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On the relative homology of cleft extensions of rings and abelian categories

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Abstract

We study the relative homological behaviour of the omnipresent class of *cleft extensions* of abelian categories. This class of extensions is a natural generalization of the trivial extensions studied in detail by Fossum, Griffith and Reiten and by Palmer and Roos. We apply our results to the relative homology of cleft extensions of rings. © 2000 Elsevier Science B.V. All rights reserved.

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1. Introduction

Cleft extensions of abelian categories were introduced by the author in an earlier paper [6] as a generalization of the same concept of rings (definitions and examples are recalled in Section 2). They provide a natural setting in which to study relative homology which is the topic of the present paper. Indeed, for simplicity and generality reasons, for the homological study of cleft ring extensions, it is natural to work instead in the context of cleft extensions of abelian categories. This leads to general theorems which can be applied back to the germinating case of rings, where they provide new information on global dimension, answer a question of Auslander–Reiten, give generalizations of some familiar results of Reiten and others, and yield new proofs of some well-known theorems of Mitchell.

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The (homological) study of cleft extensions of rings is quite essential and has interesting applications to classical ring theory, since this class of ring extensions is very large: it includes free rings, polynomial rings, exterior rings, positively graded rings, basic semiperfect rings, group rings, supplemented rings, universal enveloping algebras of Lie algebras, trivial extension rings and in particular triangular matrix rings, truncated extension rings, quotients of quiver algebras, and all rings satisfying some version of the Principal Theorem of Wedderburn. From the point of view of representation theory, a cleft extension of an abelian category is a natural generalization of the module category of the ring $k\mathcal{Q}/\langle\rho\rangle$, where $\mathcal{Q}$ is a quiver, k is a (semisimple) ring and ρ is a set of relations.

The paper is organized as follows. Each time we prove a result about general cleft extensions of abelian categories, e.g. "*Theorem n.m*", we usually include its ring-theoretic interpretation. The ring-theoretic results are labelled as "*Corollary For Rings n.m* + 1". Now we describe the contents of the paper section by section.

In Section 2, we review the basic facts from [6] about cleft extensions of abelian categories and cleft extensions of rings, we present a large list of constructions and examples of cleft extensions and we fix some notation.

The natural setting in which we study the homological behaviour of a cleft extension is that of relative homological algebra as covered in Mac Lane's book [24]. In Section 3 we lift proper classes of short exact sequences $\mathfrak{R}$ in an abelian category $\mathscr{D}$, to proper classes of short exact sequences $\mathfrak{F}$ in a cleft extension $\mathscr{C}$ of $\mathscr{D}$; we lift always the minimal proper class of split short exact sequences in $\mathscr{D}$ to obtain the proper class Σ in $\mathscr{C}$. We construct relative projective resolutions in $\mathscr{C}$ with respect to the proper classes $\mathfrak{F}, \Sigma$ and we analyse the $\mathfrak{F}$-projective resolutions of the important class of primitive objects of $\mathscr{C}$, which includes the "induced" from $\mathscr{D}$ objects of $\mathscr{C}$. These objects are the Σ-proper extensions of nonzero objects of $\mathscr{D}$. Our main aim in the remaining sections is to study the $\mathfrak{F}$-, and Σ-relative homological structure of $\mathscr{C}$, using information directly available from $\mathscr{D}$.

In Section 4 we study relative derived functors with respect to $\mathfrak{F}, \Sigma$ and we obtain preliminary formulas for the relative extension functors, and for the relative derived functors of a particular functor $\mathbf{C}: \mathscr{C} \to \mathscr{D}$, which is part of the structure of a cleft extension. This functor plays an important role in the homological behaviour of $\mathscr{C}$. We construct a natural morphism between the $\mathfrak{F}$-, Σ-relative derived functors of $\mathbf{C}$ and we study its properties. As an application we study when a relative (co-)tilting object in $\mathscr{D}$ can be lifted to a relative (co-)tilting object in $\mathscr{C}$. As a consequence of the lifting of (co-)tilting objects, we prove that the trivial extension of a Cohen–Macaulay Artin algebra by a dualizing bimodule is Gorenstein, giving a positive answer to a question of Auslander and Reiten [2].

In Section 5, we characterize under various conditions, the finiteness of the relative (finitistic) global dimension of a cleft extension, giving a variety of bounds, and we prove generalized versions of the Hilbert Syzygy Theorem and the Hilbert Basis Theorem. Then we apply these results to some specific examples from Section 2. In this respect we compute the relative global dimension and the relative extension functors in free, polynomial, symmetric and exterior categories over an abelian category, equipped with a family of right exact endofunctors, generalizing most of the results of Mitchell [28, Chapter IX].

In Section 6 we study cleft extensions of small relative global dimension and relative Frobenius cleft extensions. More precisely, we give necessary and sufficient conditions for the relative global dimension of $\mathscr{C}$ to be (less than or equal to) 1 or 2 and for $\mathscr{C}$ to be a (relative) Frobenius category.

In Section 7, we introduce some vanishing conditions which permit more precise formulas for the relative extension functors of $\mathscr{C}$ and the relative derived functors of the functor $\mathbf{C}$ mentioned above. These formulas become exact in the case of trivial extensions, comma-categories, categories of morphisms and in particular in the case of generalized triangular matrix rings. Finally, we analyse the relations between the relative derived functors of $\mathbf{C}$ with respect to $\mathfrak{F}, \Sigma$, by studying the Butler–Horrocks spectral sequence for $\mathbf{C}$, induced by the inclusion of proper classes $\Sigma \subseteq \mathfrak{F}$ in $\mathscr{C}$. The results of this and the previous sections generalize to the nontrivial case all the analogous results of Palmer and Roos [31] and Fossum et al. [13].

Section 8 is devoted to the homological study of truncated extensions. We apply our previous results to a truncated extension $\mathscr{C}$ of an abelian category $\mathscr{D}$, and under some mild vanishing condition we obtain exact formulas for the relative global dimension, the relative derived functors of $\mathbf{C}$, and the relative extension functors of $\mathscr{C}$, generalizing all the results of Marmaridis and Papistas [27]. In particular, we prove the generalized form of the Strong No Loops Conjecture for truncated extensions [27], under much more general conditions and using different and more simple methods. As a final application we prove the Cartan Determinant Conjecture for finite-dimensional truncated algebras over a field.

A general convention used in this paper is that the composition of morphisms in a given category is meant in the diagrammatic order: if f, g are composable, then $f \circ g$, means first f then g. There are two exceptions: we use the usual anti-diagrammatic order when we compose functors and when we apply elements to (compositions of) morphisms in concrete categories. If $f_1 : X \to A$, $f_2 : Y \to A$ and $g_1 : A \to X$, $g_2 : A \to Y$ are morphisms in an additive category, then we denote by ${}^t(f_1, f_2) : X \oplus Y \to A$ and $(g_1, g_2) : A \to X \oplus Y$, the uniquely induced morphisms.

2. Cleft extensions of rings and of abelian categories

In this section we review from [6], some basic facts concerning cleft extensions of rings and abelian categories, we present constructions and examples of such extensions (mainly from ring theory), and we fix the notation.

2.1. Cleft and ϑ-extensions of rings

Let Γ be an associative ring. A *cleft extension* of Γ is a triple $(\Lambda, \varepsilon, \mu)$ consisting of a ring Λ and ring morphisms $\varepsilon : \Lambda \to \Gamma$, $\mu : \Gamma \to \Lambda$, such that $\mu \circ \varepsilon = \mathrm{Id}_\Gamma$. We call

ε the *projection* and μ the *cleaving* of $(\Lambda,\varepsilon,\mu)$. Cleft extensions of Γ are the objects of a category $\mathscr{CE}(\Gamma)$. A morphism in $\mathscr{CE}(\Gamma)$ between $(\Lambda,\varepsilon_1,\mu_1)$ and $(\Xi,\varepsilon_2,\mu_2)$, is a ring morphism $\rho:\Lambda\to\Xi$ such that $\rho\circ\varepsilon_2=\varepsilon_1$ and $\mu_1\circ\rho=\mu_2$. For a better description of a cleft extension Λ of a ring Γ we need the notion of a ϑ-extension of Γ by a Γ–Γ-bimodule M, [26,6]. Let ${}_\Gamma M_\Gamma$ be a Γ–Γ-bimodule and let $\vartheta:M\otimes_\Gamma M\to M$ be an *associative* Γ–Γ-bimodule morphism, i.e. the diagram below commutes:

$$\begin{tikzcd}
M\otimes_\Gamma M\otimes_\Gamma M \arrow[r, "1_M\otimes\vartheta"] \arrow[d, "\vartheta\otimes 1_M"'] & M\otimes_\Gamma M \arrow[d, "\vartheta"] \\
M\otimes_\Gamma M \arrow[r, "\vartheta"] & M.
\end{tikzcd}$$

We call the pair (M,ϑ) a *multiplicative* Γ–Γ-bimodule. The ϑ-*extension ring* $\Lambda=\Gamma\ltimes_\vartheta M$ of Γ by M, is defined as follows: $\Lambda=\Gamma\oplus M$ in the category of abelian groups $\mathscr{A}b$. The multiplication of Λ is defined by the formula:

$$(r_1,m_1)(r_2,m_2)=(r_1r_2,r_1m_2+m_1r_2+\vartheta(m_1\otimes m_2)).$$

If $\vartheta=0$, we obtain the familiar *trivial extension* of Γ by M. It is clear that $\Gamma\ltimes_\vartheta M$ is a cleft extension of Γ with projection defined by $\pi(\gamma,m)=\gamma$ and cleaving defined by $\iota(\gamma)=(\gamma,0)$. We define the category $\mathscr{E}_\vartheta(\Gamma)$ of ϑ-extensions of Γ as follows. The objects are triples $(\Gamma\ltimes_\vartheta M,\pi,\iota)$ as above. A morphism $\alpha:\Gamma\ltimes_\vartheta M\to\Gamma\ltimes_\varphi N$ is a ring morphism α which commutes with π and ι. Let $(\Lambda,\varepsilon,\mu)$ be a cleft extension of Γ and set $M:=\mathrm{Ker}(\varepsilon)$. The ideal M is a multiplicative Γ–Γ-bimodule with actions $\Gamma\times M\to M:(\gamma,m)\mapsto\mu(\gamma)\cdot m$ and $M\times\Gamma\to M:(m,\gamma)\mapsto m\cdot\mu(\gamma)$ and multiplication $\vartheta(m_1\otimes m_2)=m_1m_2$ (multiplication inside Λ). Hence, we can form the ϑ-extension $\Gamma\ltimes_\vartheta M$ and in this way we obtain a functor $\mathbb{F}:\mathscr{CE}(\Gamma)\to\mathscr{E}_\vartheta(\Gamma)$. Now consider the category $\mathscr{MB}(\Gamma)$ with objects multiplicative Γ–Γ-bimodules (M,ϑ). A morphism $\alpha:(M,\vartheta)\to(N,\varphi)$ in $\mathscr{MB}(\Gamma)$ is a Γ–Γ-bimodule morphism $\alpha:M\to N$ such that $\vartheta\circ\alpha=(\alpha\otimes\alpha)\circ\varphi$. Then we define a functor $\mathbb{G}:\mathscr{MB}(\Gamma)\to\mathscr{E}_\vartheta(\Gamma)$ by setting $\mathbb{G}(M,\vartheta)=\Gamma\ltimes_\vartheta M$ and $\mathbb{G}(\alpha)(\gamma,m)=(\gamma,\alpha(m))$.

The following result [6], which is a direct consequence of the definitions, allows us to reduce the study of cleft extensions to the more concrete class of ϑ-extensions.

Proposition 2.1. *The functors $\mathbb{F}:\mathscr{CE}(\Gamma)\to\mathscr{E}_\vartheta(\Gamma)$ and $\mathbb{G}:\mathscr{MB}(\Gamma)\to\mathscr{E}_\vartheta(\Gamma)$ defined above are equivalences of categories. The equivalence $\mathbb{G}$ restricts to an equivalence $\mathbb{G}:\mathrm{Mod}(\Gamma^{\mathrm{e}})\xrightarrow{\approx}\mathscr{E}_{\vartheta=0}(\Gamma)$, where Γ^{e} is the enveloping ring of Γ.*

If $\Lambda=\Gamma\ltimes_\vartheta M$ is a cleft ϑ-extension of Γ, then the Γ–Γ-bimodule M is an ideal of Λ. The morphism ϑ is called *nilpotent* if the ideal M of Λ is nilpotent. This happens iff $\exists i\in\mathbb{N}:(1_M\otimes 1_M\otimes\cdots\otimes 1_M\otimes\vartheta)\circ\cdots\circ(1_M\otimes\vartheta)\circ\vartheta=0:M^{\otimes_{i+1}}\to M$. Also ϑ is called *right* T-*nilpotent* if the ideal M is right T-nilpotent [5] and this happens iff $\forall(m_i)_{i\in\mathbb{N}}\subseteq M$, $\exists j\in\mathbb{N}$: $\vartheta(m_j\otimes\vartheta(m_{j-1}\otimes\vartheta(m_{j-2}\otimes\cdots\otimes\vartheta(m_3\otimes\vartheta(m_2\otimes m_1))\cdots)))=0$. Left T-nilpotency of ϑ is defined similarly.

2.2. Cleft and η-extensions of abelian categories

Throughout this section we fix an abelian category $\mathscr{D}$.

Definition 2.2 (Beligiannis [6]). A *cleft extension* of $\mathscr{D}$ is an abelian category $\mathscr{C}$ together with additive functors $\mathbf{U}:\mathscr{C}\to\mathscr{D}$ (*projection*) and $\mathbf{Z}:\mathscr{D}\to\mathscr{C}$ (*cleaving*) such that:

(i) $\mathbf{U}$ is faithful exact and admits a left adjoint $\mathbf{T}:\mathscr{D}\to\mathscr{C}$.

(ii) $\mathbf{UZ}=\mathrm{Id}_{\mathscr{D}}$.

Example 2.3. The motivating example of a cleft extension is the following. Let $(\Lambda,\varepsilon,\mu)$ be a cleft extension of the ring Γ and let $\mathrm{Mod}(\Lambda)$, resp. $\mathrm{Mod}(\Gamma)$, be the category of right Λ-, resp. Γ-, modules. Set $\mathbf{U}=\mathrm{Hom}_\Lambda[{}_\Gamma\Lambda_\Lambda,-]=-\otimes_\Lambda\Lambda_\Gamma:\mathrm{Mod}(\Lambda)\to\mathrm{Mod}(\Gamma)$, $\mathbf{T}=-\otimes_\Gamma\Lambda_\Lambda:\mathrm{Mod}(\Gamma)\to\mathrm{Mod}(\Lambda)$ and $\mathbf{Z}=\mathrm{Hom}_\Gamma[{}_\Lambda\Gamma_\Gamma,-]:\mathrm{Mod}(\Gamma)\to\mathrm{Mod}(\Lambda)$. It is easy to see that $\mathrm{Mod}(\Lambda)$ is a cleft extension of $\mathrm{Mod}(\Gamma)$.

For an internal description of cleft extensions of $\mathscr{D}$, we need the notion of an η-*extension* $\mathscr{D}_F(\eta)$ of $\mathscr{D}$ by a right exact endofunctor $F:\mathscr{D}\to\mathscr{D}$, as introduced in [26]. Here $\eta:F^2\to F$ is an *associative* natural morphism, i.e. $\eta F\circ\eta=F\eta\circ\eta$, and $\mathscr{D}_F(\eta)$ has as objects pairs (X,f) where $f:FX\to X$ is a morphism of $\mathscr{D}$ such that: $Ff\circ f=\eta_X\circ f$. A morphism $a:(X,f)\to(Y,g)$ in $\mathscr{D}_F(\eta)$ is a morphism $a:X\to Y$ in $\mathscr{D}$ such that $f\circ a=Fa\circ g$. The above commutativities are shown in the following diagrams:

$$\begin{tikzcd} F^3 \arrow[r, "F\eta"] \arrow[d, "\eta F"'] & F^2 \arrow[d, "\eta"] \\ F^2 \arrow[r, "\eta"] & F \end{tikzcd}\qquad \begin{tikzcd} F^2(X) \arrow[r, "Ff"] \arrow[d, "\eta_X"'] & FX \arrow[d, "f"] \\ FX \arrow[r, "f"] & X \end{tikzcd}\qquad \begin{tikzcd} FX \arrow[r, "Fa"] \arrow[d, "f"'] & FY \arrow[d, "g"] \\ X \arrow[r, "a"] & Y. \end{tikzcd}$$

Then $\mathscr{D}_F(\eta)$ is abelian and there are adjoint pairs of functors $(\mathbf{T},\mathbf{U})$, $(\mathbf{C},\mathbf{Z})$:

$$\mathbf{T}:\mathscr{D}\to\mathscr{D}_F(\eta),\quad \mathbf{U}:\mathscr{D}_F(\eta)\to\mathscr{D},\quad \mathbf{C}:\mathscr{D}_F(\eta)\to\mathscr{D},\quad \mathbf{Z}:\mathscr{D}\to\mathscr{D}_F(\eta)$$

defined as follows. If $a:X\to Y$ is a morphism in $\mathscr{D}$, and $\alpha:(X,f)\to(Y,g)$ is a morphism in $\mathscr{D}_F(\eta)$, then:

$$\mathbf{U}(X,f)=X,\quad \mathbf{U}(\alpha)=\alpha\quad\text{and}\quad \mathbf{Z}(X)=(X,0),\quad \mathbf{Z}(a)=a,$$

$\mathbf{T}(X)=(X\oplus FX,t_X)$, where

$$t_X=\begin{pmatrix}0 & 1_{FX}\\ 0 & \eta_X\end{pmatrix},\quad\text{and}\quad \mathbf{T}(a)=\begin{pmatrix}a & 0\\ 0 & Fa\end{pmatrix},$$

$\mathbf{C}(X,f)=\mathit{Coker}(f)$, $\mathbf{C}(\alpha)=$ the unique morphism : $\mathbf{C}(X,f)\to\mathbf{C}(Y,g)$ in $\mathscr{D}$, such that $\alpha\circ\mathit{coker}(g)=\mathit{coker}(f)\circ\mathbf{C}(\alpha)$.

Obviously, $\mathbf{U}$ is faithful exact, $\mathbf{Z}$ is fully faithful exact, the following relations hold: $\mathbf{CZ}=\mathrm{Id}_{\mathscr{D}}$, $\mathbf{CT}=\mathrm{Id}_{\mathscr{D}}$, $\mathbf{UZ}=\mathrm{Id}_{\mathscr{D}}$ and $\mathscr{D}_F(\eta)$ is a cleft extension of $\mathscr{D}$. We denote by

$\varepsilon : \mathbf{TU} \to \mathrm{Id}_{\mathscr{D}_F(\eta)}$ the counit and by $\delta : \mathrm{Id}_{\mathscr{D}} \to \mathbf{UT} = \mathrm{Id}_{\mathscr{D}} \oplus F$ the unit of the adjoint pair $(\mathbf{T}, \mathbf{U})$. ε is defined by $\varepsilon_{(X,f)} = {}^t(1_X, f)$, and $\delta = (1_{\mathrm{Id}_{\mathscr{D}}}, 0)$. The unit $\lambda : \mathrm{Id}_{\mathscr{D}_F(\eta)} \to \mathbf{ZC}$ of the adjoint pair $(\mathbf{C}, \mathbf{Z})$ is defined by $\lambda_{(X,f)} = coker(f)$ and the counit is the identification $\mathbf{CZ} = \mathrm{Id}_{\mathscr{D}}$. By [6] we have the following classification:

Theorem 2.4. *For an abelian category $\mathscr{C}$, the following are equivalent*:

(i) *$\mathscr{C}$ is a cleft extension of $\mathscr{D}$.*

(ii) *There exists an equivalence $\mathscr{C} \xrightarrow{\approx} \mathscr{D}_F(\eta)$, where $\eta : F^2 \to F$ is an associative natural morphism and $F : \mathscr{D} \to \mathscr{D}$ is a right exact functor.*

In this case the functor $\mathbf{Z}$ has a left adjoint $\mathbf{C} : \mathscr{C} \to \mathscr{D}$ and the following relations hold: $\mathbf{CZ} = \mathrm{Id}_{\mathscr{D}}$, $\mathbf{CT} = \mathrm{Id}_{\mathscr{D}}$, $\mathbf{UZ} = \mathrm{Id}_{\mathscr{D}}$.

