marked points on boundary components (without punctures).

The main idea behind these combinatorial geometric models is that they give rise to a stable translation quiver $\Gamma = (\Gamma,\tau)$ which is isomorphic to the Auslander-Reiten quiver of the corresponding cluster category. The vertices of Γ are defined as the arcs (diagonals, m-diagonals or m-arcs, respectively) in the figure and the arrows arise from simple rotations between these arcs (diagonals, m-diagonals or m-arcs, respectively).

Properties of the geometric models. Let $\mathcal{C}$ be a cluster category of any of these types and let S be the corresponding surface (disc, infinity-gon, or a more general surface). We will use "arc" to denote the lines between the marked points (up to homotopy). Then the correspondence we have recalled above tell us that arcs in S correspond to (isomorphism classes of) indecomposable objects of $\mathcal{C}$. In particular, we have the following

$$\{\text{simple arcs in } S\} \longleftrightarrow \{\text{rigid indecomposable objects of } \mathcal{C}\}/\sim$$

If γ is an arc in S we write M_γ for the corresponding indecomposable object of $\mathcal{C}$. Then we can describe the dimensions of the Ext groups in terms of intersection numbers of arcs:
$\dim \mathrm{Ext}^1_{\mathcal{C}}(M_\gamma, M_\delta)$ is the minimal intersection number of γ and δ.

New models. Our goal is to provide geometric models for more general categories of modules. In cluster categories, we have the symmetry dim $\mathrm{Ext}^1_{\mathcal{C}}(M,N)$ = dim $\mathrm{Ext}^1_{\mathcal{C}}(N,M)$ thanks to the Calabi-Yau property. In more general categories of modules, this symmetry does not hold. Thus we cannot expect that simple arcs in surfaces will provide an adequate model. We propose to use orientations of arcs and to introduce winding numbers. To illustrate how this works, we discuss the case of a tube, [BM4], and show how our approach leads to new combinatorial geometric models.

Tube categories. A stable translation quiver (Γ,τ) is of *tube type* if $\Gamma = \mathbb{Z}\,A_\infty/n$. In this case, we say that Γ has rank n. If Γ has rank n, we denote the additive hull of the mesh category of (Γ,τ) by $\mathcal{T}_n$ and call $\mathcal{T}_n$ a *tube of rank* n.

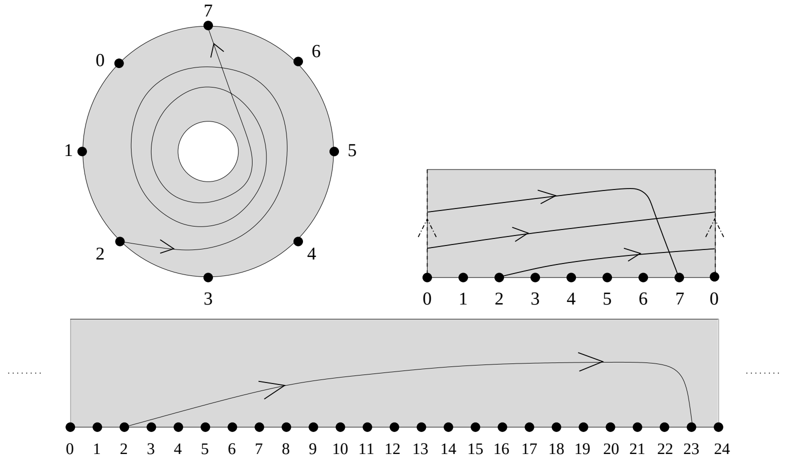
We recall that $\mathcal{T}_n$ is equivalent to the category of nilpotent representations of kC_n[1], cf. [R, 3.6 (6)] and [RVdB, III.1.1].

$\mathcal{T}_n$ has a number of nice properties: it is Hom-finite, abelian, hereditary, uniserial and $\mathrm{Ext}^1_{\mathcal{T}_n}(X,Y) \cong \mathrm{DHom}_{\mathcal{T}_n}(Y,\tau X)$ (τ denotes the AR-translate). We can form ist cluster category as in [BMRRT], $\mathcal{C}_{\mathcal{T}_n} := \mathrm{D}^b(\mathcal{T}_n)/\tau^{-1}[1]$. Observe that $\mathrm{AR}(\mathcal{T}_n) \cong \mathrm{AR}(\mathcal{C}_{\mathcal{T}_n})$.

Combinatorial model. Let $A(n) := A(n,0)$ be an annulus with n marked points on the outer boundary. Arcs in $A(n)$ start at a vertex i, wind around the inner boundary counterclockwise and, after a number of revolutions, end at a vertex j. We best describe arcs using the universal cover of $A(n)$.

$$A(n) \quad \longleftrightarrow \quad \mathrm{Cyl}(n) \quad \xleftarrow{\pi} U = (U,\pi_n)$$

Here, we show an arc from 2 to 7 in $A(8)$ and in Cyl(8), winding around twice. The arc from 2 to 23 in the universal cover (U,π_8) is a preimage of them.



The map π_n sends (x,y) to $(x \mod n, y) \in \mathrm{Cyl}(n)$ or to its image in $A(n)$.

Definition.

1) If $i < j-1$ we call the curve $[i,j] : (i,0) \to (j,0)$ in U an *admissible arc* in U.

2) An *admissible arc* in $A(n)$ is the image $\pi_n([i,j])$ of an admissible arc in U.

[1] for $k = \overline{k}$ and C_n a cyclic equioriented quiver with n vertices.

Results. The arc model in the annulus gives rise to a stable translation quiver $\Gamma(A(n))$ and that this quiver is isomorphic to the AR-quiver $\mathrm{AR}(\mathcal{T}_n)$, [BM4]. This isomorphism gives rise to a bijection

$$\{\text{admissible arcs in } \mathrm{A}(n)\} \longleftrightarrow \{\text{indec. obj. of } \mathcal{T}_n\}/\text{iso}$$

Let $M[a,b] \in \mathrm{ind}(\mathcal{T}_n)$ be the image of $\pi_n[a,b]$ under this bijection. Then we have:
Theorem. [BM4]
Let $\pi_n([a,b])$ and $\pi_n([c,d])$ be admissible arcs in $A(n)$.
(i) $\dim \mathrm{Ext}^1_{\mathcal{T}_n}(M[a,b],M[c,d]) = I^-(\pi_n[a,b],\pi_n[c,d])$
(ii) $\dim \mathrm{Ext}^1_{\mathcal{T}_n}(M[c,d],M[a,b]) = I^+(\pi_n[a,b],\pi_n[c,d])$
(iii) $\dim \mathrm{Ext}^1_{\mathcal{C}_{\mathcal{T}_n}}(M[a,b],M[c,d]) = I^+(\pi_n[a,b],\pi_n[c,d]) + I^-(\pi_n[a,b],\pi_n[c,d])$

In particular, if we restrict to the case of unoriented arcs, we obtain the dimension of Ext^1 as the sum of the positive and the negative intersections.
Remarks.
a) We can describe explicitly how to calculate the signed intersection numbers $I^\pm$ ([BM4, §3]).
b) In independent work, Warkentin has given a bijection between string modules over a quiver of type $\widetilde{A}_n$ and certain oriented arcs in the annulus, [W].
c) In [FST], cluster algebras are associated to annuli $A(r,s)$ for $r,s \geq 1$. We can view our model $A(n,0)$ as a limit case of $A(r,s)$.
d) Oriented arcs also have been used in [BM3] to obtain a geometric model for the root category [H] of type A.

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Relative homology and higher cluster tilting theory

Apostolos Beligiannis

Let throughout $\mathcal{T}$ be a triangulated category with split idempotents and suspension functor $A \mapsto A[1]$.

1. Cluster-tilting subcategories. We fix a full subcategory $\mathcal{X}$ of $\mathcal{T}$. Recall that $\mathcal{X}$ is called *contravariantly finite* in $\mathcal{T}$ if any object $A \in \mathcal{T}$ admits a right $\mathcal{X}$-approximation, i.e. a map $X_A \to A$, where $X_A \in \mathcal{X}$, such that the induced map $\mathcal{T}(\mathcal{X}, X_A) \to \mathcal{T}(\mathcal{X}, A)$ is surjective. *Covariant finiteness* is defined dually, and $\mathcal{X}$ is *functorially finite* if $\mathcal{X}$ is both contravariantly and covariantly finite.

For an integer $n \geq 1$, we say that $\mathcal{X}$ is $(n+1)$-**cluster tilting**, see [2,4], if:

(1) $\mathcal{X}$ is functorially finite in $\mathcal{T}$.
(2) $\mathcal{X} = \big\{A \in \mathcal{T} \mid \mathcal{T}(\mathcal{X}, A[i]) = 0,\ \ 1 \leq i \leq n\big\} := \mathcal{X}_n^{\perp}$.
(3) $\mathcal{X} = \big\{A \in \mathcal{T} \mid \mathcal{T}(A, \mathcal{X}[i]) = 0,\ \ 1 \leq i \leq n\big\} := {}_n^{\perp}\mathcal{X}$.

In particular $\mathcal{X}$ is *n-rigid* in the sense that: $\mathcal{T}(\mathcal{X}, \mathcal{X}[i]) = 0$, $1 \leq i \leq n$.

Let $\mathcal{X}$ be an $(n+1)$-cluster tilting subcategory of $\mathcal{T}$. The **cluster tilted category** associated to $\mathcal{X}$ is the category $\mathsf{mod}\text{-}\mathcal{X}$ of coherent functors over $\mathcal{X}$, where an additive functor $F \colon \mathcal{X}^{\mathsf{op}} \to \mathcal{A}b$ is coherent if there is an exact sequence $\mathcal{X}(-, X^1) \to \mathcal{X}(-, X^0) \to F \to 0$. An easy consequence of contravariant finiteness of $\mathcal{X}$ is that the category $\mathsf{mod}\text{-}\mathcal{X}$ is abelian, and then we have a homological functor

$$\mathsf{H} \,:\, \mathcal{T} \,\longrightarrow\, \mathsf{mod}\text{-}\mathcal{X}, \quad \mathsf{H}(A) = \mathcal{T}(-, A)|_{\mathcal{X}}.$$

In case $n = 1$, Keller and Reiten in [4,5] proved, under certain additional assumptions (removed later by Koenig and Zhu [6] who also proved that $\mathcal{X}$ is 2-cluster tilting iff $\mathcal{X}$ is contravariantly finite and $\mathcal{X} = \mathcal{X}_1^{\perp}$ iff $\mathcal{X}$ is covariantly finite and $\mathcal{X} = {}_1^{\perp}\mathcal{X}$), the following basic result concerning the structure of $\mathcal{T}$ in connection with certain homological properties of the cluster tilted category $\mathsf{mod}\text{-}\mathcal{X}$.

Theorem 1. [4,5] (see also [6]). *Let $\mathcal{X}$ be a 2-cluster tilting subcategory of $\mathcal{T}$.*

1. *The cluster tilted category $\mathsf{mod}\text{-}\mathcal{X}$ has enough projectives and injectives. The functor $\mathsf{H} : \mathcal{T} \longrightarrow \mathsf{mod}\text{-}\mathcal{X}$ induces equivalences between $\mathcal{T}/\mathcal{X}[1]$ and $\mathsf{mod}\text{-}\mathcal{X}$, between $\mathcal{X}$ and $\mathsf{Proj}\,\mathsf{mod}\text{-}\mathcal{X}$, and between $\mathcal{X}[2]$ and $\mathsf{Inj}\,\mathsf{mod}\text{-}\mathcal{X}$.*
2. *The cluster tilted category $\mathsf{mod}\text{-}\mathcal{X}$ is 1-Gorenstein.*
3. *If $\mathcal{T}$ is 2-Calabi-Yau over a field k, then the stable triangulated category*

$$\underline{\mathsf{CM}}(\mathsf{mod}\text{-}\mathcal{X})$$

of Cohen-Macaulay objects of $\mathsf{mod}\text{-}\mathcal{X}$ is 3-Calabi-Yau.

Recall that $\mathcal{T}$ is called d-Calabi-Yau over a field k, $d \geq 1$, if $\mathcal{T}$ is k-linear with finite-dimensional Hom spaces over k and there exist natural isomorphisms

$$\mathsf{D}\mathcal{T}(A,B) \xrightarrow{\approx} \mathcal{T}(B, A[d]), \quad \forall A, B \in \mathcal{T}$$

where D denotes duality with respect to the base field k. Also recall that the *Gorenstein dimension*, $\mathsf{G\text{-}dim}\mathcal{A}$, of an abelian category $\mathcal{A}$ is defined as follows. First let $\mathsf{spli}\mathcal{A} = \sup\{\mathsf{p.d}I \mid I \in \mathsf{Inj}\mathcal{A}\}$ and $\mathsf{silp}\mathcal{A} = \sup\{\mathsf{id}P \mid P \in \mathsf{Proj}\mathcal{A}\}$. Then:

$$\mathsf{G\text{-}dim}\mathcal{A} = \max\{\mathsf{spli}\mathcal{A}, \mathsf{silp}\mathcal{A}\}$$

and $\mathcal{A}$ is called *Gorenstein*, resp. *k-Gorenstein*, if $\mathsf{G\text{-}dim}\mathcal{A} < \infty$, resp. $\mathsf{G\text{-}dim}\mathcal{A} \leq k$.

In view of Theorem 1, we are interested in the problem of whether there exists an analogous result for $(n+1)$-cluster tilting subcategories $\mathcal{X}$ of $\mathcal{T}$, if $n > 1$.

2. Ghosts, Extensions and Homological Dimension. Let $\mathcal{X}$ be a full subcategory of $\mathcal{T}$. For any $k \geq 0$, we consider all maps $f \colon A \to B$ in $\mathcal{T}$ such that $\mathcal{T}(\mathcal{X}[k], f) = 0$. The set of all such maps forms a subgroup $\mathsf{Gh}_{\mathcal{X}[k]}(A,B)$ of $\mathcal{T}(A,B)$, called the subgroup of $\mathcal{X}[k]$-*ghost maps*. Clearly then we obtain an ideal $\mathsf{Gh}_{\mathcal{X}[k]}(\mathcal{T})$ of $\mathcal{T}$ and the product ideal $\mathsf{Gh}^{[k]}_{\mathcal{X}}(\mathcal{T})$ of $\mathcal{X}$-**ghost maps of depth** k is defined by

$$\mathsf{Gh}^{[k]}_{\mathcal{X}}(\mathcal{T}) = \mathsf{Gh}_{\mathcal{X}}(\mathcal{T}) \circ \mathsf{Gh}_{\mathcal{X}[1]}(\mathcal{T}) \circ \mathsf{Gh}_{\mathcal{X}[2]}(\mathcal{T}) \circ \cdots \circ \mathsf{Gh}_{\mathcal{X}[k-1]}(\mathcal{T})$$

i.e. $\mathsf{Gh}^{[k]}_{\mathcal{X}}(A,B)$ consists of all maps $f : A \to B$ which can be written as a composition $f = f_0 \circ f_1 \circ \cdots \circ f_{k-1}$, where $f_0 : A \to B_0$ is $\mathcal{X}$-ghost, $f_1 : B_0 \to B_1$ is $\mathcal{X}[1]$-ghost, $\cdots$, $f_{k-1} : B_{k-2} \to B$ is $\mathcal{X}[k-1]$-ghost. The structure of the ideal $\mathsf{Gh}^{[k]}_{\mathcal{X}}(\mathcal{T})$ of $\mathcal{X}$-ghost maps of depth k is related to the **category of extensions** $\bigstar_{i=0}^{k}\mathcal{X}[i] = \mathcal{X} \star \mathcal{X}[1] \star \cdots \star \mathcal{X}[k]$ of $\mathcal{X}$, see below. Recall that if $\mathcal{A}_i \subseteq \mathcal{T}$, $i = 1, 2$, then $\mathcal{A}_1 \star \mathcal{A}_2$ is the full subcategory of $\mathcal{T}$ consisting of all direct summands of objects $C \in \mathcal{T}$ for which there exists a triangle $A_1 \to C \to A_2 \to A_1[1]$, where $A_i \in \mathcal{A}_i$. The full subcategory $\mathcal{A}_1 \star \mathcal{A}_2 \star \cdots \star \mathcal{A}_k$ is defined inductively for $k \geq 3$.

Now let $\mathcal{X}$ be contravariantly finite in $\mathcal{T}$. Then $\forall A \in \mathcal{T}$ there exists a triangle

$$\Omega^1_{\mathcal{X}}(A) \longrightarrow X^0_A \longrightarrow A \longrightarrow \Omega^1_{\mathcal{X}}(A)[1]$$

where the middle map is a right $\mathcal{X}$-approximation of A. Note that the object $\Omega^1_{\mathcal{X}}(A)$ is uniquely determined in the stable category $\mathcal{T}/\mathcal{X}$. Inductively we define the object $\Omega^k_{\mathcal{X}}(A)$, $\forall k \geq 1$, and we set $\Omega^0_{\mathcal{X}}(A) = A$. The minimum $k \geq 0$ such that $\Omega^k_{\mathcal{X}}(A)$ lies in $\mathcal{X}$, or ∞ if no such k exists, is well-defined, it is denoted by $\mathsf{p.d}_{\mathcal{X}}A$ and is called the $\mathcal{X}$-**projective dimension** of A. Then the $\mathcal{X}$-**global dimension** of $\mathcal{T}$ is defined by $\mathsf{gl.dim}_{\mathcal{X}}\mathcal{T} = \sup\{\mathsf{p.d}_{\mathcal{X}}A \mid A \in \mathcal{T}\}$.

The ideal of $\mathcal{X}$-ghost maps, the category of extensions of $\mathcal{X}$ and the $\mathcal{X}$-projective dimension are related via the following version of the Ghost Lemma:

Ghost Lemma. [1] *If $\mathcal{X}$ is contravariantly finite in $\mathcal{T}$, then $\forall A \in \mathcal{T}$ and $\forall n \geq 0$, we have: $\mathsf{p.d}_{\mathcal{X}}A \leq n \Longrightarrow \mathsf{Gh}^{[n+1]}_{\mathcal{X}}(A,-) = 0 \Longleftrightarrow A \in \mathcal{X} \star \mathcal{X}[1] \star \cdots \star \mathcal{X}[n]$. If $\mathcal{X}$ is in addition n-rigid, then we also have: $\mathsf{Gh}^{[n+1]}_{\mathcal{X}}(A,-) = 0 \Longrightarrow \mathsf{p.d}_{\mathcal{X}}A \leq n$.*

Then we have the following characterization of cluster tilting subcategories.

Theorem 2. [1] *For a subcategory $\mathcal{X} \subseteq \mathcal{T}$ the following are equivalent, $\forall n \geq 1$:*

1. *$\mathcal{X}$ is an $(n+1)$-cluster tilting subcategory of $\mathcal{T}$.*
2. *$\mathcal{X}$ is contravariantly finite and $\mathcal{X} = \mathcal{X}_n^{\perp}$.*
3. *$\mathcal{X}$ is covariantly finite and $\mathcal{X} = {}_n^{\perp}\mathcal{X}$.*
4. *$\mathcal{X}$ is contravariantly (or covariantly) finite n-rigid and: $\mathsf{gl.dim}_{\mathcal{X}}\mathcal{T} = n$.*
5. *$\mathcal{X}$ is contravariantly (or covariantly) finite n-rigid and: $\mathcal{T} = \bigstar_{k=0}^{n}\mathcal{X}[k]$.*
6. *$\mathcal{X}$ is contravariantly (or covariantly) finite n-rigid and: $\mathsf{Gh}_{\mathcal{X}}^{[n+1]}(\mathcal{T}) = 0$.*

If $\mathcal{X}$ is an $(n+1)$-cluster subcategory of $\mathcal{T}$, then $\mathsf{mod}\text{-}\mathcal{X}$ has enough projectives and enough injectives, and the functor $\mathsf{H}\colon \mathcal{T} \to \mathsf{mod}\text{-}\mathcal{X}$ is surjective on objects and induces equivalences between $\mathcal{X}$ and $\mathsf{Proj}\,\mathsf{mod}\text{-}\mathcal{X}$ and between $\mathcal{X}[n+1]$ and $\mathsf{Inj}\,\mathsf{mod}\text{-}\mathcal{X}$.

3. The Gorenstein Condition. Let as before $\mathcal{T}$ be a triangulated category and $\mathcal{X}$ an $(n+1)$-cluster tilting subcategory of $\mathcal{T}$. If $n = 1$, then, by Theorem 1, the cluster tilted category $\mathsf{mod}\text{-}\mathcal{X}$ is Gorenstein. However this fails if $n > 1$ by an example of Iyama [4] who constructed a 3-cluster tilting subcategory $\mathcal{X}$ in a 3-Calabi-Yau triangulated category $\mathcal{T}$ such that the cluster tilted category $\mathsf{mod}\text{-}\mathcal{X}$ is not Gorenstein, see [4, 5.3]. To remedy this failure we need the notion of a t-strong subcategory, $t \geq 1$, in the following sense: $\mathcal{X}$ is t-**strong** if $\mathcal{X} \subseteq \mathcal{X}_t^{\perp}[t+1]$, i.e. $\mathcal{T}(\mathcal{X}, \mathcal{X}[-i]) = 0$, $1 \leq i \leq t$. E.g. the $(n+1)$-cluster category associated to a finite dimensional hereditary algebra H contains H as an $(n-1)$-strong $(n+1)$-cluster tilting object, see [5, 4.1]. The following result generalizes [5, 4.6].

Theorem 3. [1] *Assume that $\mathcal{X}$ is $(n-k)$-strong for some k with $0 \leq k \leq \frac{n+1}{2}$.*

1. *The cluster tilted category $\mathsf{mod}\text{-}\mathcal{X}$ is k-Gorenstein.*
2. *If $0 \leq k \leq \frac{n}{2}$, $n \geq 2$, then $\mathsf{mod}\text{-}\mathcal{X}$ has finite global dimension if and only if the functor H induces an equivalence:*

$$\mathsf{H}:\ \big(\mathcal{X} \star \mathcal{X}[1] \star \cdots \star \mathcal{X}[k]\big) \cap \mathcal{X}_k^{\perp}[k+1] \ \xrightarrow{\approx}\ \mathsf{mod}\text{-}\mathcal{X}$$

Note that fullness of H implies 1-Gorensteinnes of $\mathsf{mod}\text{-}\mathcal{X}$:

Proposition 4. [1] *Assume that $n \geq 2$ and the functor $\mathsf{H}\colon \mathcal{T} \to \mathsf{mod}\text{-}\mathcal{X}$ is full. Then $\mathcal{X}$ is $(n-k)$-strong for any k with $0 \leq k \leq n-1$ and $\mathsf{mod}\text{-}\mathcal{X}$ is 1-Gorenstein.*

Using this and Iyama's example mentioned above, we see that for $n \geq 2$, the functor $\mathsf{H}\colon \mathcal{T} \to \mathsf{mod}\text{-}\mathcal{X}$, in contrast to the case $n = 1$, is not full in general.

4. The Calabi-Yau Condition. Finally assume that the triangulated category $\mathcal{T}$ is $(n+1)$-Calabi-Yau over a field k.

Theorem 5. [1] *Let $\mathcal{X}$ be an $(n+1)$-cluster tilting subcategory of $\mathcal{T}$. Assume:*

1. *$\mathcal{X}$ is $(n-k)$-strong, for some integer k with $0 \leq k \leq \frac{n+1}{2}$.*
2. *Any object $\mathsf{H}(A)$ of $\mathsf{mod}\text{-}\mathcal{X}$, where A lies in $\mathcal{X}[-n+1] \star \cdots \star \mathcal{X}[-1]$, has finite projective dimension.*

Then the stable triangulated category $\underline{\mathsf{CM}}\,\mathsf{mod}\text{-}\mathcal{X}$ of Cohen-Macaulay objects of the cluster tilted category $\mathsf{mod}\text{-}\mathcal{X}$ is $(n+2)$-Calabi-Yau.

The case $k = 1$ of Theorem 3 and of Theorem 5, where $\mathsf{H}(A) = 0$, $\forall A \in \mathcal{X}[-n+1] \star \cdots \star \mathcal{X}[-1]$, was obtained independently by Iyama-Oppermann, see [3].

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Some connections between abelian varieties and representation theory

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This talk is based on the preprints [3] and [4]. Its main objects are the *homogeneous* vector bundles over an abelian variety A, that is, those vector bundles E over A such that $\tau_a^*(E) \cong E$ for all $a \in A$, where τ_a denotes the translation $x \mapsto x + a$ in A. We work over a fixed algebraically closed field k.

By a classical result of Rosenlicht and Serre (see [9]), a line bundle L over A is homogeneous if and only if it is algebraically trivial. Thus, the homogeneous line bundles are parametrized by the dual abelian variety $\widehat{A} = \mathrm{Pic}^0(A)$.

The homogeneous vector bundles have been described by Miyanishi and Mukai (see [5, 6]). They showed that the following conditions are equivalent for a vector bundle E over A:
(i) E is homogeneous.
(ii) E admits a filtration $0 = E_0 \subset E_1 \subset \cdots \subset E_r = E$ by sub-bundles such that $E_i/E_{i-1} \in \hat{A}$ for $i = 1, \ldots, r$.
(iii) E admits a decomposition $E \cong \bigoplus_i L_i \otimes U_i$ such that the L_i are pairwise distinct line bundles in $\widehat{A}$, and the U_i are *unipotent* vector bundles, i.e., they admit a filtration with subquotients being trivial bundles.

As a consequence of (ii), any extension of homogeneous vector bundles is again homogeneous. Also, note that the decomposition in (iii) is orthogonal in the sense that $\mathrm{Ext}^n_A(L_i \otimes U_i, L_j \otimes U_j) = 0$ for all n and $i \neq j$, since $H^n(A, L) = 0$ for all n whenever $L \in \hat{A}$ is non-trivial.

By a result of Mukai in [6], the category of homogeneous (resp. unipotent) vector bundles over A is equivalent to that of coherent sheaves on $\widehat{A}$ with finite support (resp. with support at the origin). This is obtained by assigning to a coherent sheaf $\mathcal{F}$ with finite support on $\widehat{A}$, the sheaf $p_*(\mathcal{P} \otimes q^*\mathcal{F})$ on A, where $\mathcal{P}$ denotes the Poincaré sheaf on $A \times \widehat{A}$, and $p : A \times \hat{A} \to A$, $q : A \times \widehat{A} \to \widehat{A}$ denote the projections. (This defines the Fourier-Mukai transform, which yields in turn an equivalence of derived categories of coherent sheaves on A and $\widehat{A}$; see [7]).

We present a structure result for homogeneous bundles that yields another approach to these results. Assume for simplicity that k has characteristic 0. Then the following hold:

(a) A vector bundle E over A is homogeneous if and only if there exist an extension of commutative algebraic groups

$$0 \longrightarrow H \longrightarrow G \longrightarrow A \longrightarrow 0 \tag{1}$$

and a faithful representation

$$\rho : H \longrightarrow \mathrm{GL}(V) \tag{2}$$

such that E is the associated bundle $G \times^H V$ over $G/H = A$. Then there exists a unique minimal extension (1), where G is *anti-affine* (i.e., $\mathcal{O}(G) = k$).

(b) E is unipotent if and only if we may take for (1) the *universal extension of E by a vector group*,

$$0 \longrightarrow H(A) \longrightarrow E(A) \longrightarrow A \longrightarrow 0,$$

where $H(A) := H^1(A, \mathcal{O}_A)^*$ is a vector group of dimension $g := \dim(A)$.

