GEOMETRIC APPLICATIONS OF THE GENERALIZED OMORI-YAU MAXIMUM PRINCIPLE

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Abstract

Following the terminology introduced by Pigola, Rigoli and Setti in [5], the Omori-Yau maximum principle is said to hold on an n-dimensional Riemannian manifold Σ if, for any smooth function $u \in \mathcal{C}^\infty(\Sigma)$ with $u^* = \sup_\Sigma u < +\infty$ there exists a sequence of points $\{p_k\}_{k\in\mathbb{N}}$ in Σ with the properties: (i) $u(p_k) > u^* - \frac{1}{k}$, (ii) $|\nabla u(p_k)| < \frac{1}{k}$, and (iii) $\Delta u(p_k) < \frac{1}{k}$. In this sense, the classical result given by Omori [4] and Yau [6] states that the Omori-Yau maximum principle holds on every complete Riemannian manifold with Ricci curvature bounded from below. More generally, as shown by Pigola, Rigoli and Setti [5], a sufficiently controlled decay of the radial Ricci curvature of the form $\mathrm{Ric}_\Sigma(\nabla\varrho, \nabla\varrho) \geq -C^2 G(\varrho)$, where ϱ is the distance function on Σ to a fixed point, C is a positive constant, and $G : [0, +\infty) \to \mathbb{R}$ is a smooth function satisfying (i) $G(0) > 0$, (ii) $G'(t) \geq 0$, (iii) $\int_0^{+\infty} 1/\sqrt{G(t)} = +\infty$, and (iv) $\limsup_{t\to+\infty} tG(\sqrt{t})/G(t) < +\infty$, suffices to imply the validity of the Omori-Yau maximum principle. On the other hand, as observed also in [5], the validity of the Omori-Yau maximum principle on Σ does not depend on curvature bounds as much as one would expect. For instance, the Omori-Yau maximum principle holds on every Riemannian manifold admitting a non-negative C^2 function φ satisfying the following requirements: (i) $\varphi(p) \to +\infty$ as $p \to \infty$; (ii) there exists $A > 0$ such that $|\nabla\varphi| \leq A\sqrt{\varphi}$ off a compact set; and (iii) there exists $B > 0$ such that $\Delta\varphi \leq B\sqrt{\varphi}\sqrt{G(\sqrt{\varphi})}$ off a compact set, where G is as above.

In this lecture we will introduce some geometric applications of the generalized Omori-Yau maximum principle to the study of hypersurfaces with constant mean curvature both in Riemannian and Lorentzian ambient spaces. The results in this lecture are part of our recent research work developed jointly with Bessa and Dajczer [1], Bessa and Montenegro [2], and Hurtado and Palmer [3].

References

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