Remark 2.5. (1) η-extensions are the analogues of ϑ-extensions in the class of cleft extensions. One can define the category of cleft extensions of $\mathscr{D}$ and the category of η-extensions of $\mathscr{D}$ and prove a version of Proposition 2.1.

(2) By Theorem 2.4, the study of cleft extensions is equivalent to the study of η-extensions. Although one can describe the homological properties of a cleft extension intrinsically, we prefer to work with η-extensions for the following reasons. First the defining data of an η-extension depends only on information directly available from $\mathscr{D}$, so our aim to reduce the study of $\mathscr{D}_F(\eta)$ to the study of the triad $\{\mathscr{D}, F, \eta\}$, is more reasonable: we think of F as an "insertion of arrows" in $\mathscr{D}$ and η as "relations". Our spirit is close to the spirit of Gabriel [15] who first described finite-dimensional algebras by quivers and relations. Second working with general cleft extensions is rather impractical for our purposes and offers no real advantages; for example the notation becomes more complicated. Finally, most familiar cleft extensions of $\mathscr{D}$ are constructed using specific $F : \mathscr{D} \to \mathscr{D}$ and $\eta : F^2 \to F$.

2.3. The module-theoretic intrepretation

The link between cleft extensions of abelian categories and cleft extensions of rings (equivalently ϑ-extensions of rings), is as nice as possible [6]: $\mathscr{D}_F(\eta) \approx \mathrm{Mod}(\Lambda)$ iff $\mathscr{D} \approx \mathrm{Mod}(\Gamma)$ and $\mathbf{U}$ (equivalently F), preserves coproducts. In this case by Watt's Theorem, $F \cong - \otimes_\Gamma M$ for a Γ–Γ-bimodule ${}_\Gamma M_\Gamma$ and by Theorem 2.4, there exists an associative natural morphism $\eta : - \otimes_\Gamma M \otimes_\Gamma M \to - \otimes_\Gamma M$. Setting $\vartheta := \eta_\Gamma$, we obtain an associative Γ–Γ-bimodule morphism $\vartheta : M \otimes_\Gamma M \to M$ and an isomorphism of rings $\Lambda \cong \Gamma \ltimes_\vartheta M$. Conversely if $\Lambda = \Gamma \ltimes_\vartheta M$ is a cleft extension of Γ, then $\mathrm{Mod}(\Lambda) = \mathrm{Mod}(\Gamma)_F(\eta)$. Here $F = - \otimes_\Gamma M$ and $\eta_X = 1_X \otimes \vartheta$, where $\vartheta : M \otimes_\Gamma M \to M$ is defined by $\vartheta(m_1 \otimes m_2) = (0, m_1)(0, m_2)$. The above identification sends a right Λ-module X_Λ to the object $(X_\Gamma, f) \in \mathrm{Mod}(\Gamma)_F(\eta)$, where $f : X \otimes_\Gamma M \to X_\Gamma$ is defined as follows: $f(x \otimes m) = x \cdot (0, m)$. Conversely we identify the object (X, f) of $\mathrm{Mod}(\Gamma)_F(\eta)$ with the Λ-module X_Γ, with right Λ-action: $x \cdot (\gamma, m) = x \cdot \gamma + f(x \otimes m)$; e.g. the ideal $M = (M, \vartheta)$ as an object of $\mathrm{Mod}(\Gamma)_F(\eta)$.

2.4. Cleft coextensions

All the above definitions and constructions can be dualized in the following way. A *cleft coextension* of $\mathscr{D}$ is an abelian category $\mathscr{C}$ together with additive functors $\mathbf{U}:\mathscr{C}\to\mathscr{D}$ and $\mathbf{Z}:\mathscr{D}\to\mathscr{C}$, such that $\mathbf{U}$ is faithful exact and admits a right adjoint $\mathbf{H}$ and $\mathbf{UZ}=\mathrm{Id}_{\mathscr{D}}$. The primal examples of cleft coextensions of the abelian category $\mathscr{D}$ are the ζ-coextensions $\mathscr{D}^G(\zeta)$ of $\mathscr{D}$ by a left exact endofunctor $G:\mathscr{D}\to\mathscr{D}$. Here $\zeta:G\to G^2$ is a *coassociative* natural morphism, i.e. $\zeta\circ\zeta G=\zeta\circ G\zeta$, and $\mathscr{D}^G(\zeta)$ has as objects pairs (X,f) where $f:X\to GX$ is a morphism of $\mathscr{D}$ such that $f\circ Gf=f\circ\zeta_X$. A morphism $a:(X,f)\to(Y,g)$ in $\mathscr{D}^G(\zeta)$ is a morphism $a:X\to Y$ in $\mathscr{D}$, such that $a\circ g=f\circ Ga$. The category $\mathscr{D}^G(\zeta)$ is abelian and there are adjoint pairs of functors $(\mathbf{U},\mathbf{H})$, $(\mathbf{Z},\mathbf{K})$:

$$\mathbf{H}:\mathscr{D}\to\mathscr{D}^G(\zeta),\quad \mathbf{U}:\mathscr{D}^G(\zeta)\to\mathscr{D},\quad \mathbf{K}:\mathscr{D}^G(\zeta)\to\mathscr{D},\quad \mathbf{Z}:\mathscr{D}\to\mathscr{D}^G(\zeta)$$

defined as follows. If $a:X\to Y$ is a morphism in $\mathscr{D}$ and $\alpha:(X,f)\to(Y,g)$ is a morphism in $\mathscr{D}^G(\zeta)$, then $\mathbf{H}(X)=(GX\oplus X,h_X)$, where

$$h_X=\begin{pmatrix}\zeta_X & 1_{GX}\\ 0 & 0\end{pmatrix}\quad\text{and}\quad \mathbf{H}(a)=\begin{pmatrix}Ga & 0\\ 0 & a\end{pmatrix}.$$

$\mathbf{K}(X,f)=\mathit{Ker}(f)$, and $\mathbf{K}(\alpha)=$ the unique morphism : $\mathbf{K}(X,f)\to\mathbf{K}(Y,g)$ in $\mathscr{D}$ such that $\mathbf{K}(\alpha)\circ \mathit{ker}(g)=\mathit{ker}(f)\circ\alpha$. The functors $\mathbf{U},\mathbf{Z}$ are defined as in the case of cleft extensions. Trivially $\mathbf{U}$ is faithful exact, $\mathbf{Z}$ is fully faithful exact and the following relations hold: $\mathbf{KZ}=\mathrm{Id}_{\mathscr{D}}$, $\mathbf{KH}=\mathrm{Id}_{\mathscr{D}}$, $\mathbf{UZ}=\mathrm{Id}_{\mathscr{D}}$. In particular $\mathscr{D}^G(\zeta)$ is a cleft coextension of $\mathscr{D}$. The dual of Theorem 2.4 is also true: *the category $\mathscr{C}$ is a cleft coextension of $\mathscr{D}$ iff $\mathscr{C}\approx\mathscr{D}^G(\zeta)$ for a coassociative natural morphism $\zeta:G\to G^2$, where $G:\mathscr{D}\to\mathscr{D}$ is a left exact functor. In this case $\mathbf{Z}$ has a right adjoint $\mathbf{K}$ and the following are true*: $\mathbf{KZ}=\mathrm{Id}_{\mathscr{D}}$, $\mathbf{KH}=\mathrm{Id}_{\mathscr{D}}$, $\mathbf{UZ}=\mathrm{Id}_{\mathscr{D}}$.

Since the concepts of cleft extension and cleft coextension are dual, we shall study only cleft extensions, noting that when it is necessary we shall state and use the dual results concerning cleft coextensions, leaving their proofs to the reader. We will see that the cleft extensions are suitable for the study of projective dimension, and the cleft coextensions are suitable for the study of injective dimension.

A nice situation occurs when a cleft extension $\mathscr{D}_F(\eta)$ of $\mathscr{D}$ is also a cleft coextension. This happens if (and only if) the functor F has a right adjoint G. In this case fixing counit $\rho:FG\to\mathrm{Id}_{\mathscr{D}}$ and unit $\sigma:\mathrm{Id}_{\mathscr{D}}\to GF$ of the adjoint pair (F,G), and defining $\zeta:=\sigma G\circ G\sigma FG\circ G^2\eta G\circ G^2\rho$, we obtain a coassociative natural morphism $\zeta:G\to G^2$ and an isomorphism of categories $\mathbb{D}:\mathscr{D}_F(\eta)\xrightarrow{\cong}\mathscr{D}^G(\zeta)$, defined by $\mathbb{D}(X,f)=(X,\sigma_X\circ Gf)$, and $\mathbb{D}(a)=a$. By the above remarks and Watt's Theorem, a cleft extension of a module or Grothendieck category by a colimit preserving endofunctor F, is also a cleft coextension. Note that cleft coextensions can be used to study cleft extensions of coalgebras, using the cotensor product functor. Finally, observe that if $\eta=0$ ($\zeta=0$) then $\mathscr{D}_F(0)$ ($\mathscr{D}^G(0)$) is the *trivial extension* (*coextension*) $\mathscr{D}\ltimes F$ ($G\ltimes\mathscr{D}$) of $\mathscr{D}$ studied in detail in [13].

If the cleft extension $\mathscr{D}_F(\eta)$ of $\mathscr{D}$ is induced by a cleft extension of rings $\Lambda = \Gamma \ltimes_{\vartheta} M$, then by [6] the morphism ϑ is right, resp. left, T-nilpotent iff $\mathbf{C}(X,f)=0$ implies that $(X,f)=0$, resp. $\mathbf{K}(X,f)=0$ implies that $(X,f)=0$. The functors $F,G,\mathbf{T},\mathbf{U},\mathbf{C},\mathbf{Z},\mathbf{H},\mathbf{K}$ introduced above, can be realized as follows: $F = -\otimes_\Gamma M_\Gamma$, $G = \mathrm{Hom}_\Gamma[{}_\Gamma M_\Gamma, -]$, $\mathbf{T} = -\otimes_\Gamma \Lambda_\Lambda$, $\mathbf{U} = \mathrm{Hom}_\Lambda[{}_\Gamma \Lambda_\Lambda, -] \cong -\otimes_\Lambda \Lambda_\Gamma$, $\mathbf{C} = -\otimes_\Lambda \Gamma_\Gamma$, $\mathbf{Z} = \mathrm{Hom}_\Gamma[{}_\Lambda \Gamma_\Gamma, -] \cong -\otimes_\Gamma \Gamma_\Lambda$, $\mathbf{H} = \mathrm{Hom}_\Gamma[{}_\Lambda \Lambda_\Gamma, -]$, $\mathbf{K} = \mathrm{Hom}_\Lambda[{}_\Gamma \Gamma_\Lambda, -]$.

2.5. Constructions and examples of cleft extensions

In this subsection we present some constructions of cleft extensions and we indicate how familiar categories can be viewed as cleft η-extensions. The description of the functors $F,\mathbf{T},\mathbf{U},\mathbf{Z},\mathbf{C}$ and of the morphism $\eta : F^2 \to F$, will be clear from the context.

2.5.1. Free categories

The *free category* $\mathscr{D}\langle F_i,\ i\in I\rangle$ over $\mathscr{D}$ with respect to a family $\{F_i\}_{i\in I}$ of right exact endofunctors of $\mathscr{D}$ has as objects, pairs $(X, f_i)_{i\in I}$ where $f_i : F_i X \to X$ are morphisms in $\mathscr{D}$ indexed by I. A morphism $\alpha : (X,f_i)_{i\in I} \to (Y,g_i)_{i\in I}$ in $\mathscr{D}\langle F_i,\ i\in I\rangle$, is a morphism $\alpha : X \to Y$ in $\mathscr{D}$, such that $f_i \circ \alpha = F\alpha \circ g_i$, $\forall i \in I$. If $J = max(\aleph_0, |I|)$, and if $\mathscr{D}$ has coproducts indexed by J which are preserved by all the functors F_i, we get from [6] that $\mathscr{D}\langle F_i,\ i\in I\rangle = \mathscr{D}(F)$ is a cleft extension of $\mathscr{D}$, where $F = \bigoplus_{i\in I} F_i$. The definition of the functors $\mathbf{U},\mathbf{Z}$ is the obvious one and $\mathbf{T}(X) = (\bigoplus_{i\in\mathbb{N}} F^i X, t^X)$, where t^X has components $t^X_{i,j}=0$, $\forall j \neq i+1$ and $t^X_{i,i+1} = 1_{F^i(X)}$; for $\alpha : X \to Y$ a morphism in $\mathscr{D}$, $\mathbf{T}(\alpha) = \bigoplus_{i\in\mathbb{N}} F^i(\alpha)$. If $F_i = \mathrm{Id}_{\mathscr{D}}$, $\forall i \in I$, then $\mathscr{D}\langle F_i,\ i\in I\rangle := \mathscr{D}\langle I\rangle$ is the free category over $\mathscr{D}$ in $|I|$ noncommuting variables [28]. In case $\mathscr{D} \approx \mathrm{Mod}(\Gamma)$, we have that $\mathscr{D}\langle F_i,\ i\in I\rangle \approx \mathrm{Mod}(\mathrm{T}_\Gamma(M))$, where $\mathrm{T}_\Gamma(M)$ is the tensor ring over Γ of the unique Γ–Γ-bimodule M, with $F \cong -\otimes_\Gamma M$, where $F = \bigoplus_{i\in I} F_i$. In particular if $F_i = \mathrm{Id}_{\mathscr{D}}$, $\forall i \in I$, then $\mathscr{D}\langle I\rangle \approx \mathrm{Mod}(\Gamma\langle I\rangle)$ is the module category of the free Γ-ring in $|I|$ noncommuting variables. Note that categories of representations of quivers [15,3], and more generally categories of diagrams, are free cleft extensions. In general, any cleft extension of $\mathscr{D}$ can be realized as a full exact reflective subcategory of a free cleft extension $\mathscr{D}(F)$, which is a model of the module category of a tensor ring. The morphisms $f : FX \to X$ can be considered as generators, and the relations are of the form $\eta_X \circ f - Ff \circ f$. In the case of a cleft extension Λ of a ring Γ, it is very useful to view the Λ-modules as Γ-morphisms, satisfying relations given by the multiplication.

2.5.2. Polynomial and symmetric categories

If $\{F_i\}_{i=1}^n$ is a set of *commuting* right exact endofunctors of $\mathscr{D}$: $F_iF_j = F_jF_i$, $\forall i,j$, then the *symmetric category* $\mathscr{D}[F_1,F_2,\ldots,F_n]$ of $\mathscr{D}$ with respect to the set $\{F_i\}_{i=1}^n$, is defined inductively as follows: $\mathscr{D}[F_1,F_2] = \mathscr{D}(F_1)(F_2^*)$, where $F_2^* : \mathscr{D}(F_1) \to \mathscr{D}(F_1)$ is defined by $F_2^*(X,f) = (F_2X, F_2f)$, $F_2^*(a) = F_2 a$, and so on. If $\mathscr{D}$ has countable coproducts and each F_i preserves them, then each $\mathscr{D}[F_1,F_2,\ldots,F_i]$ is a cleft extension of $\mathscr{D}$ and of $\mathscr{D}[F_1,F_2,\ldots,F_{i-1}]$. If $F_i = \mathrm{Id}_{\mathscr{D}}, \forall i$, then $\mathscr{D}[F_1,F_2,\ldots,F_n] := \mathscr{D}[n]$ is the *polynomial* category of $\mathscr{D}$ in n-variables [28], and a typical object of $\mathscr{D}[n]$ consists of

n-endomorphisms $f_i : X \to X$, $X \in \mathscr{D}$, with $f_i \circ f_j = f_j \circ f_i$. Of course if $\mathscr{D} = \mathrm{Mod}(\Gamma)$, then $\mathscr{D}[n] = \mathrm{Mod}(\Gamma[\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n])$.

2.5.3. Exterior categories

Let $\{F_i\}_{i=1}^n$ be a set of commuting right exact endofunctors of $\mathscr{D}$: $F_iF_j = F_jF_i$, $\forall i,j$. The *exterior category* $\bigwedge(\mathscr{D})(F_1, F_2, \ldots, F_n)$ of $\mathscr{D}$ with respect to the set $\{F_i\}_{i=1}^n$ is constructed inductively as follows. First, we set $\bigwedge(\mathscr{D})(F_1) = \mathscr{D}_{F_1}(0)$. Then $\bigwedge(\mathscr{D})(F_1, F_2) = [\bigwedge(\mathscr{D})(F_1)]_{F_2^!}(0)$, where $F_2^! : \bigwedge(\mathscr{D})(F_1) \to \bigwedge(\mathscr{D})(F_1)$ is defined by $F_2^!(X, f) = (F_2X, -F_2f)$, $F_2^!(a) = F_2a$, and so on. Each category $\bigwedge(\mathscr{D})(F_1, F_2, \ldots, F_i)$ is a cleft (trivial) extension of $\mathscr{D}$ and of $\bigwedge(\mathscr{D})(F_1, F_2, \ldots, F_{i-1})$. If $F_i = \mathrm{Id}_{\mathscr{D}}$, $\forall i$, then $\bigwedge(\mathscr{D})(F_1, F_2, \ldots, F_n) := \bigwedge^n(\mathscr{D})$ is the exterior category of $\mathscr{D}$ in n-variables [29], and a typical object of $\bigwedge^n(\mathscr{D})$ consists of n-endomorphisms $f_i : X \to X$, $X \in \mathscr{D}$, with $f_i^2 = 0$, $f_i \circ f_j + f_j \circ f_i = 0$. If $\mathscr{D} \approx \mathrm{Mod}(\Gamma)$, then $\bigwedge^n(\mathscr{D}) \approx \mathrm{Mod}(\bigwedge_\Gamma^n(\Gamma))$, where $\bigwedge_\Gamma^n(\Gamma)$ is the exterior ring of Γ.

2.5.4. Trivial and truncated extensions

Let $F : \mathscr{D} \to \mathscr{D}$ be a right exact endofunctor. We view the trivial extension $\mathscr{D}_F(0)$ of $\mathscr{D}$ as the full subcategory of the free category $\mathscr{D}(F)$ consisting of objects $f : FX \to X$ such that $Ff \circ f = 0$. Special cases of the trivial extension construction are the comma-categories, the category of composable morphisms of any finite length and categories of complexes.

Let $G : \mathscr{D} \to \mathscr{E}$ be a right exact functor between abelian categories, and let $\mathscr{C} = (\mathscr{D}, G, \mathscr{E})$ be the induced *comma-category*. Recall that an object of $\mathscr{C}$ is a triple (A, f, B) where $f : GA \to B$ is a morphism in $\mathscr{E}$. A morphism $\gamma : (A, f, B) \to (A', f', B')$ in $\mathscr{C}$ consists of two morphisms $\alpha : A \to A'$ in $\mathscr{D}$ and $\beta : B \to B'$ in $\mathscr{E}$ such that $f \circ \beta = G\alpha \circ f'$. The category $\mathscr{C}$ can be considered as a trivial extension of $\mathscr{D} \times \mathscr{E}$ by the right exact functor $F : \mathscr{D} \times \mathscr{E} \to \mathscr{D} \times \mathscr{E}$ given by $F(A, B) = (0, GA)$. This construction gives a nice description of the module category of a triangular matrix ring

$$\begin{pmatrix} R & {}_RM_S \\ 0 & S \end{pmatrix}.$$

Here $\mathscr{D} = \mathrm{Mod}(R)$, $\mathscr{E} = \mathrm{Mod}(S)$ and $G = - \otimes_R M_S$.

Let $\mathscr{D}^n$ be the category of *composable morphisms* of length $n - 1$, over $\mathscr{D}$, in particular $\mathscr{D}^2$ is the category of morphisms of $\mathscr{D}$. Obviously $\mathscr{D}^2$ is a trivial extension of $\mathscr{D} \times \mathscr{D}$ since it is the comma-category $(\mathscr{D}, \mathrm{Id}_{\mathscr{D}}, \mathscr{D})$. Inductively $\mathscr{D}^{n+1}$ is a trivial extension of $\mathscr{D}^n \times \mathscr{D}$. Similarly, the category of (bounded, bounded above or below, unbounded or of any given finite length) complexes over $\mathscr{D}$ is a trivial extension of the underlying graded category by the shift functor.