As a consequence, given two unipotent vector bundles E_1, E_2, we have for the associated representations V_1, V_2 of the vector group $H(A)$:

$$\mathrm{Hom}_A(E_1, E_2) = \big(\mathcal{O}(G) \otimes \mathrm{Hom}(V_1, V_2)\big)^{H(A)} = \mathrm{Hom}^{H(A)}(V_1, V_2).$$

It follows that *the category of unipotent vector bundles over A is equivalent to the category* $\mathrm{Rep}\, H(A)$ of finite-dimensional representations of the vetor group $H(A)$. This implies in turn Mukai's result (in characteristic 0), since $\mathrm{Rep}\, H(A)$ is equivalent to the category of finite-dimensional representations of the polynomial ring $k[t_1, \ldots, t_g]$ on which each t_i acts nilpotently, i.e., to that of finite-dimensional modules over the local ring $\mathcal{O}_{\widehat{A},0}$. But the latter is equivalent to the category of coherent sheaves on $\widehat{A}$ with support at 0.

Also, our structure results readily implies the existence of a filtration as in (i) (since the representation V of the commutative group H admits a filtration with one-dimensional subquotients) and of a decomposition as in (ii) (by decomposing V into a sum of generalized weight spaces for H).

The anti-affine algebraic groups are classified in [2, 8], and the extensions (1) in [3]. In the latter preprint, the structure of homogeneous vector bundles is generalized to that of homogeneous principal bundles under an arbitrary algebraic group; recall that vector bundles of rank n correspond to principal bundles under the general linear group GL_n.

Also, principal bundles under the projective linear group PGL_n correspond to *projective bundles* of rank n, i.e., smooth proper morphisms with fibers isomorphic to the projective space $\mathbb{P}^{n-1}$. The structure of homogeneous projective bundles is described similarly as in (a) and (b), by replacing the representation (2) with a projective representation.

A new feature of these bundles is the existence of many *irreducible* bundles. Here we say that a homogeneous projective bundle is irreducible, if it has no proper homogeneous sub-bundle; equivalently, the associated projective representation $\rho : H \to \mathrm{PGL}(V)$ is irreducible. (The analogous notion for vector bundles just gives the homogeneous line bundles).

In fact, *the irreducible homogeneous projective bundles on A, up to bundle isomorphism, correspond bijectively to the pairs (H, e), where $H \subset \widehat{A}$ is a finite subgroup, and $e : H \times H \to \mathbb{G}_m$ a non-degenerate alternating bilinear map.*

To construct a projective bundle from such a pair, consider the exact sequence

$$0 \longrightarrow H \longrightarrow \widehat{A} \longrightarrow B \longrightarrow 0,$$

where B is some abelian variety. The dual exact sequence reads

$$0 \longrightarrow \widehat{H} \longrightarrow \widehat{B} \longrightarrow A \longrightarrow 0,$$

where $\widehat{H}$ denotes the character group of H. Now e yields an isomorphism $H \cong \widehat{H}$, and also a central extension

$$0 \longrightarrow \mathbb{G}_m \longrightarrow \tilde{H} \longrightarrow H \longrightarrow 0,$$

where $\tilde{H}$ denotes the associated *Heisenberg group.* Moreover, $\tilde{H}$ has a unique irreducible representation on which the center $\mathbb{G}_m$ acts by scalars; its dimension n satisfies $n^2 = \#(H)$. Thus, H acts irreducibly on $\mathbb{P}^{n-1}$; the desired projective bundle is the associated bundle

$$p : P = \widehat{B} \times^H \mathbb{P}^{n-1} \longrightarrow \widehat{B}/H = A.$$

For example, if A is an elliptic curve, then the subgroups H as above are exactly the n-torsion subgroups A_n; then the pairing e is uniquely determined up to multiplication by a primitive n-th root of unity. Thus, the homogeneous irreducible projective bundles on A are parametrized by the pairs (n, m) where n is a positive integer, and m an integer such that $0 \leq m < n$ and $(n, m) = 1$. In fact, every such bundle is the projectivization of an indecomposable vector bundle E of rank n and degree m (then E is uniquely determined up to tensoring with a line bundle, see [1]). But if $\dim(A) \geq 2$, then many homogeneous projective bundles cannot be obtained as projectivizations of vector bundles; namely, one can show that the Brauer group of A is generated by classes of homogeneous projective bundles.

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Classes of quiver cycles and quiver coefficients

Anders S. Buch

Let Q be a quiver with vertex set $\{1, 2, \dots, n\}$, and let $e = (e_1, \dots, e_n)$ be a dimension vector for Q. Set $E_i = \mathbb{C}^{e_i}$ for each i. The affine space of quiver representations $V = \bigoplus_{i \to j} \operatorname{Hom}(E_i, E_j)$ has a natural conjugation action of the group $G = \prod_{i=1}^n \operatorname{GL}(E_i)$. A **quiver cycle** is any G-stable closed irreducible subvariety $\Omega \subset V$. For example, any G-orbit closure is a quiver cycle. A quiver cycle Ω determines a G-equivariant cohomology class $[\Omega] \in H^*_G(V)$ and a G-equivariant Grothendieck class $[\mathcal{O}_\Omega] \in K_G(V)$. Notice that

$$H^*_G(V) = H^*_G(\text{point}) = \mathbb{Z}[c_{i,j}]_{1 \leq i \leq n \text{ and } 1 \leq j \leq e_i}$$

is a polynomial ring, where the variables $c_{i,1}, c_{i,2}, \dots, c_{i,e_i}$ are the Chern classes of $\operatorname{GL}(E_i)$. The cohomology class $[\Omega] \in H^*_G(V)$ is a polynomial in these variables. The K-theory ring $K_G(V)$ can be identified with the Grothendieck ring $\operatorname{Rep}(G)$ of virtual representations of G.

The classes $[\Omega]$ and $[\mathcal{O}_\Omega]$ can be interpreted as formulas for degeneracy loci as follows. Let X be a non-singular variety and let $\mathcal{E}_\bullet$ be a representation of Q on vector bundles over X, i.e. a collection of vector bundles $\mathcal{E}_i$ corresponding to the vertices $i \in \{1, 2, \dots, n\}$ together with vector bundle maps $\mathcal{E}_i \to \mathcal{E}_j$ corresponding to the arrows $i \to j$ of Q. Assume that $\operatorname{rank}(\mathcal{E}_i) = e_i$ for each i. For each point $x \in X$, the fiber $\mathcal{E}_\bullet(x)$ is representation of Q of dimension vector e. We define a degeneracy locus $\Omega(\mathcal{E}_\bullet) \subset X$ by

$$\Omega(\mathcal{E}_\bullet) = \{x \in X \mid \mathcal{E}_\bullet(x) \in \Omega\}.$$

This degeneracy locus has a natural structure of subscheme of X. Examples of degeneracy loci of this type include determinantal varieties and Schubert varieties in flag manifolds GL_m / P.

Proposition 1. *Assume that Ω is Cohen-Macaulay and that*

$$\operatorname{codim}(\Omega(\mathcal{E}_\bullet); X) = \operatorname{codim}(\Omega; V).$$

Assume also that X admits an ample line bundle. Then the (Chow) cohomology class $[\Omega(\mathcal{E}_\bullet)] \in H^*(X)$ is obtained from $[\Omega] \in H^*_G(V)$ by setting $c_{i,j} = c_j(\mathcal{E}_i)$ for all i, j.

The simplest interesting example is when $Q = \{1 \to 2\}$ has two vertices and one arrow. In this case any quiver cycle is a G-orbit closure defined by

$$\Omega = \{\phi \in \operatorname{Hom}(E_1, E_2) \mid \operatorname{rank}(\phi) \leq r\}$$

for some non-negative integer r. To describe the class $[\Omega]$ we need the following notation. Given an integer partition $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_\ell \geq 0)$ we define the Schur polynomial

$$S_\lambda(E_2 - E_1) = \det \left(h_{\lambda_i+j-i}\right)_{\ell\times\ell} \in H_G^*(V)$$

where the classes h_i are determined by the identity of power series

$$\sum_{i\geq 0} h_i T^i := \frac{1 - c_{1,1}T + c_{1,2}T^2 - \cdots \pm c_{1,e_1}T^{e_1}}{1 - c_{2,1}T + c_{2,2}T^2 - \cdots \pm c_{2,e_2}T^{e_2}}\,.$$

The classical Thom-Porteous formula states that $[\Omega] = S_\lambda(E_2 - E_1)$ in $H_G^*(V)$ for the partition $\lambda = (e_1 - r)^{e_2-r} = (e_1 - r, \ldots, e_1 - r)$ consisting of $e_2 - r$ copies of $e_1 - r$. The Grothendieck class of Ω is given by the analogous formula $[\mathcal{O}_\Omega] = \mathcal{G}_\lambda(E_2 - E_1) \in K_G(V)$ where $\mathcal{G}_\lambda$ denotes a stable Grothendieck polynomial. This formula is proved in [5].

Let Q be a quiver without oriented cycles, and let $\Omega \subset V = \bigoplus_{i\to j} \mathrm{Hom}(E_i, E_j)$ be a quiver cycle. For each vertex i, let $M_i = \bigoplus_{j\to i} E_j$ be the sum of all vertex vector spaces mapping to E_i. For example, the quiver $Q = \{1 \rightrightarrows 2 \leftarrow 3\}$ gives $M_2 = E_1 \oplus E_1 \oplus E_3$. The length $\ell(\lambda)$ of a partition λ is the number of non-zero parts of λ, and its weight is the sum $|\lambda| = \sum \lambda_i$ of its parts.

Definition 1. *The* **cohomological quiver coefficients** *of Ω are the unique integers $c_\mu(\Omega) \in \mathbb{Z}$, indexed by sequences $\mu = (\mu^1, \ldots, \mu^n)$ of partitions μ^i with $\ell(\mu^i) \leq e_i$, such that*

$$[\Omega] = \sum_\mu c_\mu(\Omega) \prod_{i=1}^n S_{\mu^i}(E_i - M_i) \in H_G^*(V)\,.$$

More generally, the K**-theoretic quiver coefficients** *of Ω are given by*

$$[\mathcal{O}_\Omega] = \sum_\mu c_\mu(\Omega) \prod_{i=1}^n \mathcal{G}_{\mu^i}(E_i - M_i) \in K_G(V)\,. \tag{1}$$

Since $H_G^*(V)$ is a graded ring, it follows that the cohomological quiver coefficients of Ω are indexed by sequences μ for which $|\mu| := \sum |\mu^i| = \mathrm{codim}(\Omega; V)$. These coefficients are a subset of the K-theoretic quiver coefficients, which are defined for sequences μ with $|\mu| \geq \mathrm{codim}(\Omega; V)$. The cohomological quiver coefficients for equioriented quivers of type A were introduce in [8]. This was extended to K-theory and more general quivers in [5,7]. Examples of quiver coefficients include the Littlewood-Richardson coefficients, Stanley coefficients, the monomial coefficients of Schubert polynomials, and the analogous K-theoretic constants [10,11].

Conjecture 1. *Let $\Omega \subset V$ be any quiver cycle.*

(a) *The cohomological quiver coefficients of Ω are non-negative, i.e. $c_\mu(\Omega) \geq 0$ for $|\mu| = \mathrm{codim}(\Omega; V)$.*

(b) *The K-theoretic coefficient $c_\mu(\Omega)$ is non-zero for only finitely many sequences μ, i.e. the sum (1) is finite.*

(c) *If Ω has rational singularities, then the K-theoretic quiver coefficients of Ω have alternating signs, i.e.* $(-1)^{|\mu|-\operatorname{codim}(\Omega;V)}\, c_\mu(\Omega) \geq 0$.

This conjecture is motivated in part by Schubert calculus on flag varieties G/P. If $Y \subset G/P$ is any closed irreducible subvariety, then the cohomology class $[Y] \in H^*(G/P)$ can be uniquely written as a linear combination of Schubert classes, and the coefficients in this combination are non-negative integers. Furthermore, a result of Brion states that if Y has rational singularities, then its Grothendieck class $[\mathcal{O}_Y] \in K(G/P)$ is a linear combination of Schubert structure sheaves with alternating signs [3].

The Conjecture is known when $Q = \{1 \to 2 \to \cdots \to n\}$ is an equioriented quiver of type A. Special cases of (a) were proved Buch, Kresch, Tamvakis, and Yong [4, 10] after which the general case was proved by Knutson, Miller, and Shimozono [14]. Part (b) was proved by Buch [5], and part (c) was proved by Buch [6] and by Miller [17].

Now suppose that Q is a quiver of Dynkin type. In this case Fehér and Rimányi have given a set of linear equations that uniquely determine the cohomology class $[\Omega] \in H^*_G(V)$ [13]. These equations simply say that the restriction of $[\Omega]$ to any disjoint G-orbit in V is zero. Reineke has given an explicit resolution of the singularities of Ω [18]. Under the assumption that Ω has rational singularities, this resolution has been used to prove formulas for the K-theory class $[\mathcal{O}_\Omega]$, by Knutson and Shimozono [15] and by Buch [7]. The latter paper expresses the class $[\mathcal{O}_\Omega]$ in terms of quiver coefficients and proves part (b) of the conjecture, as well as part (c) when Q is of type A_3. All quiver cycles of Dynkin type A or D are known to have rational singularities by results of Bobiński and Zwara [1, 2] (see also [16, 19] for the case of equioriented quivers of type A).

We refer to [6, 9, 12, 14] for a different type of positivity of quiver cycle classes, which has been proved for the cohomology class of any quiver cycle of type A and for the K-theory class of equioriented quiver cycles of type A.

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Complete resolutions over complete intersections

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(joint work with Thuy Pham and Collin Roberts)

In his groundbreaking paper [6], Eisenbud elucidated the structure of projective resolutions over complete intersections and introduced, in particular, matrix factorizations as a tool to study maximal Cohen–Macaulay (MCM) modules over hypersurface rings. In [1] it was shown how to perform his analysis in the category of DG modules over the Koszul complex.

Here we extend that approach to construct complete resolutions and show that MCM modules over complete intersection rings are as well determined by matrix factorizations, albeit over a larger ring. For a conceptual geometric interpretation of this latter point of view, see Isik's recent work [7].

Complete Resolutions over Quasi–Frobenius Algebras. We fix a commutative ring K, and unadorned tensor products are taken over it. In the graded context, K will be concentrated in degree zero and $(-)^\vee$ denotes the (graded) K–dual. Without further qualification, modules are right modules.

Definition 1. *A K–algebra A is* Quasi–Frobenius *if A is finite*[1] *projective as K–module and the A–bimodule $\omega = A^\vee$ is invertible, equivalently, for every ring homomorphism from K to a field k, the Artin algebra $A\otimes k$ is self-injective.*

If A is a (graded) Quasi–Frobenius algebra, so are its (graded) opposite algebra A^{op} and its (graded) enveloping algebra $A^{\mathrm{ev}} = A^{\mathrm{op}} \tilde{\otimes} A$, where $\tilde{\otimes}$ records the (graded) tensor product algebra structure. Note that $\mathrm{Hom}_K(A^{\mathrm{ev}}, K) \cong \omega\otimes\omega$ and $\mathrm{Hom}_{A^{\mathrm{ev}}}(A, A^{\mathrm{ev}}) \cong \omega^{-1}$ as A–bimodules.

[1] *finite*=finitely generated=finitely presented for projective modules.

Definition 2. *A* complete resolution C *over* A *(of* $M = \mathsf{Im}(\partial : C_0 \to C_{-1})$*) is an acyclic complex of finite projective* A*–modules with its dual complex* $C^* = \mathrm{Hom}_A(C, A)$ *still acyclic, then necessarily a complete resolution of* M^* *over* A^{op}*.*

Complete resolutions are stable under translation, thereby completely resolving any (positive or negative) syzygy module of M*.*

Complete resolutions are unique up to homotopy, and so the *stable* or *complete* extension groups $\widehat{\mathrm{Ext}}^i_A(M, N) = H^i\big(\mathrm{Hom}_A(C, N)\big)$ are well defined.

For a *Frobenius algebra* the first part of the following is due to Nakayama [8].

Theorem 1. *Let* $\mathcal{P}_\bullet \xrightarrow{\mu} A$ *be a resolution of* A *by finite projective* A^{ev}*–modules, set* $(-)^\star = \mathrm{Hom}_{A^{\mathrm{ev}}}(-, A^{\mathrm{ev}})$ *and abbreviate further* ${}^\star(-) = (-)^\star \otimes_A \omega$*.*

(1) *The complex* $\mathbb{CR}(A) = \mathsf{cone}({}^\star\mu \circ \mu)[-1]$ *of* A*–bimodules,*

$$\begin{tikzcd} \cdots \arrow[r] & \mathcal{P}_1 \arrow[r] & \mathcal{P}_0 \arrow[rr, "{}^\star\mu\circ\mu"] \arrow[dr, "\mu"'] & & {}^\star\mathcal{P}_0 \arrow[r] & {}^\star\mathcal{P}_1 \arrow[r] & \cdots \\ & & & A \arrow[ur, "{}^\star\mu"'] & & & \end{tikzcd}$$

is a complete resolution of A *as a bimodule over itself.*

(2) *For* M *an* A*–module that is finite projective over* K*, the complex* $\mathbb{CR}_A(M) = M \otimes_A \mathbb{CR}(A)$ *constitutes a complete resolution of* M *over* A*.*

Corollary 1. *Over a (graded) Quasi–Frobenius algebra* A *any (graded) module* M *that is finite projective over* K *defines a homomorphism of (bi-)graded rings* $M \otimes_A (-) : \widehat{\mathrm{HH}}^\bullet(A/K) = \widehat{\mathrm{Ext}}^\bullet_{A^{\mathrm{ev}}}(A, A) \to \widehat{\mathrm{Ext}}^\bullet_A(M, M)$*, whose source is the stable Hochschild cohomology of* A *over* K *and whose image is in the graded centre (for the total degree) of the stable* Ext*–algebra of* M *over* A*.*

Complete Resolutions over Hopf Algebras. (cf. [4, Ch.XII]) If A is a (graded) Hopf K–algebra that is finite projective as K–module, then it is Quasi–Frobenius. Given a (graded) A–module N, the comultiplication $\Delta : A \to A \tilde{\otimes} A$ endows for any A–module M the $A \tilde{\otimes} A$–module $M \otimes N$ with the *twisted* A*–module structure* $M \boxtimes N = \Delta_*(M \otimes N)$. For a (graded) projective A–module P, $M \boxtimes P$ is isomorphic to $M \otimes P$ viewed as A–module through the second factor; see [2, Prop.3.1.5]. From a (graded) complete resolution $\mathbb{C}$ of the augmentation module K over A, one recovers a (graded) complete resolution of A over A^{ev} as $A \boxtimes \mathbb{C}$, and then $\mathbb{CR}_A(M) \cong M \boxtimes \mathbb{C}$ for every A–module M. Accordingly, $A \boxtimes (-) : \widehat{H}^\bullet(A) = \widehat{\mathrm{Ext}}^\bullet_A(K, K) \to \widehat{\mathrm{Ext}}^\bullet_{A^{\mathrm{ev}}}(A, A)$ defines on $\widehat{\mathrm{HH}}^\bullet(A/K)$ the structure of an augmented graded algebra over the graded commutative complete cohomology ring $\widehat{H}^\bullet(A)$.

Complete Resolutions over an Exterior Algebra. Let F be a finite projective K–module, $\Lambda^\bullet = \oplus_i \Lambda^i F$ the exterior algebra over it. This is a graded Hopf algebra over K with $\omega \cong (\det F)^\vee \otimes \Lambda$. A (graded) projective resolution of the augmentation or co-unit was first exhibited by H. Cartan [3] and is given by

$$\cdots \longrightarrow \Gamma_i \otimes \Lambda \xrightarrow{\ \partial\ } \cdots \xrightarrow{\ \partial\ } F \otimes \Lambda \xrightarrow{\ \partial\ } \Lambda(\xrightarrow{\ \epsilon\ } K \longrightarrow 0)$$

where $\Gamma_\bullet = \oplus_i \Gamma_i F$ is the divided power algebra on F over K and ∂ is the Λ–linear derivation on $\Gamma\otimes\Lambda$ induced by $\mathrm{id}_F : \Gamma_1 = F \to F = \Lambda^1$. Now $\Gamma_i^\vee \cong S^i = S^i(F^\vee)$, the symmetric power on the K–dual of F, and the preceding results yield:

Theorem 2. *A Λ–module E that is finite projective as K–module admits a complete resolution of the form*

$$\begin{tikzcd}
\cdots \arrow[r] & E\otimes\Gamma_1\otimes\Lambda \arrow[r] & E\otimes\Lambda \arrow[rr, "{}^\vee\rho\circ\rho"] \arrow[dr, "\rho"'] & & E\otimes\omega \arrow[r] & E\otimes S^1\otimes\omega \arrow[r] & \cdots \\
& & & E \arrow[ur, "{}^\vee\rho"'] & & &
\end{tikzcd}$$

where $\rho = E\boxtimes\epsilon$ represents the right module structure on E and ${}^\vee\rho = E\boxtimes\epsilon^\vee$.

Corollary 2. *The cohomology ring $H^\bullet(\Lambda) = \mathrm{Ext}^\bullet_\Lambda(K,K)$ is the polynomial ring*[2] *$K[F^\vee]$, its Hochschild cohomology is*[3] *$\mathrm{HH}^\bullet(\Lambda/K) \cong H^\bullet(\Lambda)\otimes\Lambda$. The complete cohomology ring is naturally isomorphic to the total cohomology ring of $\mathbb{P} = \mathbb{P}_K(F)$, the projective space defined by F over K, that is, $\widehat{H}^\bullet(\Lambda) \cong \bigoplus_{a+b=\bullet} H^a(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(b))$.*

The complete Hochschild cohomology ring satisfies $\widehat{\mathrm{HH}}^\bullet(\Lambda/K) \cong \widehat{H}^\bullet(\Lambda)\otimes\Lambda$.

Complete Resolutions over Complete Intersections. Assume $\lambda : F \to K$ is a K–linear form on the finite projective K–module F such that the associated Koszul complex $\Lambda = (\Lambda, \partial = \partial_\lambda)$ resolves $R = H^0(\Lambda, \partial)$. In other words, R is a (Koszul-regular) *complete intersection* in K.

If now E is a DG module over Λ with underlying finite projective K–module, then the complete resolution just exhibited becomes a complex of projective DG Λ–modules that is still acyclic.

Theorem 3. *The total complex of the tensor product of a complete Λ–resolution of such E over Λ with R results in a complete R–resolution*

$$\cdots \longrightarrow E\otimes\Gamma_1\otimes R \longrightarrow E\otimes R \longrightarrow E\otimes\omega_{R/K} \longrightarrow E\otimes S^1\otimes\omega_{R/K} \longrightarrow \cdots$$

where $\omega_{R/K} \cong \mathbb{R}\,\mathrm{Hom}_K(R,K) \cong \det F^\vee\otimes R[-\operatorname{rk} F]$ is the relative dualizing complex of R over K. It resolves completely any sufficiently high syzygy R–module in a projective R–resolution of $E\otimes^{\mathbb{L}}_\Lambda R$, in turn quasiisomorphic to E.

1. *For a yet more concrete description, let $F = \oplus_{i=1}^c K\sigma_i$ be a free K–module and $\lambda(\sigma_i) = f_i$ a regular sequence in the maximal ideal of the local noetherian regular (or just Gorenstein) ring K, with Λ the Koszul complex over λ, and $R = K/(f_1,...,f_c)$ its homology. Let $E = \oplus_{i=0}^c E_i$ be a graded, finite free K–module, endowed with a differential $A = A_\bullet : E \to E[1]$, and with K–linear maps $B^i = B^i_\bullet : E \to E[1], i = 1,...,c$. Further, denote $\Gamma_R = \Gamma_R(\zeta_1,...,\zeta_c)$ the free divided power algebra over R, with ζ_i in (homological) degree 2, $S \cong R[s_1,...,s_c]$ its graded R–dual, and observe that $\omega_R \cong R\frac{1}{\sigma_1\cdots\sigma_c}$ is the dualizing module of R, based on the dual of the volume form $\sigma_1\cdots\sigma_c$ that in turn bases $\det F = \Lambda^c$ over K.*

Theorem 4. *Set $M = \mathsf{cok}(A_0 : E_1 \to E_0)$. The following are equivalent.*

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[2]Note that $F^\vee$ is concentrated in cohomological degree 1, but total degree 2.

[3]Exactly as for group algebras over finite abelian groups; see [5].

(1) *M is a maximal Cohen–Macaulay (MCM) R–module.*
(2) *Data (E, A, B) define on E the structure of a DG Λ–module, with B_i representing multiplication by $\sigma_i \in \Lambda^1 = F$. The complex (E, A) then resolves M over K.*
(3) *Viewing $\alpha = A + \sum_{i=1}^{c} B^i s_i$ as an endomorphism of the graded S–module $E \otimes S$, one has $\alpha^2 = \big(\sum_{i=1}^{c} f_i s_i\big)\,\mathrm{id}_{E\otimes S}$.*

The MCM R–module M, the DG module (E, A, B), and the graded matrix factorization (α, α) of $\sum_{i=1}^{c} f_i s_i$ determine each other up to the appropriate notion of homotopy equivalence. Furthermore, when (1) through (3) hold,

(a) *$(E\otimes\Gamma_R, \alpha)$ is a projective resolution of M over R; see* [1, Thm.2.4].
(b) *Shifting by $[-1]$ the mapping cone over the composition of*

$$(E\otimes\Gamma_R, \alpha) \xrightarrow{proj} E_0 \otimes R \xrightarrow{\beta = \frac{B^1 \cdots B^c}{\sigma_1 \cdots \sigma_c}} E_c \otimes \omega_R \xrightarrow{incl} (E\otimes S\otimes\omega_R, \alpha\otimes\omega_R)$$

produces a complete resolution of the R–module $M \cong \operatorname{Im}\beta$.
(c) *$\widehat{\operatorname{Ext}}^{\bullet}_{R\otimes^{\mathbb{L}}_K R}(R, R) \cong \bigoplus_{a+b=\bullet} H^a(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(b))$, the total cohomology ring of the projectivized normal bundle $\mathbb{P} = \mathbb{P}_R(F\otimes R)$ of R relative to K, and this graded ring acts naturally on the stable graded extension groups $\widehat{\operatorname{Ext}}^{\bullet}_R(M, N)$.*

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Generic bases and cluster character identities for unipotent groups

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(joint work with Bernard Leclerc and Jan Schröer)

1. Cluster Algebras and Cluster Characters

For our purpose, a *seed* is a tuple $(B, \underline{f})$ with $B \in \mathbb{Z}^{m\times m}$ antisymmetric, and $\underline{f} = (f_1, \ldots, f_m)$ a transcendence base of the rational function field $\mathbb{C}(x_1, \ldots, x_m)$. We consider the equivalence relation $\sim$ on seeds which is generated by seed mutations.