A natural generalization of the trivial extension is the *t-truncated extension* $\mathscr{D} \ltimes_t F$ of $\mathscr{D}$ by the right exact endofunctor $F : \mathscr{D} \to \mathscr{D}$, $t \geq 0$. We recall from [27] that $\mathscr{D} \ltimes_t F$ is the full subcategory of the free cleft extension $\mathscr{D}(F)$ with objects (X, f) where $F^{t+1}f \circ F^t f \circ \cdots \circ F^2 f \circ Ff \circ f = 0$. There are adjoint pairs $(\mathbf{T}, \mathbf{U}) : \mathscr{D} \to \mathscr{D} \ltimes_t F$, $(\mathbf{C}, \mathbf{Z}) : \mathscr{D} \ltimes_t F \to \mathscr{D}$, such that $\mathbf{U}$ is faithful exact, and $\mathbf{UZ} = \mathrm{Id}_{\mathscr{D}}$, [27]; hence

$\mathscr{D} \ltimes_t F$ is a cleft extension of $\mathscr{D}$. We need only the description of $\mathbf{T}$ from [27]. If $X \in \mathscr{D}$, then $\mathbf{T}(X) = (\bigoplus_{i=0}^{t+1} F^i X, \tau_X)$, where $\tau_X : \bigoplus_{i=1}^{t+2} F^i X \to \bigoplus_{i=0}^{t+1} F^i X$ is represented by a $(t+1) \times (t+1)$ matrix, with all its entries zero, except the entries above the main diagonal, which are the identities $1_{F^i X}$, $i = 1, \ldots, t+1$. If $a : X \to Y$ is a morphism in $\mathscr{D}$, then $\mathbf{T}(a) = \bigoplus_{i=0}^{t+1} F^i a$. Setting $\tilde{F} := \bigoplus_{i=1}^{t+1} F^i$, it is easy to see that there exists an associative natural morphism $\tilde{\eta} : \tilde{F}^2 \to \tilde{F}$, and an isomorphism of categories $\mathbb{A} : \mathscr{D} \ltimes_t F \xrightarrow{\cong} \mathscr{D}_{\tilde{F}}(\tilde{\eta})$, defined by $\mathbb{A}(X, f) = (X, {}^t(f_0, f_1, \ldots, f_{t-1}, f_t))$, where $f_i := F^i f \circ F^{i-1} f \circ \cdots \circ F^2 f \circ F f \circ f : F^{i+1} X \to X$, and $\mathbb{A}(a) = a$. If $(\tilde{\mathbf{T}}, \tilde{\mathbf{U}}), (\tilde{\mathbf{C}}, \tilde{\mathbf{Z}})$ are the standard adjoint pairs defined on $\mathscr{D}_{\tilde{F}}(\tilde{\eta})$, then $\tilde{\mathbf{U}}\mathbb{A} = \mathbf{U}$, $\mathbb{A}\mathbf{Z} = \tilde{\mathbf{Z}}$, and moreover $\mathbb{A}\mathbf{T} = \tilde{\mathbf{T}}$, $\tilde{\mathbf{C}}\mathbb{A} = \mathbf{C}$; so $\mathbb{A}$ is an isomorphism over $\mathscr{D}$. Note that $\mathscr{D} \ltimes_t \mathrm{Id}_{\mathscr{D}}$ is the category of $(t+1)$-nilpotent endomorphisms of $\mathscr{D}$, with objects $\mathscr{D}$-morphisms $f : X \to X$ such that $f^{t+1} = 0$. The dual notion of an *s-truncated coextension* $G \ltimes_s \mathscr{D}$ of $\mathscr{D}$ by a left exact functor G is defined dually. If $\mathscr{D} = \mathrm{Mod}(\Gamma)$ and $F = - \otimes_\Gamma M$, then $\mathscr{D} \ltimes_t F = \mathrm{Mod}(\Gamma \ltimes_t M)$, where $\Gamma \ltimes_t M$ is the t-truncated extension of Γ by M, i.e. the factor ring $\mathrm{T}_\Gamma(M)/\langle \otimes_\Gamma^{t+2} M \rangle$, where $\langle \otimes_\Gamma^{t+2} M \rangle$ is the ideal of the tensor ring $\mathrm{T}_\Gamma(M)$ generated by $\otimes_\Gamma^{t+2} M$.

Example 2.6. Consider the quiver $\mathscr{Q} : \bullet \underset{\beta_1}{\overset{\alpha_1}{\rightleftarrows}} \bullet \underset{\beta_2}{\overset{\alpha_2}{\rightleftarrows}} \bullet \cdots \bullet \underset{\beta_m}{\overset{\alpha_m}{\rightleftarrows}} \bullet$, with relations

$$\mathscr{R}_{t,s} = \{\alpha_1\beta_1, \alpha_{k+1}\alpha_{k+2}\cdots\alpha_{k+t+1}, \beta_l\beta_{l-1}\cdots\beta_{l-s}, \beta_i\alpha_i - \alpha_{i+1}\beta_{i+1}\},$$

where $0 \le k \le m-t-1$, $s+1 \le l \le m$, $i = 1, 2, \ldots, m-1$, $0 \le t$, $s \le m-1$.

Let k be a field and consider the family of algebras $\Lambda_{t,s}^m = k\mathscr{Q}/\langle \mathscr{R}_{t,s} \rangle$. Set $\mathscr{D} = \mathrm{Mod}(k)$ and let $\mathscr{D}^{(m+1)}$ be the product category of $m+1$ copies of $\mathscr{D}$. Define a functor $F : \mathscr{D}^{(m+1)} \to \mathscr{D}^{(m+1)}$ as follows: $F(X_1, X_2, \ldots, X_{m+1}) = (0, X_1, X_2, \ldots, X_m)$. Let $\mathscr{C} = \mathscr{D}^{(m+1)} \ltimes_t F$ be the t-truncated extension of $\mathscr{D}^{(m+1)}$ by F. A typical object of $\mathscr{C}$ consists of k-linear maps $f_i : X_i \to X_{i+1}$, $1 \le i \le m$, $X_i \in \mathrm{Mod}(k)$, such that $f_{k+1} \circ \cdots \circ f_{k+t+1} = 0$, $0 \le k \le m-t-1$. Define a functor $G : \mathscr{C} \to \mathscr{C}$ as follows: $F(X_1 \xrightarrow{f_1} X_2 \to \cdots \to X_m \xrightarrow{f_m} X_{m+1}) = (0 \to X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} X \to \cdots \to X_{m-1} \xrightarrow{f_{m-1}} X_m)$. Let $\mathscr{E} = G \ltimes_s \mathscr{C}$ be the s-truncated coextension of $\mathscr{C}$ by G. It is easy to see that $\mathscr{E}$ is isomorphic to the module category $\mathrm{Mod}(\Lambda_{t,s}^m)$. Note that if $char(k) = p > 0$, then the algebras $\Lambda_{0,0}^m$ obtained above for various m, are exactly the nonsimple components of the basic algebra of the Schur algebra $\mathscr{S}_k(p,p)$ [39].

A host of (rather complicated) examples is obtained, by combining the free, symmetric, polynomial, exterior, truncated constructions and their duals.

2.6. *Other examples from ring theory*

The previous examples of cleft extensions are of constructive nature. In ring theory there are also examples for structural reasons. The main source of such examples comes from the Hochschild version and its generalizations, of the classical theorem of Wedderburn: let R be a commutative ring, let Λ be an R-algebra and let $\mathscr{J}ac(\Lambda)$ be the Jacobson radical of Λ. Then Λ is a cleft extension of $\Lambda/\mathscr{J}ac(\Lambda)$ in the following cases:

(i) Λ is a basic semiperfect ring or $\Lambda/\mathscr{J}ac(\Lambda)$ is R-projective, $\mathscr{J}ac(\Lambda)$ is nilpotent and the Hochschild dimension $\mathrm{Dim}(\Lambda/\mathscr{J}ac(\Lambda)) \leq 1$, see [35,32]. (ii) R is a field, $\dim_R \Lambda < \infty$ and either R is a perfect field or gl.dim $\Lambda < \infty$ [11,24]. (iii) R is a von Neumann regular or Henselian ring, Λ_R is f. g. and $\Lambda/\mathscr{J}ac(\Lambda)$ is separable [37,4]. In particular if $\mathscr{Q}$ is a finite quiver, k is a field and I is an admissible ideal of the path-algebra $k\mathscr{Q}$, then the quotient algebra $k\mathscr{Q}/I$ is a cleft extension of the semisimple subalgebra $k^{|\mathscr{Q}_0|}$, where $\mathscr{Q}_0$ is the set of vertices of $\mathscr{Q}$. Also the repetitive algebra of an algebra [17] is a (trivial) cleft extension. If $U(L)$ is the universal enveloping algebra of a finite-dimensional Lie algebra L over a field k, then by the Levi-Malcev Theorem [1], $U(L)$ is a cleft extension of $U(L/J)$ where J is the radical of L. Note that the class of cleft extensions of a ring Γ is closed under free products. For more examples see [6,16,21–23,26,37,40].

The above plethora of constructions and examples of cleft extensions indicates also the difficulty of their study. The main dificulty is of course the complexity of the multiplication $\eta : F^2 \to F$ in $\mathscr{D}_F(\eta)$, or the multiplication $\vartheta : M \otimes_\Gamma M \to M$ in $\Lambda = \Gamma \ltimes_\vartheta M$. In case η or ϑ has a simple form, for example in the case of free, symmetric, exterior, trivial or truncated extensions, we obtain satisfactory results.

The notion of a cleft extension of rings is quite classical. Perhaps its first appearence is in the construction of a unital ring Λ from a ring Γ without unit. Here Λ is just the cleft ϑ-extension $\mathbb{Z} \ltimes_\vartheta \Gamma$ of $\mathbb{Z}$, where $\vartheta : \Gamma \otimes_{\mathbb{Z}} \Gamma \to \Gamma$ is the multiplication. To the author's best knowledge the concept goes back at least to Wedderburn and his celebrated Principal Theorem (stating that any finite-dimensional algebra over a field of characteristic zero is cleft). The terminology "cleft" it seems to be due to Vinograde [36] (in 1944). In the literature cleft extensions are also known as Everett extensions. Hochschild in his seminal paper [19] relates cleft extensions of algebras with their cohomology and he uses the term segregated extension. Cartan–Eilenberg in their book [8] use the term inessential extension. In Mac Lane's book [24] the term cleft is used and Pierce in his book [32] uses the term split extension. Recently, Cuntz and Quillen [10] studied $\mathbb{C}$-algebras Γ such that any nilpotent extension of Γ is cleft, under the name of quasi-free algebras [38]. They show that quasi-free algebras are the noncommutative analogues of smooth algebras and they provide a natural setting for noncommutative versions of certain aspects of manifolds, in connection with cyclic homology.

3. Proper classes, resolutions and primitive objects

3.1. Proper classes

In this subsection we recall some definitions and concepts from relative homology. The setting in which we study the homological structure of $\mathscr{D}_F(\eta)$ is that of proper exact sequences as presented in Mac Lane's book [24], to which we refer for details. In this subsection $\mathscr{D}$ can be an arbitrary abelian category. Let $\mathfrak{R}$ be a class of short exact sequences in $\mathscr{D}$. The corresponding class of monics, resp. epics, is denoted by $\mathfrak{R}_{\mathrm{m}}$, resp. $\mathfrak{R}_{\mathrm{e}}$. A morphism in $\mathfrak{R}_{\mathrm{m}}$, resp. $\mathfrak{R}_{\mathrm{e}}$ is called an $\mathfrak{R}$-*proper monic*, resp. $\mathfrak{R}$-*proper epic*. We denote by $\mathfrak{R}_0$ the class of split short exact sequences. We recall from [24] that $\mathfrak{R}$ is called a *proper class* if (α) $\mathfrak{R}$ is closed under isomorphisms and $\mathfrak{R}_0 \subseteq \mathfrak{R}$. ($\beta$) The classes $\mathfrak{R}_{\mathrm{e}}$ and $\mathfrak{R}_{\mathrm{m}}$ are closed under composition. (γ) If f,g are monics and $f \circ g \in \mathfrak{R}_{\mathrm{m}}$, then $f \in \mathfrak{R}_{\mathrm{m}}$; if f,g are epics and $f \circ g \in \mathfrak{R}_{\mathrm{e}}$, then $g \in \mathfrak{R}_{\mathrm{e}}$. We recall that an object $P \in \mathscr{D}$ is called $\mathfrak{R}$-*projective* if $0 \to \mathscr{D}(P,X) \to \mathscr{D}(P,Y) \to \mathscr{D}(P,W) \to 0$ is exact for any $0 \to X \to Y \to W \to 0$ in $\mathfrak{R}$. The full subcategory of $\mathfrak{R}$-projectives is denoted by $\mathscr{P}(\mathfrak{R})$. The full subcategory $\mathscr{I}(\mathfrak{R})$ of $\mathfrak{R}$-*injective* objects is defined dually. We say that $\mathscr{D}$ *has enough* $\mathfrak{R}$-*projectives* if any object $X \in \mathscr{D}$ is included in a sequence $0 \to K \to P \to X \to 0$ in $\mathfrak{R}$ with $P \in \mathscr{P}(\mathfrak{R})$; such a sequence is called an $\mathfrak{R}$-*projective presentation* of X. If $\mathscr{D}$ has enough $\mathfrak{R}$-projectives, then the class of epics $\mathfrak{R}_{\mathrm{e}}$ is a projective class in $\mathscr{D}$ in the sense of [12], see also [28,18]. Obviously $\mathscr{D}$ has enough $\mathfrak{R}_0$-projectives for the proper class $\mathfrak{R}_0$ and $\mathscr{P}(\mathfrak{R}_0) = \mathscr{D}$. The class $\mathfrak{R}_1$ of all short exact sequences is always proper and $\mathscr{D}$ has enough $\mathfrak{R}_1$-projectives iff $\mathscr{D}$ has enough projectives in the usual sense. For any proper class $\mathfrak{R}$ we have the relations: $\mathfrak{R}_0 \subseteq \mathfrak{R} \subseteq \mathfrak{R}_1$ and $\mathscr{P}(\mathfrak{R}_1) \subseteq \mathscr{P}(\mathfrak{R}) \subseteq \mathscr{P}(\mathfrak{R}_0) = \mathscr{D}$.

A morphism $\alpha : X \to Y$ is called $\mathfrak{R}$-*proper* if $coker(\alpha) : Y \to Coker(\alpha)$ is in $\mathfrak{R}_{\mathrm{e}}$ and $ker(\alpha) : Ker(\alpha) \to X$ is in $\mathfrak{R}_{\mathrm{m}}$, or equivalently if $im(\alpha) : Im(\alpha) \to Y$ is in $\mathfrak{R}_{\mathrm{m}}$ and $coim(\alpha) : X \to Im(\alpha)$ is in $\mathfrak{R}_{\mathrm{e}}$. A complex in $\mathscr{D}$ is called $\mathfrak{R}$-*proper* if each of its differentials is an $\mathfrak{R}$-proper morphism. An $\mathfrak{R}$-proper exact sequence is called $\mathfrak{R}$-*exact*. Clearly, any $\mathfrak{R}$-exact sequence is a Yoneda composition of members of $\mathfrak{R}$. An additive functor $\mathbb{K} : \mathscr{D} \to \mathscr{A}$ with abelian range, is called $\mathfrak{R}$-exact if $\mathbb{K}$ sends $\mathfrak{R}$-exact sequences to exact sequences in $\mathscr{A}$. An $\mathfrak{R}$-*projective resolution* of the object $X \in \mathscr{D}$ is an $\mathfrak{R}$-exact sequence $\cdots \to P_1 \to P_0 \to X \to 0$ such that $P_i \in \mathscr{P}(\mathfrak{R})$, $\forall i \geq 0$. Such a resolution with X deleted, is called a *deleted* $\mathfrak{R}$-projective resolution of X. If $\mathscr{D}$ has enough $\mathfrak{R}$-projectives and if $\mathbb{K} : \mathscr{D} \to \mathscr{A}$, $\mathbb{L} : \mathscr{D}^{op} \to \mathscr{A}$ are additive functors to an abelian category $\mathscr{A}$, then we denote by $\mathscr{L}_*^{\mathfrak{R}}\mathbb{K}$ the left $\mathfrak{R}$-derived functors of $\mathbb{K}$ and by $\mathscr{R}_{\mathfrak{R}}^*\mathbb{L}$ the right $\mathfrak{R}$-derived functors of $\mathbb{L}$. In particular we denote by $\mathscr{E}xt_{\mathfrak{R}}^*[-,-]$ the $\mathfrak{R}$-proper extension functors. These derived functors are computed in the usual way, by means of $\mathfrak{R}$-projective resolutions. We denote by $\mathfrak{R}$-p.dX the $\mathfrak{R}$-projective dimension of $X \in \mathscr{D}$ and by $\mathfrak{R}$-gl.dim$\mathscr{D}$ the $\mathfrak{R}$-global dimension of $\mathscr{D}$.

Finally we recall that a *resolvent pair* for $\mathfrak{R}$ in $\mathscr{D}$ is a pair $(\mathbb{R},\rho)$, where $\mathbb{R} : \mathscr{D} \to \mathscr{D}$ is an additive functor and $\rho : \mathbb{R} \to \mathrm{Id}_{\mathscr{D}}$ is a natural epimorphism such that $\forall X \in \mathscr{D}$: $\mathbb{R}(X) \in \mathscr{P}(\mathfrak{R})$ and $0 \to \mathbb{K}(X) \xrightarrow{\kappa_X} \mathbb{R}(X) \xrightarrow{\rho_X} X \to 0$ is in $\mathfrak{R}$, where $\kappa : \mathbb{K} \to \mathbb{R}$ is the kernel of ρ. The endofunctor $\mathbb{K}$ is called the first $\mathfrak{R}$-*syzygy* functor of $\mathscr{D}$ with respect to $(\mathbb{R},\rho)$. Clearly if a proper class admits a resolvent pair, then $\mathscr{D}$ admits functorial $\mathfrak{R}$-projective resolutions.

3.2. Proper resolutions in $\mathscr{D}_F(\eta)$

We fix throughout the paper a cleft η-extension $\mathscr{D}_F(\eta)$ of the abelian category $\mathscr{D}$ by the right exact functor $F : \mathscr{D} \to \mathscr{D}$, where $\eta : F^2 \to F$ is an associative natural morphism. We follow always the notation introduced in the previous section.

Fix a proper class $\mathfrak{R}$ of short exact sequences in $\mathscr{D}$ and assume throughout that $\mathscr{D}$ has enough $\mathfrak{R}$-projectives. We define a class of sequences $\mathfrak{F}$ in $\mathscr{D}_F(\eta)$ as follows:

$$\mathfrak{F} := \mathbf{U}^{-1}(\mathfrak{R}) = \{(E): 0 \to (X,f) \xrightarrow{\alpha} (Y,g) \xrightarrow{\beta} (W,h) \to 0 \text{ is a sequence in } \mathscr{D}_F(\eta) \text{ such that } \mathbf{U}(E): 0 \to X \xrightarrow{\alpha} Y \xrightarrow{\beta} W \to 0 \in \mathfrak{R}\}.$$

Using the adjoint pair $(\mathbf{T},\mathbf{U})$, the Adjoint Theorem [12,28], and the fact that $\mathbf{U}$ is faithful we have that $\mathfrak{F}$ is a proper class of short exact sequences in $\mathscr{D}_F(\eta)$. Moreover $\mathscr{D}_F(\eta)$ has enough $\mathfrak{F}$-projectives and an object $(X,f) \in \mathscr{P}(\mathfrak{F})$ iff (X,f) is a direct summand of $\mathbf{T}(P)$ for an object $P \in \mathscr{P}(\mathfrak{R})$.