The *cluster algebra* $\mathcal{A}(B, \underline{x})$ is the $\mathbb{C}$-subalgebra of $\mathbb{C}(x_1, \ldots, x_m)$ which is generated by the (certainly redundant) set of *cluster monomials*

$$\bigcup_{(B',\underline{x}')\sim(B,\underline{x})} \{(\underline{x}')^{\mathbf{e}} := \prod_{i=1}^{m} {x'_i}^{\mathbf{e}(i)} \mid \mathbf{e} \in \mathbb{N}^m\}.$$

By the Laurent phenomenon, $\mathcal{A}(B, \underline{x})$ is contained in the *upper cluster algebra*

$$\mathcal{A}^+(B, \underline{x}) := \bigcap_{(B',\underline{x}')\sim(B,\underline{x})} \mathbb{C}[{x'}_1^{\pm}, \ldots, {x'}_m^{\pm}] \subset \mathbb{C}(x_1, \ldots, x_m).$$

We identify B to a quiver with vertices $\{1, \ldots, m\}$ and $\max\{B_{ji}, 0\}$ arrows from i to j. Thus, as a quiver, B has no loops or oriented 2-cycles. Fix a generic potential W and consider the corresponding Jacobian algebra $J := \mathrm{Jac}(B, W)$. We assume for this talk that J is finite-dimensional. Thus, the *E-invariant* of a J-module M is $E(M) := \dim \mathrm{Hom}_J(\tau^- M, M)$, see [5, Sec. 10].

The following "cluster character" for J-modules M has been studied in slightly different versions by several authors in the context of categorification of cluster algebras:

$$C_M^J := \underline{x}^{\mathbf{g}_M} \sum_{\mathbf{e}\in\mathbb{N}^m} \chi(\mathrm{Gr}_{\mathbf{e}}^J(M))\widehat{\underline{x}}^{\mathbf{e}} \text{ for } M \in J\text{-mod},$$

where

$$\underline{x}^{\mathbf{g}_M} := \prod_{i=1}^{m} x_i^{\dim \mathrm{Ext}^1_J(S_i,M) - \dim \mathrm{Hom}_J(S_i,M)}, \quad \widehat{\underline{x}}^{\mathbf{e}} := \prod_{i=1}^{m} x_i^{(B\cdot\mathbf{e})_i},$$

and $\mathrm{Gr}_{\mathbf{e}}^J(M)$ is the projective variety whose points are the submodules of M with dimension vector $\mathbf{e}$, and χ denotes the (topological) Euler characteristic.

Remark 1. *Let $(B', \underline{x}') = \mu_k(B, x)$ be the seed obtained by seed mutation in direction k. Since W is generic, we can find a potential W' for B' such that $J' = \mathrm{Jac}(B', W') = \mu_k(J)$. Following [4] we can consider $\mu_k(M) \in J'$-mod. Now, the key-lemma [5, Lemma 5.2] can be interpreted as the equation*

$$C_M^J(x_1, \ldots, x_m) = C_{\mu_k(M)}^{J'}({x'}_1, \ldots, {x'}_m) \in \mathcal{A}^+(B, \underline{x}).$$

2. Strongly reduced components and generic basis conjecture

Consider $\mathrm{Rep}_{\mathbf{d}}(J)$, the (affine) variety of representations of J which have dimension vector $\mathbf{d} \in \mathbb{N}^m$. The group $\mathrm{GL}_{\mathbf{d}} := \times_{i=1}^m \mathrm{GL}_{\mathbf{d}(i)}(\mathbb{C})$ acts on it by conjugation, thus orbits are in bijection with isoclasses of modules. We note, that $C_?\colon \mathrm{Rep}_{\mathbf{d}}(J) \to \mathbb{C}[x_1^{\pm}, \ldots, x_m^{\pm}]$ is a $\mathrm{GL}_{\mathbf{d}}$-invariant constructible function. In particular, it makes sense to consider the generic value $C_{\mathcal{Z}}$ of $C_?$ on an irreducible component $\mathcal{Z} \in \mathrm{Irr}(\mathrm{Rep}_{\mathbf{d}}(J))$.

Definition 1. *An irreducible component $\mathcal{Z} \in \mathrm{Irr}(\mathrm{Rep}_{\mathbf{d}}(J))$ is called* strongly reduced *if the subset*

$$\{M \in \mathcal{Z} \mid \mathrm{codim}_{\mathcal{Z}}(\mathrm{GL}_{\mathbf{d}} .M) = E(M)\}$$

is dense in $\mathcal{Z}$.

Remark 2. *A strongly reduced component is (scheme theoretically) generically reduced by Voigt's Lemma, since $\dim \mathrm{Hom}_J(\tau^- M, M) \geq \dim \mathrm{Ext}_J^1(M, M)$. The converse is not true in general. However, in case B is acyclic (and thus $W = 0$), $\mathrm{Rep}_{\mathbf{d}}(\mathbb{C}B)$ is irreducible and strongly reduced. We denote by $\mathrm{Irr}_{sr}(\mathrm{Rep}_{\mathbf{d}}(J))$ the set of strongly reduced irreducible components of $\mathrm{Rep}_{\mathbf{d}}(\mathbb{C}B)$.*

Conjecture 1. *The set*

$$\mathcal{G} := \{\underline{x}^{\mathbf{a}} C_{\mathcal{Z}} \mid \mathcal{Z} \in \mathrm{Irr}_{sr}(\mathrm{Rep}_{\mathbf{d}}(J)),\ \mathbf{a}, \mathbf{d} \in \mathbb{N}^m, \mathbf{a} \cdot \mathbf{d} = 0\},$$

is a basis of the cluster algebra $\mathcal{A}(B, \underline{x})$.

Remark 3. *(1) $E(M) = 0$ implies that the closure of the $\mathrm{GL}_{\underline{\dim}(M)}$-orbit of M is a strongly reduced component. Thus, it follows from [5] that the set of cluster monomials is contained in $\mathcal{G}$.*

(2) Exploiting a bit further Remark 1 one can see that $\mathcal{G}$ is invariant under mutation of quivers with (generic) potential. However, it is conceivable that $\mathcal{G}$ depends on the choice of the potential W itself.

3. The dual semicanonical basis is generic

We show in this section that (part of) Lusztig's dual semicanonical basis can be viewed (after specializing coefficients to 1) as a generic basis in the above sense for a large class of cluster algebras coming from representation theory.

Definition 2. *Let $A \in \mathbb{Z}^{n \times n}$ be a symmetric generalized Cartan matrix, and $\mathbf{i} = (i_r, \ldots, i_2, i_1)$ a reduced expression for an element w of the corresponding Weyl group. We may assume $\{1, 2, \ldots, n\} = \{i_1, \ldots, i_r\}$. Following [2], we associate to this data a quiver $\widetilde{Q}_{\mathbf{i}}$ with vertices $\{1, 2, \ldots, r\}$. For $k \in \{1, \ldots, r\}$ set $k^+ := \min(\{k < l \leq r+1 \mid i_l = i_k\} \cup \{r+1\})$. Then the vertices are $\widetilde{Q}_{\mathbf{i},0} := \{1 \leq k \leq r \mid k^+ \neq r+1\}$. For each k with $k^+ \neq r+1$ there is an arrow $k^+ \to k$. Moreover, there are $-A_{i_k, i_l}$ arrows from l to k if $k^+ > l^+ > k > l$. Finally, $\underline{\widetilde{Q}}_{\mathbf{i}}$ is the full subquiver of $\widetilde{Q}_{\mathbf{i}}$ with vertices $\{1 \leq k \leq r \mid k^+ \neq r+1\}$. We call the elements of this class of quivers* Coxeter quivers.

Remark 4. *It is easy to see that different reduced expressions for the same Weyl group element yield mutation equivalent quivers. Moreover, each acyclic quiver is mutation equivalent to some $\widetilde{Q}_{\underline{\mathbf{i}}}$.*

The quivers $\widetilde{Q}_{\underline{\mathbf{i}}}$ admit a rigid potential, and the same is true for all quivers which are mutation equivalent to some $\widetilde{Q}_{\underline{\mathbf{i}}}$, see [3].

Theorem 1 ([8, Thm. 5])**.** *Let $\widetilde{Q}$ be a quiver which is mutation equivalent to a Coxeter quiver $\widetilde{Q}_{\underline{\mathbf{i}}}$, then $\mathcal{G}$ is a basis of the cluster algebra $\mathcal{A}(\widetilde{Q}, \underline{x})$. In particular, the cluster monomials are linearly independent.*

The idea behind the proof is to identify $\mathcal{G}$ with $\mathcal{S}^*$, the dual of Lusztig's semicanonical basis. Let us explain this, for the technically easier special case, when A is positive and $w = w_0$ is the longest element of W. The starting point is to dualize Lusztig's lagrangian construction of the positive part $U(\mathfrak{n})$ of the enveloping algebra $U(\mathfrak{g})$ of the simple, simply-laced Lie algebra $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}$ associated with A, together with the corresponding semicanonical basis: Let Q be a Dynkin quiver such that the underlying graph $|Q|$ corresponds to the Cartan matrix A, and $\mathbf{i}$ a reduced expression for w_0 which is adapted to Q.

Now, let $\Lambda := \Pi_1(\mathbb{C}Q)$ be the corresponding preprojective algebra, and consider for each dimension vector $\mathbf{d} \in \mathbb{N}^n$ the affine variety $\Lambda_{\mathbf{d}}$ of Λ-modules with dimension vector $\mathbf{d}$. One obtains for each dimension vector $\mathbf{d}$ a constructible $\mathrm{GL}_{\mathbf{d}}$-invariant function

$$\varphi_? \colon \Lambda_{\mathbf{d}} \to \mathbb{C}[N] := U(\mathfrak{n})^*_{\mathrm{gr}}.$$

For $Y \in \Lambda$-mod the regular function $\varphi_Y \in \mathbb{C}[N]$ can be described as follows: For each sequence $\mathbf{j} = (j_1, \dots, j_l) \in \{1, \dots, n\}^l$ we have

$$\varphi_Y(x_{j_1}(t_1) \cdot \dots \cdot x_{j_l}(t_l)) = \sum_{\mathbf{a} \in \mathbb{N}^l} \chi(\mathcal{P}(\mathbf{j}^{\mathbf{a}}, Y))\, \underline{t}^{\mathbf{a}},$$

where $x_j \colon \mathbb{C} \to N$ is the standard one-parameter subgroup associated to the simple root α_j, and $\mathcal{P}(\mathbf{j}^{\mathbf{a}}, Y)$ is the (projective) variety of flags of submodules

$$\{Y_\bullet = (0 = Y_0 \subset Y_1 \subset \dots \subset Y_l = Y) \mid Y_k/Y_{k-1} \cong S_{j_k}^{a_k}, k = 1, \dots, l\}.$$

If we denote by $\varphi_{\mathcal{Z}}$ the generic value of $\varphi_?$ on a component $\mathcal{Z} \in \mathrm{Irr}(\Lambda_{\mathbf{d}})$, then

$$\mathcal{S}^* := \{\varphi_{\mathcal{Z}} \mid \mathcal{Z} \in \mathrm{Irr}(\Lambda_{\mathbf{d}}), \mathbf{d} \in \mathbb{N}^n\}$$

is the dual semicanonical basis of the coordinate ring $\mathbb{C}[N]$ of the unipotent group N with $\mathrm{Lie}(N) = \mathfrak{n}$.

The category Λ-mod has a cluster structure in the sense of [2], with a canonical cluster tilting object $V = \oplus_{k=1}^r V_k$. Using that $\varphi_?$ is a cluster character, one can show that $\mathbb{C}[N]$ is a cluster algebra with initial seed $(\widetilde{Q}_{\mathbf{i}}, (\varphi_{V_1}, \dots, \varphi_{V_r})))$. Here, the φ_{V_k} are certain (generalized) flag minors, and the projective injective summands of V correspond to coefficients in this cluster algebra. Moreover, all cluster monomials are of the form φ_X for some rigid Λ-module X. In particular, all cluster monomials belong by Voigt's lemma to $\mathcal{S}^*$.

Now, we can relate this result with the generic basis setup by the following key facts:

- The projective variety $\mathcal{P}(\mathbf{i}^{\mathbf{a}}, Y)$ is isomorphic to a quiver grassmannian $\mathrm{Gr}^{J'}_{\mathbf{d}(\mathbf{a})}(\mathrm{Ext}^1_\Lambda(W, Y))$ where the cluster tilting object W is obtained from V by Heller's loop functor, and $J' := \underline{\mathrm{End}}_\Lambda(W)^{\mathrm{op}}$ is given by a quiver $\underline{\widetilde{Q}}_{\mathbf{i}}$ with rigid potential.
- Since W is reachable by mutations from V, we obtain from this by the Key-Lemma [5, 5.2] a Fu-Keller [6] like cluster expansion

$$\varphi_Y = \underline{\varphi}_T^{\left(\underline{\dim}_{\tilde{J}_T} \mathrm{Hom}_\Lambda(T,Y)\right)\cdot B_T} \sum_{\mathbf{e}} \chi\left(\mathrm{Gr}^{J_T}_{\mathbf{e}}\left(\mathrm{Ext}^1_\Lambda(T, Y)\right)\right) \widehat{\underline{\varphi}}_T^{\mathbf{e}}$$

  for each cluster tilting object T reachable from V, and each Λ-module Y. Here, B_T is the matrix of the Ringel bilinear form of $\tilde{J}_T := \mathrm{End}_\Lambda(T)$, and the stable endomorphism ring $J_T := \underline{\mathrm{End}}_\Lambda(T)^{\mathrm{op}}$ is given in this situation by a quiver with rigid potential [3].
- Next, we have a natural bijection between the generically $\mathrm{add}(T)$-free irreducible components of the varieties $\Lambda_{\mathbf{d}}$ and the strongly reduced components of the varieties $\mathrm{Rep}_{\mathbf{a}}(J_T)$ thanks to the observation that for $X \in \Lambda_{\mathbf{d}}$ we have

$$\mathrm{codim}_{\Lambda_{\mathbf{d}}}(\mathrm{GL}_{\mathbf{d}} .X) = \frac{1}{2} \dim \mathrm{Ext}^1_\Lambda(X, X) = E_{J_T}(\mathrm{Ext}^1_\Lambda(T, X)).$$

  Here, the second equality holds, since the stable module category $\Lambda\text{-}\underline{\mathrm{mod}}$ is a generalized cluster category in the sense of Amiot [1]. Thus we can apply [1, Prp. 2.12], see [8, Prp. 6.1] for details.
- Finally, we recall, that the dual semicanonical basis of $\mathbb{C}[N]$ specializes to a basis with similar properties for the corresponding cluster algebra with trivial coefficients [7, Sec. 15].

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Quiver representations and the cohomology of Hitchin fibres

Tamás Hausel

In our recent work with Letellier and Villegas [4] a picture is emerging relating the representations of quivers with character varieties and the representation theory of finite groups and algebras of Lie type. For Γ a comet-shaped quiver (a quiver with k-legs and g-loops on a central vertex) and dimension vector μ we associate a character variety $\mathcal{M}_\mu$ which can always be arranged to be non-singular and when μ is indivisible a non-singular quiver variety $\mathbb{Q}_\mu$. When additionally $g = 0$ we have the Riemann-Hilbert monodromy map $\mathbb{Q}_\mu \to \mathcal{M}_\mu$ and the induced map $H^*(\mathcal{M}_\mu) \to H^*(\mathbb{Q}_\mu)$. The purity conjecture says that this map is an isomorphism on pure parts. Taking Poincaré polynomials gives:

$$PP_c(\mathcal{M}_\mu; t) = P_c(\mathbb{Q}_\mu; t) \overset{[3]}{=} A_\Gamma(\mu, t^2),$$

where $A_\Gamma(\mu, q))$ is Kac's number for μ-dimensional absolutely indecomposable representations of Γ over $\mathbb{F}_q$. When μ is no longer indivisible and g not necessarily 0 the purity conjecture takes the form

Conjecture 1. $PP_c(\mathcal{M}_\mu; t) = t^{2d_\mu} A_\Gamma(\mu, t^2)$

One implication of this conjecture would be

Conjecture 2 ([5]). $A_\Gamma(\mu, q) \in \mathbb{N}[q]$.

This is known for μ indivisible by [3] but open for any wild quiver. Here we have a more general conjecture for the mixed Hodge polynomial $H(\mathcal{M}_\mu; q, t) := \sum_k \dim(Gr^W_{2k} H^l(\mathcal{M}_\mu)) q^k t^l$

Conjecture 3 ([4]).

$$H(\mathcal{M}_\mu; z^2, -\frac{1}{zw}) w^{d_\mu}$$
$$= (z^2-1)(1-w^2) \left\langle \operatorname{Log}\left(\sum_{\lambda\in\mathcal{P}} \mathcal{H}_\lambda(z,w) \prod_{i=1}^{k} \tilde{H}_\lambda(\mathbf{x}_i; z^2, w^2) \right), h_\mu) \right\rangle, \quad (1)$$

where $\tilde{H}_\lambda(\mathbf{x}_i; q, t)$ are Macdonald polynomials.

Theorem 1 ([4]). $PH_c(\mathcal{M}_\mu; q) = A_\Gamma(\mu, q)$.

Let $\mathfrak{M}_\mu$ be the moduli space of parabolic Higgs bundles on the Riemann surface with generic parabolic weights and quasi-parabolic structure given by μ. Then by non-Abelian Hodge theorem $H^*(\mathcal{M}_\mu) \cong H^*(\mathfrak{M}_\mu)$. Let $\chi_\mu : \mathfrak{M}_\mu \to \mathbb{A}_\mu$ be the Hitchin fibration. Let $W_k(H^*(\mathcal{M}_\mu))$ denote the weight filtration and $P_k(H^*(\mathfrak{M}_\mu))$ denote the perverse filtration associated to the Hitchin map.

Conjecture 4 ([1]). $W_{2k}(H^*(\mathcal{M}_\mu)) \cong P_k(H^*(\mathfrak{M}_\mu))$

If so this would give an alternative geometrical meaning for

$$H_c(\mathcal{M}_\mu; q, t) = \sum_k \dim(Gr^P_k H^l(\mathfrak{M}_\mu)) q^k t^l$$

which conjecturally are given by Macdonald polynomials as in Conjecture 3 and whose "pure" parts should be given by $A_\Gamma(\mu, q)$ by Conjecture 1.

An overarching aim is to find cohomological meaning for $A_\Gamma(\mu, q)$ for Γ not necessarily comet-shaped. For this we offer the following: for any quiver Γ and dimension vector $\mathbf{v}$ construct the projective curve $C_{\Gamma,\mathbf{v}}$ for which $C_{\mathbf{v}}^{red}$ has non-singular rational components, with a component for each vertex of Γ, nodal intersection of components for edges of Γ and multiplicity v_i of the ith component. One can construct a certain compactified Jacobian $\mathfrak{M}_{\mathbf{v}} := \overline{J(C_{\mathbf{v}})}$. Its cohomology will be equipped with a perverse filtration $P_k(H^*(\mathfrak{M}_{\mathbf{v}}))$ induced from some universal Hitchin system. Finally let $\mathcal{M}_{\mathbf{v}}$ be the Crawley-Boevey-Shaw generalized character variety of [2] attached to $(\Gamma, \mathbf{v})$.

Conjecture 5. *There are graphical non-Abelian Hodge theory isomorphisms $H^*(\mathcal{M}_{\mathbf{v}}) \cong H^*(\mathfrak{M}_{\mathbf{v}})$ inducing*

$$W_{2k}(H^*(\mathcal{M}_{\mathbf{v}})) \cong P_k(H^*(\mathfrak{M}_{\mathbf{v}}));$$

and for the mixed Hodge polynomial

$$H(\mathcal{M}_{\mathbf{v}}; q, t) = \sum_k \dim(Gr_k^P H^l(\mathfrak{M}_{\mathbf{v}}))q^k t^l.$$

As a generalization of Theorem 1 we should have

$$PH(\mathcal{M}_{\mathbf{v}}; t) = PPH(\mathfrak{M}_{\mathbf{v}}; t) = t^{2d_\mu} A_\Gamma(\mathbf{v}, 1/t^2),$$

giving a cohomological interpration for the latter for every quiver $(\Gamma, \mathbf{v})$.

If we allow more complicated singularities for the curve than nodes, one may expect a similar circle of ideas, although the quiver will have to be enhanced to reflect the singularities. The cohomology of the Hitchin fibre will lead to orbital integrals on p-adic groups and Khovanov cohomology of links.

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Rational surfaces and tilting bundles

Lutz Hille
(joint work with Markus Perling)

1. The Main Result

We consider a smooth and projective algebraic variety X over an algebraically closed field k. A *tilting bundle* on X is a vector bundle T satisfying two conditions

(1) $\mathrm{Ext}^q(T,T) = 0$ for all $q > 0$ and
(2) the vector bundle T generates the bounded derived category of coherent sheaves on X.

Theorem 1. *Let X be a rational surface. Then there exists a tilting bundle T on X.*

As an immediate consequence we get an equivalence of derived categories between the bounded derived category of coherent sheaves on X and the bounded derived category of finitely generated modules over the endomorphism algebra A of T:

$$\mathbb{R}\mathrm{Hom}\,(T, -) : \mathcal{D}^b(X) \longrightarrow \mathcal{D}^b(A - \mathrm{mod})$$

2. Proof

The aim of this talk was to prove the above theorem. The proof is constructive and consists of three steps. In a first step we prove the result for minimal rational surfaces. On any Hirzebruch surface we construct a tilting bundle whose direct summands consist of line bundles. For the projective plane there exists a well-known tilting bundle $\mathcal{O} \oplus \mathcal{O}(1) \oplus \mathcal{O}(2)$. In a second step we consider a blow up. In this way we obtain a full exceptional sequence consisting of line bundles. The direct summands of these line bundles may have non-vanishung Ext^1-groups. In a final step we use universal (co–)extensions of these line bundles to obtain a tilting bundle. At the end we relate the result to quasi-hereditary algebras.

Theorem 2. *Let X be a rational surface then there exists a tilting bundle T on X so that the endomorphism algebra A of T is a quasi-hereditary algebra. Moreover, we can choose T in such a way that under the derived equivalence in the theorem above the line bundles of a full strongly exceptional sequence are mapped to the Δ–modules over A.*

The sequence of line bundles in the result we construct explicitly in the second step of the proof. If we use universal extensions to construct the tilting bundle in the third step then the endomorphism algebra of T is quasi-hereditary with the properties claimed in the theorem.

3. Line Bundles on Hirzebruch Surfaces

We consider the Hirzebruch surface $\mathbb{F}_a$ defined by the equation $x_0^n y_0 - x_1^n y_1$ in $\mathbb{P}^1 \times \mathbb{P}^2$, where the coordinates are $(x_0, x_1; y_0, y_1, y_2)$. We denote by P the divisor of a fiber under the first projection $\mathbb{F}_a \longrightarrow \mathbb{P}^1$. Moreover, we also need a divisor Q with selfintersection a. By definition $P^2 = 0$. Then there exists a full, strongly exceptional sequence of line bundles on $\mathbb{F}_a$ of the form $\varepsilon = (\mathcal{O}, \mathcal{O}(P), \mathcal{O}(Q), \mathcal{O}(P+Q))$. In fact one can even classify all full (strongly) exceptional sequences consisting of line bundles on $\mathbb{F}_a$ (see [1,3]).

4. Full Exceptional Sequences and Blow Up

There exist several rational surfaces admitting full strongly exceptional sequences of line bundles. However, we know that most of the rational surfaces (if one blows up one point three times from a Hirzebruch surface) do not admit such a sequence (see [2] for a counterexample). On the positive side, any rational surface admits a full exceptional sequence of line bundles ([3]) with only non-vanishing Ext^1–groups. We start with a full exceptional sequence $\varepsilon = (L_1, \dots, L_t)$ of line bundles on X and construct one on the blow up Y in one point of X. If we denote by E the exceptional divisor on Y then

$$(L_1(E), \dots, L_{i-1}(E), L_i, L_i(E), L_{i+1}, \dots, L_t)$$

is full exceptional on Y for any $i = 1, \dots, t$. In this way we can, starting with a full (strongly) exceptional sequence of line bundles on a Hirzebruch surface $\mathbb{F}_a$ construct a full exceptional sequence of line bundles on any rational surface.

Theorem 3. *On any rational surface there exists a full exceptional sequence of line bundles, so that all groups Ext^2 between these line bundles vanish.*

5. Universal Extensions

In the last step we can even consider any exceptional sequence and then perform universal extensions or coextensions. In the situation we are interested in we start with an exceptional sequence where the only non-vanishing Ext–groups are Ext^1–groups. Then applying the following construction recursively through all objects in the sequence we get a new sequence that might have non-vanishing Hom–groups in both directions (so it is no longer an exceptional sequence) but all Ext–groups vanish.

We only define the construction for pairs of objects (E, F) (which are vector bundles in our situation, we start with line bundles and in any non-trivial step we construct new vector bundles). We only need finite dimensional groups $\mathrm{Ext}^1(E, F)$ to define it.

Universal Coextension

Let (E, F) be a pair of objects. Then there is a canonical map in the derived category $\mathrm{Ext}^1(E, F) \otimes E \longrightarrow F[1]$ which defines an exact sequence

$$0 \longrightarrow F \longrightarrow \overline{F} \longrightarrow \mathrm{Ext}^1(E, F) \otimes E \longrightarrow 0.$$

So for any pair of objects (E, F) we can define a new pair $(E, \overline{F})$. If the pair (E, F) was an exceptional sequence with all higher Ext –groups vanishing, then the new pair has no non-vanishing Ext –group at all. However, we obtain nontrivial homomorphisms from $\overline{F}$ to E (if $\mathrm{Ext}^1(E, F)$ is non-trivial).

Universal Extension
Let (E, F) be a pair of objects as above. Then there is a canonical dual map in the derived category $E \longrightarrow \mathrm{Ext}^1(E, F)^* \otimes F[1]$ which defines an exact sequence

$$0 \longrightarrow \mathrm{Ext}^1(E, F)^* \otimes F \longrightarrow \overline{E} \longrightarrow E \longrightarrow 0.$$

So for any pair of objects (E, F) we can define a new pair $(\overline{E}, F)$. If the pair (E, F) was an exceptional sequence with all higher Ext –groups vanishing, then the new pair has no non-vanishing Ext –group at all. However, in this case we obtain nontrivial homomorphisms from F to $\overline{E}$ (if $\mathrm{Ext}^1(E, F)$ is non-trivial).

Finally we apply either universal extensions or universal coextensions recursively to the full exceptional sequence of line bundles on X. Then the direct sum of these objects is a tilting bundle. To obtain a quasi-hereditary algebra one has to apply universal extensions all the time.

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Classification of torsion pairs in cluster categories of Dynkin type

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(joint work with Peter Jørgensen and Martin Rubey)

We presented in the talk a complete classification of torsion pairs in the cluster category of Dynkin type A_n, based on [6].

Torsion pairs. The concept of torsion pairs in abelian categories has been introduced by Dickson [4] in 1966. In the context of tilting theory it has had a lasting and fundamental impact on the representation theory of finite-dimensional algebras. While one of the primary aims of representation theory remains to study the module categories (which are abelian categories) the focus in modern representation theory has shifted towards related categories like derived module categories, stable module categories or cluster categories. These are no longer abelian, but carry the structure of a triangulated category. The following triangulated version of the classical notion of a torsion pair has been proposed fairly recently by Iyama and Yoshino [7]: A *torsion pair* in a triangulated category C is a pair (X, Y) of full subcategories closed under direct sums and direct summands such that

(i) the morphism space $\mathrm{Hom}_{\mathsf{C}}(x, y)$ is zero for $x \in \mathsf{X}$, $y \in \mathsf{Y}$,

(ii) each object $c \in \mathsf{C}$ sits in a triangle $x \to c \to y \to \Sigma x$ with $x \in \mathsf{X}$, $y \in \mathsf{Y}$.

Examples of such torsion pairs in the triangulated situation are given by the t-structures of Beilinson, Bernstein, and Deligne [1] where, additionally, one assumes $\Sigma\mathsf{X} \subseteq \mathsf{X}$, and by the co-t-structures of Bondarko and Pauksztello [2], [10] where, additionally, one assumes $\Sigma^{-1}\mathsf{X} \subseteq \mathsf{X}$. It is easily deduced from the definition that a torsion pair (X, Y) is determined by one of its entries, namely we then have

$$\mathsf{Y} = \mathsf{X}^{\perp} := \{\, c \in \mathsf{C} \mid \mathrm{Hom}_{\mathsf{C}}(x, c) = 0 \text{ for each } x \in \mathsf{X} \,\},$$

$$\text{and } \mathsf{X} = {}^{\perp}\mathsf{Y} := \{\, c \in \mathsf{C} \mid \mathrm{Hom}_{\mathsf{C}}(c, y) = 0 \text{ for each } y \in \mathsf{Y} \,\}.$$

If the triangulated category C is Hom-finite over a field and Krull-Schmidt (conditions which are satisfied for all categories considered below) we have the following characterisation, see [7, Prop. 2.3]. Let X be a contravariantly finite full subcategory of C which is closed under direct sums and summands. Then $(\mathsf{X}, \mathsf{X}^{\perp})$ is a torsion pair if and only if $\mathsf{X} = {}^{\perp}(\mathsf{X}^{\perp})$.

Cluster categories. Let C be the cluster category of Dynkin type A_n; this is a 2-Calabi-Yau triangulated category with finitely many indecomposable objects. There is a beautiful combinatorial model for C, due to Caldero, Chapoton and Schiffler [3]: there is a bijection between indecomposable objects of C and diagonals of a regular $(n+3)$-gon P such that the suspension Σ acts by rotation on P and

$$\dim \mathrm{Ext}^1_{\mathsf{C}}(a, b) = \begin{cases} 1 & \text{if } \mathfrak{a} \text{ and } \mathfrak{b} \text{ cross,} \\ 0 & \text{otherwise,} \end{cases}$$

where we denote by $\mathfrak{a}, \mathfrak{b}, \ldots$ the diagonals corresponding to objects of C. This extends to bijections between the following sets: clusters in the cluster algebra of type A_n, cluster tilting objects in C, and triangulations of P by non-crossing diagonals. These give rise to torsion pairs: if u is a cluster tilting object in C, then $(\mathrm{add}(u), \Sigma\, \mathrm{add}(u))$ is a torsion pair [8, Section 2.1]. However, C admits many other torsion pairs, as we shall see in our classification below.

Ptolemy diagrams. Full subcategories X of C closed under direct sums and direct summands correspond to collections $\mathfrak{X}$ of diagonals of the regular $(n+3)$-gon P. For characterising those collections $\mathfrak{X}$ yielding torsion pairs $(\mathsf{X}, \mathsf{X}^{\perp})$, the following notation is useful: let $\mathrm{nc}\,\mathfrak{X}$ be the set of diagonals of P which do not cross any diagonal from $\mathfrak{X}$. Then the perpendicular subcategory

$$\mathsf{X}^{\perp} = \{\, c \in \mathsf{C} \mid \mathrm{Ext}^1_{\mathsf{C}}(\Sigma x, c) = \mathrm{Hom}_{\mathsf{C}}(x, c) = 0 \text{ for each } x \in \mathsf{X} \,\}$$

corresponds to $\mathrm{nc}\,\Sigma\mathfrak{X}$, and similarly

$${}^{\perp}\mathsf{X} = \{\, c \in \mathsf{C} \mid \mathrm{Ext}^1_{\mathsf{C}}(c, \Sigma^{-1}x) = \mathrm{Hom}_{\mathsf{C}}(c, x) = 0 \text{ for each } x \in \mathsf{X} \,\}$$

corresponds to $\mathrm{nc}\,\Sigma^{-1}\mathfrak{X}$. Since Σ acts by rotation, it commutes with nc and we obtain: $(\mathsf{X}, \mathsf{X}^{\perp})$ is a torsion pair if and only if the corresponding collection of diagonals satisfies $\mathfrak{X} = \mathrm{nc}\,\Sigma^{-1}(\mathrm{nc}\,\Sigma\mathfrak{X}) = \mathrm{nc}\,\mathrm{nc}\,\mathfrak{X}$.

Definition. A set $\mathfrak{X}$ of diagonals in P is a *Ptolemy diagram* if it has the following property: when $\mathfrak{a}$ and $\mathfrak{b}$ are diagonals in $\mathfrak{X}$ which cross, then any diagonal connecting endpoints of $\mathfrak{a}$ and $\mathfrak{b}$ must also be in $\mathfrak{X}$.

Simple examples of Ptolemy diagrams are: a polygon with no diagonals (an *empty cell*), a polygon with all diagonals (a *clique*), and triangulations of a polygon by non-crossing diagonals.

Theorem A. *Let X be a subcategory of C closed under direct sums and direct summands, and let $\mathfrak{X}$ be the corresponding collection of diagonals.*

(1) We have $\mathfrak{X} = \mathrm{nc}\,\mathrm{nc}\,\mathfrak{X}$ if and only if $\mathfrak{X}$ is a Ptolemy diagram.
(2) There is a bijection between torsion pairs $(\mathsf{X}, \mathsf{X}^{\perp})$ in the cluster category C of type A_n and Ptolemy diagrams $\mathfrak{X} = \mathrm{nc}\,\mathrm{nc}\,\mathfrak{X}$ of the $(n+3)$-gon P.

We also presented in the talk enumerative results on the Ptolemy diagrams, and hence on torsion pairs. For this, an alternative recursive description of Ptolemy diagrams turns out to be very useful, see [6, Section 2] for details.

Theorem B.

(1) Each Ptolemy diagram can be decomposed into (smaller) Ptolemy diagrams which are either empty cells or cliques. Moreover, in such a decomposition the operator nc *exchanges empty cells and cliques.*
(3) The number of Ptolemy diagrams of a regular $(n+3)$-gon is equal to

$$\frac{1}{n+2}\sum_{\ell\geq 0} 2^{\ell}\binom{n+1+\ell}{\ell}\binom{2n+2}{n+1-2\ell}.$$

The first values (starting with $n=0$) are 1, 4, 17, 82, 422, 2274, 12665, 72326, 421214, 2492112, 14937210, 90508256, 553492552,...

Concluding remarks and work in progress. (a) Recall that t-structures are examples of torsion pairs, namely those for which $\Sigma\mathsf{X} = \mathsf{X}$. However, using that Σ acts by rotation, one can deduce from our classification the well-known fact that the cluster category C only admits the trivial t-structures $(0, \mathsf{C})$ and $(\mathsf{C}, 0)$. The analogous result holds for co-t-structures.

(b) We also enumerate in [6] the Ptolemy diagrams up to rotation (i.e. up to the action of Σ, which is the Auslander-Reiten translation since C is 2-Calabi-Yau). The first few values here are 1, 3, 5, 19, 62, 301, 1413, 7304, 38294, 208052, 1149018,... Both sequences of numbers of Ptolemy diagrams (up to rotation) do not seem to have been encountered before in other contexts. They are now items A181517 and A181519 in the Online Encyclopedia of Integer Sequences [12].

(c) Remarkably, the Ptolemy diagrams up to rotation exhibit a cyclic sieving phenomenon [9]. This might indicate further interesting connections to combinatorics, see e.g. [11, Note added in proof].

(d) Based on a well-known combinatorial model for the cluster category of Dynkin type D_n (see e.g. [5, Section 3.5]) we have also obtained a complete classification and enumeration of torsion pairs in the cluster category of type D_n; details will appear in a subsequent publication.

(e) For Dynkin type A_n, a combinatorial model for d-cluster categories is given via d-admissible diagonals in an $(d(n+1)+2)$-gon. However, the 'd-Ptolemy condition' characterising those sets of diagonals corresponding to torsion pairs is more subtle. One structural reason is that for $d > 1$, vanishing of Ext^1 is no longer symmetric since the d-cluster category is $(d+1)$-Calabi-Yau. Details on the classification of torsion pairs in higher cluster categories of type A_n will appear in yet another subsequent publication.

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The Bousfield lattice of the stable module category of a finite group

Srikanth B. Iyengar

Let G be a finite group, k a field whose characteristic divides the order of G, and $\mathsf{StMod}\, kG$ the stable module category of all (and not only the finite dimensional) kG-modules, with its natural structure of a triangulated category. Benson, Krause, and I [3, 4, 6] have been investigating global structural properties of $\mathsf{StMod}\, kG$; to be precise, the classification of its localizing subcategories and its colocalizing subcategories. The aim of my talk was to cast our results in a different light, by using them to discover the structure of certain lattices naturally associated to the stable module category. For a more systematic treatment, in the context of tensor triangulated categories, see [10]. This line of development is inspired by Bousfield's work [7] in stable homotopy theory; see also [9].

For any kG-modules M, N, the k-vectorspace $M \otimes_k N$ has a diagonal kG-action:

$$g(m \otimes n) = gm \otimes gn \quad \text{for } g \in G \text{ and } m \otimes n \text{ in } M \otimes_k N.$$

This induces a tensor product on $\mathsf{StMod}(kG)$ as well.

Definition 1. *The* Bousfield class *of a kG-module M is the full subcategory*

$$\mathrm{A}(M) = \{X \in \mathsf{StMod}(kG) \mid M \otimes_k X = 0 \text{ in } \mathsf{StMod}(kG)\}$$

*Recall that $M \otimes_k X$ is zero in $\mathsf{StMod}(kG)$ precisely when it is projective. Modules in $\mathrm{A}(M)$ are said to be M-*acyclic*, whence the notation. Modules M and N are* Bousfield equivalent *if $A(M) = A(N)$.*

A basic problem is to classify kG-modules, up to Bousfield equivalence. To this end we mimic [7], and endow the collection of all Bousfield classes, $\mathrm{A}(\mathsf{StMod}\, kG)$, with the following partial order:

$$\mathrm{A}(M) \leq \mathrm{A}(N) \quad \text{if} \quad \mathrm{A}(M) \supseteq \mathrm{A}(N)\,.$$

A priori, it is not even clear that $\mathrm{A}(\mathsf{StMod}\, kG)$ is a set. That it is so, and much more, is contained in the following:

Theorem 1. *The collection $\mathrm{A}(\mathsf{StMod}\, kG)$ with partial order $\leq$ is a lattice, with supremum and infimum given by*

$$\mathrm{A}(M) \vee \mathrm{A}(N) = \mathrm{A}(M \oplus N) \quad \text{and} \quad \mathrm{A}(M) \wedge \mathrm{A}(N) = \mathrm{A}(M \otimes_k N)\,.$$

Moreover, the lattice $\mathrm{A}(\mathsf{StMod}\, kG)$ is distributive and complete.

Assume for the moment that $\mathrm{A}(\mathsf{StMod}\, kG)$ is a set. It is clear that it is partially ordered under $\leq$. Moreover, since $- \otimes_k X$ commutes with (arbitrary) direct sums, any set $\{M_i\}$ of kG-modules has a supremum:

$$\bigvee_i \mathrm{A}(M_i) = \mathrm{A}(\bigoplus_i M_i)\,.$$

It then follows from general principles, see [8], that any subset of $\mathrm{A}(\mathsf{StMod}\, kG)$ also has a infimum; that is to say, the lattice $\mathsf{StMod}\, kG$ is complete. The non-trivial part in Theorem 1 is the explicit identification of the infimum; given that, it is clear also that the lattice is distributive.

Localizing subcategories. The tensor product on $\mathsf{StMod}\, kG$ is compatible with its structure as a triangulated category. A subcategory S is *tensor closed* if whenever M is in S so is $M \otimes_k X$ for any kG-module X. A *localizing* subcategory is a triangulated subcategory that is closed under all set-indexed coproducts. We write $\mathrm{L}(M)$ for the smallest (with respect to inclusion) tensor closed localizing subcategory of $\mathsf{StMod}\, kG$ containing M, and $\mathrm{L}(\mathsf{StMod}\, kG)$ for the collection of all such subcategories, with the (natural !) partial order:

$$\mathrm{L}(M) \leq \mathrm{L}(N) \quad \text{if} \quad \mathrm{L}(M) \subseteq \mathrm{L}(N)\,.$$

There is an analogue of Theorem 1 for this collection. There is a map of lattices from $\mathrm{L}(\mathsf{StMod}\, kG)$ and $\mathrm{A}(\mathsf{StMod}\, kG)$, the key point being the following:

Lemma 1. *If* $\mathrm{L}(M) \leq \mathrm{L}(N)$, *then* $\mathrm{A}(M) \leq \mathrm{A}(N)$. □

Corollary 3 contains the converse to the preceding lemma. Its proof uses the theory of support, which we now recall.

Support. Let $H^*(G,k)$ be the cohomology algebra, $\mathrm{Ext}^*_{kG}(k,k)$, of G. This is a k-algebra which is graded-commutative, because kG is a Hopf algebra, and also finitely generated; the last statement is due to Evens and Venkov, and the starting point of the cohomology study of modular representations of finite groups; see, for instance, [1] for details. Set

$$\mathcal{V}_G = \text{homogeneous prime ideals in } H^*(G,k)\text{, except } H^{\geqslant 1}(G,k).$$

For each $\mathfrak{p} \in \mathcal{V}_G$ Benson, Carlson, and Rickard [2] (see also [3]) construct certain idempotent exact functors on $\mathsf{StMod}\,kG$, which we denote $\Gamma_{\mathfrak{p}}$. A crucial property of these functors is that

$$\Gamma_{\mathfrak{p}} M \cong \Gamma_{\mathfrak{p}} k \otimes_k M\,.$$

The *support* of a kG-module is the subset

$$\mathrm{supp}_G\, M = \{\mathfrak{p} \in \mathcal{V}_G \mid \Gamma_{\mathfrak{p}} k \otimes_k M \neq 0\}$$

For finite dimensional modules, this coincides with the usual cohomological support; see [3]. We remark that when M is non-zero $\mathrm{supp}_G\, M$ is non-empty. The relevance of support to us is that there are maps:

$$\begin{tikzcd}
\mathrm{L}(\mathsf{StMod}\,kG) \arrow[dr, "\iota"'] & & \{\text{subsets of } \mathcal{V}_G\} \arrow[ll, "\sigma"'] \\
& \left\{\begin{array}{c}\text{tensor closed localizing}\\ \text{subcategories of } \mathsf{StMod}\,kG\end{array}\right\} \arrow[ur, "\tau"'] &
\end{tikzcd}$$

where ι is the obvious inclusion, and τ and σ are defined as follows:

$$\tau(\mathsf{S}) = \bigcup_{M \in \mathsf{S}} \mathrm{supp}_G\, M \quad \text{and} \quad \sigma(\mathcal{U}) = \mathrm{L}\,\Big(\bigoplus_{\mathfrak{p}\in\mathcal{U}} \Gamma_{\mathfrak{p}} k\Big)$$

It is not hard to see that [4, Theorem 10.3] is equivalent to the following:

Theorem 2. *The composition of any three consecutive maps in the diagram above is the identity. In particular, the maps are all bijections.* □

From this one can deduce the 'tensor product theorem'; see [4, Theorem 11.1].

Corollary 1. *For any kG-modules M and N one has*

$$\mathrm{supp}_G(M \otimes_k N) = \mathrm{supp}_G\, M \cap \mathrm{supp}_G\, N\,.$$

In particular, $\mathrm{A}(M) = \{N \mid \mathrm{supp}_G\, N \cap \mathrm{supp}_G\, M = \varnothing\}$. □

Using this result one can prove Theorem 1 without much ado. The next corollary extends Lemma 1 and characterizes Bousfield equivalent modules.

Corollary 2. *One has* $\mathrm{L}(M) \leq \mathrm{L}(N)$ *if and only if* $\mathrm{A}(M) \leq \mathrm{A}(N)$, *if and only if* $\mathrm{supp}_R\, M \subseteq \mathrm{supp}_R\, N$. □

Local objects. In what follows, the set of morphisms in $\mathsf{StMod}\,kG$ between kG-modules M and N is denoted $\underline{\mathrm{Hom}}_G(M,N)$. Once again inspired by the work in [7], we consider the right orthogonal of the M-acyclic modules:

$$\mathrm{A}(M)^{\perp} = \{N \in \mathsf{StMod}\,kG \mid \underline{\mathrm{Hom}}_G(X,N) = 0 \text{ for all } X \in \mathrm{A}(M)\}\,.$$

The modules in this subcategory are said to be *M-local*. Note that the subcategory of M-local objects is equivalent to the Verdier quotient of $\mathsf{StMod}\,kG$ by $\mathrm{A}(M)$. Again, one is faced with the problem of classifying such subcategories. To address it, we consider the right adjoint $\Lambda^{\mathfrak{p}} = \mathrm{Hom}_k(\Gamma_{\mathfrak{p}}k, -)$ to $\Gamma_{\mathfrak{p}}$. In [6] we introduced the *cosupport* of a kG-module M to be the subset

$$\operatorname{cosupp}_R M = \{\mathfrak{p} \in \mathcal{V}_G \mid \Lambda^{\mathfrak{p}} M \neq 0\}\,.$$

The cosupport of M is non-empty when $M \neq 0$; see [6, Theorem 4.5].

In what follows $\mathrm{Hom}_k(M,N)$ is viewed as a kG-module with diagonal action. The theorem below is a consequence of [6, Theorem 9.5] and [4, Theorem 10.3], which are the central results of the corresponding articles. Theorem 2, and the other results described above, can be easily deduced from it.

Theorem 3. *For any kG-modules M and N one has*

$$\operatorname{cosupp}_G \mathrm{Hom}_k(M,N) = \operatorname{supp}_G M \cap \operatorname{cosupp}_G N\,.$$

In particular, $\mathrm{Hom}_k(M,N) = 0$ *if and only if* $\operatorname{supp}_G M \cap \operatorname{cosupp}_G N = \varnothing$. □

This result and Corollary 1 yield

Corollary 3. *One has* $\mathrm{A}(M)^{\perp} = \{N \mid \operatorname{cosupp}_G N \subseteq \operatorname{supp}_G M\}\,.$ □

Using this result and [6, Theorem 11.3], one can prove an analogue of Theorem 1, yielding bijections between subcategories of form $\mathrm{A}(M)^{\perp}$, the Hom closed colocalizing subcategories of $\mathsf{StMod}\,kG$, and the set of subsets of $\mathcal{V}_G$.

In all this the cosupport of modules plays a central role, but we do not yet have a good understanding of its significance. In my lecture, I mentioned some examples from commutative algebra where we have been able to compute the cosupport of all finitely generated modules. These are discussed in detail in [6], where it is also explained that the functor $\Lambda^{\mathfrak{p}}$ is akin to completion at $\mathfrak{p}$, in the sense of commutative algebra.

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On stratifications of derived module categories

Steffen Koenig

(joint work with Lidia Angeleri Hügel and Qunhua Liu)

Let R be a ring and $M \neq 0$ an R-module. If M is not simple, then it is the middle term of a non-trivial short exact sequence $0 \to M_1 \to M \to M_2 \to 0$, where non-trivial means both M_1 and M_2 are not zero. If M_1 is not simple, it is the middle term of a non-trivial short exact sequence. And so on. If the process stops after finitely many steps, then the module M has been 'stratified', and the simple end terms of the short exact sequences occuring in the process are the composition factors ('strata') of M. A version of the Jordan-Hölder theorem asserts that for artinian modules a stratification exists, and that the strata are unique up to isomorphism and reordering.

As this example illustrates, a Jordan-Hölder theorem can be stated - but not necessarily verified - as soon as there is a concept of short exact sequences. Then an object may be called simple, if it is not the middle term of a non-trivial short exact sequence. Repeatedly forming short exact sequences yields a (possibly infinite) stratification. Additional assumptions are needed to guarantee finiteness and uniqueness of the composition factors.

In the following, the objects to be stratified are derived module categories of rings or algebras. Short exact sequences are *recollements* in the sense of Beilinson, Bernstein and Deligne [5].

The concept of recollement is of geometric origin, providing a natural habitat for Grothendieck's six functors, which relate coherent sheaves on a topological space with coherent sheaves on an open subspace and on its closed complement. Stratifications can be used to define perverse sheaves and intersection homology. Such recollements pass to recollements of derived module categories in the context of the proof of Kazhdan-Lusztig conjecture, which is based on a derived equivalence, and abstractly in the context of quasi-hereditary or, more generally, stratified algebras.

Recollements have been used in representation theory for instance to connect homological or K-theoretic data of the middle term with similar data of the outer terms (finiteness of global or finitistic dimension, Hochschild cohomology, Grothendieck groups). Recently, Angeleri Hügel and co-authors, in particular

Javier Sánchez, have started a programme of classifying tilting modules by recollements, or vice versa. See also [1] for a discussion of connections between recollements and tilting modules.

Recollements of derived module categories can be seen as semi-orthogonal decompositions; they have been characterised in terms of perpendicular categories and of torsion theoretic data. See [8] for the first result in this context and [10] for a more general and technologically much more advanced result.

Theorem 1 ([2, 3]). *Let A be a finite-dimensional piecewise hereditary algebra over a field k, $S_1, \dots, S_n$ its simple modules (up to isomorphism) and $E_i := End_A(S_i)$ their endomorphism rings. Then:*
(I) The unbounded derived category of A-modules, $D(A - Mod)$, has a stratification with factors $D(E_i - Mod)$, $i = 1, \dots, n$, and this stratification is unique up to ordering and Morita equivalence of the simple factors.
(II) The bounded derived category of finitely generated A-modules, $D^b(A - mod)$, has a stratification with factors $D^b(E_i - mod)$, $i = 1, \dots, n$, and this stratification is unique up to ordering and Morita equivalence of the simple factors.

Theorem 1 has been proven in [2] for unbounded derived categories of hereditary artinian algebras. Here, a stronger uniqueness result is shown: There is a 'normal form' of the stratification, associated with a sequence of homological epimorphisms $A = A_1 \to \cdots \to A_n$.
In [3], the geometric case - canonical algebras or weighted projective lines over a field - has been settled as well and a bijection with stratifications of bounded derived categories of finitely generated modules.
Theorem 1 fails completely when arbitrary triangulated categories are allowed as factors, see [1, 2]. It also fails, when allowing derived categories of dg algebras as factors.
For A a ring, there need not exist a finite stratification, see [2].
Recollements of bounded derived categories induce recollements of unbounded derived categories, but the converse fails, in general, see [4, 8].
In general, uniqueness fails even for finite stratifications of unbounded derived module categories. Chen and Xi [6] have constructed non-artinian algebras, whose derived categories have two stratifications of different lengths and with different composition factors.

Simple algebras are obviously derived simple with respect to any choice of derived categories. Algebras studied by Happel [7] provide examples of algebras of finite global dimension with two simples each - in particular, the Grothendieck group decomposes - which are derived simple with respect to the bounded derived category. Local algebras and polynomial rings in one variable over a field are derived simple with respect to the unbounded derived category. More examples and counterexamples in this context will be given in [4].

A large class of derived simple algebras is convincingly provided by a result of Liu and Yang:

Theorem 2 (Liu and Yang [9]). *Let G be a finite group, k a field and B a block (indecomposable algebra summand) of the group algebra kG. Then B is derived simple with respect to any choice of bounded or unbounded derived category and of finitely generated or arbitrary module category.*

The results in [9] are more general and also cover symmetric algebras of finite representation type.

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Quantum cluster algebras and dual canonical bases

Philipp Lampe

Let $\mathfrak{g} = \mathfrak{n} \oplus \mathfrak{h} \oplus \mathfrak{n}_-$ be a Kac-Moody Lie algebra with its triangular decomposition. The talk concerned the quantized universal enveloping algebra $U_v(\mathfrak{n})$ of $\mathfrak{n}$. It is a graded algebra generated by elements $E_1, E_2, \ldots, E_n$ subject to the quantum Serre relations. The algebra is graded by the root lattice if we let $\deg(E_i) = \alpha_i$ be the corresponding simple root. With every Weyl group element $w \in W$ Lusztig [5] associates a subalgebra $U_v(w) \subset U_v(\mathfrak{n})$.

The algebra $U_v(w)$ possesses several bases. First of all, for every reduced decomposition $w = s_{i_1} s_{i_2} \cdots s_{i_r}$ there is a Poincaré-Birkhoff-Witt basis consisting of ordered monomials in the generators $T_{i_1} T_{i_2} \ldots T_{i_{k-1}}(E_{i_k})$ of $U_v(w)$ (for $1 \leq k \leq r$). Furthermore, Lusztig [5] introduced a canonical basis $\mathcal{B}$ that does not depend on the choice of the reduced expression for w.

Cluster algebras have been introduced by Fomin-Zelevinsky [2] to study the dual of Lusztig's canonical basis of $U_v(w)$. In fact, it is conjectured that $U_v(w)$ carries the structure of a quantum cluster algebra (in the sense of Berenstein-Zelevinsky [1]) and that all quantum cluster monomials belong (up to a power of v) to the dual canonical basis $\mathcal{B}^*$ of $U_v(w)$.

The non-quantized version is related to Geiß-Leclerc-Schröer's cluster algebra structure [3] on the coordinate ring $\mathbb{C}[N(w)]$ of the unipotent group $N(w)$ attached to w.

We study a special case: Let $\mathfrak{g} = \mathfrak{sl}_{n+1}$, and let Q be a Dynkin quiver of type A_n with alternating orientation. The element $w = s_{i_1}s_{i_2}\cdots s_{i_{2n}}$ of length $2n$ is choosen so that the $s_{i_1}s_{i_2}\cdots s_{i_{k-1}}(\alpha_{i_k}) \in \Delta^+$ (for $1 \leq k \leq 2n$) become (under the bijection provided by Gabriel's theorem) the dimension vectors of the n indecomposable injective kQ-modules and their Auslander-Reiten translates.

The main theorem of our preprint [4] is the following:

Theorem 1. *There are $z_i, p_i \in v^{\mathbb{Z}}\mathcal{B}^*$ (for 1$\leq i \leq n$) and $\Delta_{i,j} \in v^{\mathbb{Z}}\mathcal{B}^*$ (for $1 \leq i \leq j \leq n$) such that $U_v(w)$ carries the structure of a quantum cluster algebra of type A_n; the p_i (for $1 \leq i \leq n$) are frozen quantum cluster variables, the z_i (for $1 \leq i \leq n$) are mutable quantum cluster variables in an initial seed whose prinicipal part is given by an alternating quiver $z_1 \to z_2 \leftarrow z_3 \to \cdots z_n$, and the $\Delta_{i,j}$ (for $1 \leq i \leq j \leq n$) are all further quantum cluster variables.*

The proof of the theorem features the embedding of $U_v(w)$ in the quantized shuffle algebra (compare Rosso [6]). Canonical basis elements can the described combinatorially. In our case, the occuring shuffles are related to alternating permutations and Euler numbers.