Choosing $\mathfrak{R} = \mathfrak{R}_0$ to be the proper class of split short exact sequences in $\mathscr{D}$ we obtain the proper class $\Sigma = \mathbf{U}^{-1}(\mathfrak{R}_0)$ in $\mathscr{D}_F(\eta)$. If $\mathfrak{R}_1$ is the proper class of all short exact sequences in $\mathscr{D}$ then $\mathfrak{F}_1 = \mathbf{U}^{-1}(\mathfrak{R}_1)$ is the proper class of all short exact sequences in $\mathscr{D}_F(\eta)$. Obviously, we have the relations: $\mathfrak{R}_0 \subseteq \mathfrak{R} \subseteq \mathfrak{R}_1$, and $\mathfrak{F}_0 \subseteq \Sigma \subseteq \mathfrak{F} \subseteq \mathfrak{F}_1$, where $\mathfrak{F}_0$ is the proper class of all split short exact sequences in $\mathscr{D}_F(\eta)$. Then from the above inclusions we have $\mathscr{P}(\mathfrak{R}_1) \subseteq \mathscr{P}(\mathfrak{R}) \subseteq \mathscr{P}(\mathfrak{R}_0) = \mathscr{D}$ and $\mathscr{P}(\mathfrak{F}_1) \subseteq \mathscr{P}(\mathfrak{F}) \subseteq \mathscr{P}(\Sigma) \subseteq \mathscr{P}(\mathfrak{F}_0) = \mathscr{D}_F(\eta)$. Denoting as usual by $\varepsilon: \mathbf{TU} \to \mathrm{Id}_{\mathscr{D}}$ the counit of $(\mathbf{T},\mathbf{U})$, it is easy to see [6] that $(\mathbf{TU},\varepsilon)$ is a resolvent pair for the proper class Σ in $\mathscr{D}_F(\eta)$ with corresponding Σ-syzygy functor $\Phi = ker(\varepsilon): \mathscr{D}_F(\eta) \to \mathscr{D}_F(\eta)$ given by $\Phi(X,f) = (FX, \eta_X - Ff)$ and $\Phi(a) = Fa$. In particular, $\mathscr{D}_F(\eta)$ has enough Σ-projectives and an object $(X,f) \in \mathscr{P}(\Sigma)$ iff the epic $\varepsilon_{(X,f)}: \mathbf{T}(X) \to (X,f)$ splits. Fixing the above notation we'll construct $\mathfrak{F}$-, Σ-projective resolutions of objects of $\mathscr{D}_F(\eta)$.

Let $(X,f) \in \mathscr{D}_F(\eta)$ and choose an $\mathfrak{R}$-proper epic $a_0: P_0 \to X$ with $P_0 \in \mathscr{P}(\mathfrak{R})$, in such a way that if $X \in \mathscr{P}(\mathfrak{R})$ then $P_0 = X$ and $a_0 = 1_X$. We form the pull-back diagram (0) below, and we consider the morphism $Fv_0 \circ \eta_{P_0} - F\mu_0 : FQ_0 \to FP_0$.

$$\begin{tikzcd} Q_0 \arrow[r, "v_0"] \arrow[d, "\mu_0"'] & FP_0 \arrow[d, "Fa_0 \circ f"] \\ P_0 \arrow[r, "a_0"] & X \end{tikzcd} \tag{0}$$

Then $(Fv_0 \circ \eta_{P_0} - F\mu_0) \circ Fa_0 \circ f = Fv_0 \circ F^2 a_0 \circ \eta_X \circ f - Fv_0 \circ F^2 a_0 \circ Ff \circ f = 0$, thus since (0) is a pullback diagram, there exists a unique morphism $g_0: FQ_0 \to Q_0$ with $g_0 \circ \mu_0 = 0$ and $g_0 \circ v_0 = Fv_0 \circ \eta_{P_0} - F\mu_0$.

Now we proceed by induction on the following data:

(i) $\forall i \geq 0$, $a_i: P_i \to Q_{i-1}$ is an $\mathfrak{R}$-proper epic with $P_i \in \mathscr{P}(\mathfrak{R})$, such that if $Q_{i-1} \in \mathscr{P}(\mathfrak{R})$ then $P_i = Q_{i-1}$ and $a_i = 1_{Q_{i-1}}$.

(ii) Every square (i) below is a pullback diagram in $\mathscr{D}$.

$$\begin{tikzcd} Q_i \arrow[r, "v_i"] \arrow[d, "\mu_i"'] & FP_i \arrow[d, "Fa_i \circ g_{i-1}"] \\ P_i \arrow[r, "a_i"] & Q_{i-1}. \end{tikzcd} \tag{i}$$

(iii) $\forall i \geq 0$, $g_i : FQ_i \to Q_i$ is the unique morphism in $\mathscr{D}$ such that

$$(*) \qquad g_i \circ \mu_i = 0, \quad g_i \circ v_i = Fv_i \circ \eta_{P_i} - F\mu_i.$$

By the first step of the above construction it follows that the following sequence in $\mathscr{D}_F(\eta)$ is in $\mathfrak{F}$, where $\alpha_0 := {}^t(a_0, Fa_0 \circ f)$, and $\kappa_0 := (-\mu_0, v_0)$:

$$0 \to (Q_0, g_0) \xrightarrow{\kappa_0} \mathbf{T}(P_0) \xrightarrow{\alpha_0} (X, f) \to 0.$$

Since by construction $\mathbf{T}(P_0) \in \mathscr{P}(\mathfrak{F})$, the above exact sequence is an $\mathfrak{F}$-projective presentation of (X, f). Similarly setting $\alpha_i := {}^t(a_i, Fa_i \circ g_{i-1})$ and $\kappa_i := (-\mu_i, v_i)$ we obtain short exact sequences in $\mathfrak{F}$:

$$0 \to (Q_i, g_i) \xrightarrow{\kappa_i} \mathbf{T}(P_i) \xrightarrow{\alpha_i} (Q_{i-1}, g_{i-1}) \to 0$$

which are $\mathfrak{F}$-projective presentations of the objects (Q_i, g_i), $\forall i \geq 0$, where $(Q_{-1}, g_{-1}) = (X, f)$. Hence the Yoneda composition of the above short exact sequences is an $\mathfrak{F}$-projective resolution of (X, f), which we denote by $\mathscr{P}_{\mathfrak{F}}^{(X,f)}$:

$$\cdots \to \mathbf{T}(P_i) \xrightarrow{\phi_i^{(X,f)}} \mathbf{T}(P_{i-1}) \to \cdots \to \mathbf{T}(P_1) \xrightarrow{\phi_1^{(X,f)}} \mathbf{T}(P_0) \xrightarrow{\alpha_0} (X, f) \to 0.$$

Usually, we denote by $\Omega_{\mathfrak{F}}^i(X, f)$ the $\mathfrak{F}$-syzygy object (Q_{i-1}, g_{i-1}) of (X, f). Also denoting by $\omega_i^{(X,f)} = -a_{i+1} \circ \mu_i : P_{i+1} \to P_i$, we compute easily that

$$\phi_i^{(X,f)} = \begin{pmatrix} \omega_{i-1}^{(X,f)} & a_i \circ v_{i-1} \\ 0 & F\omega_{i-1}^{(X,f)} + F(a_i \circ v_{i-1}) \circ \eta_{P_{i-1}} \end{pmatrix}, \quad i \geq 1.$$

Applying the functor $\mathbf{C}$ to the resolution $\mathscr{P}_{\mathfrak{F}}^{(X,f)}$, we obtain the following complex of $\mathfrak{R}$-projective objects in $\mathscr{D}$, which we denote by $\mathfrak{F}_{(X,f)}^{\bullet}$:

$$\cdots \to P_{i+1} \xrightarrow{\omega_i^{(X,f)}} P_i \xrightarrow{\omega_{i-1}^{(X,f)}} P_{i-1} \to \cdots \to P_2 \xrightarrow{\omega_1^{(X,f)}} P_1 \xrightarrow{\omega_0^{(X,f)}} P_0 \to 0.$$

We call $\mathfrak{F}_{(X,f)}^{\bullet}$ the $\mathfrak{F}$-*associated complex* of (X, f) with respect to the proper class $\mathfrak{F}$. The above construction applied to the proper class Σ, gives the following Σ-projective resolution of (X, f), which we denote by $\mathscr{P}_{\Sigma}^{(X,f)}$:

$$\cdots \to \mathbf{T}(F^i X) \xrightarrow{\theta_i^{(X,f)}} \mathbf{T}(F^{i-1} X) \to \cdots \to \mathbf{T}(X) \xrightarrow{\varepsilon_{(X,f)}} (X, f) \to 0.$$

Then

$$\theta_i^{(X,f)} = \begin{pmatrix} -f_{i-1}^{(X,f)} & 1_{F^i X} \\ 0 & f_i^{(X,f)} \end{pmatrix},$$

$f_i^{(X,f)} : F^{i+1} X \to F^i X$ and the morphism $f_i^{(X,f)}$ is defined by the formula

$$f_i^{(X,f)} = \sum_{k=0}^{i-1} (-1)^k F^k \eta_{F^{i-k-1} X} + (-1)^i F^i f, \quad \forall i \geq 1.$$

The Σ-*associated complex* $\Sigma_{(X,f)}^{\bullet}$ of (X, f) is of the form

$$\cdots \to F^{i+1} X \xrightarrow{-f_i^{(X,f)}} F^i X \to \cdots \to F^2 X \xrightarrow{-f_1^{(X,f)}} FX \xrightarrow{-f} X \to 0.$$

In case $\eta=0$, $\Sigma^\bullet_{(X,f)}$ is the complex constructed in an ad hoc manner in [13]. Observe that the resolution $\mathscr{P}^{(X,f)}_{\Sigma}$ is the functorial normalized bar resolution [24], induced by the resolvent pair $(\mathbf{TU},\varepsilon)$ in $\mathscr{D}_F(\eta)$. The corresponding Σ-syzygy functors are given by $\Phi^i(X,f)=(F^iX,f_i^{(X,f)})$, $\Phi^i(a)=F^ia$, and the morphisms $f_i^{(X,f)}$ satisfy the recursive relation: $f_i^{(X,f)}=\eta_{F^{i-1}X}-Ff_{i-1}^{(X,f)}, i\geq 1,\ f_0=f$.

If $(Y,g)\in\mathscr{D}_F(\eta)$, then define two complexes $\bar{\mathfrak{F}}^\bullet_{(X,f),(Y,g)}$ and $\bar{\Sigma}^\bullet_{(X,f),(Y,g)}$ in $\mathscr{A}b$, as follows:

$$\bar{\mathfrak{F}}^\bullet_{(X,f),(Y,g)}=\mathscr{D}_F(\eta)[\mathscr{P}^{(X,f)}_{\mathfrak{F}},(Y,g)],\quad \bar{\Sigma}^\bullet_{(X,f),(Y,g)}=\mathscr{D}_F(\eta)[\mathscr{P}^{(X,f)}_{\Sigma},(Y,g)].$$

Using the adjoint pair $(\mathbf{T},\mathbf{U})$, a direct calculation shows that: $\bar{\mathfrak{F}}^i_{(X,f),(Y,g)}=\mathscr{D}(P_i,Y)$ and the differential is given by $\mathscr{D}(P_{i-1},Y)\to\mathscr{D}(P_i,Y), m\mapsto \omega_{i-1}\circ m+a_i\circ v_{i-1}\circ Fm\circ g$. Similarly $\bar{\Sigma}^i_{(X,f),(Y,g)}=\mathscr{D}(F^iX,Y)$ with differential given by $\mathscr{D}(F^{i-1}X,Y)\to\mathscr{D}(F^iX,Y), m\mapsto -f_{i-1}^{(X,f)}\circ m+Fm\circ g$. Observe that if $g=0$, i.e. if $(Y,g)=\mathbf{Z}(Y)$, then: $\bar{\mathfrak{F}}^\bullet_{(X,f),\mathbf{Z}(Y)}=\mathscr{D}(\mathfrak{F}^\bullet_{(X,f)},Y)$ and $\bar{\Sigma}^\bullet_{(X,f),\mathbf{Z}(Y)}=\mathscr{D}(\Sigma^\bullet_{(X,f)},Y)$. Using the above resolutions and the formulas for $\bar{\mathfrak{F}}^\bullet_{(X,f),(Y,g)}$ and $\bar{\Sigma}^\bullet_{(X,f),(Y,g)}$, one can compute and compare $\mathfrak{F}$-, and Σ-relative derived functors in the standard way.

Convention 3.1. From now on we assume that $F(\mathfrak{R}_e)\subseteq\mathfrak{R}_e$, i.e. the functor F preserves $\mathfrak{R}$-proper epics. Also we assume that for any object $(X,f)\in\mathscr{D}_F(\eta)$, the morphism $f:FX\to X$ is $\mathfrak{R}$-proper. These assumptions are made in order to simplify the exposition and they are not essential for all parts of the paper. Also if there is no confusion we write $\phi_i=\phi_i^{(X,f)}$ and $\omega_i=\omega_i^{(X,f)}$.

Remark 3.2. (1) Consider the unit $\lambda:\mathrm{Id}_{\mathscr{D}_F(\eta)}\to\mathbf{ZC}$ of the adjoint pair $(\mathbf{C},\mathbf{Z})$. Since $\lambda_{(X,f)}=coker(f)$, it follows that λ is an $\mathfrak{R}$-proper epic, i.e. $\lambda_{(X,f)}$ is an $\mathfrak{R}$-proper epic, $\forall(X,f)\in\mathscr{D}_F(\eta)$. The morphism λ is always $\mathfrak{F}$-proper if $\mathfrak{F}=\mathfrak{F}_1$ and λ is always Σ-proper if $\mathscr{D}$ is semisimple.

(2) The functor $\mathbf{T}$ sends $\mathfrak{R}$-proper epics to $\mathfrak{F}$-proper epics, i.e. $\mathbf{T}(\mathfrak{R}_e)\subseteq\mathfrak{F}_e$. The functor $\mathbf{C}$ sends $\mathfrak{F}$-proper epics to $\mathfrak{R}$-proper epics, i.e. $\mathbf{C}(\mathfrak{F}_e)\subseteq\mathfrak{R}_e$.

(3) The $\mathfrak{F}$-associated complex $\mathfrak{F}^\bullet_{(X,f)}$ of (X,f) is an $\mathfrak{R}$-proper complex. This follows without difficulty from the construction of $\mathfrak{F}^\bullet_{(X,f)}$ and our conventions. Observe that from the pull-back diagram (i) above, we have $Coker(\omega_i)\cong Coker(a_{i+1}\circ\mu_i)\cong Coker(\mu_i)\cong Coker(Fa_i\circ g_{i-1})\cong Coker(g_{i-1})$ which by definition of the functor $\mathbf{C}$ is isomorphic to $\mathbf{C}(Q_{i-1},g_{i-1})$.

The next proposition is a kind of Universal Coefficient Theorem for $\mathscr{D}_F(\eta)$.

Proposition 3.3. *If* $\mathfrak{R}$-gl.dim $\mathscr{D}\leq 1$, *then* $\forall(X,f)\in\mathscr{D}_F(\eta)$, $\forall Y\in\mathscr{D}$:

(i) $\mathscr{E}xt^n_{\mathfrak{F}}[(X,f),\mathbf{Z}(Y)]\cong\mathscr{E}xt^1_{\mathfrak{R}}[\mathscr{L}^{\mathfrak{F}}_{n-1}\mathbf{C}(X,f),Y]\oplus\mathscr{D}(\mathscr{L}^{\mathfrak{F}}_n\mathbf{C}(X,f),Y)$, $\forall n\geq 1$.

(ii) *If* $\Theta:\mathscr{D}\to\mathscr{C}$ *is an additive functor to an abelian category* $\mathscr{C}$, *then*

$$\mathscr{L}^{\mathfrak{F}}_n\Theta\mathbf{C}(X,f)\cong\mathscr{L}^{\mathfrak{R}}_0\Theta(\mathscr{L}^{\mathfrak{F}}_n\mathbf{C}(X,f))\oplus\mathscr{L}^{\mathfrak{R}}_1\Theta(\mathscr{L}^{\mathfrak{F}}_{n-1}\mathbf{C}(X,f)),\quad\forall n\geq 1.$$

Proof. By Remark 3.2, the $\mathfrak{F}$-associated complex $\mathfrak{F}^{\bullet}_{(X,f)}$ of (X,f) is an $\mathfrak{R}$-proper complex consisting of $\mathfrak{R}$-projective objects. It is easy to see that the Universal Coefficient Theorems for homology and cohomology of complexes [35] are applied in our situation. Using these tools the proof is left to the reader. $\square$

3.3. Resolutions of primitive objects

We close this section studying the $\mathfrak{F}$-associated complex of an important class of objects of $\mathscr{D}_F(\eta)$. We call an object $(X,f)\in\mathscr{D}_F(\eta)$ *primitive* if (X,f) is a Σ-proper extension of a nonzero object of the form $\mathbf{Z}(Y)$, i.e. there exists a Σ-proper exact sequence $0\to(W,h)\to(X,f)\to\mathbf{Z}(Y)\to 0$. If (X,f) is an object of $\mathscr{D}_F(\eta)$ of the form

$$(X,f)=\left(Y\oplus W,\begin{pmatrix}0 & g\\ 0 & h\end{pmatrix}\right),$$

where $Y\neq 0$, then it is easy to see that we have $\mathbf{C}(X,f)=Y\oplus Coker({}^t(g,h))$, $g:\Phi\mathbf{Z}(Y)\to(W,h)$ is a morphism in $\mathscr{D}_F(\eta)$, and there is a push-out diagram:

$$\begin{tikzcd}
0 \arrow[r] & \Phi\mathbf{Z}(Y) \arrow[r, "{(0,1_{FY})}"] \arrow[d, "g"'] & \mathbf{T}(Y) \arrow[r, "{{}^t(1_Y,0)}"] \arrow[d, "g^!"'] & \mathbf{Z}(Y) \arrow[r] \arrow[d, equal] & 0\\
0 \arrow[r] & (W,h) \arrow[r, "{(0,1_W)}"] & (X,f) \arrow[r, "{{}^t(1_Y,0)}"] & \mathbf{Z}(Y) \arrow[r] & 0
\end{tikzcd}$$

where

$$g^!=\begin{pmatrix}1_Y & 0\\ 0 & g\end{pmatrix}.$$

Hence any object of the above form is primitive and conversely it is easy to see that any primitive object is of this form. Observe that if $\mathscr{D}$ is semisimple, then any object of $\mathscr{D}_F(\eta)$ is primitive, as follows from the short exact sequence $0\to Ker(\lambda_{(X,f)})\to(X,f)\xrightarrow{\lambda_{(X,f)}}\mathbf{ZC}(X,f)\to 0$.

Any primitive object (X,f) as above, defines two new objects (X_*,f_*) and (X^*,f^*) by the following exact sequence in $\mathscr{D}_F(\eta)$:

$$0\to(X_*,f_*)\xrightarrow{k_g}\Phi Z(Y)\xrightarrow{g}(W,h)\xrightarrow{\pi_g}(X^*,f^*)\to 0.$$

The most important primitive objects of $\mathscr{D}_F(\eta)$, are of course the objects of the form $\mathbf{Z}(X),\mathbf{T}(X)$. For $(X,f)=\mathbf{Z}(X)$, we have $(X_*,f_*)=\Phi\mathbf{Z}(X)$, $(X^*,f^*)=0$, and for $(X,f)=\mathbf{T}(X)$, we have $(X_*,f_*)=(X^*,f^*)=0$.

Theorem 3.4. Let

$$(X,f)=\left(Y\oplus W,\begin{pmatrix}0 & g\\ 0 & h\end{pmatrix}\right)$$

be a primitive object in $\mathscr{D}_F(\eta)$.

(i) *The* $\mathfrak{F}$*-associated complex of* (X,f), *is isomorphic to the direct sum of a deleted* $\mathfrak{R}$*-projective resolution of* Y *and of a complex*

$$P^{\bullet}_{(X,f)}: \cdots \to P_t \to P_{t-1} \to \cdots \to P_2 \to P_1 \to P_0 \to 0$$

consisting of $\mathfrak{R}$*-projective objects. Further* $\mathfrak{R}\text{-p.d}Y \le \mathfrak{F}\text{-p.d}(X,f)$.