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Quiver varieties and the character ring of general linear groups over finite fields

Emmanuel Letellier

Given three complex irreducible characters $\mathcal{X}_1, \mathcal{X}_2, \mathcal{X}_3$ of $GL_n(\mathbb{F}_q)$ one standard problem is to compute the multiplicity $\langle \mathcal{X}_1 \otimes \mathcal{X}_2, \mathcal{X}_3 \rangle$. This problem is equivalent to the computation of multiplicities $\langle \mathcal{X}_1 \otimes \mathcal{X}_2 \otimes \mathcal{X}_3, 1 \rangle$.

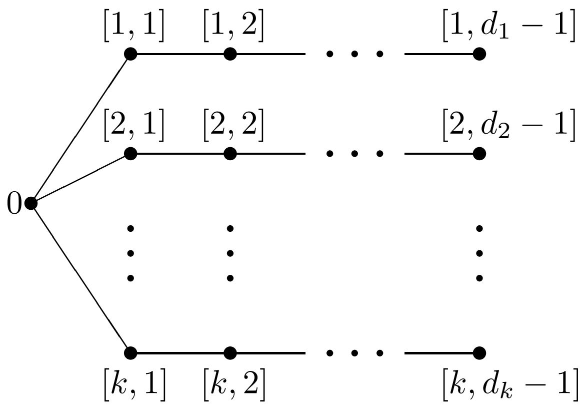
Although the character table of $GL_n(\mathbb{F}_q)$ is known since 1955 by the work of Green, this problem does not seem to have been much studied in the literature.

In this talk we study multiplicities of the form $\langle \mathcal{X}_1 \otimes \mathcal{X}_2 \otimes \cdots \otimes \mathcal{X}_k, 1 \rangle$ where $(\mathcal{X}_1, \ldots, \mathcal{X}_k)$ is a *generic tuple* of irreducible characters of $GL_n(\mathbb{F}_q)$.

Let us first give the definition of *generic tuple* in the case where $\mathcal{X}_1, \ldots, \mathcal{X}_k$ are characters of unipotent type. Assume that for each $i = 1, \ldots, k$, we have

$\mathcal{X}_i = (\alpha_i \circ \det) \cdot R_i$ with R_i a unipotent character and α_i a linear character of $\mathbb{F}_q^\times$. Then the tuple $(\mathcal{X}_1, \ldots, \mathcal{X}_k)$ is said to be generic if the subgroup generated by the linear character $\rho = \alpha_1 \cdots \alpha_k$ is of size n.

Given any tuple $(\mathcal{X}_1, \ldots, \mathcal{X}_k)$ of irreducible characters (not necessarily generic) we can define a star-shaped quiver Γ



together with a dimension vector $\mathbf{v}$ of Γ.

Let us illustrate this with characters of unipotent type. We parameterize the unipotent characters of $GL_n(\mathbb{F}_q)$ by partitions of n so that the trivial character corresponds to the trivial partition (1^n) and the Steinberg character St to the partition (n). We denote by R_λ the unipotent character corresponding to the partition λ. Assume that $\mathcal{X}_i = (\alpha_i \circ \det) \cdot R_{\lambda_i}$. Then the length of the i-th leg of Γ is the length $\ell(\lambda_i')$ of the dual partition λ_i' minus 1, i.e., $d_i = \ell(\lambda_i')$. The coordinate of the vector $\mathbf{v}$ on the i-th leg are $v_0 = n$, $v_{[i,j]} = n - \sum_{r=1}^{j} n_{i,r}$ where $\lambda_i' = (n_{i,1} \geq n_{i,2} \geq \cdots)$.

Assume now that the tuple $(\mathcal{X}_1, \ldots, \mathcal{X}_k)$ is generic.

We have the following conjecture.

Conjecture *We have*

(a) The multiplicity $\langle \mathcal{X}_1 \otimes \mathcal{X}_2 \otimes \cdots \otimes \mathcal{X}_k, 1\rangle$ is a polynomial in q with integer coefficients. If moreover the characters $\mathcal{X}_1, \ldots, \mathcal{X}_k$ are split, then the coefficients are positive.

(b) The coefficient of the highest power of q in $\langle \mathcal{X}_1 \otimes \mathcal{X}_2 \otimes \cdots \otimes \mathcal{X}_k, 1\rangle$ is 1.

(c) $\langle \mathcal{X}_1 \otimes \mathcal{X}_2 \otimes \cdots \otimes \mathcal{X}_k, 1\rangle \neq 0$ if and only if $\mathbf{v}$ is a root of the Kac-Moody algebra associated to Γ. Moroever $\langle \mathcal{X}_1 \otimes \mathcal{X}_2 \otimes \cdots \otimes \mathcal{X}_k, 1\rangle = 1$ if and only if $\mathbf{v}$ is a real root.

It is a theorem of Hausel, Rodriguez-Villegas and the author [3] that when $\mathcal{X}_1, \ldots, \mathcal{X}_k$ are split semisimple irreducible characters then the multiplicity $\langle \mathcal{X}_1 \otimes \mathcal{X}_2 \otimes \cdots \otimes \mathcal{X}_k, 1\rangle$ coincides with the so-called A-polynomial $A_{\Gamma,\mathbf{v}}(q)$ which counts the number of absolutely indecomposable representations of Γ of dimension $\mathbf{v}$ over $\mathbb{F}_q$. By a well-known theorem of V. Kac, we know that $A_{\Gamma,\mathbf{v}}(q)$ is a polynomial in q with integer coefficients and that it is non-zero if and only if $\mathbf{v}$ is a root of Γ (with $A_{\Gamma,\mathbf{v}}(q) = 1$ if and only if $\mathbf{v}$ is real). Moreover it is was proved by Crawley-Boevey and van den Bergh [2] that $A_{\Gamma,\mathbf{v}}(q)$ has positive coefficients when $\mathbf{v}$ is indivisible.

Hence in the split semisimple case the conjecture is true (except the positivity in the divisible case).

Let us now discuss on the non-semisimple case.

We introduce quiver varieties which will provide a geometrical interpretation of $\langle \mathcal{X}_1 \otimes \cdots \otimes \mathcal{X}_k, 1\rangle$ for a large class of generic tuples $(\mathcal{X}_1, \ldots, \mathcal{X}_k)$ which we call *admissible.*

Let P be a parabolic subgroup of $GL_n(\mathbb{C})$, L a Levi factor of P and let $\Sigma = \sigma + C$ where C is a nilpotent orbit of the Lie algebra $\mathfrak{l}$ of L and where σ is an element of the center $z_{\mathfrak{l}}$ of $\mathfrak{l}$. Put

$$\mathbb{X}_{L,P,\Sigma} := \{(X, gP) \in gl_n \times (GL_n/P) \,|\, g^{-1}Xg \in \overline{\Sigma} + \mathfrak{u}_P\}$$

where $\mathfrak{u}_P$ is the Lie algebra of the unipotent radical of P.

It is known that the image of the projection $\mathbb{X}_{L,P,\Sigma} \to gl_n$ on the first coordinate is the Zariski closure of an adjoint orbit.

Consider triples $\{(L_i, P_i, \Sigma_i)\}_{i=1,\ldots,k}$, with $\Sigma_i = \sigma_i + C_i$, as above and put $\mathbf{L} := L_1 \times \cdots \times L_k$, $\mathbf{P} := P_1 \times \cdots \times P_k$, $\mathbf{\Sigma} := \Sigma_1 \times \cdots \times \Sigma_k$ and $\mathbf{C} := C_1 \times \cdots \times C_k$.

Let $(\mathcal{O}_1, \ldots, \mathcal{O}_k)$ be the tuple of adjoint orbits of $gl_n(\mathbb{C})$ such that the image of $\mathbb{X}_{L_i,P_i,\Sigma_i} \to gl_n$ is $\overline{\mathcal{O}}_i$.

We assume now that $(\mathcal{O}_1, \ldots, \mathcal{O}_k)$ is generic.

Define

$$\mathbb{V}_{\mathbf{L},\mathbf{P},\mathbf{\Sigma}} := \left\{ (X_1, \ldots, X_k, g_1P_1, \ldots, g_kP_k) \in \mathbb{X}_{\mathbf{L},\mathbf{P},\mathbf{\Sigma}} \,|\, \sum_i X_i = 0 \right\}.$$

Put $\mathbf{O} := \overline{\mathcal{O}}_1 \times \cdots \times \overline{\mathcal{O}}_k$ and define

$$\mathcal{V}_{\mathbf{O}} := \left\{ (X_1, \ldots, X_k) \in \mathbf{O} \,|\, \sum_i X_i = 0 \right\}.$$

Let $\rho : \mathbb{V}_{\mathbf{L},\mathbf{P},\mathbf{\Sigma}} \to \mathcal{V}_{\mathbf{O}}$ be the projection on the first k coordinates.

The group GL_n acts on $\mathbb{V}_{\mathbf{L},\mathbf{P},\mathbf{\Sigma}}$ (resp. on $\mathcal{V}_{\mathbf{O}}$) diagonally by conjugating the first k coordinates and by left multiplication of the last k-coordinates (resp. diagonally by conjugating the k coordinates). Since the tuple $(\mathcal{O}_1, \ldots, \mathcal{O}_k)$ is generic, this action induces a free action of PGL_n on both $\mathbb{V}_{\mathbf{L},\mathbf{P},\mathbf{\Sigma}}$ and $\mathcal{V}_{\mathbf{O}}$. The PGL_n-orbits of these two spaces are then all closed. Consider the affine GIT quotient

$$\mathcal{Q}_{\mathbf{O}} := \mathcal{V}_{\mathbf{O}}/PGL_n = \mathrm{Spec}\left(\mathbb{C}[\mathcal{V}_{\mathbf{O}}]^{PGL_n}\right).$$

The variety $\mathcal{Q}_{\mathbf{O}}$ can be identified with the orbit space $\mathcal{V}_{\mathbf{O}}/PGL_n$. We prove in [4] that we can identify

$$\mathbb{Q}_{\mathbf{L},\mathbf{P},\mathbf{\Sigma}} := \mathbb{V}_{\mathbf{L},\mathbf{P},\mathbf{\Sigma}}/PGL_n$$

with some GIT quotient $X//_{\chi}G$. In Nakajima's notation, the varieties $X//G$ and $X//_{\chi}G$ are quiver varieties $\mathfrak{M}_{\xi}(\mathbf{v})$ and $\mathfrak{M}_{\xi,\theta}(\mathbf{v})$ associated to some star-shaped quiver Γ together with some dimension vector $\mathbf{v}$ of Γ. The pair $(\Gamma, \mathbf{v})$ is obtained from $(\mathbf{L}, \mathbf{C})$ similarly from partitions as explained above.

There is an action (similar to Springer action) of

$$W(\mathbf{L},\mathbf{\Sigma}) = W(L_1,\Sigma_1)\times\cdots\times W(L_k,\Sigma_k)$$

with $W(L,\Sigma) := N_{GL_n}(L,\Sigma)/L$ on the complex $(\rho/_{PGL_n})_*\left(IC^\bullet_{\mathbb{Q}_{\mathbf{L},\mathbf{P},\mathbf{\Sigma}}}\right)$ and so on the compactly supported intersection cohomology $IH^i_c\left(\mathbb{Q}_{\mathbf{L},\mathbf{P},\mathbf{\Sigma}},\mathbb{C}\right)$.

From the theory of quiver varieties, we have $IH^i_c\left(\mathbb{Q}_{\mathbf{L},\mathbf{P},\mathbf{\Sigma}},\mathbb{C}\right)=0$ for odd i. Let us then consider the polynomials

$$P_c^{\mathbf{w}}\left(\mathbb{Q}_{\mathbf{L},\mathbf{P},\mathbf{\Sigma}},q\right) := \sum_i \mathrm{Tr}\,\left(\mathbf{w}\;\middle|\; IH^{2i}_c\left(\mathbb{Q}_{\mathbf{L},\mathbf{P},\mathbf{\Sigma}},\mathbb{C}\right)\right)q^i,$$

with $\mathbf{w}\in W(\mathbf{L},\mathbf{\Sigma})$.

Let $(\mathcal{X}_1,\ldots,\mathcal{X}_k)$ be a generic tuple of irreducible characters of $GL_n(\mathbb{F}_q)$ of same *type* as $(\mathbf{L},\mathbf{C},\mathbf{w})$ in which case we say that $(\mathcal{X}_1,\ldots,\mathcal{X}_k)$ is admissible.
Theorem (see [4]) *We have:*

$$P_c^{\mathbf{w}}\left(\mathbb{Q}_{\mathbf{L},\mathbf{P},\mathbf{\Sigma}},q\right) = q^{\frac{1}{2}\dim\mathbb{Q}_{\mathbf{L},\mathbf{P},\mathbf{\Sigma}}}\langle\mathcal{X}_1\otimes\cdots\otimes\mathcal{X}_k,1\rangle.$$

Using some result of Crawley-Boevey [1] we prove that the theorem implies the conjecture for admissible generic tuples $(\mathcal{X}_1,\ldots,\mathcal{X}_k)$.

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Coloured quivers for rigid objects and partial triangulations: the unpunctured case.

Robert J. Marsh
(joint work with Yann Palu)

This talk was an exposition of [8]. Let (S,M) be a *marked surface*, i.e. a pair consisting of a Riemann surface S with boundary and a finite set M of marked points on the boundary. We assume S has no component homeomorphic to a monogon, digon or triangle and that each boundary component has at least one marked point. Let $\mathcal{R}$ be a *partial triangulation* of S, i.e. a collection of noncrossing simple curves in S whose end-points lie in M. We shall refer to such curves as *arcs*. We define the *mutation* of $\mathcal{R}$ as a generalisation of the usual flip of triangulations (see [5, Defn. 3.5]), in order to associate a coloured quiver $Q(\mathcal{R})$ to $\mathcal{R}$.

The usual mutation of cluster-tilting objects in a Hom-finite, 2-Calabi-Yau Krull-Schmidt triangulated category over a field k extends to rigid objects. We associate a coloured quiver $Q(R)$ to a rigid object R in a way generalising that of Buan-Thomas [3]. Note that, in the situation of [2], where, in particular, R is maximal rigid, $Q(R)$ contains the same information as the matrix associated there to R. Brüstle-Zhang [4] have associated a generalised cluster category $\mathcal{C}_{(S,M)}$ in the sense of Amiot [1] to a surface (S, M) as above, using the quiver [5] associated to (S, M) with potential from [7]. Brüstle-Zhang show that there is a bijection $\alpha \mapsto X_\alpha$ between the indecomposable rigid objects in $\mathcal{C}_{(S,M)}$ and the arcs in (S, M) which extends to a bijection between partial triangulations in (S, M) and rigid objects in $\mathcal{C}_{(S,M)}$. Our main result is:

Theorem. *Let (S, M) be a marked surface and $\mathcal{R}$ a partial triangulation of (S, M). Let R be the corresponding rigid object in $\mathcal{C}_{(S,M)}$. Then $Q(\mathcal{R})$ and $Q(R)$ are isomorphic coloured quivers. Furthermore, if $\mathcal{R}'$ denotes the mutation of $\mathcal{R}$ at an arc $\alpha \in \mathcal{R}$ and R' denotes the mutation of R at X_α then $\mathcal{R}'$ and R' correspond under the bijection above (and thus their coloured quivers are isomorphic).*

In the type A case (a disk with marked points on its boundary), we give an explicit description of mutation of the coloured quiver by analysing the combinatorics.

Example: We consider the case of a disk with 10 marked points on its boundary. Figure 1 gives a partial triangulation of the disk and the associated coloured quiver. It also depicts the partial triangulation resulting from mutating the arc numbered 2 and the corresponding coloured quiver.

Suppose that α is an arc in (S, M) as above and X_α is the corresponding rigid indecomposable object in $\mathcal{C} = \mathcal{C}_{(S,M)}$. A special case of a result of Iyama-Yoshino [6] means that the quotient $\mathcal{C}_{X_\alpha}$ of the full subcategory of $\mathcal{C}$ consisting of objects Ext-orthogonal to X_α by the additive subcategory generated by X_α is triangulated. We show:

Theorem. The Iyama-Yoshino reduction $\mathcal{C}_{X_\alpha}$ is equivalent to the cluster category attached to the surface $(S, M)/\alpha$ obtained by cutting (S, M) along the arc α.

Thus Iyama-Yoshino reduction can be regarded as a categorification of cutting along an arc in a Riemann surface.

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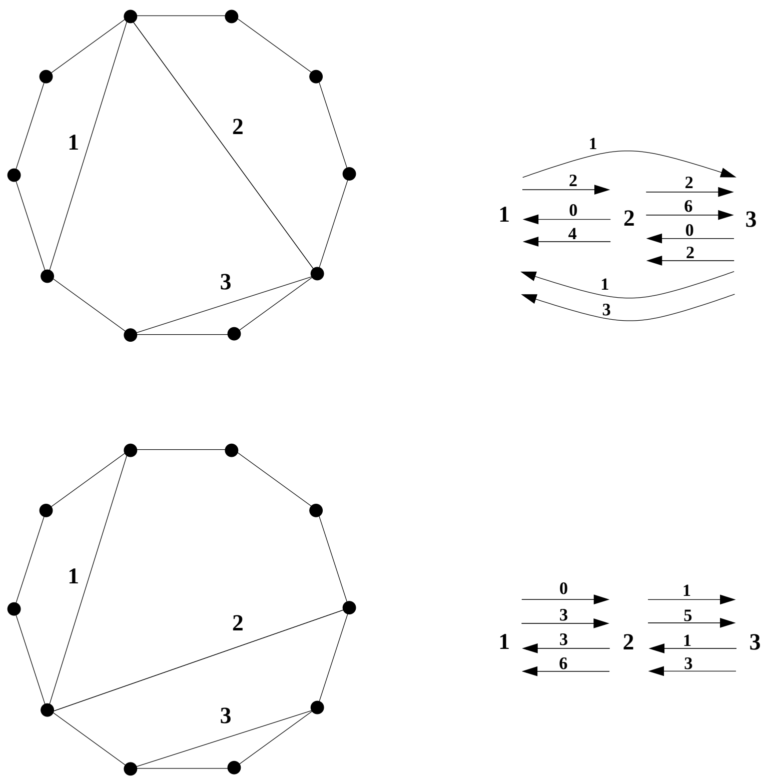

FIGURE 1. A coloured quiver of type A_7 and (below) its mutation at 2.

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Positivity questions related to quiver moduli spaces

SERGEY MOZGOVOY

0.1. **Hall algebra and quantum torus.** Let $Q = (Q_0, Q_1)$ be a quiver. Let χ be the corresponding Euler-Ringel form. Let

$$\langle \alpha, \beta \rangle = \chi(\alpha, \beta) - \chi(\beta, \alpha), \qquad \alpha, \beta \in \mathbb{Z}^{Q_0}$$

be the anti-symmetric form of Q and let $T(\alpha) = \chi(\alpha, \alpha)$ be the Tits form of Q. Let H be the (opposite) Ringel-Hall algebra of Q over a finite field $\mathbb{F}_q$. Let $\widehat{H}$ be its completion. Let $\widehat{\mathbb{T}} = \widehat{\mathbb{T}}_Q$ be the quantum torus associated with Q. As a vector space it is $\mathbb{Q}(q^{\frac{1}{2}})[\![x_1,\dots,x_r]\!]$, where $r=\#Q_0$ (q will be either a power of prime number or a new variable, depending on the context). Multiplication is given by

$$x^\alpha \circ x^\beta = (-q^{\frac{1}{2}})^{\langle\alpha,\beta\rangle} x^{\alpha+\beta}.$$

It was proved by Reineke [7] that the map

$$I:\widehat{H}\to\widehat{\mathbb{T}}, \qquad [M]\mapsto \frac{(-q^{\frac{1}{2}})^{T(\underline{\dim}M)}}{\#\operatorname{Aut}M}x^{\underline{\dim}M}$$

is an algebra homomorphism.

0.2. **Semistable representations.** Given $\theta\in\mathbb{R}^{Q_0}$, we define the slope function μ_θ on $\mathbb{N}^{Q_0}\backslash\{0\}$ by the formula

$$\mu_\theta(\alpha)=\frac{\theta\cdot\alpha}{\sum\alpha_i}.$$

For any Q-representation M, we define $\mu_\theta(M)=\mu_\theta(\underline{\dim}M)$, where $\underline{\dim}M=(\dim M_i)_{i\in Q_0}\in\mathbb{N}^{Q_0}$ is the dimension vector of M. Using this slope function we define the notions of θ-stability and θ-semistability of Q-representations in the usual way. For any $\mu\in\mathbb{R}$, we define

$$A^\theta_\mu=\sum_{\mu_\theta(\alpha)=\mu}A^\theta_\alpha x^\alpha := \sum_{\overline{M \text{ is } \theta-sst\mu_\theta(M)=\mu}} I(M)\in\widehat{\mathbb{T}}.$$

It was proved by Markus Reineke that $A_\alpha(q)$ are rational functions in the variable $q^{\frac{1}{2}}$ [6]. For $\theta=0$, $\mu=0$, we denote A^θ_μ just by A.

0.3. **Plethystic operations.** Consider $\widehat{\mathbb{T}}$ as an algebra endowed with the usual commutative multiplication. Consider $q^{\frac{1}{2}}$ as a new variable. For any function $f(q^{\frac{1}{2}},x_1,\dots,x_r)$ in $\widehat{\mathbb{T}}$, we define the Adams operations

$$\psi_n(f(q^{\frac{1}{2}},x_1,\dots,x_r))=f(q^{\frac{1}{2}n},x_1^n,\dots,x_r^n), \qquad n\geq 1.$$

We define the plethystic exponential $\operatorname{Exp}:\widehat{\mathbb{T}}^+\to 1+\widehat{\mathbb{T}}^+$ (here $\widehat{\mathbb{T}}^+$ is the maximal ideal of $\widehat{\mathbb{T}}$) by the rule

$$\operatorname{Exp}(f)=\exp\left(\sum_{n\geq 1}\frac{1}{n}\psi_n(f)\right).$$

Define the operator $T:\widehat{\mathbb{T}}\to\widehat{\mathbb{T}}$, $x^\alpha\mapsto(-q^{\frac{1}{2}})^{T(\alpha)}x^\alpha$. Then we can write the element $A\in\widehat{\mathbb{T}}$ defined earlier as

$$A=\sum_\alpha\frac{(-q^{\frac{1}{2}})^{-T(\alpha)}}{(q^{-1})_\alpha}x^\alpha=T^{-1}\left(\sum_\alpha\frac{x^\alpha}{(q^{-1})_\alpha}\right)=T^{-1}\operatorname{Exp}\left(\frac{\sum x_i}{1-q^{-1}}\right),$$

where $(q)_\alpha=\prod_i(q)_{\alpha_i}$ and $(q)_n=\prod_{k=1}^n(1-q^k)$ for $n\geq 0$.

0.4. **Donaldson-Thomas invariants.** Assume that Q is a symmetric quiver, i.e. the anti-symmetric bilinear form $\langle -,-\rangle$ is zero. Then $\widehat{\mathbb{T}}$ (with the twisted multiplication) is a commutative algebra.

Definition 1. *For any $\mu \in \mathbb{R}$, we define the Donaldson-Thomas invariants $\Omega^\theta_\mu = \sum_{\mu_\theta(\alpha)=\mu} \Omega^\theta_\alpha x^\alpha \in \widehat{\mathbb{T}}$ by the formula*

$$A^\theta_\mu = \operatorname{Exp}\left(\frac{\Omega^\theta_\mu}{q-1}\right).$$

For the trivial stability $\theta = 0$, we denote Ω^θ_α by Ω_α.

Remark 1. *The classical Donaldson-Thomas invariants*

$$\overline{\Omega}^\theta_\mu = \sum_{\mu(\alpha)=\mu} \overline{\Omega}^\theta_\alpha x^\alpha \in \mathbb{Q}[\![x_i, i \in Q_0]\!]$$

are defined by the formula

$$\lim_{q\to 1}(q-1)\log A^\theta_\mu = \sum_\alpha \overline{\Omega}^\theta_\alpha \operatorname{Li}_2(x^\alpha) = \sum_{n\geq 1}\frac{1}{n^2}\sum_\alpha \overline{\Omega}^\theta_\alpha x^{n\alpha},$$

where the dilogarithm function Li_2 is defined by $\operatorname{Li}_2(x) = \sum_{n\geq 1}\frac{x^n}{n^2}$. If we "quantize" this formula, we obtain

$$A^\theta_\mu = \exp\left(\frac{1}{q-1}\sum_{n\geq 1}\frac{1}{n}\frac{q-1}{q^n-1}\sum_\alpha \Omega^\theta_\alpha(q^n)x^{n\alpha}\right) = \operatorname{Exp}\left(\frac{1}{q-1}\sum_\alpha \Omega^\theta_\alpha(q)x^\alpha\right).$$

This coincides with Definition 1.

The following statement is a consequence of [4, Conjecture 1]

Conjecture 1. *For any $\alpha \in \mathbb{N}^{Q_0}$, the functions $\Omega_\alpha(-q^{\frac{1}{2}})$ are polynomials in $q^{\pm\frac{1}{2}}$ with non-negative integer coefficients.*

Remark 2. *A slightly weaker statement of Conjecture 1 for the quivers with one vertex and several loops was recently proved by Markus Reineke [8]. The complete proof of [4, Conjecture 1] and thus of Conjecture 1 was recently obtained by Efimov [1].*

Conjecture 2. *For any stability parameter $\theta \in \mathbb{R}^{Q_0}$ and for any $\alpha \in \mathbb{N}^{Q_0}$, the functions $\Omega^\theta_\alpha(-q^{\frac{1}{2}})$ are polynomials in $q^{\pm\frac{1}{2}}$ with non-negative integer coefficients.*

The fact that $\Omega^\theta_\alpha \in \mathbb{Z}[q^{\pm\frac{1}{2}}]$ follows from [4].

0.5. **Combinatorial positivity conjecture.** Let C be an $r\times r$ matrix with non-negative coefficients (not necessarily symmetric). We define the operator $\overline{T} : \widehat{\mathbb{T}} \to \widehat{\mathbb{T}}$ by $\overline{T}(x^\alpha) = q^{\alpha^t C\alpha}x^\alpha$.

Conjecture 3. *Assume that*

$$\operatorname{Exp}\left(\frac{\sum b_\alpha(q)x^\alpha}{q-1}\right) = \overline{T}\operatorname{Exp}\left(\frac{\sum a_\alpha(q)x^\alpha}{q-1}\right),$$

where $a_\alpha \in \mathbb{N}[q]$, $\alpha \in \mathbb{N}^r$. *Then* $b_\alpha \in \mathbb{N}[q]$, $\alpha \in \mathbb{N}^r$.

Theorem 1. *If Conjecture 1 is true then Conjecture 3 is also true.*

0.6. **Kac positivity conjecture.** Let Q be an arbitrary quiver with r vertices and let $\alpha \in \mathbb{N}^{Q_0}$. It was proved by Kac [3] that there exists a polynomial $a_\alpha \in \mathbb{Z}[q]$, such that the number of absolutely stable representations of Q of dimension α over a finite field $\mathbb{F}_q$ equals $a_\alpha(q)$. Kac conjectured that $a_\alpha \in \mathbb{N}[q]$.