(ii) *If* g *is monic and* $\mathscr{L}_i^{\mathfrak{R}}F(Y)=0,\ 1\le i\le t$, *then the* $\mathfrak{F}$*-associated complex of the object* (X^*,f^*) *is of the form*

$$\cdots \to P'_{t+2} \to P_{t+1} \to P_t \to \cdots \to P_2 \to P_1 \to P_0 \to 0.$$

(iii) *If* g *is epic and* $\mathscr{L}_i^{\mathfrak{R}}F(Y)=0, 1\le i\le t$, *then the* $\mathfrak{F}$*-associated complex of the object* (X_*,f_*) *is homotopy equivalent to a complex of the form*

$$\cdots \to P'_{t+2} \to P_{t+1} \to P_t \to \cdots \to P_2 \to P_1 \to 0.$$

(iv) *If* g *is monic and* $\mathscr{L}_i^{\mathfrak{R}}F(Y)=0,\ \forall i\ge 1$, *then* $P^{\bullet}_{(X,f)}$ *is the* $\mathfrak{F}$*-associated complex of the object* (X^*,f^*).

(v) *If* g *is epic and* $\mathscr{L}_i^{\mathfrak{R}}F(Y)=0,\ \forall i\ge 1$, *then the complex*

$$P^{\bullet}_{(X,f)}[1]: \cdots \to P_t \to P_{t-1} \to \cdots \to P_2 \to P_1 \to 0$$

is homotopy equivalent to the $\mathfrak{F}$*-associated complex of the object* (X_*,f_*).

Proof. (i) We will analyse the $\mathfrak{F}$-projective resolution $\mathscr{P}_{\mathfrak{F}}^{(X,f)}$ of (X,f) constructed before. We begin by considering an $\mathfrak{R}$-projective resolution $\mathscr{P}^{\bullet}_{\mathfrak{R}}(Y)$ of Y:

$$\cdots \to R_{n+1} \xrightarrow{a_{n+1}} R_n \to \cdots \to R_1 \xrightarrow{a_1} R_0 \xrightarrow{a_0} Y \to 0.$$

Let $a_{n+1}=l_{n+1}\circ k_n$ be the canonical factorization of a_{n+1}, and set $K_n=Im(a_{n+1})$, $\forall n\ge 0$. Let $b_0: P_0\to W$ be an $\mathfrak{R}$-proper epic with $P_0\in\mathscr{P}(\mathfrak{R})$. By the construction of $\mathscr{P}_{\mathfrak{F}}^{(X,f)}$, we have the following $\mathfrak{F}$-projective presentation of $(X,f): 0\to(Q_0,g_0)\xrightarrow{\kappa_0}\mathbf{T}(R_0)\oplus\mathbf{T}(P_0)\xrightarrow{\alpha_0}(X,f)\to 0$, where $Q_0=K_0\oplus L_0$,

$$g_0=\begin{pmatrix} 0 & x_0\\ 0 & y_0\end{pmatrix},\quad \kappa_0=\begin{pmatrix} k_0 & 0\\ 0 & s_0\end{pmatrix},\quad \alpha_0=\begin{pmatrix} a_0 & 0\\ 0 & p_0\end{pmatrix},$$

and the object L_0 is defined by the short exact sequence in $\mathscr{D}_F(\eta)$, $(1): 0\to(L_0,y_0)\xrightarrow{s_0}\Phi\mathbf{Z}(R_0)\oplus\mathbf{T}(P_0)\xrightarrow{p_0}(W,h)\to 0$, where $s_0=(c_0,d_0,e_0)$, and $p_0={}^t(Fa_0\circ g, b_0, Fb_0\circ h)$. From the above construction it follows that the first $\mathfrak{F}$-syzygy (Q_0,g_0) of (X,f) is a primitive object. Since $\mathfrak{F}$-syzygies of primitive objects are primitive, we can proceed inductively. In the ith-step of this construction we have the following $\mathfrak{F}$-projective presentation of $(X,f): 0\to(Q_{i-1},g_{i-1})\xrightarrow{\kappa_{i-1}}\mathbf{T}(R_{i-1})\oplus\mathbf{T}(P_{i-1})\xrightarrow{\alpha_{i-1}}(Q_{i-2},g_{i-2})\to 0$, where

$$Q_{i-1}=K_{i-1}\oplus L_{i-1},\quad g_{i-1}=\begin{pmatrix} 0 & x_{i-1}\\ 0 & y_{i-1}\end{pmatrix},\quad \kappa_{i-1}=\begin{pmatrix} k_{i-1} & 0\\ 0 & s_{i-1}\end{pmatrix},$$

$$\alpha_{i-1}=\begin{pmatrix} l_{i-1} & 0\\ 0 & p_{i-1}\end{pmatrix},$$

and the object L_{i-1} is defined by the short exact sequence in $\mathfrak{F}$: $0\to(L_{i-1},y_{i-1})\xrightarrow{s_{i-1}}\Phi\mathbf{Z}(R_{i-1})\oplus\mathbf{T}(P_{i-1})\xrightarrow{p_{i-1}}(L_{i-2},y_{i-2})\to 0$, where $s_{i-1}=(c_{i-1},d_{i-1},e_{i-1})$, and $p_{i-1} = {}^t(Fl_{i-1} \circ x_{i-2}, b_{i-1}, Fb_{i-1} \circ y_{i-2})$, and $b_{i-1} : P_{i-1} \to L_{i-2}$ is an $\mathfrak{R}$-proper epic with $P_{i-1} \in \mathscr{P}(\mathfrak{R})$. Composing the above $\mathfrak{F}$-projective presentations we get an $\mathfrak{F}$-projective resolution $\mathscr{P}_{\mathfrak{F}}^{(X,f)}$ of (X, f):

$$\cdots \to \mathbf{T}(R_{i+1}) \oplus \mathbf{T}(P_{i+1}) \xrightarrow{\phi_{i+1}} \mathbf{T}(R_i) \oplus \mathbf{T}(P_i) \xrightarrow{\phi_i} \mathbf{T}(R_{i-1}) \oplus \mathbf{T}(P_{i-1}) \to \cdots$$
$$\cdots \to \mathbf{T}(R_1) \oplus \mathbf{T}(P_1) \xrightarrow{\phi_1} \mathbf{T}(R_0) \oplus \mathbf{T}(P_0) \xrightarrow{\alpha_0} (X, f) \to 0.$$

We see easily that

$$\mathbf{C}(\phi_i) := \omega_i = \begin{pmatrix} a_i & 0 \\ 0 & b_i \circ d_{i-1} \end{pmatrix} : R_i \oplus P_i \to R_{i-1} \oplus P_{i-1}.$$

Hence the $\mathfrak{F}$-associated complex $\mathfrak{F}^\bullet_{(X,f)}$ of (X, f) is the following:

$$\cdots \to R_i \oplus P_i \xrightarrow{\omega_i} R_{i-1} \oplus P_{i-1} \to \cdots \to R_2 \oplus P_2 \xrightarrow{\omega_2} R_1 \oplus P_1 \xrightarrow{\omega_1} R_0 \oplus P_0 \to 0.$$

Thus $\mathfrak{F}^\bullet_{(X,f)}$ is the direct sum of the complex $\mathscr{P}^\bullet_{\mathfrak{R}}(Y) : \cdots \to R_n \xrightarrow{a_n} R_{n-1} \to \cdots \to R_1 \xrightarrow{a_1} R_0 \to 0$, and of the complex $P^\bullet_{(X,f)} : \cdots \to P_{i+1} \xrightarrow{\delta_{i+1}} P_i \to \cdots \to P_1 \xrightarrow{\delta_1} P_0 \to 0$, where $\delta_i = b_i \circ d_{i-1}$, $\forall i \geq 1$. Observe that $\mathscr{P}^\bullet_{\mathfrak{R}}(Y)$ is a deleted $\mathfrak{R}$-projective resolution of Y and $P^\bullet_{(X,f)}$ is a complex of $\mathfrak{R}$-projective objects. The above proof also shows that $\mathfrak{R}\text{-p.d}Y \leq \mathfrak{F}\text{-p.d}(X, f)$.

(ii) Suppose that g is monic and $(+) : \mathscr{L}_i^{\mathfrak{R}} F(Y) = 0, 1 \leq i \leq t$. Keep the notation of part (i) and consider the morphism $\zeta_0 := {}^t(b_0 \circ \pi_g, Fb_0 \circ h \circ \pi_g) : \mathbf{T}(P_0) \to (X^*, f^*)$ which is an $\mathfrak{F}$-proper epic. Let $\xi_0 := ker(\zeta_0) : (N_0, n_0) \to \mathbf{T}(P_0)$ be the kernel of ζ_0 in $\mathscr{D}_F(\eta)$; then ξ_0 is of the form (z_0, w_0), where $z_0 : N_0 \to P_0$ and $w_0 : N_0 \to FP_0$. Using condition $(+)$ and the $\mathfrak{F}$-exact sequence $0 \to (N_0, n_0) \xrightarrow{\xi_0} \mathbf{T}(P_0) \xrightarrow{\zeta_0} (X^*, f^*) \to 0$, we have the following $\mathfrak{R}$-exact commutative diagram, except the lower left square:

$$\begin{tikzcd}
0 \arrow[r] & \Phi\mathbf{Z}(K_0) \arrow[r, "Fk_0"] \arrow[d, "x_0"'] & \Phi\mathbf{Z}(R_0) \arrow[r, "Fa_0"] \arrow[d, "t_0"'] & \Phi\mathbf{Z}(Y) \arrow[r] \arrow[d, "g"'] & 0 \\
0 \arrow[r] & (L_0, y_0) \arrow[r, "s_0"] \arrow[d, "\exists! \rho_0"'] & \Phi\mathbf{Z}(R_0) \oplus \mathbf{T}(P_0) \arrow[r, "p_0"] \arrow[d, "\tau_0"'] & (W, h) \arrow[r] \arrow[d, "\pi_g"'] & 0 \\
0 \arrow[r] & (N_0, n_0) \arrow[r, "\xi_0"] & \mathbf{T}(P_0) \arrow[r, "\zeta_0"] & (X^*, f^*) \arrow[r] & 0
\end{tikzcd}$$

where $t_0 = (1_{\Phi\mathbf{Z}(R_0)}, 0)$, $\tau_0 = {}^t(0, 1_{\mathbf{T}(P_0)})$. Since the lower right square of the above diagram commutes, $\exists! \rho_0 : (L_0, y_0) \to (N_0, n_0)$, such that the above diagram is commutative and has $\mathfrak{F}$-exact rows and columns, and then $\rho_0 = coker(x_0)$. Observe that x_0 is monic $\Leftrightarrow$ $\mathscr{L}_1^{\mathfrak{R}} F(Y) = 0$. Now we proceed inductively, noting that this inductive procedure continues for $t \geq 0$, as long as the morphisms x_i are monics for $0 \leq i \leq t-1$ or equivalently $\mathscr{L}_i^{\mathfrak{R}} F(Y) = 0$, $1 \leq i \leq t$. Then we get for $1 \leq i \leq t$, an $\mathfrak{F}$-exact sequence $0 \to (N_i, n_i) \xrightarrow{\xi_i} \mathbf{T}(P_i) \xrightarrow{\zeta_i} (N_{i-1}, n_{i-1}) \to 0$, where $\xi_i = (z_i, w_i) = ker(\zeta_i)$, and there exists a unique morphism $\rho_i : (L_i, y_i) \to (N_i, n_i)$, such that $(*) : \rho_i = coker(x_i)$

and $\rho_i \circ z_i = d_i, \rho_i \circ w_i = e_i$. Observing also that there exists an $\mathfrak{F}$-epic $\mathbf{T}(P_{t+1}) \to (N_t, n_t)$, and forming the Yoneda composition of the above $\mathfrak{F}$-exact sequences, we get an $\mathfrak{F}$-projective resolution $\mathscr{P}_{\mathfrak{F}}^{(X^*, f^*)}$ of (X^*, f^*):

$$\cdots \to \mathbf{T}(P'_{t+2}) \to \mathbf{T}(P_{t+1}) \xrightarrow{\tilde{\phi}_{t+1}} \mathbf{T}(P_t) \to \cdots \to \mathbf{T}(P_0) \xrightarrow{\zeta_0} (X^*, f^*) \to 0.$$

Using relations $(*)$, we compute easily that

$$\tilde{\phi}_i = \begin{pmatrix} b_i \circ d_{i-1} & b_i \circ e_{i-1} \\ 0 & Fb_i \circ y_{i-1} \circ e_{i-1} \end{pmatrix}, \quad \mathbf{C}(\tilde{\phi}_i) = b_i \circ d_{i-1} : P_i \to P_{i-1}.$$

Hence the $\mathfrak{F}$-associated complex of (X^*, f^*), has the desired form.

(iii) Suppose that g is epic and $(+) : \mathscr{L}_i^{\mathfrak{R}} F(Y) = 0,\ 1 \le i \le t$. Then using $(+)$, we have the following exact commutative diagram:

$$\begin{tikzcd} 0 \arrow[r] & \Phi\mathbf{Z}(K_0) \arrow[r, "Fk_0"] \arrow[d, "x_0"] & \Phi\mathbf{Z}(R_0) \arrow[r, "Fa_0"] \arrow[d, "t_0"] & \Phi\mathbf{Z}(Y) \arrow[r] \arrow[d, "g"] & 0 \\ 0 \arrow[r] & (L_0, y_0) \arrow[r, "s_0"] & \Phi\mathbf{Z}(R_0) \oplus \mathbf{T}(P_0) \arrow[r, "p_0"] & (W, h) \arrow[r] & 0 \end{tikzcd}.$$

From the above diagram, using the snake lemma, we get a short exact sequence: $0 \to (X_*, f_*) \to (N_0, n_0) \to \mathbf{T}(P_0) \to 0$ in $\mathfrak{F}$ which of course splits. Using the same arguments as in (ii), we get an $\mathfrak{F}$-projective resolution $\mathscr{P}_{\mathfrak{F}}^{(N_0, n_0)}$ of (N_0, n_0):

$$\cdots \to \mathbf{T}(P'_{t+2}) \to \mathbf{T}(P_{t+1}) \xrightarrow{\tilde{\phi}_{t+1}} \mathbf{T}(P_t) \to \cdots \to \mathbf{T}(P_1) \xrightarrow{\zeta_1} (N_0, n_0) \to 0.$$

Since $(N_0, n_0) \cong (X_*, f_*) \oplus \mathbf{T}(P_0)$, the above resolution is homotopy equivalent to the $\mathfrak{F}$-projective resolution of (X_*, f_*). Hence the $\mathfrak{F}$-associated complex of (X_*, f_*) is homotopy equivalent to the complex of the desired form.

Conditions (iv) and (v) are direct consequences of (ii) and (iii). $\square$

4. $\mathfrak{F}$-derived functors and Σ-derived functors

4.1. Homology of primitive objects

We are interested in the homology of primitive objects of the form $\mathbf{T}(X), \mathbf{Z}(X)$.

Theorem 4.1. *Suppose that $\Theta : \mathscr{D} \to \mathscr{E},\ \Xi : \mathscr{D}^{op} \to \mathscr{E}$, are additive functors to an abelian category $\mathscr{E}$. Then we have the following:*

(α) *The following are true $\forall i \ge 0$:*

(i) *$\mathscr{L}_i^{\mathfrak{R}} \Theta$ is a direct summand of the functors $\mathscr{L}_i^{\mathfrak{F}}(\Theta\mathbf{C})\mathbf{Z}$, $\mathscr{L}_i^{\mathfrak{F}}(\Theta\mathbf{C})\mathbf{T}$.*

(ii) *$\mathscr{R}_{\mathfrak{R}}^i \Xi$ is a direct summand of the functors $\mathscr{R}_{\mathfrak{F}}^i(\Xi\mathbf{C})\mathbf{Z}$, $\mathscr{R}_{\mathfrak{F}}^i(\Xi\mathbf{C})\mathbf{T}$.*

(β) *There are natural isomorphisms:*

$$\mathscr{L}_1^{\mathfrak{F}}(\Theta\mathbf{C})\mathbf{Z} \cong \mathscr{L}_1^{\mathfrak{R}} \Theta \oplus \mathscr{L}_0^{\mathfrak{R}} \Theta(coker(\eta)),$$
$$\mathscr{R}_{\mathfrak{F}}^1(\Xi\mathbf{C})\mathbf{Z} \cong \mathscr{R}_{\mathfrak{R}}^1 \Xi \oplus \mathscr{R}_{\mathfrak{R}}^0 \Xi(coker(\eta)).$$

(γ) *There are natural isomorphisms*:

$$\mathscr{L}_1^{\mathfrak{F}}(\Theta\mathbf{C})\mathbf{T} \cong \mathscr{L}_1^{\mathfrak{R}}\,\Theta, \quad \mathscr{R}^1_{\mathfrak{F}}(\Xi\mathbf{C})\mathbf{T} \cong \mathscr{R}^1_{\mathfrak{R}}\,\Xi,$$

and exact sequences, $\forall X \in \mathscr{D}$:

$$\mathscr{L}_2^{\mathfrak{R}}\,\Theta(X) \to \mathscr{L}_2^{\mathfrak{F}}(\Theta\mathbf{C})(\mathbf{T}(X)) \to \mathscr{L}_0^{\mathfrak{F}}(\Theta\mathbf{C})(\mathscr{L}_1^{\mathfrak{R}}\,\mathbf{T}(X)) \to 0,$$

$$0 \to \mathscr{R}^0_{\mathfrak{F}}(\Xi\mathbf{C})(\mathscr{L}_1^{\mathfrak{R}}\,\mathbf{T}(X)) \to \mathscr{R}^2_{\mathfrak{F}}(\Xi\mathbf{C})(\mathbf{T}(X)) \to \mathscr{R}^2_{\mathfrak{R}}\,\Xi(X).$$

(δ) *Suppose that* $\mathscr{L}_i^{\mathfrak{R}}F(X) = 0,\ 1 \leq i \leq t$. *Then* $\forall i \in \mathbb{N}$, *with* $1 \leq i \leq t+1$:

$$\mathscr{L}_i^{\mathfrak{F}}(\Theta\mathbf{C})\mathbf{Z}(X) \cong \mathscr{L}_i^{\mathfrak{R}}\,\Theta(X) \oplus \mathscr{L}_{i-1}^{\mathfrak{F}}(\Theta\mathbf{C})\Phi\mathbf{Z}(X),$$

$$\mathscr{R}^i_{\mathfrak{F}}(\Xi\mathbf{C})\mathbf{Z}(X) \cong \mathscr{R}^i_{\mathfrak{R}}\,\Xi(X) \oplus \mathscr{R}^{i-1}_{\mathfrak{F}}(\Xi\mathbf{C})\Phi\mathbf{Z}(X),$$

$$\mathscr{L}_i^{\mathfrak{F}}(\Theta\mathbf{C})\mathbf{T}(X) \cong \mathscr{L}_i^{\mathfrak{R}}\,\Theta(X), \quad \mathscr{R}^i_{\mathfrak{F}}(\Xi\mathbf{C})\mathbf{T}(X) \cong \mathscr{R}^i_{\mathfrak{R}}\,\Xi(X).$$

(ε) *If the functor F is $\mathfrak{R}$-exact, there are natural isomorphisms*, $\forall i \geq 1$:

$$\mathscr{L}_i^{\mathfrak{F}}(\Theta\mathbf{C})\mathbf{Z} \cong \mathscr{L}_i^{\mathfrak{R}}\,\Theta \oplus \mathscr{L}_{i-1}^{\mathfrak{F}}(\Theta\mathbf{C})\Phi\mathbf{Z},$$

$$\mathscr{R}^i_{\mathfrak{F}}(\Xi\mathbf{C})\mathbf{Z} \cong \mathscr{R}^i_{\mathfrak{R}}\,\Xi \oplus \mathscr{R}^{i-1}_{\mathfrak{F}}(\Xi\mathbf{C})\Phi\mathbf{Z},$$

$$\mathscr{L}_i^{\mathfrak{F}}(\Theta\mathbf{C})\mathbf{T} \cong \mathscr{L}_i^{\mathfrak{R}}\,\Theta, \quad \mathscr{R}^i_{\mathfrak{F}}(\Xi\mathbf{C})\mathbf{T} \cong \mathscr{R}^i_{\mathfrak{R}}\,\Xi.$$

Proof. Since $\forall X \in \mathscr{D}$, $\mathbf{Z}(X)$ and $\mathbf{T}(X)$ are primitive objects we can apply Theorem 3.4. We consider only the assertions involving the functor Θ since the other parts are similar. Now part (α) follows easily from Theorem 3.4 by standard arguments.

(β) Using Theorem 3.4 it is not difficult to see that $\mathscr{L}_1^{\mathfrak{F}}(\Theta\mathbf{C})\mathbf{Z}(X) \cong \mathscr{L}_1^{\mathfrak{R}}\,\Theta(X) \oplus \mathrm{H}_1[\Theta P^{\bullet}_{\mathbf{Z}(X)}]$. Using the notation of (the proof of) Theorem 3.4, we have $P_0 = 0$, and $\mathrm{H}_1[P^{\bullet}_{\mathbf{Z}(X)}] = Coker(\delta_1) = Coker(b_2 \circ d_1) \cong Coker(d_1)$, since b_2 is epic. A simple calculation shows that $Coker(d_1) \cong Coker(\eta_X)$, via the epimorphism $b_1 \circ Fa_0 \circ coker(\eta_X): P_1 \to Coker(\eta_X)$. This shows that $\mathrm{H}_1[\Theta P^{\bullet}_{\mathbf{Z}(X)}] \cong \mathscr{L}_0^{\mathfrak{R}}\,\Theta(Coker(\eta_X))$.

(γ) Let (1) $0 \to K_0 \xrightarrow{k_0} R_0 \xrightarrow{a_0} X \to 0$ be an $\mathfrak{R}$-projective presentation of X in $\mathscr{D}$. Then we have the $\mathfrak{F}$-projective presentation of $\mathbf{T}(X)$ in $\mathscr{D}_F(\eta)$: $0 \to (Y,g) \xrightarrow{\kappa_0} \mathbf{T}(R_0) \xrightarrow{\mathbf{T}(a_0)} \mathbf{T}(X) \to 0$. The object (Y,g) is primitive of the form

$$(Y,g) = \left(K_0 \oplus W, \begin{pmatrix} 0 & x \\ 0 & y \end{pmatrix}\right),$$

where $Fk_0 = x \circ \iota$ is the canonical factorization of Fk_0 and $\iota: W \hookrightarrow FR_0$ is the kernel of Fa_0. Then $\mathscr{L}_1^{\mathfrak{F}}(\Theta\mathbf{C})\mathbf{T}(X) \cong Ker(\mathscr{L}_0^{\mathfrak{F}}(\Theta\mathbf{C})(\kappa_0))$. But $\mathscr{L}_0^{\mathfrak{F}}(\Theta\mathbf{C})(Y,g) \cong \mathscr{L}_0^{\mathfrak{R}}\,\Theta(K_0), \mathscr{L}_0^{\mathfrak{F}}(\Theta\mathbf{C})(\kappa_0) = \mathscr{L}_0^{\mathfrak{R}}\,\Theta(k_0)$ and $\mathscr{L}_0^{\mathfrak{F}}(\Theta\mathbf{C})(\mathbf{T}(R_0)) \cong \mathscr{L}_0^{\mathfrak{R}}\,\Theta(R_0)$. Hence $\mathscr{L}_1^{\mathfrak{F}}(\Theta\mathbf{C})\mathbf{T}(X) \cong \mathscr{L}_1^{\mathfrak{R}}\,\Theta(X)$. Now from (1) we have a short exact sequence (2): $0 \to \mathscr{L}_1^{\mathfrak{R}}\,\mathbf{T}(X) \xrightarrow{\mu} \mathbf{T}(K_0) \xrightarrow{\nu} (Y,g) \to 0$. Applying the functor $\mathscr{L}_0^{\mathfrak{F}}(\Theta\mathbf{C})$ to (2), we have the exact sequence

$$\cdots \to \mathscr{L}_1^{\mathfrak{F}}(\Theta\mathbf{C})(\mathbf{T}(K_0)) \to \mathscr{L}_1^{\mathfrak{F}}(\Theta\mathbf{C})(Y,g) \to \mathscr{L}_0^{\mathfrak{F}}(\Theta\mathbf{C})(\mathscr{L}_1^{\mathfrak{R}}\,\mathbf{T}(X))$$

$$\to \mathscr{L}_0^{\mathfrak{F}}(\Theta\mathbf{C})(\mathbf{T}K_0) \xrightarrow{\mathscr{L}_0^{\mathfrak{F}}(\Theta\mathbf{C})(\nu)} \mathscr{L}_0^{\mathfrak{F}}(\Theta\mathbf{C})(Y,g)) \to 0.$$

Since $\mathscr{L}_0^{\mathfrak{F}}(\Theta\mathbf{C})(v)$ is (isomorphic to) the identity of $\mathscr{L}_0^{\mathfrak{R}}\Theta(K_0), \mathscr{L}_0^{\mathfrak{F}}(\Theta\mathbf{C})(\mu)=0$. Now the result follows from the isomorphisms $\mathscr{L}_1^{\mathfrak{F}}(\Theta\mathbf{C})(\mathbf{T}(K_0)) \cong \mathscr{L}_1^{\mathfrak{R}}\Theta(K_0) \cong \mathscr{L}_2^{\mathfrak{R}}\Theta(X)$ and $\mathscr{L}_1^{\mathfrak{F}}(\Theta\mathbf{C})(Y,g) \cong \mathscr{L}_2^{\mathfrak{F}}(\Theta\mathbf{C})(\mathbf{T}(X))$.

(δ)–(ε) If $(X,f) = \mathbf{Z}(X)$, then $(X_*, f_*) = \Phi\mathbf{Z}(X)$, and if $(X,f) = \mathbf{T}(X)$, then $(X^*, f^*) = 0$. Setting $(X,f) = \mathbf{Z}(X)$ and $(X,f) = \mathbf{T}(X)$ in Theorem 3.4, and using induction we get the result, noting that the case $i = 1$ is always true by part (β) and the first two isomorphisms in (γ). □

The next corollary of the previous theorem, contains a lot of information.

Corollary 4.2. (1) *The functors* $\mathscr{E}xt^i_{\mathfrak{F}}[\mathbf{Z}(-),\mathbf{Z}(-)]$, $\mathscr{E}xt^i_{\mathfrak{F}}[\mathbf{T}(-),\mathbf{Z}(-)]$ *contain as a direct summand the functor* $\mathscr{E}xt^i_{\mathfrak{R}}[-,-]$, $\forall i \geq 0$.

(2) *There are natural isomorphisms*:

$$\mathbf{U}\mathscr{L}_i^{\mathfrak{R}}\mathbf{T} \cong \mathscr{L}_i^{\mathfrak{R}}F, \quad \forall i \geq 1,$$

$$\mathscr{E}xt^1_{\mathfrak{F}}[\mathbf{T}(-),\mathbf{Z}(-)] \cong \mathscr{E}xt^1_{\mathfrak{R}}[-,-],$$

$$\mathscr{E}xt^2_{\mathfrak{F}}[\mathbf{T}(-),\mathbf{Z}(-)] \cong \mathscr{E}xt^2_{\mathfrak{R}}[-,-] \oplus \mathscr{D}[\mathbf{C}\mathscr{L}_1^{\mathfrak{R}}\mathbf{T}-,-],$$

$$\mathscr{E}xt^1_{\mathfrak{F}}[\mathbf{Z}(-),\mathbf{Z}(-)] \cong \mathscr{E}xt^1_{\mathfrak{R}}[-,-] \oplus \mathscr{D}[Coker(\eta_-),-],$$

$$\mathscr{L}_1^{\mathfrak{F}}\mathbf{CZ} = Coker(\eta), \quad \mathscr{L}_1^{\mathfrak{F}}\mathbf{CT} = 0, \quad \mathscr{L}_1^{\mathfrak{F}}\mathbf{C} \cong \mathscr{L}_1^{\Sigma}\mathbf{C}, \quad \mathscr{L}_2^{\mathfrak{F}}\mathbf{CT} \cong \mathbf{C}\mathscr{L}_1^{\mathfrak{R}}\mathbf{T}.$$

(3) *If* $\mathscr{L}_i^{\mathfrak{R}}F(X) = 0$, $1 \leq i \leq n$, *then* $\forall (Y,g) \in \mathscr{D}_F(\eta)$:

$$\mathscr{E}xt^i_{\mathfrak{F}}[\mathbf{T}(X),(Y,g)] \cong \mathscr{E}xt^i_{\mathfrak{R}}[X,Y], \quad 1 \leq i \leq n$$

and for all $i \in \mathbb{N}$, *with* $1 \leq i \leq n+1$:

$$\mathscr{L}_i^{\mathfrak{F}}\mathbf{CT}(X) = 0, \quad \mathscr{E}xt^i_{\mathfrak{F}}[\mathbf{Z}(X),\mathbf{Z}(Y)] \cong \mathscr{E}xt^i_{\mathfrak{R}}[X,Y] \oplus \mathscr{E}xt^{i-1}_{\mathfrak{F}}[\Phi\mathbf{Z}(X),\mathbf{Z}(Y)].$$

(4) *If* $\mathscr{L}_i^{\mathfrak{R}}F(X) = 0$, $\forall i \geq 1$, *then*

(i) $\mathfrak{F}$-p.d $\mathbf{T}(X) = \mathfrak{R}$-p.d X *and* $\forall (Y,g) \in \mathscr{D}_F(\eta)$:

$$\mathscr{E}xt^n_{\mathfrak{F}}[\mathbf{T}(X),(Y,g)] \cong \mathscr{E}xt^n_{\mathfrak{R}}[X,Y].$$

(ii) *There are isomorphisms* $\forall i \geq 1$:

$$\mathscr{L}_i^{\mathfrak{F}}\mathbf{CT}(X) = 0, \quad \mathscr{E}xt^i_{\mathfrak{F}}[\mathbf{Z}(X),\mathbf{Z}(-)] \cong \mathscr{E}xt^i_{\mathfrak{R}}[X,-] \oplus \mathscr{E}xt^{i-1}_{\mathfrak{F}}[\Phi\mathbf{Z}(X),\mathbf{Z}(-)].$$

(5) *If the functor* F *is* $\mathfrak{R}$*-exact, there are natural isomorphisms*, $\forall i \geq 1$:

$$\mathscr{L}_i^{\mathfrak{F}}\mathbf{CT} = 0, \quad \mathscr{E}xt^i_{\mathfrak{F}}[\mathbf{T}(-),-] \cong \mathscr{E}xt^i_{\mathfrak{R}}[-,\mathbf{U}(-)],$$

$$\mathscr{E}xt^i_{\mathfrak{F}}[\mathbf{Z}(-),\mathbf{Z}(-)] \cong \mathscr{E}xt^i_{\mathfrak{R}}[-,-] \oplus \mathscr{E}xt^{i-1}_{\mathfrak{F}}[\Phi\mathbf{Z}(-),\mathbf{Z}(-)].$$

Proof. (1) In Theorem 4.1, choose $\forall Y \in \mathscr{D}$: $\Xi_Y = \mathscr{D}[-,Y]$, and use that $\Xi_Y\mathbf{C} = \mathscr{D}[-,Y]\mathbf{C} = \mathscr{D}[\mathbf{C}-,Y] \cong \mathscr{D}_F(\eta)[-,\mathbf{Z}(Y)]$.

(2) For the first isomorphism, let $0 \to K \xrightarrow{k} P \xrightarrow{a} X \to 0$ be an $\mathfrak{R}$-projective presentation of $X \in \mathscr{D}$. Then $\mathscr{L}_1^{\mathfrak{R}} F(X) = Ker(Fk)$, $\mathscr{L}_1^{\mathfrak{R}} \mathbf{T}(X) = Ker(\mathbf{T}(a))$, and $\mathbf{U}\mathscr{L}_1^{\mathfrak{R}} \mathbf{T}(X) = \mathbf{U}(Ker(\mathbf{T}(a))) \cong Ker(\mathbf{U}(\mathbf{T}(a))) \cong Ker(Fk) = \mathscr{L}_1^{\mathfrak{R}} F(X)$. Hence: $\mathbf{U}\mathscr{L}_1^{\mathfrak{R}} \mathbf{T}(X) \cong \mathscr{L}_1^{\mathfrak{R}} F(X)$, $\forall X \in \mathscr{D}$. This implies trivially that $\mathbf{U}\mathscr{L}_i^{\mathfrak{R}} \mathbf{T} \cong \mathscr{L}_i^{\mathfrak{R}} F$, $\forall i \geq 1$. Consider now the functorial Σ-projective presentation of (X, f) : $(*) 0 \to \Phi(X, f) \to \mathbf{T}(X) \to (X, f) \to 0$.

By definition $\mathscr{L}_1^{\Sigma} \mathbf{C}(X, f) = Ker[\mathbf{C}(\Phi(X, f) \to X)]$. Applying $\mathbf{C}$ to $(*)$ and using that $\mathscr{L}_1^{\mathfrak{F}} \mathbf{CT} = 0$, we get $\mathscr{L}_1^{\mathfrak{F}} \mathbf{C}(X, f) \cong Ker[\mathbf{C}(\Phi(X, f) \to X)] = \mathscr{L}_1^{\Sigma} \mathbf{C}(X, f)$, hence $\mathscr{L}_1^{\mathfrak{F}} \mathbf{C} \cong \mathscr{L}_1^{\Sigma} \mathbf{C}$. The remaining isomorphisms are derived from Theorem 4.1, choosing $\forall Y \in \mathscr{D}$: $\Xi_Y = \mathscr{D}[-, Y]$ and $\Theta = \mathrm{Id}_{\mathscr{D}}$.

(3) The last two isomorphisms follow from Theorem 4.1. For the first, choose an $\mathfrak{R}$-projective resolution $\cdots \to P_i \to P_{i-1} \to \cdots \to P_0 \to X \to 0$ of X. Then since $\mathscr{L}_i^{\mathfrak{R}} F(X) = 0$, $1 \leq i \leq n$, by the first isomorphism in (ii) we have $\mathscr{L}_i^{\mathfrak{R}} \mathbf{T}(X) = 0, 1 \leq i \leq n$, because $\mathbf{U}$ is faithful. Hence $\cdots \to \mathbf{T}(P_{n+1}) \to \mathbf{T}(P_n) \to \cdots \to \mathbf{T}(P_0) \to \mathbf{T}(X) \to 0$ is part of an $\mathfrak{F}$-projective resolution of $\mathbf{T}(X)$. Then $\forall (Y, g) \in \mathscr{D}_F(\eta)$, applying the functor $\mathscr{D}_F(\eta)[-, (Y, g)]$ to the above resolution and using the adjoint pair $(\mathbf{T}, \mathbf{U})$, we see that the resulting complex is of the form $0 \to \mathscr{D}[X, Y] \to \mathscr{D}[P_0, Y] \to \cdots \to \mathscr{D}[P_n, Y] \to \mathscr{D}[P_{n+1}, Y] \to \cdots$, and the second isomorphism follows. Parts (4) and (5) are consequences of (3). $\square$

From the above corollary it follows that $\forall X, Y \in \mathscr{D}$ we have an isomorphism $\mathscr{E}xt_{\mathfrak{F}}^1[\mathbf{Z}(X), \mathbf{Z}(Y)] \cong \mathscr{E}xt_{\mathfrak{R}}^1[X, Y] \oplus \mathscr{E}xt_{\Sigma}^1[\mathbf{Z}(X), \mathbf{Z}(Y)]$, since trivially there is an isomorphism: $\mathscr{E}xt_{\Sigma}^1[\mathbf{Z}(X), \mathbf{Z}(Y)] \cong \mathscr{D}[Coker(\eta_X), Y]$. So it is natural to ask, if the above formula extends to higher dimensions:

Corollary 4.3. *Suppose that* $\mathscr{L}_i^{\mathfrak{R}} F(X) = 0$, $1 \leq i \leq n-1$, *and let* $X, Y \in \mathscr{D}$ *satisfy* $\mathscr{E}xt_{\mathfrak{R}}^p[F^q X, Y] = 0$, $\forall p + q = n$, $0 < p < n$. *Then*

$$\mathscr{E}xt_{\mathfrak{F}}^i[\mathbf{Z}(X), \mathbf{Z}(Y)] \cong \mathscr{E}xt_{\mathfrak{R}}^i[X, Y] \oplus \mathscr{E}xt_{\Sigma}^i[\mathbf{Z}(X), \mathbf{Z}(Y)], \quad 1 \leq i \leq n.$$

Proof. Consider the natural Σ-projective presentations:

$$0 \to \Phi^{i+1}\mathbf{Z}(X) \to \mathbf{T}(F^i X) \to \Phi^i \mathbf{Z}(X) \to 0, \quad 0 < i < n.$$

Then $\forall Y \in \mathscr{D}$, applying to these presentations the functor $\mathscr{D}_F(\eta)[-, \mathbf{Z}(Y)]$, the induced long exact sequences and the hypothesis produce isomorphisms:

$$\mathscr{E}xt_{\mathfrak{F}}^1[\Phi^i \mathbf{Z}(X), \mathbf{Z}(Y)] \cong \mathscr{E}xt_{\Sigma}^1[\Phi^i \mathbf{Z}(X), \mathbf{Z}(Y)] \cong \mathscr{E}xt_{\Sigma}^{i+1}[\mathbf{Z}(X), \mathbf{Z}(Y)],$$

$$\mathscr{E}xt_{\mathfrak{F}}^1[\Phi^i \mathbf{Z}(X), \mathbf{Z}(Y)] \cong \mathscr{E}xt_{\mathfrak{F}}^i[\Phi \mathbf{Z}(X), \mathbf{Z}(Y)], \quad 0 < i < n.$$

The assertion now follows from the third formula of Corollary 4.2(3). $\square$

We continue with other consequences of Corollary 4.2.

Corollary 4.4. *If $\eta = 0$, the following are true for the cleft extension $\mathscr{D}_F(0)$:*

$$\mathscr{L}_i^{\mathfrak{R}}\mathbf{T} \cong \mathbf{Z}\mathscr{L}_i^{\mathfrak{R}}F, \quad \forall i \geq 1,$$

$$\mathscr{E}xt^1_{\mathfrak{F}}[\mathbf{Z}(-),\mathbf{Z}(-)] \cong \mathscr{E}xt^1_{\mathfrak{R}}[-,-] \oplus \mathscr{D}[F-,-],$$

$$\mathscr{E}xt^2_{\mathfrak{F}}[\mathbf{T}(-),\mathbf{Z}(-)] \cong \mathscr{E}xt^2_{\mathfrak{R}}[-,-] \oplus \mathscr{D}[\mathscr{L}_1^{\mathfrak{R}}F-,-],$$

$$\mathscr{L}_1^{\mathfrak{F}}\mathbf{CZ} \cong F, \quad \mathscr{L}_2^{\mathfrak{F}}\mathbf{CZ} \cong F^2 \oplus \mathscr{L}_1^{\mathfrak{R}}F, \quad \mathscr{L}_2^{\mathfrak{F}}\mathbf{CT} \cong \mathscr{L}_1^{\mathfrak{R}}F.$$

If in addition the functor F is $\mathfrak{R}$-exact, then

$$\mathscr{E}xt^i_{\mathfrak{F}}[\mathbf{Z}(-),\mathbf{Z}(-)] \cong \bigoplus_{t=0}^{i} \mathscr{E}xt^t_{\mathfrak{R}}[F^{i-t}-,-], \quad \forall i \geq 0.$$

Proof. Since $\eta=0$ implies $\Phi\mathbf{Z}=\mathbf{Z}F$, all assertions are consequences of Corollary 4.2, except for the formula: $\mathscr{L}_2^{\mathfrak{F}}\mathbf{CZ} \cong F^2 \oplus \mathscr{L}_1^{\mathfrak{R}}F$. But $\mathscr{L}_2^{\mathfrak{F}}\mathbf{CZ}(X) \cong \mathscr{L}_1^{\mathfrak{F}}\mathbf{C}\Omega_{\mathfrak{F}}\mathbf{Z}(X) \cong \mathscr{L}_1^{\Sigma}\mathbf{C}\Omega_{\mathfrak{F}}\mathbf{Z}(X)$. The Σ-associated complex of $\Omega_{\mathfrak{F}}\mathbf{Z}(X)$ is of the form

$$\cdots \to F^2K \oplus F^3P \xrightarrow{z} FK \oplus F^2P \xrightarrow{w} K \oplus FP \to 0,$$

where

$$z = \begin{pmatrix} 0 & F^2k \\ 0 & 0 \end{pmatrix}, \quad w = \begin{pmatrix} 0 & -Fk \\ 0 & 0 \end{pmatrix},$$

and $0 \to K \xrightarrow{k} P \xrightarrow{a} X \to 0$ is an $\mathfrak{R}$-projective presentation of X. A direct calculation of the first homology of this complex shows that $\mathscr{L}_2^{\mathfrak{F}}\mathbf{CZ}(X) \cong \mathscr{L}_1^{\Sigma}\mathbf{C}\Omega_{\mathfrak{F}}\mathbf{Z}(X) \cong F^2X \oplus \mathscr{L}_1^{\mathfrak{R}}F(X)$. $\Box$

Corollary 4.5. (i) $\mathfrak{R}$-p.d $X \leq \mathit{min}\{\mathfrak{F}\text{-p.d}\,\mathbf{Z}(X), \mathfrak{F}\text{-p.d}\,\mathbf{T}(X)\}$.
(ii) $\mathfrak{R}$-gl.dim $\mathscr{D} \leq \mathfrak{F}$-gl.dim $\mathscr{D}_F(\eta)$.