There is a rather explicit formula due to Hua [2, 5] that allows to compute the polynomials $a_\alpha \in \mathbb{N}[q]$ for an arbitrary quiver. Using a thorough analysis of this formula together with Theorem 1 we can prove

Theorem 2. *If Conjecture 1 is true then the Kac positivity conjecture is true for quivers having at least one loop at every vertex.*

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Donaldson-Thomas theory and cluster algebras

Kentaro Nagao

This is an extended abstract of my talk at the Oberwolfach workshop "Representation Theory of Quivers and Finite Dimensional Algebras" (February 20–26, 2011). We study cluster algebras from the viewpoint of Donaldson-Thomas theory. Consequently, we get a description of a composition of cluster transformations in terms of quiver Grassmannians.

0.1. **Cluster algebras.** Cluster algebras were introduced by Fomin and Zelevinsky ([FZ02]) in their study of dual canonical bases and total positivity in semisimple groups. Although the initial aim has not been established, it has been discovered that the theory of cluster algebras has many links with a wide range of mathematics (see [Kel, §1.1] and the references there). Since a cluster transformation helps us to understand the whole structure in an inductive way, study of compositions of cluster transformations is important.

In the case of a quiver of finite type, Caldero and Chapoton ([CC06]) described a composition of cluster transformations in terms of quiver Grassmannians of the original quiver. This result is generalized

- for an acyclic quiver by Caldero and Keller ([CK06]),
- for a Jacobi-finite quiver with a potential by Fu and Keller ([FK]) using the result of [Ami09] and [Pal08], and
- and for an arbitrary quiver without loops and 2-cycles by Derksen-Weyman-Zelevinsky ([DWZ]), Plamondon ([Pla]) and the author ([Naga]).

In [DWZ] and [Pla], they prove six conjectures given in [FZ07] for cluster algebras associated to quivers [1].

0.2. **Donaldson-Thomas theory.** Donaldson-Thomas invariants , introduced in [Tho00,MNOP06], are defined as the topological Euler characteristics (more precisely, the weighted Euler characteristics weighted by Behrend function [Beh09]) of the moduli spaces of sheaves on a Calabi-Yau 3-fold (more generally, the moduli spaces of objects in a 3-Calabi-Yau category [Sze08, Joy08, KS, JS]). Dominic Joyce introduced the motivic Hall algebra for an Abelian category in his study of generalized Donaldson-Thomas invariants ([Joy07]). One of the important results is that for a 3-Calabi-Yau category there exists a Poisson algebra homomorphism, so called the *integration map*, from the motivic Hall algebra to a power series ring ([Joy07, JS, Bri]). The integration map is given by taking the (weighted) Euler characteristic of an element in the motivic Hall algebra. Due to the integration map, we get the following powerful method in Donaldson-Thomas theory for 3-Calabi-Yau categories, which originates with Reineke's computation of the Betti numbers of the spaces of stable quiver representations ([Rei03]):

> Starting from a simple categorical statement, provide an identity in the motivic Hall algebra. Pushing it out by the integration map, we get a power series identity for the generating functions of Donaldson-Thomas invariants.

0.3. **DT theory and cluster algebras.** Konstevich and Soibelman ([KS]) observed that the cluster transformation appears in the transformation formula of non-commutative Donaldson-Thomas invariants under a mutation.

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[1] Cluster algebras are associated not only with quivers without loops and oriented 2-cycles (equivalently, with skew-symmetric integer matrices) but also with *skew-symmetrizable* matrices.

In [Naga], the author provides a transformation formula of non-commutative Donaldson-Thomas invariants under a sequence of mutations. The formula is described in terms of Euler characteristics of quiver Grassmannians.

As a consequence, we get a description of compositions of cluster transformations in terms of Euler characteristics of quiver Grassmannians. Similar results were given by [DWZ] and [Pla].

0.4. **Motivic DT theory and quantum cluster algebras.** The quantum cluster transformation defined in [BZ06] is the automorphism of the quantum torus obtained by taking adjoint with respect to the quantum dilogarithm ([FG09]). In [KS], Kontsevich-Soibelman proposed a generalization of the DT invariants which are called the *motivic DT invariants* and observed that the quantum cluster transformation should appear in the transformation formula of motivic non-commu-tative DT invariants under a mutation. It is natural to expect that we can study the quantum cluster algebras from the view points of the motivic DT theory.

After the workshop, I submitted a paper on the wall-crossing formula of the motivic DT invariants to the arXiv ([Nagb]). Unfortunately, the arguments work only in a very special setting and are not strong enough to study the quantum cluster algebras. I hope to provide the quantum version of [Naga] in the future.

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Generating the bounded derived category and perfect ghosts

Steffen Oppermann
(joint work with Jan Šťovíček)

Bondal and van den Bergh [2] have introduced the notion of being strongly finitely generated for triangulated categories. They show that this property is useful when studying the representability of certain cohomological functors. Their definition is as follows:

Definition 1. *Let $\mathcal{T}$ be a triangulated category, T an object in $\mathcal{T}$. Set*

$$\operatorname{thick}_1 T = \operatorname{add}\{T[i] \mid i \in \mathbb{Z}\},$$

and inductively

$$\operatorname{thick}_{n+1} T = \operatorname{add}\{\operatorname{Cone} f \mid f \colon \operatorname{thick}_n T \to \operatorname{thick}_1 T\}.$$

A triangulated category $\mathcal{T}$ is called strongly finitely generated *if there is an object T in $\mathcal{T}$ and some $n \in \mathbb{N}$ such that $\mathcal{T} = \operatorname{thick}_n T$.*

Note that being strongly finitely generated is equivalent to having finite dimension in the sense of Rouquier [5].

Probably the most widely used class of triangulated categories in algebra are derived categories. Here we look at subcategories of bounded derived categories in the following two setups:

Setup 1. *Let Λ be a finite dimensional algebra. We denote by $\mathrm{D^b}(\operatorname{mod}\Lambda)$ its bounded derived category, and by $\mathrm{D^b}(\operatorname{proj}\Lambda)$ the full subcategory whose objects are bounded complexes of projectives. Assume*

$$\mathrm{D^b}(\operatorname{proj}\Lambda) \subseteq \mathcal{T} \overset{\text{thick}}{\subseteq} \mathrm{D^b}(\operatorname{mod}\Lambda).$$

Setup 2. *Let* $\mathbb{X}$ *be a projective scheme of finite type over some commutative noetherian ring. We denote by* $\mathrm{D^b}(\mathrm{coh}\,\mathbb{X})$ *its bounded derived category, and by* $\mathrm{D^b}(\mathrm{vect}\,\mathbb{X})$ *the full subcategory whose objects are bounded complexes of vector bundles. Assume*

$$\mathrm{D^b}(\mathrm{vect}\,\mathbb{X}) \subseteq \mathcal{T} \overset{\text{thick}}{\subseteq} \mathrm{D^b}(\mathrm{coh}\,\mathbb{X}).$$

In these two setups we obtain the following result, which, somewhat surprisingly, says that there are almost no strongly finitely generated triangulated categories $\mathcal{T}$ satisfying the above setups.

Theorem 1 (O-Šťovíček [4])**.** *Let* $\mathcal{T}$ *be a triangulated category as in Setup 1 or 2 above. Assume that* $\mathcal{T}$ *is strongly finitely generated. Then* $\mathcal{T} = \mathrm{D^b}(\mathrm{mod}\,\Lambda)$ *or* $\mathcal{T} = \mathrm{D^b}(\mathrm{coh}\,\mathbb{X})$*.*

In Setup 1 it is immediate that $\mathrm{D^b}(\mathrm{mod}\,\Lambda)$ is strongly finitely generated: Since any module has Loewy length at most the Loewy length $\ell\ell\ \Lambda$ of the algebra, one sees that $\mathrm{thick}_{\ell\ell\ \Lambda}(\Lambda/\mathrm{Rad}\Lambda) = \mathrm{D^b}(\mathrm{mod}\,\Lambda)$.

Similarly, in Setup 2 if moreover the scheme is of finite type over a perfect field, Rouquier [5, Theorem 7.38] has shown that the category $\mathrm{D^b}(\mathrm{coh}\,\mathbb{X})$ is strongly finitely generated.

Thus in these setups the statement of the theorem immediately strenghtens to an equivalence.

In the setup of finite dimensional algebras we immediately obtain the following corollary.

Corollary 1. *Let* Λ *be a finite dimensional algebra,* $T \in \mathrm{D^b}(\mathrm{mod}\,\Lambda)$ *such that* $\Lambda \in \mathrm{add}\,T$*. Then either*

(1) $\mathrm{thick}_1\,T \subsetneq \mathrm{thick}_2\,T \subsetneq \cdots$*, and* $\cup_{n\in\mathbb{N}}\,\mathrm{thick}_n\,T \subsetneq \mathrm{D^b}(\mathrm{mod}\,\Lambda)$*, or*

(2) $\exists n \in \mathbb{N}$ *such that* $\mathrm{thick}_n\,T = \mathrm{D^b}(\mathrm{mod}\,\Lambda)$*.*

The main ingredient for proving the theorem is to strengthen the *Ghost Lemma.*

A (to our knowledge first) version of this lemma has appeared in work of Kelly [3]. Subsequently the lemma has been generalized by many authors, see for instance [1].

In the notation here the idea of the Ghost Lemma is that, given an object in $X \in \mathrm{thick}_n\,T$, any sequence of n ghost maps ending in X must compose to zero. Here a map f is called ghost map if "it cannot be seen from T", that is more precisely if $\mathrm{Hom}(f, \mathrm{thick}_1\,T) = 0$.

We show that, in the situations of Setup 1 and 2, the Ghost Lemma can be strengthened to an equivalence, that is we show that it is enough to check if certain sequences of ghosts compose to 0 in order to know if an object lies in $\mathrm{thick}_n\,T$.

We then further strengthen this result by showing that it is even enough to consider "perfect ghosts", that is ghost maps between objects in $\mathrm{D^b}(\mathrm{proj}\,\Lambda)$ or $\mathrm{D^b}(\mathrm{vect}\,\mathbb{X})$.

Using this version of the ghost lemma, the proof of the main theorem can be sketched as follows:

$$\begin{aligned} &\mathcal{T} = \operatorname{thick}_n T \\ \iff &\text{all sequences of } n \text{ perfect ghosts compose to zero} \\ \iff &\mathrm{D^b}(\mod \Lambda) = \operatorname{thick}_n T \text{ resp. } \mathrm{D^b}(\operatorname{coh} \mathbb{X}) = \operatorname{thick}_n T \end{aligned}$$

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Indices and generic bases for cluster algebras

Pierre-Guy Plamondon

In their paper [9] published in 2002 , S. Fomin and A. Zelevinsky laid the foundations of the theory of cluster algebras. One of their aims was to give a combinatorial approach to the study of Kashiwara/Lusztig's canonical bases in quantum groups. For this reason, the search for suitable bases of cluster algebras is an important problem in the theory. The aim of this extended abstract is to put forward a method, using the additive categorification of cluster algebras by means of 2-Calabi–Yau triangulated categories (see [1], [13]), which will hopefully lead to the construction of such bases. This method is based on that of C. Geiss, B. Leclerc and J. Schröer [11], who obtained "generic" bases for a large class of cluster algebras. Previously, such bases were constructed explicitly by G. Dupont [7], see also the work of M. Ding, J. Xiao and F. Xu [6], for cluster algebras associated with affine quivers.

1. Setting : 2-Calabi–Yau triangulated categories

We will work over the field $\mathbb{C}$ of complex numbers. Let $\mathcal{C}$ be a triangulated $\mathbb{C}$-category with suspension functor Σ. We will make the following assumptions:

- $\mathcal{C}$ is Hom-finite (*i.e.* all morphism spaces are finite-dimensional);
- $\mathcal{C}$ is 2-Calabi–Yau (*i.e.* for any objects X and Y of $\mathcal{C}$, there exists a bifunctorial isomorphism $\operatorname{Hom}_{\mathcal{C}}(X,Y) \cong D\operatorname{Hom}_{\mathcal{C}}(Y,\Sigma^2 X)$, where $D = \operatorname{Hom}_{\mathbb{C}}(?,\mathbb{C})$ is the standard duality);
- $\mathcal{C}$ admits a cluster-tilting object T (*i.e.* for any object X of $\mathcal{C}$, the space $\operatorname{Hom}_{\mathcal{C}}(T,\Sigma X)$ vanishes iff X is in $\operatorname{add} T$) which is non-degenerate (*i.e.* if T' is obtained from T by iterated mutations, then the Gabriel quiver of $\operatorname{End}_{\mathcal{C}}(T')$ has no oriented cycles of length ≤ 2).

Example 1. *C. Amiot's generalized cluster category [1] associated to a quiver with potential [5] satisfies the above assumptions, provided that the quiver with potential is Jacobi-finite.*

Example 2. *Let Q be a finite quiver without oriented cycles. To any element w of the Weyl group associated to Q, A. Buan, O. Iyama, I. Reiten and J. Scott [3] (see also [10]) associated a category $\underline{\mathcal{C}}_w$ which satisfies the above assumptions. To any reduced expression $\mathbf{i}$ of w, they associated a cluster-tilting object $V_{\mathbf{i}}$ of $\underline{\mathcal{C}}_w$.*

2. Generic basis of C. Geiss, B. Leclerc and J. Schröer

For any finite-dimensional $\mathbb{C}$-algebra A with Gabriel quiver Q, we denote by $\mathrm{rep}_{\mathbf{d}}(A)$ the variety of finite-dimensional representations of A of dimension vector $\mathbf{d}$; it is a closed subvariety of $\bigoplus_{\alpha:i\to j}\mathrm{Hom}_{\mathbb{C}}(\mathbb{C}^{d_i},\mathbb{C}^{d_j})$. Let $\mathrm{rep}(A)$ be the union of all the $\mathrm{rep}_{\mathbf{d}}(A)$. For any irreducible component $\mathcal{Z}$ of $\mathrm{rep}(A)$, we let $\underline{\dim}\,\mathcal{Z}=\mathbf{d}$ if $\mathcal{Z}$ is contained in $\mathrm{rep}_{\mathbf{d}}(A)$.

A significant result in the search for a good basis of cluster algebras is the following theorem of C. Geiss, B. Leclerc and J. Schröer, which requires the definition [11, Section 7.1] of strongly reduced components of $\mathrm{rep}(A)$.

Theorem 1 (Theorem 5 of [11])**.** *Let $\underline{\mathcal{C}}_w$ be as in the setting of Example 2, with a cluster-tilting object $V_{\mathbf{i}}$. Let $A=\mathrm{End}_{\underline{\mathcal{C}}_w}(V_{\mathbf{i}})$, and let Q be the Gabriel quiver of A. Then a basis of the cluster algebra $\mathcal{A}_Q$ is given by the set*

$$\Big\{\mathbf{x}^{\mathbf{e}}\psi_{\mathcal{Z}}\,\Big|\ \mathcal{Z}\text{ is strongly reduced}, \mathbf{e}\in\mathbb{N}^{Q_0}, \mathbf{e}\text{ and }\underline{\dim}\,\mathcal{Z}\text{ have disjoint support}\Big\},$$

where $\psi_{\mathcal{Z}}$ is the generic value taken by Y. Palu's cluster character [13] inside $\mathcal{Z}$.

Remarks 1.
1. *(1) The generic basis obtained is dual to the semicanonical basis of G. Lusztig [12].*
2. *(2) The authors of [11] conjecture that the set defined in Theorem 1 is a basis of the associated cluster algebra, even outside of the setting of Example 2.*

3. Results

Let $\mathcal{C}$ be a category which satisfies the assumptions of section 1. Let T be a cluster-tilting object of $\mathcal{C}$, let $A=\mathrm{End}_{\mathcal{C}}(T)$ and let Q be the Gabriel quiver of A. Denote by $K_0(\mathrm{proj}\,A)$ the Grothendieck group of the category of finite-dimensional projective A-modules ; its elements are called *indices*, as in [13].

Theorem 2. *There exists a canonical map $I: K_0(\mathrm{proj}\,A)\to\mathcal{A}_Q^+$, where $\mathcal{A}_Q^+$ is the upper cluster algebra of [2]. In the setting of Example 2, the image of I is the generic basis of [11].*

Remarks 2.
1. *(1) Let T_1 and T_0 be objects of $\mathrm{add}\,T$; then $P_i=\mathrm{Hom}_{\mathcal{C}}(T,T_i)$ is in $\mathrm{add}\,A$, for $i=1,2$. In that case, I sends the index $[P_0]-[P_1]$ to the generic value taken by Y. Palu's cluster character on the cones of morphisms in $\mathrm{Hom}_{\mathcal{C}}(T_1,T_0)$.*

(2) *The canonical decomposition of H. Derksen and J. Fei [4] of a morphism in* $\mathrm{Hom}_A(P_1, P_0)$ *yields a factorization of* $I([P_0] - [P_1])$.

Theorem 3. *The map* I *commutes with mutations.*

Remark 3. *This theorem links results of [11] to a conjecture of V. Fock and A. Goncharov [8, Conjecture 4.1]. Together with the results of G. Lusztig and of C. Geiss, B. Leclerc and J. Schröer, the theorem allows to prove this conjecture in the setting of Example 2.*

Theorem 4. *There is a commutative diagram*

$$\begin{tikzcd} \{\textit{strongly reduced comp. of } \mathrm{rep}(A)\} \arrow[r, hook] \arrow[dr, "\psi"'] & K_0(\mathrm{proj}\, A) \arrow[d, "I"] \\ & \mathcal{A}_Q^+, \end{tikzcd}$$

where ψ *is the map described in Theorem 1.*

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Cohomological Hall algebra and positivity conjectures

Markus Reineke

Let Q be a finite quiver with set of vertices I, and let $d \in \mathbf{N}I$ be a dimension vector. We consider the following polynomials:

(1) $A_d(q)$, the polynomial counting isomorphism classes of absolutely indecomposable representations of Q of dimension vector d over the finite field $\mathbf{F}_q$. It is known to belong to $\mathbf{Z}[q]$, and conjectured to actually belong to $\mathbf{N}[q]$ by [4]. This conjecture is proved when d is indivisible in [1].

(2) $S_d(q)$, the polynomial counting isomorphism classes of absolutely simple representations of Q of dimension vector d over the finite field $\mathbf{F}_q$. It is known to belong to $\mathbf{Z}[q]$, and conjectured to actually belong to $\mathbf{N}[q-1]$ in [7].

(3) (in case Q is a symmetric quiver) $\mathrm{DT}_d(q)$, the quantized Donaldson-Thomas invariant of Q (see below for the definition). It is known to belong to $\mathbf{Q}[q,q^{-1}]$, and conjectured to belong to $\mathbf{N}[q]$ by [5].

Define the so-called plethystic exponential of a formal series in one variable z by

$$\mathrm{Exp}(\sum_{n\geq 1} a_n z^n) = \prod_{n\geq 1}(1-z^n)^{-a_n}.$$

(with an obvious extension to series in several variables). Using this notation, there are the following explicit formulas for the above polynomials (in the third case, this is actually the definition):

(1) We have

$$\mathrm{Hua}^Q(q,t) = \sum_{\lambda} \frac{q^{-\langle\lambda,\lambda\rangle}}{\prod_i \prod_r (q^{-1})_{\lambda^i_r}} t^{|\lambda|} = \mathrm{Exp}(\frac{1}{q-1}\sum_{d\neq 0} A_d(q)t^d),$$

where $\lambda = (\lambda^i)_i)$ denotes a tuple of partitions $(\lambda^i = \lambda^i_1 \geq \ldots \geq \lambda^i_{l_i})_{i\in I}$, $|\lambda| = (|\lambda_i|)_{i\in I} \in \mathbf{N}I$, $\langle\lambda,\lambda\rangle = \sum_r \langle\lambda_r,\lambda_r\rangle$ for $\lambda_r = (\lambda^i_r)_{i\in I} \in \mathbf{N}I$ and $(z)_n = (1-z)\cdot\ldots\cdot(1-z^n)$.

(2) We have

$$\sum_{d\geq 0} \frac{q^{-\langle d,d\rangle}}{\prod_i (q^{-1})_{d_i}} t^d \circ \mathrm{Exp}(\frac{1}{1-q}\sum_{d\neq 0} S_d(q)t^d) = 1,$$

where $t^d \circ t^e = q^{-\langle e,d\rangle} t^{d+e}$.

(3) We have

$$\sum_{d\geq 0} \frac{q^{-\langle d,d\rangle}}{\prod_i (q^{-1})_{d_i}} t^d = \mathrm{Exp}(\frac{1}{1-q}\sum_{d\neq 0} \mathrm{DT}_d(q)(\pm t)^d),$$

where the sign only depends on Q.

Potential proofs of positivity properties usually involve linking the above polynomials to some geometry, like the geometry of preprojective varieties in the case of [1], or Higgs moduli in [3]. Here we use an algebraic approach to the polynomials $\mathrm{DT}_d(q)$ using the Cohomological Hall algebra of [5] and the geometry of noncommutative Hilbert schemes (see [6]) in the case of the m-loop quiver.

Theorem 1. *Let* $H(q,t)=\sum_{n\geq 0}\frac{q^{(m-1)\binom{n}{2}}}{(1-q^{-1})\cdot\ldots\cdot(1-q^{-n})}t^n$, *and write*

$$H(q,(-1)^{m-1}t)=\mathrm{Exp}(\frac{1}{1-q^{-1}}\sum_{n\geq 1}\mathrm{DT}_n(q)t^n).$$

Then

(1) $\mathrm{DT}_n(q)\in\mathbf{Z}[q]$,
(2) $\mathrm{DT}_n(1)=\frac{1}{n^2}\sum_{d|n}\mu(\frac{n}{d})(-1)^{(m-1)(n-d)}\binom{md-1}{d-1}\in\mathbf{N}$,
(3) $\mathrm{DT}_n(q)=q^{n-1}\frac{1}{1+q+\ldots+q^{n-1}}\sum_C q^{\mathrm{wt}(C)}$,
(4) $\mathrm{DT}_n(1)=|\{C\,|\,\mathrm{wt}(C)\equiv d \bmod n\}|$,

where in the last two statements, C denotes the set of cyclic equivalence classes of sequences

$$(a_1,\ldots,a_n)\in\mathbf{N}^n\,:\,\sum_i a_i=(m-1)n$$

which are primitive in the sense that the sequence is different from all its proper cyclic shifts (a slight modification is needed in case m even, $n\equiv 2 \bmod 4$), and $\mathrm{wt}(C)$ *is the maximum of* $\sum_{i=1}^n(m-i)(m-1-a_i)$ *along the cyclic class C.*

The algebraic approach towards positivity of $\mathrm{DT}_d(q)$ pursued in [5] is to construct a bigraded $\mathbf{Q}$-algebra A whose Poincaré-Hilbert series

$$P_A(q,t)=\sum_{n\geq 0}\sum_k \dim_{\mathbf{Q}} A_{n,k}q^kt^n$$

equals $H(q,t)$, and to prove that A is isomorphic to the symmetric algebra of a bigraded supervectorspace $C\otimes\mathbf{Q}[z]$ with z homogeneous of bidegree $(0,-1)$. Namely, in this case $H(q,t)=$

$$=P_A(q,t)=P_{\mathrm{Sym}(C\otimes\mathbf{Q}[z]}(q,t)=\mathrm{Exp}(P_{C\otimes\mathbf{Q}[z]}(q,t))=\mathrm{Exp}(\frac{1}{1-q^{-1}}P_C(q,t)).$$

The candidate proposed in [5] is the Cohomological Hall algebra of a quiver: for $d\in\mathbf{N}I$, denote by $R_d(Q)$ the variety of (complex) representations of Q of dimension vector d with the action of the base change group G_d. There exists an analogue of the Hall algebra construction in the G_d-equivariant cohomology of $R_d(Q)$, defining an algebra structure on

$$\bigoplus_d H^*_{G_d}(R_d(Q),\mathbf{Q})$$

with a natural bigrading by d and cohomological degree (suitably normalized). A proof of positivity using this algebra was announced recently [2].

The above theorem is proved using a degenerate version of the Cohomological Hall algebra, which has a purely combinatorial definition:

Define A as the $\mathbf{Q}$-algebra with basis parametrized by all partitions

$$\lambda = (0 \leq \lambda_1, \ldots, \lambda_{l(\lambda)}),$$

and with multiplication

$$\lambda * \mu = \underbrace{\lambda_1, \ldots, \lambda_{l(\lambda)}, \mu_1 + (m-1)l(\lambda), \ldots, \mu_{l(\mu)} + (m-1)l(\lambda)}_{\text{resorted in ascending order}}.$$

This algebra is bigraded by $(l(\lambda), (m-1)\binom{n}{2} - |\lambda|)$. The above theorem follows from an isomorphism

$$A \simeq \mathrm{Sym}(\bigoplus_{n\geq 0} S^n B^L),$$

where $S\lambda = (\lambda_1 + 1, \ldots, \lambda_{l(\lambda)} + 1)$, and B^l is spanned by a subset of certain partitions (Lyndon words in Dyck partitions; see [8] for details); the definition of this class of partitions is motivated by partitions naturally arizing in the study of Noncommutative Hilbert schemes, varieties parametrizing left ideals in the path algebra of the m-loop quiver (see [6]).

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Some Auslander-Reiten quilts

Claus Michael Ringel

The lecture was dealing with the module categories of some special biserial algebras. Special biserial algebras were first studied by Gelfand and Ponomarev in 1968, they have provided the methods in order to classify all the indecomposable representations of such an algebra (the string modules and the band modules), and there is also known a precise recipe for obtaining all the irreducible maps (adding or deleting hooks and cohooks). The algebras which were considered in the lecture are the wind wheel algebras, they are obtained from the hereditary algebras of type $\widetilde{\mathbb{A}}_n$ by identifying suitable pairs of linearly oriented subquivers, the bars. The wind wheel algebras are minimal representation-infinite algebras.

The study of minimal representation-infinite k-algebras with k an algebraically closed field was one of the central themes of the representation theory around 1984 with contributions by Bautista, Gabriel, Roiter, Salmeron, Bongartz, Fischbacher and many others. Recent investigations of Bongartz [1] provide a new impetus for analyzing the module category of such an algebra and even seem to yield a basis for a classification of these algebras. Here is a short summary of this development. First of all, there are algebras with a non-distributive ideal lattice, such algebras have been studied already 1957 by Jans. Second, there are algebras with a good universal cover $\widetilde{\Lambda}$ and such that $\widetilde{\Lambda}$ has a convex subcategory which is a tame concealed algebra of type $\widetilde{\mathbb{D}}_n, \widetilde{\mathbb{E}}_6, \widetilde{\mathbb{E}}_7$ or $\widetilde{\mathbb{E}}_8$; these were the algebras which have been discussed by Bautista, Gabriel, Roiter and Salmeron in 1984 (we say that the universal cover is *good* provided it is a Galois cover with free Galois group and is interval-finite). As Bongartz now has shown, the remaining minimal representation-infinite algebras also have a good cover $\widetilde{\Lambda}$, but all finite convex subcategories of $\widetilde{\Lambda}$ are representation-finite. These algebras can be shown to be special biserial and can be classified completely: After the separation of nodes, there are three different kinds: the hereditary algebras of type $\widetilde{\mathbb{A}}_n$, the wind wheel algebras, as well as the barbell algebras with non-serial bars. Whereas the barbell algebras are algebras with non-polynomial growth, the hereditary ones and the wind wheels are 1-domestic: this means that there is precisely one primitive 1-parameter family of indecomposable modules (and of course additional isolated indecomposables).