Corollary for Rings 4.6. *Let $\Lambda = \Gamma \ltimes_{\vartheta} M$ be a cleft extension of rings.*
(i) *The functor $\mathscr{T}or_n^{\Gamma}[-,-]$ is a direct summand of the functors $\mathscr{T}or_n^{\Lambda}[\mathbf{Z}-,\mathbf{Z}-]$, $\mathscr{T}or_n^{\Lambda}[\mathbf{T}-,\mathbf{Z}-]$, the functor $\mathscr{E}xt^n_{\Gamma}[-,-]$ is a direct summand of the functors $\mathscr{E}xt^n_{\Lambda}[\mathbf{Z}-,\mathbf{Z}-]$, $\mathscr{E}xt^n_{\Lambda}[\mathbf{T}-,\mathbf{Z}-]$, $\forall n \geq 0$, and there are isomorphisms*:

$$\mathscr{T}or_1^{\Lambda}[\mathbf{T}-,\mathbf{Z}-] \cong \mathscr{T}or_1^{\Gamma}[-,-], \quad \mathscr{E}xt^1_{\Lambda}[\mathbf{T}-,\mathbf{Z}-] \cong \mathscr{E}xt^1_{\Gamma}[-,-],$$

$$\mathscr{T}or_1^{\Lambda}[\mathbf{Z}-,\mathbf{Z}-] \cong \mathscr{T}or_1^{\Gamma}[-,-] \oplus - \otimes_{\Gamma} M/M^2 \otimes_{\Gamma} -,$$

$$\mathscr{E}xt^1_{\Lambda}[\mathbf{Z}-,\mathbf{Z}-] \cong \mathscr{E}xt^1_{\Gamma}[-,-] \oplus \mathrm{Hom}_{\Gamma}[-\otimes_{\Gamma} M/M^2,-],$$

$$\mathscr{T}or_2^{\Lambda}[\mathbf{T}-,\mathbf{Z}-] \cong \mathscr{T}or_2^{\Gamma}[-,-] \oplus \mathscr{T}or_1^{\Gamma}[-,\Lambda] \otimes_{\Lambda} \Gamma \otimes_{\Gamma} -,$$

$$\mathscr{E}xt^2_{\Lambda}[\mathbf{T}-,\mathbf{Z}-] \cong \mathscr{E}xt^2_{\Gamma}[-,-] \oplus \mathrm{Hom}_{\Gamma}[\mathscr{T}or_1^{\Gamma}[-,\Lambda] \otimes_{\Lambda} \Gamma,-].$$

(ii) w.gl.dim $\Gamma \leq$ w.gl.dim Λ *and* r.gl.dim $\Gamma \leq$ r.gl.dim Λ.

(iii) *If M is left Γ-flat then there are isomorphisms,* $\forall n \geq 1$:

$$\mathscr{T}or_n^{\Lambda}[\mathbf{T}-,-] \cong \mathscr{T}or_n^{\Gamma}[-,\mathbf{U}-], \quad \mathscr{E}xt_{\Lambda}^n[\mathbf{T}-,-] \cong \mathscr{E}xt_{\Gamma}^n[-,\mathbf{U}-],$$

$$\mathscr{T}or_n^{\Lambda}[\mathbf{Z}-,\mathbf{Z}-] \cong \mathscr{T}or_n^{\Gamma}[-,-] \oplus \mathscr{T}or_{n-1}^{\Lambda}[\Phi\mathbf{Z}-,\mathbf{Z}-],$$

$$\mathscr{E}xt_{\Lambda}^n[\mathbf{Z}-,\mathbf{Z}-] \cong \mathscr{E}xt_{\Gamma}^n[-,-] \oplus \mathscr{E}xt_{\Lambda}^{n-1}[\Phi\mathbf{Z}-,\mathbf{Z}-].$$

(iv) *If M is left Γ-flat and $\vartheta = 0$, then there are isomorphisms,* $\forall n \geq 0$:

$$\mathscr{T}or_n^{\Lambda}[\mathbf{Z}-,\mathbf{Z}-] \cong \bigoplus_{i=0}^{n} \mathscr{T}or_n^{\Gamma}[-\otimes_{\Gamma}(\otimes_{\Gamma}^{n-i}M),-],$$

$$\mathscr{E}xt_{\Lambda}^n[\mathbf{Z}-,\mathbf{Z}-] \cong \bigoplus_{i=0}^{n} \mathscr{E}xt_{\Gamma}^n[-\otimes_{\Gamma}(\otimes_{\Gamma}^{n-i}M),-].$$

4.2. The natural morphism: $\mathscr{L}_*^{\mathfrak{F}}\mathbf{C} \to \mathscr{L}_*^{\Sigma}\mathbf{C}$

As we will see in the next section, the behaviour of $\mathfrak{F}$-global dimension of $\mathscr{D}_F(\eta)$ depends essentially on the behaviour of the sequence of functors $\mathscr{L}_i^{\mathfrak{F}}\mathbf{C}$, $i \geq 0$. It follows from the Σ-associated complex $\Sigma^{\bullet}_{(X,f)}$ of (X,f), especially when $f = 0$, that the study of the sequence of functors $\mathscr{L}_i^{\Sigma}\mathbf{C}$, $i \geq 0$, requires information which is more close to the triad $\{\mathscr{D}, F, \eta\}$. Hence it is useful to reduce the study of the sequence $\mathscr{L}_i^{\mathfrak{F}}\mathbf{C}$, $i \geq 0$, to the study of the sequence $\mathscr{L}_i^{\Sigma}\mathbf{C}$, $i \geq 0$. In this respect we have the following result, in which we denote by $\operatorname{Ker}\mathbf{C}$ the full subcategory of $\mathscr{D}_F(\eta)$ with objects (X,f), such that $\mathbf{C}(X,f) = 0$.

Theorem 4.7. *There are natural morphisms*:

$\xi^i : \mathscr{L}_i^{\mathfrak{F}}\mathbf{C} \to \mathscr{L}_i^{\Sigma}\mathbf{C}$, $\forall i \geq 0$, *such that*

(i) ξ^0, ξ^1 *are isomorphisms and ξ^2 is an epimorphism.*

(ii) *Consider the following statements,* $\forall n \geq 2$:

  (a) *The morphisms $\xi^i_{(X,f)}$ are isomorphisms, $1 \leq i \leq n$, and $\xi^{n+1}_{(X,f)}$ is an epimorphism.*

  (b) $\mathscr{L}_i^{\mathfrak{F}}\mathbf{C}\mathbf{T}(X) = 0$, $2 \leq i \leq n$.

  (c) $\mathscr{L}_i^{\mathfrak{R}}F(X) = 0$, $1 \leq i \leq n-1$.

  Then (a) $\Leftrightarrow$ (b) $\Leftarrow$ (c). *If* $\operatorname{Ker}\mathbf{C} = 0$, *then all these statements are equivalent.*

(iii) *Consider the following statements*:

  (a) *The morphisms $\xi^i_{(X,f)}$ are isomorphisms,* $\forall i \geq 0$.

  (b) $\mathscr{L}_i^{\mathfrak{F}}\mathbf{C}\mathbf{T}(X) = 0$, $\forall i \geq 2$.

  (c) $\mathscr{L}_i^{\mathfrak{R}}F(X) = 0$, $\forall i \geq 1$, *i.e. F is $\mathfrak{R}$-exact.*

  Then (a) $\Leftrightarrow$ (b) $\Leftarrow$ (c). *If* $\operatorname{Ker}\mathbf{C} = 0$, *then all these statements are equivalent.*

Proof. (i) By Corollary 4.2: $\mathscr{L}_1^{\mathfrak{F}}\mathbf{C}\mathbf{T} = 0$. Let now $(X,f) \in \mathscr{D}_F(\eta)$, and consider the natural Σ-projective presentation $(1): 0 \to \Phi(X,f) \xrightarrow{m} \mathbf{T}(X) \xrightarrow{\varepsilon(X,f)} (X,f) \to 0$ of (X,f). The sequence (1) is $\mathfrak{F}$-exact since it is Σ-exact and $\Sigma \subseteq \mathfrak{F}$. Applying the functor $\mathbf{C}$ to (1) we get isomorphisms $\rho^i_{(X,f)} : \mathscr{L}^{\Sigma}_{i+1}\mathbf{C}(X,f) \to \mathscr{L}^{\Sigma}_i\mathbf{C}\Phi(X,f)$, $\forall i \geq 1$, and a long exact sequence (2):

$$\cdots \to \mathscr{L}^{\mathfrak{F}}_{i+1}\mathbf{CT}(X) \to \mathscr{L}^{\mathfrak{F}}_{i+1}\mathbf{C}(X,f) \xrightarrow{\sigma^i_{(X,f)}} \mathscr{L}^{\mathfrak{F}}_i\mathbf{C}\Phi(X,f) \to \mathscr{L}^{\mathfrak{F}}_i\mathbf{CT}(X) \to$$

$$\cdots \to \mathscr{L}^{\mathfrak{F}}_2\mathbf{C}(X,f) \xrightarrow{\sigma^1_{(X,f)}} \mathscr{L}^{\mathfrak{F}}_1\mathbf{C}\Phi(X,f) \to \mathscr{L}^{\mathfrak{F}}_1\mathbf{CT}(X) \to \mathscr{L}^{\mathfrak{F}}_1\mathbf{C}(X,f)$$

$$\to \mathbf{C}\Phi(X,f) \to X \to \mathbf{C}(X,f) \to 0.$$

Since $\mathscr{L}^{\mathfrak{F}}_1\mathbf{CT}(X)=0$ we have $\mathscr{L}^{\mathfrak{F}}_1\mathbf{C}(X,f) \cong Ker(\mathbf{C}\Phi(X,f) \to X) = \mathscr{L}^{\Sigma}_1\mathbf{C}(X,f)$ by an isomorphism $\xi^1_{(X,f)}$ and the morphism $\sigma^1_{(X,f)} : \mathscr{L}^{\mathfrak{F}}_2\mathbf{C}(X,f) \to \mathscr{L}^{\mathfrak{F}}_1\mathbf{C}\Phi(X,f)$ is an epimorphism. Since (1) is functorial, the morphisms $\rho^i_{(X,f)}$, $\xi^1_{(X,f)}$, $\sigma^1_{(X,f)}$ are natural, thus we get natural isomorphisms $\rho^i : \mathscr{L}^{\Sigma}_{i+1}\mathbf{C} \to \mathscr{L}^{\Sigma}_i\mathbf{C}\Phi$, $\xi^1 : \mathscr{L}^{\mathfrak{F}}_1\mathbf{C} \to \mathscr{L}^{\Sigma}_1\mathbf{C}$, and a natural epimorphism $\sigma^1 : \mathscr{L}^{\mathfrak{F}}_2\mathbf{C} \to \mathscr{L}^{\mathfrak{F}}_1\mathbf{C}\Phi$. We set $\xi^2 = \sigma^1 \circ (\xi^1\Phi) \circ (\rho^1)^{-1} : \mathscr{L}^{\mathfrak{F}}_2\mathbf{C} \to \mathscr{L}^{\Sigma}_2\mathbf{C}$. Then ξ^2 is an epimorphism. Now we proceed inductively setting $\xi^{i+1} = \sigma^i \circ \xi^i\Phi \circ (\rho^i)^{-1}$: $\mathscr{L}^{\mathfrak{F}}_{i+1}\mathbf{C} \to \mathscr{L}^{\Sigma}_{i+1}\mathbf{C}$, $\forall i \geq 1$. Then the natural morphisms $\xi^i : \mathscr{L}^{\mathfrak{F}}_i\mathbf{C} \to \mathscr{L}^{\Sigma}_i\mathbf{C}$, $\forall i \geq 0$, together with the obvious isomorphism ξ^0: $\mathscr{L}^{\mathfrak{F}}_0\mathbf{C} \cong \mathscr{L}^{\Sigma}_0\mathbf{C} = \mathbf{C}$, is the desired sequence of natural morphisms.

(ii), (iii) (a) $\Leftrightarrow$ (b) Follows directly from the long exact sequence (2), and the definition of the natural morphisms ξ^i. (c) $\Rightarrow$ (b) follows from Corollary 4.2(3). Suppose now that $\operatorname{Ker}\mathbf{C} = 0$, and (b) is true. Since $\mathscr{L}^{\mathfrak{F}}_2\mathbf{CT}(X) = 0$, by the last isomorphism in Corollary 4.2(2) we have $\mathbf{C}\mathscr{L}^{\mathfrak{R}}_1\mathbf{T}(X) = 0$. Since $\operatorname{Ker}\mathbf{C} = 0$, $\mathscr{L}^{\mathfrak{R}}_1\mathbf{T}(X) = 0 \Rightarrow \mathscr{L}^{\mathfrak{R}}_1F(X) = 0$. Let $0 \to K_0 \to P_0 \to X \to 0$ be an $\mathfrak{R}$-projective presentation of X. Since $\mathscr{L}^{\mathfrak{R}}_1\mathbf{T}(X) = 0$, we have the $\mathfrak{F}$-projective presentation $0 \to \mathbf{T}(K_0) \to \mathbf{T}(P_0) \to \mathbf{T}(X) \to 0$ of $\mathbf{T}(X)$. Then $0 = \mathscr{L}^{\mathfrak{F}}_3\mathbf{CT}(X) = \mathscr{L}^{\mathfrak{F}}_2\mathbf{CT}(K_0) = \mathbf{C}\mathscr{L}^{\mathfrak{R}}_1\mathbf{T}(K_0) \Rightarrow \mathscr{L}^{\mathfrak{R}}_1F(K_0) = \mathscr{L}^{\mathfrak{R}}_2F(X) = 0$. A simple induction and dimension-shifting argument completes the proof. $\square$

Remark 4.8. The above morphism $\xi^* : \mathscr{L}^{\mathfrak{F}}_*\mathbf{C} \to \mathscr{L}^{\Sigma}_*\mathbf{C}$ is a morphism of Σ-connected sequence of functors in the sense of [24,18]. For more information concerning the morphism ξ^*, see Section 7.

Corollary 4.9. *If F is $\mathfrak{R}$-exact, then for any proper class $\mathfrak{F}$ in $\mathscr{D}_F(\eta)$ of the form $\mathfrak{F} = \mathbf{U}^{-1}(\mathfrak{R})$, where $\mathfrak{R}$ is a proper class in $\mathscr{D}$, we have*

$$\mathscr{L}^{\mathfrak{F}}_i\mathbf{C} \cong \mathscr{L}^{\Sigma}_i\mathbf{C}, \quad \forall i \geq 0.$$

In particular if $\eta = 0$, then $\mathscr{L}^{\mathfrak{F}}_i\mathbf{CZ} \cong \mathscr{L}^{\Sigma}_i\mathbf{CZ} \cong F^i$, $\forall i \geq 0$.

If $\Lambda = \Gamma \ltimes_\vartheta M$ is a cleft extension of rings, then the proper class Σ is the class of short exact sequences of Λ-modules, which are split when considered as short exact sequences of Γ-modules. In this concrete case we use the notation $\Sigma := (\Lambda, \Gamma)$.

Corollary for Rings 4.10. *Let $\Lambda = \Gamma \ltimes_{\vartheta} M$ be a cleft extension of rings. Then for the following statements*:

(i) *M is left Γ-flat.*

(ii) $\mathscr{T}or_i^{\Lambda}[-\otimes_{\Gamma}\Lambda, \Gamma] = 0, \ \forall i \geq 1.$

(iii) $\mathscr{T}or_i^{\Lambda}[-, \Gamma] \cong \mathscr{T}or_i^{(\Lambda,\Gamma)}[-, \Gamma], \ \forall i \geq 1.$

we have (i) $\Rightarrow$ (ii) $\Leftrightarrow$ (iii); *if ϑ is right T-nilpotent, then they are equivalent.*

4.3. Lifting tilting objects

We close this section with an application to tilting theory. Let $\mathscr{C}$ be an abelian category and let $\mathfrak{R}$ be a proper class of short exact sequences in $\mathscr{C}$. We recall that if $X \in \mathscr{C}$, then $\mathrm{add}(X)$ is the full subcategory of $\mathscr{C}$ with objects the direct summands of all finite direct sums of copies of X. Define a full subcategory $\mathrm{add}(X)^t_{\mathfrak{R}}$ of $\mathscr{C}$ as follows: $\mathrm{add}(X)^t_{\mathfrak{R}} = \{A \in \mathscr{C}$: there exists an $\mathfrak{R}$-exact sequence $0 \to A \to X_0 \to \cdots \to X_t \to 0$, with $X_i \in \mathrm{add}(X), \forall i = 0, 1, \ldots, t\}$, where $t \geq 0$.

We call an object $X \in \mathscr{C}$, a *t–$\mathfrak{R}$-tilting object* iff (α) $\mathfrak{R}$-p.d $X = t < \infty$, (β) $\mathscr{E}xt^i_{\mathfrak{R}}[X, X] = 0, \ \forall i \geq 1$, and ($\gamma$) $\mathscr{P}(\mathfrak{R}) \subseteq \mathrm{add}(X)^t_{\mathfrak{R}}$.

Theorem 4.11. *Suppose that* $\mathrm{Ker}\,\mathbf{C} = 0$.

(1) *Assume that $X \in \mathscr{D}$ is an object with $\mathscr{L}^{\mathfrak{R}}_t F(X) = 0, \ 1 \leq t \leq n$. Then for any t with $1 \leq t \leq n$, the following are equivalent*:

(i) *$\mathbf{T}(X)$ is a t–$\mathfrak{F}$-tilting object.*

(ii) (a) *X is a t–$\mathfrak{R}$-tilting object.*

(b) $\mathscr{E}xt^i_{\mathfrak{R}}[X, FX] = 0, 1 \leq i \leq t.$

(2) *The following are equivalent*:

(i) *$\mathbf{T}(X)$ is a 1–$\mathfrak{F}$-tilting object.*

(ii) (a) *X is a 1–$\mathfrak{R}$-tilting object.*

(b) $\mathscr{E}xt^1_{\mathfrak{R}}[X, FX] = \mathscr{L}^{\mathfrak{R}}_1 F(X) = 0.$

The ring $\mathrm{End}_{\mathscr{D}_F(\eta)}[\mathbf{T}(X)]$ is a cleft ϑ-extension of the ring $\mathrm{End}_{\mathscr{D}}[X]$, by the bimodule $\mathscr{D}[X, FX]$, where ϑ is defined by $\vartheta(a \otimes b) = a \circ Fb \circ \eta_X$, for $a, b \in \mathscr{D}[X, FX]$ and the bimodule structure of $\mathscr{D}[X, FX]$ is given by $f \cdot a = f \circ a$ and $b \cdot g = b \circ Fg$, for $f, g \in \mathrm{End}_{\mathscr{D}}[X]$ and $a, b \in \mathscr{D}[X, FX]$.