The aim of the lecture was to look at a wind wheel algebra W and to describe in detail first its Auslander-Reiten quiver, but then also the Auslander-Reiten quilt Γ of W (see [3]); the quilt is obtained from the set of Auslander-Reiten components which contain string modules by inserting suitable (infinite dimensional) indecomposable algebraically compact modules. These additional modules are constructed using $\mathbb{N}$-words and $\mathbb{Z}$-words, quite similar to the construction of the string modules using finite words, but for the infinite dimensional modules often some completion is necessary (see [2]).

We denote by rad the radical of the module category $\operatorname{mod} W$, it is the ideal generated by the non-invertible homomorphisms between indecomposable W-modules. Using transfinite induction, one defines powers $\operatorname{rad}^\lambda$ for any ordinal number. For example, for the first limit ordinal ω, one takes as $\operatorname{rad}^\omega$ the intersection of all finite powers rad^n — note that the Auslander-Reiten quiver of W is meant to display the factor category $\operatorname{mod} W/\operatorname{rad}^\omega$. In the same way, the factor category $\operatorname{mod} W/\operatorname{rad}^{\omega 2}$ (or at least part of it) is exhibited by the Auslander-Reiten quilt (here $\omega 2$ is the second limit ordinal).

It turns out that *the Auslander-Reiten quilt Γ of a wind wheel algebra is a connected orientable surface with boundary, its Euler characteristic is $\chi(\Gamma) = -t$, where t is the number of bars.*

The components of the Auslander-Reiten quiver of W which contain string modules are ramified components of type $\mathbb{A}_\infty^\infty$, the ramification data being given by a permutation π. Such a component is sewn together from partial translation quivers in the same way as one constructs Riemann surfaces in complex analysis. The permutations π which arise for the wind wheel algebras with t bars are precisely the commutators of two t-cycles.

As we have mentioned, the Auslander-Reiten quilt of any wind wheel algebra W is orientable (for example, for the wind wheel with only two simple modules we obtain a torus with one hole). On the other hand, it is easy to see that the category of W-modules contains as a full subcategory the module category of a representation-finite algebra L whose Auslander-Reiten quiver is homeomorphic to a Möbius strip. In order to understand the embedding $\operatorname{mod} L \to \operatorname{mod} W$, one may analyze in which way the irreducible maps of $\operatorname{mod} L$ are factorized inside $\operatorname{mod} W$ by looking at the quilt Γ. It turns out that a curious change of direction occurs when approaching some infinite dimensional W-modules which are not part of the quilt.

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$A(1)$ and the dilogarithm

Fernando Rodriguez Villegas

In this talk I presented an approach to obtaining a formula for the value at $q = 1$ of the A-polynomial of an arbitrary quiver. The A-polynomial $A(q)$ of a quiver Γ counts the number of absolutely indecomposable representations of Γ over the field $\mathbb{F}_q$ with a given dimension vector. One one hand, by our joint work [2] with Hausel and Letellier, we expect the value $A(1)$ to be the middle Betti number of an associated character variety. On the other, it is tempting to interpret $A(1)$ as counting certain simpler combinatorial objects resulting from letting the field size become one. The formula might shed light into these questions. See [9] for the full version.

I concentrate on the following example. Let S_g be the quiver consisting of one vertex and g loops and let $A_n^g(q)$ be the A-polynomial for dimension n. We define a priori rational functions $A_\lambda(q)$ indexed by partitions λ which give a decomposition

$$A_n(q) = \sum_{|\lambda|=n} A_\lambda(q)$$

of the A-polynomial of S_g.

Computations suggest that for $g > 0$, which we assume from now on, $A_\lambda(q)$ is in fact a polynomial in q with non-negative integer coefficients. For example, for $g = 2$ and $n = 3$ we obtain

$$A_{(1,1,1)}(q) = q^{10} + q^8 + q^7, \quad A_{(2,1)} = q^6 + q^5, \quad A_{(3)} = q^4$$

with sum

$$A_3(q) = q^{10} + q^8 + q^7 + q^6 + q^5 + q^4$$

We have the following

Theorem 1. *For any non-zero partition λ we have*

$$A_\lambda(1) = \frac{1}{\rho} \sum_{d|m} \frac{\mu(d)}{d^2} \frac{1}{P_1(m/d)P_N(m/d)} \prod_{i\geq 1} \binom{\rho P_i(m/d) - 1 + m_i/d}{m_i/d} \tag{1}$$

where $\lambda = (1^{m_1}2^{m_2}\cdots N^{m_N})$ with $N = \lambda_1$, the largest part of λ,

$$P_i(m) := \sum_{j\geq 1} \min(i,j)\, m_j, \qquad m := (m_1, m_2, \ldots), \quad \rho =: 2g-2,$$

and μ is the Möbius function of number theory.

The case $\lambda = (1^n)$ was previously proved by Reineke [7] by different methods. By the conjectures of [1] the number $A_n(1) = \sum_{|\lambda|=n} A_\lambda(1)$ should equal the dimension of the middle dimensional cohomology group of the character variety $\mathcal{M}_n$ studied there. A refined version of this conjecture states that $A_\lambda(1)$ is the number of connected components of type λ of a natural $\mathbb{C}^\times$ action on the moduli space of Higgs bundles, which is diffeomorphic to $\mathcal{M}_n$. A proof of this conjecture for $\lambda = (1^n)$ was recently given by Reineke [8, Theorem 7.1]. The refined conjecture originates in [1, Remark 4.4.6] and was in fact the motivation to construct the truncated polynomials $A_\lambda(q)$ studied here.

Corollary 1. *As a function of g, the quantity $A_\lambda(1)$ is a polynomial of degree $l(\lambda)-1$; its leading coefficient in $\rho := 2g-2$ is*

$$\frac{1}{P_1(m)P_{l(\lambda)}(m)} \prod_{i\geq 1} \frac{P_i(m)^{m_i}}{m_i!}. \tag{2}$$

Remark 1. *In particular, we recover the fact (noticed numerically in [1] and proved in greater generality in [3]) that $A_n(1)$ is a polynomial in ρ of degree $n-1$ and leading coefficient $n^{n-2}/n!$. (The appearance of the term n^{n-2}, the number of spanning trees on n labelled vertices, is not a coincidence.)*

We also note the following important property (here we write A_λ^ρ with $\rho = 2g-2$ for A_λ to indicate the dependence on g), which was inspired by the interpretation of $A_\lambda(1)$ in terms of the moduli space of Higgs bundles mentioned above.

Proposition 1. *Let n be a positive integer and $\lambda = (\lambda_1, \lambda_2, \ldots)$ a non-zero partition. Define $n\lambda := (n\lambda_1, n\lambda_2, \ldots)$. Then*

$$A_{n\lambda}^\rho(q) = A_\lambda^{n\rho}(q), \qquad \rho := 2g-2.$$

In particular,

$$A_{(n)}(q) = q^{n(g-1)+1}$$

The starting point for the proof of (1) is the formula of Hua for the A-polynomial. Truncating the sum for the S_g quiver to partitions of length at most N leads to a series of the following form

$$\sum_{m=(m_1,\dots,m_N)} \frac{q^{(g-1)^t m\mathcal{H}_N m}}{\prod_{i=1}^N (q^{-1})_{m_i}} T^{\sum_i i m_i} \tag{3}$$

where $m_i \in \mathbb{Z}_{\geq 0}$ and $\mathcal{H}_N := (\min(i,j)), i,j = 1,2,\dots,N$. The asymptotic as $q \to 1$ of this type of series has been studied extensively, starting with at least Ramanujan who used it to study the validity of one of his famous formulae (now known as the Rogers-Ramanujan formulae). There are several approaches which yield an expression for its leading term as a sum of values of the dilogarithm function (see for example [5], [6], and [10]). On the other hand, by Hua's formula the leading term can also be expressed in terms of $A_n^g(1)$. Combining these two expressions yields a proof of (1).

Series like (3) arise in conformal field theory in physics (see for example Nahm's paper [6]). From this point of view, (1) is a sort of fermionic-type formula. In fact, the kind of analysis we used appears prominently in the physics literature under the heading of Q-systems, originating from the work of Kirillov–Reshetikhin on representation theory and the combinatorics of the Bethe Ansatz. There is a substantial literature on the subject. The basic application of Lagrange's inversion can be found for example in [4]. We preferred to rederive the results we needed from scratch.

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Tilted algebras and short chains of modules

Andrzej Skowroński
(joint work with Alicja Jaworska and Piotr Malicki)

Let A be a basic connected artin algebra over a commutative artin ring K. We denote by $\operatorname{mod} A$ the category of finitely generated right A-modules, by $\operatorname{ind} A$ the full subcategory of $\operatorname{mod} A$ formed by the indecomposable modules, and by $K_0(A)$ the Grothendieck group of A. Further, we denote by Γ_A the Auslander-Reiten quiver of A and by τ_A the Auslander-Reiten translation $D\mathrm{Tr}$. A module M in $\operatorname{mod} A$ is called *sincere* if every simple right A-module occurs as a composition factor of M. Following [1], [5], a chain of nonzero homomorphisms $X \to M \to \tau_A X$ in $\operatorname{mod} A$ with X being indecomposable is called a *short chain*, and M is called the middle of this short chain. It is known that if a module M in $\operatorname{mod} A$ is not the middle of a short chain, then the number of pairwise nonisomorphic indecomposable direct summands of M is less or equal to the rank of $K_0(A)$ (by [8, Lemma 2]) and the indecomposable direct summands of M are uniquely determined by their images in $K_0(A)$ (by [5, Corollary 2.2]).

Let H be a hereditary algebra, T a tilting module in $\operatorname{mod} H$, and $B = \operatorname{End}_H(T)$ the associated tilted algebra. Then the images $\operatorname{Hom}_H(T, I)$ of indecomposable injective modules in $\operatorname{mod} H$ via the functor $\operatorname{Hom}_H(T, -) : \operatorname{mod} H \to \operatorname{mod} B$ form the canonical section Δ_T of a connected component $\mathscr{C}_T$ of Γ_B, called the connecting component of Γ_B. Moreover, the direct sum M_T of all modules lying on Δ_T is a sincere module in $\operatorname{mod} B$ which is not the middle of a short chain. In [5, Section 3] the authors asked whether the existence of a sincere module in $\operatorname{mod} A$ that is not the middle of a short chain implies that A is a tilted algebra.

During the talk we announced the following two theorems from [3].

Theorem 1. *Let A be an artin algebra. Then A is a tilted algebra if and only if* $\operatorname{mod} A$ *admits a sincere module M which is not the middle of a short chain.*

Theorem 2. *Let A be an artin algebra and M be a module in* $\operatorname{mod} A$ *which is not the middle of a short chain. Then there exists a hereditary algebra H, a tilting module T in* $\operatorname{mod} H$*, and an injective module I in* $\operatorname{mod} H$ *such that the following statements hold.*

- *(i) The tilted algebra $B = \operatorname{End}_H(T)$ is a quotient algebra of A.*
- *(ii) M is isomorphic to the right B-module* $\operatorname{Hom}_H(T, I)$.
- *(iii) The indecomposable direct summands of M lie on the section Δ_T of the connecting component $\mathscr{C}_T$ of Γ_B determined by T.*

We refer to [2], [4], [6] and [7] for more results on tilted algebras as well as their characterizations.

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Fractional Euler characteristics and 3*j*-symbols

Catharina Stroppel

(joint work with Igor Frenkel and Josh Sussan)

1. Representation theory of quantum $\mathfrak{sl}_2$

Let $\mathbb{C}(q)$ be the field of rational functions in q. Let $\mathcal{U}_q = \mathcal{U}_q(\mathfrak{sl}_2)$ be the associative algebra over $\mathbb{C}(q)$ generated by E, F, K, K^{-1} satisfying the relations:

$$KK^{-1} = K^{-1}K = 1,\ KE = q^2EK,\ KF = q^{-2}FK,\ EF - FE = \frac{K - K^{-1}}{q - q^{-1}}$$

Let $[k] = \sum_{j=0}^{k-1} q^{k-2j-1}$ and $\begin{bmatrix} n \\ k \end{bmatrix} = \frac{[n]!}{[k]![n-k]!}$. Let V_n be the (unique up to iso of type I) irreducible $\mathcal{U}_q(\mathfrak{sl}_2)$-module. It has basis $\{v_0, v_1, \dots, v_n\}$ such that

$$K^{\pm 1}v_i = q^{\pm(2i-n)}v_i \qquad Ev_i = [i+1]v_{i+1} \qquad Fv_i = [n-i+1]v_{i-1}. \tag{1}$$

Recall that $\mathcal{U}_q$ is a Hopf algebra, hence it makes sense to consider tensor products $V_{d_1} \otimes V_{d_2} \otimes \cdots \otimes V_{d_r}$ of finite dimensional modules.

Question: *Is it possible to categorify these representations?*

2. Categorification using quasi-hereditary algebras

Let first $d_1 = \cdots = d_n = 1$ and consider the n-fold tensor product $V_1^{\otimes n}$ of the vector representation. Let $V_1^{\otimes n} = \bigoplus_{j=0}^{n} V_1^{\otimes n}(j)$ be the decomposition into weight spaces.

Theorem 1. *Let $1 \leq j \leq n$. Then there is a finite dimensional quasi-hereditary graded $\mathbb{C}$-algebra $A_{j,n}$ of finite global dimension such that*

- $\mathcal{C}_{j,n} := A_{j,n}$-gmod *has up to isomorphism and grading shift precisely $\binom{n}{j}$ simple objects $L(\lambda)$ naturally indexed by $\{0,1\}$-sequences λ of length n with exactly j ones.*

- *There is a natural isomorphism of* $\mathbb{Z}[q,q^{-1}]$*-modules*

$$\bigoplus_{j=0}^{n} K_0(\mathcal{C}_{j,n}) \cong \bigoplus_{j=0}^{n} V_1^{\otimes n}(j) \tag{2}$$

which sends the isomorphism class of a standard module $\Delta(\lambda)$ *with head in degree zero to the standard basis vector* $v_{\lambda_1} \otimes \cdots \otimes v_{\lambda_n}$*. The action of* $q \in \mathbb{Z}[q,q^{-1}]$ *on* $K_0(\mathcal{C}_{j,n})$ *is given by shifting the grading up by* 1*.*
- *There are exact functors satisfying the relations of* $\mathcal{U}_q$ *inducing a* $\mathcal{U}_q$*-action on the Grothendieck group which turns* (2) *and* (3) *into* $\mathcal{U}_q$*-homomorphisms.*
- *The isomorphism classes of simple objects in degree zero correspond to dual canonical basis elements.*

Here $K_0(\mathcal{C}_{j,n})$ denotes the Grothendieck group of $\mathcal{C}_{j,n}$ defined as the free group of isomorphism classes of objects in $\mathcal{C}_{j,n}$ modulo short exact sequences. The grading defines a free $\mathbb{Z}[q,q^{-1}]$-module structure on $K_0(\mathcal{C}_{j,n})$ with basis given by the isomorphism classes of simple objects concentrated in degree zero. An alternative basis is given by the isomorphism classes of indecomposable projective objects with head concentrated in degree zero (which is however wrong if the algebra has infinite global dimension).

3. Categorification using properly stratified algebras

Let now $d_1, d_2, \ldots, d_r$ arbitrary. Then $M := V_{d_1} \otimes V_{d_2} \otimes \cdots \otimes V_{d_r}$ is a direct summand of $V_1^{\otimes n}$ with $n = \sum_{i=1}^{r} d_i$.

Theorem 2. *There exists a Serre subcategory* $\mathcal{S}_j$ *of* $\mathcal{C}_{j,n}$ *for* $1 \leq j \leq n$ *(depending on* $d_1, d_2, \ldots, d_r$ *and invariant under grading shifts) such that there is an isomorphism of* $\mathbb{Z}[q,q^{-1}]$*-modules respecting the weight space decomposition* $M = \oplus_{j=0}^{n} M(j)$

$$\bigoplus_{j=0}^{n} K_0(\mathcal{C}_{j,n}/\mathcal{S}_j) \quad \cong \quad \bigoplus_{j=0}^{n} M(j) \tag{3}$$

The quotient functor

$$\bigoplus_{j=0}^{n} \mathcal{C}_{j,n} \to \bigoplus_{j=0}^{n} \mathcal{C}_{j,n}\mathcal{S}_j$$

is exact, $\mathcal{U}_q$*-equivariant, and induces a morphism on the respective Grothendieck groups which corresponds via* (2) *and* (3) *to the Jones-Wenzl projector*

$$\pi : \; V_1^{\otimes n} \to V_{d_1} \otimes V_{d_2} \otimes \cdots \otimes V_{d_r}. \tag{4}$$

$\mathcal{C}_{j,n}/\mathcal{S}_j$ *is equivalent to* gmod-$\mathrm{End}_{\mathcal{C}_{j,n}}(P_j)$ *for some projective* $P_j \in \mathcal{C}_{j,n}$*. The algebra* $\mathrm{End}_{\mathcal{C}_{j,n}}(P_j)$ *is graded properly stratified. The isomorphism* (3) *maps isoclasses of*

- *standard modules* $\Delta(\lambda)$ *with head in degree zero to the standard basis,* $v_{\lambda_1} \otimes \cdots \otimes v_{\lambda_n}$*,*

- *proper standard modules* $\overline{\Delta}(\lambda)$ *in degree zero to the dual standard basis,* $v^{\lambda_1} \otimes \cdots \otimes v^{\lambda_n}$*,*
- *simple standard modules with head in degree zero to Lusztig's dual canonical basis elements.*

Note that the dual standard basis vector $v^i \in V_n$ is defined as $\begin{bmatrix} n \\ i \end{bmatrix}^{-1} v_i$. In particular, the transformation matrix from proper standard objects to standard objects is *not integral*, but involves *rational* (quantum) numbers. Categorically this corresponds to the fact that $\overline{\Delta}(\lambda)$ has an infinite projective and Δ-resolution.

4. The smallest non-trivial example and its categorification

Consider the Jones-Wenzl projector $V_2 \to V_1 \otimes V_1$ displayed in the following picture: The horizontal arrows indicate the action of E and F, whereas the loops show the action of K. The vertical arrows indicate the projection and inclusion.

$$\begin{tikzcd}
V_2 : & v_0 \arrow[loop above, "q^{-2}"] \arrow[r, bend left, "{[1]\cdot}"] \arrow[d, bend left, "\iota_3"] & v_1 \arrow[loop above, "q^{0}"] \arrow[l, bend left, "{[2]\cdot}"] \arrow[r, bend left, "{[2]\cdot}"] \arrow[d, bend left, "\iota_3"] & v_2 \arrow[loop above, "q2"] \arrow[l, bend left, "{[1]\cdot}"] \arrow[d, bend left, "\iota_3"] \\
V_1 \otimes V_1 : & v_0 \otimes v_0 \arrow[loop below, "q^{-2}"] \arrow[u, bend left, "\pi_3"] \arrow[r, bend left, "\binom{1}{q}"] & v_1 \otimes v_0, v_0 \otimes v_1 \arrow[loop below, "q^{0}"] \arrow[u, bend left, "\pi_3"] \arrow[l, bend left, "(q^{-1}\ 1)"] \arrow[r, bend left, "(q^{-1}\ 1)"] & v_1 \otimes v_1 \arrow[loop below, "q^{2}"] \arrow[u, bend left, "\pi_3"] \arrow[l, bend left, "\binom{1}{q}"]
\end{tikzcd}$$

Let $A = \mathrm{End}_{\mathbb{C}[x]/(x^2)}(\mathbb{C} \oplus \mathbb{C}[x]/(x^2))$, the path algebra of the quiver $\overset{1}{\bullet} \leftrightarrows \overset{2}{\bullet}$ subject to the relation $1 \to 2 \to 1$. It is graded by the path length and $R = \mathrm{Hom}_A(Ae_2, Ae_2) \cong \mathbb{C}[x]/(x^2)$, hence $R - \mathrm{gmod} \cong A - \mathrm{gmod}/\mathcal{S}$, where $\mathcal{S}$ is the graded Serre subcategory generated by $\mathbb{C}e_1$. Set $\mathcal{C}_{0,2} = \mathcal{C}_{2,2} = \mathbb{C} - \mathrm{gmod}$ and $\mathcal{C}_{1,2} = A - \mathrm{gmod}$:

$$\begin{tikzcd}
V_2 : & \mathbb{C} - \mathrm{gmod} \arrow[loop above, "\langle -2 \rangle"] \arrow[r, bend left, "\mathrm{ind}"] \arrow[d, bend left, "\mathrm{id}"] & R - \mathrm{gmod} \arrow[loop above, "\mathrm{id}"] \arrow[l, bend left, "\mathrm{res}"] \arrow[r, bend left, "\mathrm{res}"] \arrow[d, bend left, "A \otimes_R - \langle 1 \rangle"] & \mathbb{C} - \mathrm{gmod} \arrow[loop above, "\langle 2 \rangle"] \arrow[l, bend left, "\mathrm{ind}"] \arrow[d, bend left, "\mathrm{id}"] \\
V_1 \otimes V_1 : & \mathbb{C} - \mathrm{gmod} \arrow[loop below, "\langle -2 \rangle"] \arrow[u, bend left, "\mathrm{id}"] \arrow[r, bend left, "Ae_2 \otimes -"] & A - \mathrm{gmod} \arrow[loop below, "\mathrm{id}"] \arrow[u, bend left, "\mathrm{Hom}_A(Ae_2, -)\langle -1 \rangle"] \arrow[l, bend left, "\mathrm{Hom}_A(Ae_2, -)\langle -1 \rangle"] \arrow[r, bend left, "\mathrm{Hom}_A(Ae_2, -)\langle -1 \rangle"] & \mathbb{C} - \mathrm{gmod} \arrow[loop below, "\langle 2 \rangle"] \arrow[u, bend left, "\mathrm{id}"] \arrow[l, bend left, "Ae_2 \otimes -"]
\end{tikzcd}$$

The tilting module $T = q^{-1}R$ corresponds to v^1 and the equality $v^1 = [2]^{-1}v_1$ is categorified via $[\mathbb{C}] = (1 - q^2 + q^4 - q^6 + \cdots)[R]$ from the infinite projective resolution $\cdots q^4R \to q^2R \to R \to \mathbb{C}$.

Problems:

- Categorify the projections into the various summands of $V_i \otimes V_j$.
- Categorify the matrix entries so-called $3j$-symbols, in different bases.

We sketch answers to both problems. In particular, we show that $3j$-symbols can be viewed as generalized Kazhdan-Lusztig polynomials and explain how this could lead to categorified 3-manifold invariants of Turaev-Viro. The categorification of tensor product is based on [2]. Details can be found in [3] and [1].

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Derived autoequivalences of singular elliptic curves and mirror symmetry

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(joint work with So Okoda)

1. Notation and conventions

Throughout k is an algebraically closed field of characteristic zero. For an algebraic variety X over k we write $D^b(X)$ for $D^b(\operatorname{coh} X)$.

2. The smooth case

If E is a smooth elliptic curve over k then there is a short exact sequence

$$0 \to \operatorname{Aut}^{\text{triv}}(D^b(E)) \to \operatorname{Aut}(D^b(E)) \to \operatorname{Sl}_2(\mathbb{Z}) \to 0$$

with

$$\operatorname{Aut}^{\text{triv}}(D^b(E)) = \{\sigma_*(- \otimes \mathcal{L})[n] \mid \sigma \in \operatorname{Aut}(E), \mathcal{L} \in \operatorname{Pic}^0(E), n \in 2\mathbb{Z}\}$$

representing the "obvious" autoequivalences. These are precisely the derived autoequivalences which act trivially on $K_0(\operatorname{coh} E)$. This result is a special case of a general result by Orlov for abelian varieties [6].

The group $\operatorname{Sl}_2(\mathbb{Z})$ is generated by so-called *Seidel-Thomas twists* [7]. To be more precise: if $C \in D^b(E)$ then we say that C is (1-)spherical if

$$\operatorname{Hom}^i_E(C, C[i]) = \begin{cases} k & i = 0, 1 \\ 0 & \text{otherwise} \end{cases}$$

The following fact is well-known

Proposition 1. *The spherical objects in $D^b(E)$ form a single orbit under* $\mathrm{Aut}(D^b(E))$*. They are all shifts of sheaves (as are all indecomposable objects).*

The Seidel-Thomas twist associated to a spherical object C is defined by

$$T_C : D^b(E) \to D^b(E) : A \mapsto \mathrm{cone}(\mathrm{Hom}^\bullet(C, A) \otimes_k C \to A)$$

with the usual caveat that we need to use the standard enhancement on $D^b(E)$ to make T_C into a functor.

It is clear that $\mathcal{O}_E$ is spherical. If $y \in E$ then $\mathcal{O}_y$ is spherical as well. One has

$$\mathrm{Aut}(D^b(E))/\,\mathrm{Aut}^{\mathrm{triv}}(D^b(E)) = \langle T_{\mathcal{O}_E}, T_{\mathcal{O}_y} \rangle$$

with y an arbitrary point of E. A crude way of summarizing this is

$$\mathrm{Aut}(D^b(E)) = \langle \mathrm{Pic}\, E, \mathrm{Aut}\, E, 2\mathbb{Z}, T_{\mathcal{O}_E}, T_{\mathcal{O}_y} \rangle$$

3. The singular case

Now we assume E is a cycle of $n \geq 2$ projective lines $\bigcup_{i=1}^n E_i$, $E_i \cong \mathbb{P}^1$ where the point 0 in E_i is identified with the point ∞ in $E_{i+1 \bmod n}$. For $n = 1$ it is natural to let $E = E_1$ be a nodal elliptic curve. This is what we will do. The results stated in §2 generalize to the case $n = 1$ [2]. However it turns out that the case $n \geq 2$ is substantially harder.

Since E is now singular we have to make a distinction between $D^b(E)$ and its full subcategory $\mathrm{Perv}(E)$ consisting of perfect complexes. By definition spherical objects ly in $\mathrm{Perv}(E)$. They are in general no longer sheaves as the following example from [1] shows.

Example 1. *Consider the case $n = 2$. Let $\mathcal{L}$ be a line bundle on E whose restriction to E_1, E_2 is respectively $\mathcal{O}_{E_1}(2)$ and $\mathcal{O}_{E_2}(-1)$. It is easy to see that such an $\mathcal{L}$ is unique up to the choice of an element of k^*. Put $C = T_{\mathcal{O}_E}(\mathcal{L})$. As $\mathcal{L}$ is a line bundle it is spherical (this is easy to see). Since $T_{\mathcal{O}_E}$ is an auto-equivalence we obtain that C is spherical as well. A simple computation shows*

$$H^i(C) = \begin{cases} \mathcal{O}_{E_2}(-1) & \text{if } i = 0 \\ \mathcal{O}_{E_2}(-2) & \text{if } i = -1 \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

Thus C is not a shifted sheaf. With a little more effort one may construct spherical objects which are arbitrary long complexes.

The indecomposable objects in $D^b(E)$ are understood [2] but from the description in loc. cit. it seems non-trivial how to recognize the spherical objects among the indecomposable objects.

In the lecture we outlined proofs of the following results.

Proposition 2. *(1) The spherical objects in* $\mathrm{Perv}(E)$ *form a single orbit under* $\mathrm{Aut}\, D^b(E)$*.*

(2) If $n \leq 3$ then

$$\mathrm{Aut}(D^b(E)) = \langle \mathrm{Pic}\, E, \mathrm{Aut}\, E, \mathbb{Z}, T_{\mathcal{O}_E}, T_{\mathcal{O}_{y_1}}, \dots, T_{\mathcal{O}_{y_n}} \rangle$$

for a choice of smooth points $y_i \in E_i$.

Remark 1.
- *As said this result is known if $n = 1$ [2].*
- *(2) is not true for $n \geq 4$. One needs extra generators (which are known).*
- *There has been work on this problem in [4, 5]. In particular (1) is proved in the case $n = 2$.*

Our proof uses mirror symmetry and the explict description of the mirror dual to E given in [3] (see also [8]).

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Localization in quiver moduli spaces and tree modules

Thorsten Weist

Let $k = \mathbb{C}$ and $Q = (Q_0, Q_1)$ be a quiver without oriented cycles. For a fixed representation X of the quiver Q we choose a basis $\mathcal{B}$ of each vector space X_i.

Definition 1. *The coefficient quiver $\Gamma(X, \mathcal{B})$ of a representation X has vertex set $\mathcal{B}$ and arrows between vertices are defined by the condition: if $(X_{\alpha,\mathcal{B}})_{b,b'} \neq 0$, there exists an arrow $(\alpha, b, b') : b \mapsto b'$.*
A representation X is called a tree module if there exists a basis $\mathcal{B}$ for X such that the corresponding coefficient quiver is a tree.

This leads us to the following problem stated by Ringel, see [8]:
Does there exist an indecomposable tree module for every root $d \in \mathbb{N}Q_0$? In particular, Ringel conjectured that there should be more than one isomorphism class for imaginary roots.
Let $\langle Q_1 \rangle$ be the non-commutative group generated by the arrows $\alpha \in Q_1$ and its

formal inverse α^{-1}. The universal covering quiver $\tilde{Q}$ of Q is given by the vertex set

$$\tilde{Q}_0 = \{(i, w) \mid i \in Q_0, w \in \langle Q_1 \rangle\}$$

and the arrow set

$$\tilde{Q}_1 = \{\alpha_{(i,w)} : (i, w) \to (j, w\alpha) \mid \alpha : i \to j \in Q_1\}.$$

Let $R_d(Q)$ be the affine variety of k-representations. The push-down functor $\Pi : R_{\tilde{d}}(\tilde{Q}) \to R_d(Q)$ preserves indecomposability and stability, see [2] and [10].
For a fixed slope $\mu : \mathbb{N}Q_0 \to \mathbb{Q}$ denote by $M_d^s(Q)$ the moduli space of stable representations of Q, see [4] and [5] for a more detailed discussion.
Let $T := (\mathbb{C}^*)^{|Q_1|}$ be the $|Q_1|$-dimensional torus. It acts on $R_d(Q)$ via $(t_\alpha)_{\alpha\in Q_1} \cdot (X_\alpha)_{\alpha\in Q_1} = (t_\alpha X_\alpha)_{\alpha\in Q_1}$ inducing an action on $M_d^s(Q)$. Since torus fixed points correspond to stable representations of the universal abelian cover, we can iterate this procedure and define

$$M_d^s(Q)^{T,n} = (\ldots(M_d^s(Q)^{T_1})\ldots)^{T_n}$$

where the tori T_i are appropriately chosen. We call a dimension vector $\tilde{d} \in \mathbb{N}\tilde{Q}_0$ compatible with $d \in \mathbb{N}Q_0$ if

$$d_i = \sum_{w\in\langle Q_1\rangle} \tilde{d}_{i,w}$$

for all $i \in Q_0$. Moreover, we consider dimension vectors of $\tilde{Q}_0$ up to the equivalence induced by the action of $\langle Q_1 \rangle$ on $\tilde{Q}_0$ given by $p \cdot (i, w) = (i, wp)$. In summary, we obtain the following Theorem, see [10] for more details:

Theorem 1. *There exists an natural number n such that*

$$M_d(Q)^{T,n'} \simeq \bigcup_{\tilde{d}} M_{\tilde{d}}^s(\tilde{Q})$$

for all $n' \geq n$ where $\tilde{d}$ ranges over all equivalence classes being compatible with d.

Moreover we immediately get the following corollary:

Corollary 1. *For the topological Euler characteristic in singular cohomology we have:*

$$\chi(M_d^s(Q)) = \sum_{\tilde{d}} \chi(M_{\tilde{d}}^s(\tilde{Q}).$$

Example 1.

- Let $K(m)$ be the generalized Kronecker quiver. We consider the dimension vector $(d, e) = (2, 3)$ with $m \geq 3$. First we observe the quiver given by

$$\begin{tikzcd} & 1 \arrow[dl, "i_1"'] \arrow[dr, "i_2"] & & 1 \arrow[dl, "i_3"'] \arrow[dr, "i_4"] & \\ 1 & & 1 & & 1 \end{tikzcd}$$

By colouring the arrows in the colours $\{1, \ldots, m\}$ such that we get a subquiver of $\tilde{Q}$, every stable representation of this quiver gives rise to a torus

fixed point. Note that each colouring is unique up to the symmetry induced by S_2.
Further torus fixed points are given by stable representations of the following quiver:

$$\begin{tikzcd} & 1 \\ 2 \arrow[ur, "i_1"] \arrow[r, "i_2"] \arrow[dr, "i_3"] & 1 \\ & 1 \end{tikzcd}$$

Here we have to take into account the symmetries of S_3. In summary, we get

$$\chi(M^m_{2,3}) = \frac{m(m-1)^3}{2} + \frac{m(m-1)(m-2)}{6}.$$

Obviously, the unique indecomposable representation of the first quiver of the preceding example is a stable tree module. Moreover, we may easily construct a factor representation of dimension type $(2, 2)$ which is also an indecomposable tree module. Investigating such torus fixed points in general, we get the following result, where the map $r : \mathbb{N}^2 \to \mathbb{N}^2$ is defined by $r(d, e) := (e, me - d)$, see [11]:

Theorem 2. *(1) For every root (d, e) of the generalized Kronecker quiver there exists an indecomposable tree module.*
(2) Let $k, l, n \in \mathbb{N}_0$. For each root $(d, e) \neq r^l(n, kn)$ there exists a stable tree module.

Following the results of [12] we now construct indecomposable tree modules for every imaginary Schur root of a quiver Q.
Therefore, fixed a pair of representations X, Y we always choose a tree-shaped basis of $\mathrm{Ext}(X, Y)$, i.e. the corresponding matrices are of type $E(s, t)_{ij} = \delta_{si}\delta_{tj}$. Based on [9], the algorithm of [1] leads us to the following statement where we also use the notation of [9]:

Proposition 1. *Let α be an imaginary Schur root. Then at least one the following cases holds:*

(1) There exist a real Schur root β and $t \in \mathbb{N}_+$ such that $\gamma = \alpha - t\beta$ is an imaginary Schur root. Moreover, we have $\beta \in \gamma^\perp$ and $\mathrm{hom}(\beta, \gamma) = 0$ or $\beta \in^\perp \gamma$ and $\mathrm{hom}(\gamma, \beta) = 0$.
(2) There exist a real Schur root β and a real or isotropic root γ and $d, e \in \mathbb{N}_+$ such that $\alpha = \beta^d + \gamma^e$. Moreover, we have $\beta \in \gamma^\perp$ and $\mathrm{hom}(\beta, \gamma) = 0$ or $\beta \in^\perp \gamma$ and $\mathrm{hom}(\gamma, \beta) = 0$ and (d, e) is a root of $K(\mathrm{ext}(\beta, \gamma))$ or $K(\mathrm{ext}(\gamma, \beta))$.
(3) There exist two imaginary Schur roots γ and δ such that $\gamma + \delta = \alpha$. Moreover, we have $\delta \in \gamma^\perp$ and $\mathrm{hom}(\delta, \gamma) = 0$.

This Proposition gives us a recipe how to decompose Schur roots in order to construct an indecomposable tree module of such a root. In the first two cases we may restrict to one of the two possible cases.

In the first case let X_β and X_γ be the corresponding indecomposable representations. Since they are exceptional, by [7] it follows that X_β and X_γ are tree modules. Since we also have $\mathrm{Ext}(X_\beta, X_\gamma) = 0$, see [9], it follows that the subcategory consisting of middle terms of sequences of the form

$$0 \to X_\gamma^d \to X_\alpha \to X_\beta^e \to 0$$

is equivalent to the category $R_{e,d}(K(\mathrm{ext}(\beta,\gamma))$. Thus by applying Theorem 2 we get that there exists an indecomposable tree module of dimension α.
In the second case by applying Ringel's reflection functor, see [6], we obtain the following diagram

$$\begin{tikzcd}
 & & 0 & 0 & \\
 & 0 \arrow[r] & \bigoplus_{i=1}^{n-k} S \arrow[u] \arrow[r, equal] & \bigoplus_{i=1}^{n-k} S \arrow[u] \arrow[r] & 0 \\
0 \arrow[r] & X \arrow[u] \arrow[r] & X^S \arrow[u] \arrow[r] & \bigoplus_{i=1}^{n} S \arrow[u] \arrow[r] & 0 \\
0 \arrow[r] & X \arrow[u, equal] \arrow[r] & Y^S \arrow[u] \arrow[r] & \bigoplus_{i=1}^{k} S \arrow[u] \arrow[r] & 0 \\
 & 0 \arrow[u] & 0 \arrow[u] & 0 \arrow[u] &
\end{tikzcd}$$

Now, since X^S is indecomposable, one checks that Y^S is indecomposable as well. In the last case, we first construct indecomposable tree modules of dimension γ and δ. By [3] it follows that $\mathrm{Hom}(X_\gamma, X_\delta) = 0$. Thus the middle terms of non-splitting exact sequences of the form

$$0 \to X_\delta \to X_\alpha \to X_\gamma \to 0$$

are indecomposable. Thus in summary we get the following result, see [12]:

Theorem 3. *For every imaginary Schur root there exists an indecomposable tree module.*

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Generic free resolutions and root systems

Jerzy Weyman

Consider the complexes

$$\mathbb{F}_\bullet : 0 \to F_3 \stackrel{d_3}{\to} F_2 \stackrel{d_2}{\to} F_1 \stackrel{d_1}{\to} F_0$$

of the free modules over commutative rings R. We assume that $d_i : F_i \to F_{i-1}$ has rank r_i, and that

$$f_i := rank\ F_i = r_i + r_{i+1}$$

with the convention $r_4 = 0, r_0 \geq 0$. We fix the numbers f_i, r_i. We will say that a resolution with these ranks has *a format* $\underline{f} = (f_3, f_2, f_1, f_0)$.

Definition 1. *Let us fix the format* $\underline{f}$. *A pair* $(R_{gen}, \mathbb{G}_\bullet)$ *is a generic resolution of format* $\underline{f}$ *if*
a) $\mathbb{G}_\bullet$ *is acyclic,*
b) For every pair $(R, \mathbb{F}_\bullet)$ *with* $\mathbb{F}_\bullet$ *acyclic of type* $\underline{f}$, *there exists a homomorphism* $\phi : R_{gen} \to R$ *such that* $\mathbb{F}_\bullet = \mathbb{G}_\bullet \otimes_{R_{gen}} R$.

In [3] I constructed candidates for the generic rings of types (f_3, f_2, f_1, f_0). The construction of the ring R_{gen} depended on certain Lie algebra.

Definition 2. *The algebra* $\mathbb{L}(p, E, F)$ *is a graded Lie algebra, with*

$$\mathbb{L}(p, E, F) = \oplus_{i>0} \mathbb{L}_i,$$

where
a) $\mathbb{L}_1 = \mathbb{C}^{r-1} \otimes \bigwedge^p \mathbb{C}^{p+q}$,
b) $\mathbb{L}(p, E, F)$ *is a factor of a universal Lie algebra generated by* $\mathbb{L}_1$ *by the quadratic relations ensuring that*

$$\mathbb{L}_2 = \bigwedge^2 \mathbb{C}^{r-1} \otimes Ker(S_2(\bigwedge^p \mathbb{C}^{p+q}) \to S_{2^p}\mathbb{C}^{p+q}) \oplus$$

$$\oplus S_2\mathbb{C}^{r-1} \otimes Ker(\bigwedge^2(\bigwedge^p \mathbb{C}^{p+q}) \to S_{2^{p-1},1^2}\mathbb{C}^{p+q}).$$

This means $\mathbb{L}(p, E, F)$ is the universal Lie algebra generated by $\mathbb{L}_1$ with the quadratic relations exhibiting $\mathbb{L}_2$ as a factor of $\bigwedge^2(\mathbb{L}_1)$.

The higher components $\mathbb{L}_m$ can be defined as cokernels of the graded components of the Koszul complex

$$(\bigwedge^3 \mathbb{L})_m \to (\bigwedge^2 \mathbb{L})_m \to \mathbb{L}_m \to 0.$$

Let $\mathbb{F}_\bullet$ be the an acyclic complex of length three over a ring R. Let $\mathbb{L} := \mathbb{L}(r_1+1, F_3, F_1)$ be the corresponding defect algebra. Finally, let

$$0 \to \bigwedge^0 \mathcal{K} \to \bigwedge^1 \mathcal{K} \to \bigwedge^2 \mathcal{K} \to \bigwedge^3 \mathcal{K}$$

be the beginning of the Koszul complex on $I(d_3)$, the ideal of maximal minors of d_3. Thus $\mathcal{K} := \bigwedge^{r_3} F_3^* \otimes \bigwedge^{r_3} F_2$.

In [3] I proved that for the acyclic complex $\mathbb{F}_\bullet$ there exists a sequence of structure maps $p_i : \mathbb{L}_i^* \to \bigwedge^1 \mathcal{K}$ satisfying the following commutative diagram

$$\begin{array}{ccccccccc}
0 & \to & \bigwedge^0 \mathcal{K} & \to & \bigwedge^1 \mathcal{K} & \to & \bigwedge^2 \mathcal{K} & \to & \bigwedge^3 \mathcal{K} \\
 & & & & \uparrow p_{m+1} & & \uparrow q_{2,m+1} & & \uparrow q_{3,m+1} \\
 & & 0 & \to & \mathbb{L}_{m+1}^* & \to & (\bigwedge^2 \mathbb{L})_{m+1}^* & \to & (\bigwedge^3 \mathbb{L})_{m+1}^*
\end{array}$$

where $q_{2,m+1} = \sum(p_i \wedge p_j)$, $q_{3,m+1} = \sum(p_i \wedge p_j \wedge p_k)$. Here the map p_1 is related to the second structure theorem of Buchsbaum-Eisenbud [1], and p_2 was defined in [3] for the first time.

One constructs the ring R_n by adding (generically) the coefficients of the structure maps $p_1, \ldots, p_n$, dividing by the relations satisfied by their realizations for acyclic complexes. Finally we define $R_{gen} := lim_{n \mapsto \infty} R_n$. The Lie algebra $\mathbb{L}$ acts on R_{gen}.

In [3] I defined some complexes of free $U(\mathbb{L})$-modules

$$K_2' \oplus K_2'' \to K_1 \to K_0 \to 0$$

where each term cosists of a single irreducible representation of the group $\mathbb{GL}_{odd}$ tensored with $U(\mathbb{L})$. I showed that if these complexes are acyclic at the middle term, then R_{gen} is indeed a generic ring.

Associate to the triple (r_1, r_2, r_3) a triple $(p, q, r) = (r_1+1, r_2-1, r_3+1)$. We associate to (p, q, r) the graph $T_{p,q,r}$.

$$\begin{array}{ccccccccccccc}
x_{p-1} & - & x_{p-2} & \ldots & x_1 & - & u & - & y_1 & \ldots & y_{q-2} & - & y_{q-1} \\
 & & & & & & | & & & & & & \\
 & & & & & & z_1 & & & & & & \\
 & & & & & & | & & & & & & \\
 & & & & & & \ldots & & & & & & \\
 & & & & & & z_{r-2} & & & & & & \\
 & & & & & & | & & & & & & \\
 & & & & & & z_{r-1} & & & & & &
\end{array}$$

We denote $\mathfrak{g}(T_{p,q,r})$ the Kac-Moody Lie algebra associated to the graph $T_{p,q,r}$. By

$$\mathfrak{g}(T_{p,q,r}) = \oplus_{i\in\mathbb{Z}}\mathfrak{g}_i$$

we denote the grading on $\mathfrak{g}(T_{p,q,r})$ associated to the simple root α corresponding to the node z_1. More precisely, $\mathfrak{g}_i$ is span of weight spaces of roots β in which α appears with coefficient 1.

Proposition 1. *The defect Lie algebra $\mathbb{L}(r_1, F_3, F_1)$ is isomorphic to the positive part $\oplus_{i>0}\mathfrak{g}_i$ of $\mathfrak{g}(T_{p.q.r})$.*

This identification allows also to identify the complexes $\mathcal{K}^*(\alpha, \beta, s)$.

Proposition 2. *The complexes $\mathcal{K}^*(\alpha, \beta, s)$ are (the graded duals) of the parts of the parabolic BGG resolutions ([2]) associated with the parabolic subalgebra $\oplus_{i\geq 0}\mathfrak{g}_i$ of $\mathfrak{g}(T_{,q,r})$, and are therefore acyclic at the middle term.*

The preceeding discussion proves.

Theorem 1. *For every format $\underline{f}$ there exists a generic pair $(R_{gen}, \mathbb{F}_{gen})$. The generic ring R_{gen} carries a multiplicity free action of $\mathfrak{g}(T_{p,q,r}) \times SL(F_2) \times GL(F_0)$, where $f_3 = r - 1, f_2 = q + r, f_1 = p + q, r_1 = p - 1$. The generic ring R_{gen} is Noetherian if and only if $T_{p,q,r}$ is a Dynkin graph.*

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Stable categories of preprojective algebras and cluster categories

Osamu Iyama

Let K be an algebraically closed field. For an integer n, we say that a Hom-finite K-linear triangulated category $\mathcal{T}$ is *n-Calabi-Yau* (*n-CY*) if there exists a functorial isomorphism $\mathrm{Hom}_{\mathcal{T}}(X, Y) \simeq D\mathrm{Hom}_{\mathcal{T}}(Y, X[n])$ for any $X, Y \in \mathcal{T}$, where $D = \mathrm{Hom}_K(-, K)$ is the K-dual. There are many important triangulated categories in representation theory, in particular cluster categories played an important role in categorification of cluster algebras.

1. Background (I) For an acyclic quiver Q, we denote by KQ the path algebra and by $\Pi = \Pi(KQ)$ the preprojective algebra of Q. The following dichotomies of representation theory of KQ and structure theory of Π are well known.

Q	KQ	Π
Dynkin	representation finite	finite dimensional selfinjective
non-Dynkin	representation infinite	infinite dimensional

The stable category $\underline{\mathrm{mod}}\,\Pi$ is 2-CY for Dynkin case, and the bounded derived category $\mathcal{D}^{\mathrm{b}}(\Pi)$ of finite dimensional Π-modules is 2-CY for non-Dynkin case.

(II) Let Q be an extended Dynkin quiver with the extending vertex e. Then $R := e\Pi e$ is a Kleinian singularity, and in particular the stable category $\underline{\mathrm{CM}}(R)$ of maximal Cohen-Macaulay R-modules is 1-CY [19].

Recently higher analogue of preprojective algebras are introduced in representation theory [10, 11, 13] and non-commutative algebraic geometry [15, 16]:

Definition 1 Let n be a positive integer and Λ be a finite dimensional K-algebra with $\mathrm{gl.dim}\,\Lambda \le n$. The $(n+1)$-*preprojective algebra* of Λ is defined as the tensor algebra of the Λ-bimodule $\mathrm{Ext}^n_\Lambda(D\Lambda, \Lambda)$:

$$\Pi = \Pi_{n+1}(\Lambda) := \mathrm{T}_\Lambda \mathrm{Ext}^n_\Lambda(D\Lambda, \Lambda).$$

For the case $n = 1$, this is a well-known description of preprojective algebras. For the case $n = 2$, this gives a description of cluster tilted algebras [3].

We will generalize CY properties in (I) and (II) above to higher cases.

The above stable categories have realizations as cluster categories defined as follows: Let n be a positive integer and Λ be a finite dimensional K-algebra with $\mathrm{gl.dim}\,\Lambda \le n$. Let $\mathcal{D}^{\mathrm{b}}(\Lambda)$ be the bounded derived category of finite dimensional Λ-modules, ν be the Nakayama functor of $\mathcal{D}^{\mathrm{b}}(\Lambda)$ and $\nu_n := \nu\circ[-n]$. The triangulated hull $\mathcal{C}_n(\Lambda)$ of the orbit category $\mathcal{D}^{\mathrm{b}}(\Lambda)/\nu_n$ is called the *n-cluster category* [1, 4, 5, 12, 18]. If $\mathcal{C}_n(\Lambda)$ is Hom-finite, then it is n-CY. Notice that $\Pi_{n+1}(\Lambda)$ is the endomorphism algebra $\mathrm{End}_{\mathcal{C}_n(\Lambda)}(\Lambda)$ of Λ in $\mathcal{C}_n(\Lambda)$.

We have the equivalences between stable categories and cluster categories:

Theorem 2 (a) [1] For a Dynkin quiver Q, we have a triangle equivalence $\underline{\mathrm{mod}}\,\Pi(KQ) \simeq \mathcal{C}_2(\underline{\Gamma})$ for the stable Auslander algebra $\underline{\Gamma}$ of KQ.

(b) [17] In (II) above, we have an equivalence $\underline{\mathrm{CM}}(R) \simeq \mathcal{C}_1(KQ')$, where Q' is the Dynkin quiver obtained by removing e from Q.

We will generalize these equivalences to higher cases.

2. Our results Throughout let n be a positive integer and Λ be a finite dimensional K-algebra with $\mathrm{gl.dim}\,\Lambda \le n$. In general, the homological behaviour of $\Pi_{n+1}(\Lambda)$ is not as nice as the case $n = 1$. So we have to restrict to the following.

Definition 3 [8] We say that Λ is *n-representation controlled* if $H^\ell(\nu_n^i(\Lambda)) = 0$ for any $i \in \mathbb{Z}$ and $\ell \in \mathbb{Z} - n\mathbb{Z}$.

We have the following dichotomy of n-representation controlled algebras, where $M \in \mathrm{mod}\,\Lambda$ is *n-cluster tilting* if $\mathrm{add}\,M$ coincides with the following subcategories:

- $\{X \in \mathrm{mod}\,\Lambda \mid \mathrm{Ext}^i_\Lambda(M, X) = 0 \text{ for any } 0 < i < n\}$.
- $\{X \in \mathrm{mod}\,\Lambda \mid \mathrm{Ext}^i_\Lambda(X, M) = 0 \text{ for any } 0 < i < n\}$.

Proposition 4 (Dichotomy) Λ is n-representation controlled if and only if precisely one of the following conditions holds.

(a) Λ has an n-cluster tilting module M. (*n-representation finite* [6, 9, 10])

(b) $\nu_n^{-i}(\Lambda) \in \mathrm{mod}\,\Lambda$ for any $i \ge 0$. (*n-representation infinite* [8])

For the case (a), the basic part of M is unique. We call $\mathrm{End}_\Lambda(M)$ and $\underline{\mathrm{End}}_\Lambda(M)$ the *n-Auslander algebra* and the *stable n-Auslander algebra* of Λ respectively.

Example 5 (a) It is clear from definition that the path algebra of an acyclic quiver is always 1-representation controlled. Moreover it is easy to check that 1-representation (in)finiteness coincides with representation (in)finiteness.

(b) [6] The tensor product $KQ_1 \otimes_K \cdots \otimes_K KQ_n$ for non-Dynkin quivers Q_i is n-representation infinite. The tensor product $KQ_1 \otimes_K \cdots \otimes_K KQ_n$ for Dynkin quivers Q_i is n-representation finite if each Q_i is stable under the canonical involution of the underlying graph and the Coxeter numbers of all Q_i's are equal.

Notice that n-representation infinite algebras are studied in non-commutative algebraic geometry [15, 16] under the name 'n-Fano algebra'.

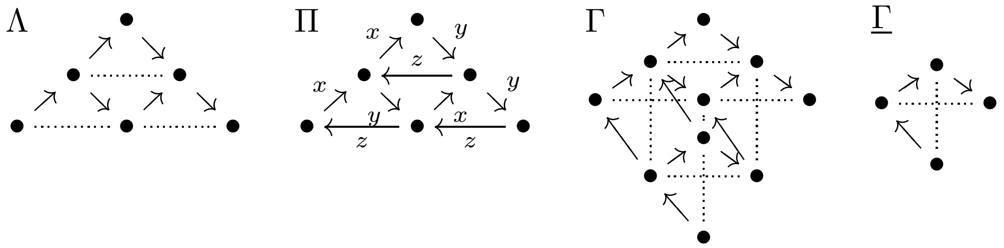
2.1. Finite case We have the results for n-representation finite algebras:

Theorem 6 [11] Let Λ be an n-representation finite algebra and $\Pi = \Pi_{n+1}(\Lambda)$.

(a) Π is a finite dimensional selfinjective algebra and $\underline{\mathrm{mod}}\,\Pi$ is $(n+1)$-CY.

(b) We have a triangle equivalence $\underline{\mathrm{mod}}\,\Pi \simeq \mathcal{C}_{n+1}(\underline{\Gamma})$ for the stable n-Auslander algebra $\underline{\Gamma}$ of Λ (e.g. Theorem 2 (a)).

Example 7 [9–11] Let $n = 2$ and Λ be an Auslander algebra of the path algebra of type A_3. Then Λ is 2-representation finite and $\Pi = \Pi_3(\Lambda)$ is the Jacobian algebra of the quiver below with potential $\sum xyz - zyx$. The 2-Auslander algebra Γ and the stable 2-Auslander algebra $\underline{\Gamma}$ are the following:



There is a general structure theorem of 2-representation finite algebras in terms of 'selfinjective quivers with potential' and their 'cuts' [7].

2.2. Infinite case We have the results for n-representation infinite algebras:

Theorem 8 Let Λ be an n-representation infinite algebra and $\Pi = \Pi_{n+1}(\Lambda)$.

(a) [13] $\mathcal{D}^{\mathrm{b}}(\Pi)$ is $(n+1)$-CY.

(b) [2] Let $e \in \Lambda$ be an idempotent. Assume $\dim_K(\Pi/(e)) < \infty$, $e\Lambda(1-e) = 0$ and that Π is noetherian. Then $\underline{\mathrm{CM}}(\Pi)$ is n-CY and we have a triangle equivalence $\underline{\mathrm{CM}}(e\Pi e) \simeq \mathcal{C}_n(\Lambda/(e))$ (e.g. Theorem 2 (b)).

Example 9 [2, 8] Let $n = 2$ and Λ be a Beilinson algebra of dimension 2. Then Λ is 2-representation infinite and $\Pi = \Pi_3(\Lambda)$ is the Jacobian algebra of the quiver below with potential $\sum xyz - zyx$.

Moreover $R = e\Pi e$ is the subring of $K[x, y, z]$ generated by all monomials whose degrees are multiples of 3. In particular we recover the equivalence $\underline{\mathrm{CM}}(R) \simeq \mathcal{C}_2(KQ)$ for $Q\ \bullet \substack{\longrightarrow \\ \longrightarrow \\ \longrightarrow} \bullet$ given in [14]. See [2] for more examples.

There is a general structure theorem of 2-representation infinite algebras in terms of 'good quivers with potential' and their 'cuts' [8].

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