Proof. (1) The hypothesis $\mathscr{L}^{\mathfrak{R}}_t F(X) = 0, \ 1 \leq t \leq n$, and Corollary 4.2(3), implies that $(*)$: $\mathscr{E}xt^t_{\mathfrak{F}}[\mathbf{T}(X), \mathbf{T}(X)] \cong \mathscr{E}xt^t_{\mathfrak{R}}[X, X \oplus FX], \ 1 \leq t \leq n.$

(a) $\Rightarrow$ (b) Let $t \in \mathbb{N}$ with $1 \leq t \leq n$, and suppose that $\mathbf{T}(X)$ is a t-$\mathfrak{F}$-tilting object. Then since $\mathfrak{F}$-p.d $\mathbf{T}(X) = t$ and $\mathfrak{R}$-p.d $X \leq \mathfrak{F}$-p.d $\mathbf{T}(X)$, by $(*)$ we have $\mathfrak{R}$-p.d $X = \mathfrak{F}$-p.d $\mathbf{T}(X) = t$, and $0 = \mathscr{E}xt^i_{\mathfrak{F}}[\mathbf{T}(X), \mathbf{T}(X)] \cong \mathscr{E}xt^i_{\mathfrak{R}}[X, X \oplus FX] \cong \mathscr{E}xt^i_{\mathfrak{R}}[X, X] \oplus \mathscr{E}xt^i_{\mathfrak{R}}[X, FX], \ \forall i \geq 1$. Thus it remains to show that $\mathscr{P}(\mathfrak{R}) \subseteq \mathrm{add}(X)^t_{\mathfrak{R}}$. If $P \in \mathscr{P}(\mathfrak{R})$, then $\mathbf{T}(P) \in \mathscr{P}(\mathfrak{F})$ and since $\mathbf{T}(X)$ is an $\mathfrak{F}$-tilting object, there exists an $\mathfrak{F}$-exact sequence $(**)$: $0 \to \mathbf{T}(P) \to (X_0, f_0) \to \cdots \to (X_t, f_t) \to 0$, with $(X_i, f_i) \in \mathrm{add}\,\mathbf{T}(X), \ \forall i = 0, 1, \ldots, t$. But then obviously $\mathbf{C}(X_i, f_i) \in \mathrm{add}(X), \ \forall i = 0, 1, \ldots, t$. We claim that the sequence $\mathbf{C}(**)$: $0 \to P \to \mathbf{C}(X_0, f_0) \to \cdots \to \mathbf{C}(X_t, f_t) \to 0$ is $\mathfrak{R}$-exact. Indeed since by Corollary 4.2, $\mathscr{L}_1^{\mathfrak{F}}\mathbf{C} \cong \mathscr{L}_1^{\Sigma}\mathbf{C}$ and since all the objects in $(*)$ are direct summands of objects in $Im\,\mathbf{T} \subseteq \mathscr{P}(\Sigma)$, it follows that any object in $(*)$ is in the kernel of the functor $\mathscr{L}_1^{\mathfrak{F}}\mathbf{C}$. Hence $\mathbf{C}(**)$ is exact and it is $\mathfrak{R}$-exact, since $\mathbf{C}$ sends $\mathfrak{F}$-proper epics in $\mathscr{D}_F(\eta)$ to $\mathfrak{R}$-proper epics in $\mathscr{D}$. Thus $\mathscr{P}(\mathfrak{R}) \subseteq \operatorname{add}(X)^t_{\mathfrak{R}}$.

(b) $\Rightarrow$ (a) If X is a t-$\mathfrak{R}$-tilting object, then $\mathfrak{R}$-p.d $X = \mathfrak{F}$-p.d $\mathbf{T}(X) = t$ and $\mathscr{E}xt^i_{\mathfrak{F}}[\mathbf{T}(X), \mathbf{T}(X)] \cong \mathscr{E}xt^i_{\mathfrak{R}}[X, X \oplus FX] \cong \mathscr{E}xt^i_{\mathfrak{R}}[X,X] \oplus \mathscr{E}xt^i_{\mathfrak{R}}[X,FX] = 0$, $\forall i \geq 1$, by hypothesis. So it remains to show that $\mathscr{P}(\mathfrak{F}) \subseteq \operatorname{add}\mathbf{T}(X)^t_{\mathfrak{F}}$. Let $(X, f) \in \mathscr{P}(\mathfrak{F})$; then by [6] or Lemma 5.3 below, $(X, f) \cong \mathbf{T}(P)$ with $P \in \mathscr{P}(\mathfrak{R})$. Since X is an $\mathfrak{R}$-tilting object, there exists an $\mathfrak{R}$-exact sequence $(!)$: $0 \to P \to X_0 \to \cdots \to X_t \to 0$, with $X_i \in \operatorname{add}(X), \forall i = 0, 1, \ldots, t$. Applying $\mathbf{T}$ we obtain the $\mathfrak{F}$-proper sequence $\mathbf{T}(!)$: $0 \to \mathbf{T}(P) \to \mathbf{T}(X_0) \to \cdots \to \mathbf{T}(X_t) \to 0$. Since $X_i \in \operatorname{add}(X)$, $\forall i = 0, 1, \ldots, t$ and $\mathscr{L}_n^{\mathfrak{R}} FX = 0$, $\forall n \geq 1$, and since $\mathscr{L}_1^{\mathfrak{R}} F \cong \mathbf{U}\mathscr{L}_1^{\mathfrak{R}}\mathbf{T}$, we see easily that $\mathbf{T}(!)$ is $\mathfrak{F}$-exact. Thus $\mathscr{P}(\mathfrak{F}) \subseteq \operatorname{add}\mathbf{T}(X)^t_{\mathfrak{F}}$.

(2) From the proof of (1), it is enough to prove that if $\mathbf{T}(X)$ is 1–$\mathfrak{F}$-tilting object, then $\mathscr{L}_1^{\mathfrak{R}} F(X) = 0$. Since $\mathfrak{F}$-p.d $\mathbf{T}(X) = 1$, we have $\mathscr{L}_2^{\mathfrak{F}}\mathbf{CT}(X) = 0$. But by Corollary 4.2(2): $\mathscr{L}_2^{\mathfrak{F}}\mathbf{CT}(X) \cong \mathbf{C}\mathscr{L}_1^{\mathfrak{R}}\mathbf{T}(X)$. Hence $\mathbf{C}\mathscr{L}_1^{\mathfrak{R}}\mathbf{T}(X) = 0$, and since $\operatorname{Ker}\mathbf{C} = 0$, we have $\mathscr{L}_1^{\mathfrak{R}}\mathbf{T}(X) = 0$. But then $\mathbf{U}\mathscr{L}_1^{\mathfrak{R}}\mathbf{T}(X) = \mathscr{L}_1^{\mathfrak{R}} F(X) = 0$.

The last assertion follows from [6]. $\square$

Now consider a cleft ϑ-extension $\Lambda = \Gamma \ltimes_\vartheta M$ of rings. In the following result which is a consequence of the above theorem and its dual, an n-tilting (n-cotilting) module is a (co-)tilting module of projective (injective) dimension n, in the sense of [29]. We note that the case (1) below is a generalization of the main result of [25].

Corollary for Rings 4.12. *Let $\Lambda = \Gamma \ltimes_\vartheta M$ be a cleft extension of rings and assume that ϑ is right T-nilpotent. Then*

(1) *The following are equivalent*:

(i) *$X \otimes_\Gamma \Lambda$ is a* 1*-tilting module.*

(ii) *X is a* 1*-tilting module, and* $\mathscr{E}xt^1_\Gamma[X, X \otimes_\Gamma M] = \mathscr{T}or^\Gamma_1[X, M] = 0$.

(2) *Suppose that* $\mathscr{T}or^\Gamma_i[X, M] = 0$, $\forall i \geq 1$. *Then the following are equivalent*:

(i) *$X \otimes_\Gamma \Lambda$ is an n-tilting module.*

(ii) *X is an n-tilting module, and* $\mathscr{E}xt^i_\Gamma[X, X \otimes_\Gamma M] = 0$, $\forall i \geq 1$.

If ϑ is left T*-nilpotent, then*

(3) *The following are equivalent*:

(i) $\operatorname{Hom}_\Gamma[\Lambda, X]$ *is a* 1*-cotilting module.*

(ii) *X is a* 1*-cotilting module, and* $\mathscr{E}xt^1_\Gamma[\operatorname{Hom}_\Gamma[M, X], X] = \mathscr{E}xt^1_\Gamma[M, X] = 0$.

(4) *Suppose that* $\mathscr{E}xt^i_\Gamma[M, X] = 0$, $\forall i \geq 1$. *Then the following are equivalent*:

(i) $\operatorname{Hom}_\Gamma[\Lambda, X]$ *is an n-cotilting module.*

(ii) *X is an n-cotilting module, and* $\mathscr{E}xt^i_\Gamma[\operatorname{Hom}_\Gamma[M, X], X] = 0$, $\forall i \geq 1$.

Our next result is a corollary of part (4) of the above result.

Corollary 4.13. *Let $\Lambda = \Gamma \ltimes_0 M$ be a trivial cleft extension of rings. Suppose that* $\mathscr{E}xt^i_\Gamma[M_\Gamma, M_\Gamma] = 0$, $\forall i \geq 1$, *and the natural morphism* $\omega_r : \Gamma \to \operatorname{End}_\Gamma(M_\Gamma)$ *is an* *isomorphism. Then* $\mathrm{Hom}_\Gamma[\Lambda, M_\Gamma]_\Lambda$ *is an n-cotilting module* $\Leftrightarrow$ M_Γ *is an n-cotilting module*; *in this case* $\Lambda \cong \mathrm{End}_\Lambda[\mathrm{Hom}_\Gamma[\Lambda, M_\Gamma]]$, *and* $\mathrm{i.d}\Lambda_\Lambda = n$.

Auslander and Reiten [2] ask if the trivial extension of a Cohen–Macaulay Artin algebra is Gorenstein (see [2] for the definition of the involved concepts). Our final result in this subsection is a direct consequence of the previous corollary and answers the question of Auslander–Reiten in the affirmative.

Corollary 4.14. *Let* Γ *be a Cohen–Macaulay Artin algebra with dualizing bimodule* ${}_\Gamma M_\Gamma$ *and consider the trivial extension* $\Lambda = \Gamma \ltimes_0 M$. *Then*

$$\mathrm{i.d}\Lambda_\Lambda = \mathrm{i.d}_\Lambda \Lambda = \mathrm{i.d}\, M_\Gamma = \mathrm{i.d}_\Gamma M < \infty$$

and Λ *is a Gorenstein algebra.*

5. $\mathfrak{F}$-, Σ-relative global dimension of cleft extensions

We continue to use the notation introduced in the previous section.

Definition 5.1. The category $\mathscr{D}_F(\eta)$ is called *admissible* if any object $(X, f) \in \mathscr{D}_F(\eta)$ has a finite filtration

$$0 = (X_n, f_n) \subseteq (X_{n-1}, f_{n-1}) \subseteq \cdots \subseteq (X_1, f_1) \subseteq (X_0, f_0) = (X, f)$$

such that all quotient objects $(X_j, f_j)/(X_{j+1}, f_{j+1})$ are in $\mathbf{Z}(\mathscr{D})$.

Hence $\mathscr{D}_F(\eta)$ is admissible iff the full subcategory $\mathbf{Z}(\mathscr{D})$ is admissible in the sense of [33]. By our conventions $\mathscr{D}_F(\eta)$ is admissible iff it is $\mathfrak{F}$-admissible in the sense that, in the filtration above all short exact sequences: $0 \to (X_{j+1}, f_{j+1}) \to (X_j, f_j) \to (X_j, f_j)/(X_{j+1}, f_{j+1}) \to 0$ are in $\mathfrak{F}$, $\forall j = 0, 2, \ldots, n-1$.

Remark 5.2. (1) By a result of [6], $\mathscr{D}_F(\eta)$ is admissible $\Leftrightarrow$ the functor $\mathscr{R} := \mathrm{Ker}(\lambda)$ is locally nilpotent, i.e. $\forall (X, f) \in \mathscr{D}_F(\eta)$, $\exists n \geq 0$: $\mathscr{R}^n(X, f) = 0$. Note that in this case by [33], the categories $\mathscr{D}$ and $\mathscr{D}_F(\eta)$ have isomorphic higher K-theory.

(2) If $\mathscr{D}_F(\eta)$ is admissible, then $\mathrm{Ker}\,\mathbf{C} = 0$ (see [6]).

(3) The morphism $\eta : F^2 \to F$ is called *locally nilpotent*, resp. *nilpotent*, if $\forall X \in \mathscr{D}$, $\exists t \in \mathbb{N}$: $\eta_{F^t X} \circ \eta_{F^{t-1}X} \circ \cdots \circ \eta_{FX} \circ \eta_X = 0$, resp. $\exists t \in \mathbb{N}$: $\eta_{F^t} \circ \eta_{F^{t-1}} \circ \cdots \circ \eta_F \circ \eta = 0$. In the last case η is called t-nilpotent. It is not difficult to see that if η is locally nilpotent, then $\mathscr{D}_F(\eta)$ is admissible, in particular $\mathrm{Ker}\,\mathbf{C} = 0$. It follows that a trivial extension or more generally a truncated extension is admissible.

(4) If η is t-nilpotent, then $\mathscr{D}_F(\eta)$ is a full exact subcategory of the truncated extension $\mathscr{D} \ltimes_{t+1} F$. This follows form the fact that $\forall t \geq 0$, $\forall (X, f) \in \mathscr{D}_F(\eta)$: $\eta_{F^t X} \circ \eta_{F^{t-1}X} \circ \cdots \circ \eta_{FX} \circ \eta_X \circ f = F^{t+1} f \circ F^t f \circ \cdots \circ F^2 f \circ F f \circ f$.

(5) If $\mathscr{D}_F(\eta)$ is admissible, then it is $\mathfrak{F}$-admissible, hence the vanishing of the extension functors $\mathscr{E}xt^n_{\mathfrak{F}}[-,-]$ is controlled by the vanishing of $\mathscr{E}xt^n_{\mathfrak{F}}[\mathbf{Z}(-),-]$.

(6) If the cleft extension is induced by a cleft extension of rings $\Lambda = \Gamma \ltimes_\vartheta M$, then [6] ϑ is nilpotent iff $\eta = 1 \otimes_\Gamma \vartheta$ is (locally) nilpotent iff $\mathrm{Mod}(\Lambda)$ is admissible. As in Section 2.1, ϑ is right, resp. left, T-nilpotent $\Leftrightarrow$ $\mathrm{Ker}\,\mathbf{C} = 0$, resp. $\mathrm{Ker}\,\mathbf{K} = 0$.

We recall that an epic $\alpha : X \to Y$ in an abelian category $\mathscr{A}$ is called *minimal* if for any morphism $\beta : Z \to X$, $\beta \circ \alpha$ is epic implies that β is epic. If the category $\mathscr{A}$ has enough $\mathfrak{F}$-projectives with respect to a proper class of short exact sequences $\mathfrak{F}$, then a morphism $\alpha : P \to X$ in $\mathfrak{F}$ is called an *$\mathfrak{F}$-projective cover*, if α is a minimal epic and P is $\mathfrak{F}$-projective. Then $\mathscr{A}$ is called *$\mathfrak{F}$-perfect* if any object $X \in \mathscr{A}$ has an $\mathfrak{F}$-projective cover. Consider now the full subcategory $\mathrm{Ker}\,\mathbf{C} = \{(X, f) \in \mathscr{D}_F(\eta) : \mathbf{C}(X, f) = 0\}$. In [6] we have shown that $\mathrm{Ker}\,\mathbf{C} = 0 \Leftrightarrow \lambda$ is a minimal epic, i.e. $\forall (X, f) \in \mathscr{D}_F(\eta)$, $\lambda_{(X,f)}$ is a minimal epic, and that the condition $\mathrm{Ker}\,\mathbf{C} = 0$ has many pleasant consequences for the structure of the simple and projective objects of $\mathscr{D}_F(\eta)$. We restate Lemma 5.4 of [6] in the present setting:

Lemma 5.3. *If* $\mathrm{Ker}\,\mathbf{C} = 0$, *then we have the following*:
(i) *$\mathscr{D}_F(\eta)$ is $\mathfrak{F}$-perfect $\Leftrightarrow$ $\mathscr{D}$ is $\mathfrak{R}$-perfect.*
(ii) *(X, f) is simple $\Leftrightarrow$ $\mathbf{C}(X, f)$ is simple and $(X, f) \cong \mathbf{ZC}(X, f)$.*
(iii) *The following are equivalent*:
(a) $(X, f) \in \mathscr{P}(\mathfrak{F})$.
(b) $\mathbf{C}(X, f) \in \mathscr{P}(\mathfrak{R})$ *and* $(X, f) \cong \mathbf{TC}(X, f)$.
(c) $\mathbf{C}(X, f) \in \mathscr{P}(\mathfrak{R})$ *and* $\mathscr{L}_1^{\Sigma}\mathbf{C}(X, f) = \mathscr{L}_1^{\mathfrak{F}}\mathbf{C}(X, f) = 0$.
(d) $\mathscr{E}xt^1_{\mathfrak{F}}[(X, f), \mathbf{Z}(-)] = 0$.

Proof. We only prove that (a) $\Leftrightarrow$ (d) in (iii). The other parts follow directly from Lemma 5.4 of [6]. Obviously (a) $\Rightarrow$ (d). Suppose that (d) is true. Consider an $\mathfrak{F}$-projective presentation of $(X, f) : 0 \to (Q_0, g_0) \to \mathbf{T}(P_0) \to (X, f) \to 0$. Then we have the exact sequence: $0 \to \mathscr{D}_F(\eta)[(X, f), \mathbf{Z}(-)] \to \mathscr{D}_F(\eta)[\mathbf{T}(P_0), \mathbf{Z}(-)] \to \mathscr{D}_F(\eta)[(Q_0, g_0), \mathbf{Z}(-)] \to 0$ which is isomorphic to the short exact sequence: $0 \to \mathscr{D}[\mathbf{C}(X, f), -] \to \mathscr{D}[P_0, -] \to \mathscr{D}[\mathbf{C}(Q_0, g_0), -] \to 0$. This implies that the exact sequence: $\mathbf{C}(Q_0, g_0) \to P_0 \to \mathbf{C}(X, f) \to 0$ is indeed a split short exact sequence and $\mathbf{C}(X, f) \in \mathscr{P}(\mathfrak{R})$ as a direct summand of P_0. The above argument shows also that $\mathscr{L}_1^{\mathfrak{F}}\mathbf{C}(X, f) = \mathrm{Ker}\,(\mathbf{C}(Q_0, g_0) \to P_0) = 0$, thus by (c), $(X, f) \in \mathscr{P}(\mathfrak{F})$. $\square$

If $\Lambda = \Gamma \ltimes_\vartheta M$ is a cleft ring extension, then the next result follows from Lemma 5.3, using the realization of the functors $F, G, \mathbf{C}, \mathbf{T}, \mathbf{H}, \mathbf{K}, \mathbf{Z}, \mathbf{U}$ from Section 2.

Corollary for Rings 5.4. (1) *If ϑ is right* T*-nilpotent, then*
(i) *Λ is right perfect $\Leftrightarrow$ Γ is right perfect.*
(ii) *(X, f) is simple in* $\mathrm{Mod}(\Lambda) \Leftrightarrow X$ *is simple in* $\mathrm{Mod}(\Gamma)$ *and* $f = 0$.
(iii) *The following are equivalent*:
(a) *(X, f) is a projective, (resp. flat, resp. free), right Λ-module.*

(b) $(X, f) \otimes_\Lambda \Gamma_\Gamma$ *is a projective*, (*resp. flat*, *resp. free*), *right* Γ-*module and* $Ker(f) = Im(1_X \otimes \vartheta\text{-}1_M \otimes f)$.

(c) $(X, f) \otimes_\Lambda \Gamma_\Gamma$ *is a projective*, (*resp. flat*, *resp. free*), *right* Γ-*module and* $(X, f) \otimes_\Lambda \Gamma \otimes_\Gamma \Lambda \cong (X, f)$.

(d) $\mathscr{E}xt^1_\Lambda[(X, f), - \otimes_\Gamma \Gamma_\Lambda] = 0$, (*resp.* $\mathscr{T}or_1^\Lambda[(X, f),_\Lambda \Gamma \otimes_\Gamma -] = 0$).

(2) *If* ϑ *is left* T-*nilpotent*, *then the following are equivalent*:

(a) (X, f) *is an injective right* Λ-*module.*

(b) $\mathrm{Hom}_\Lambda[{}_\Gamma\Gamma_\Lambda, (X, f)]$ *is an injective right* Γ-*module and* $Ker(f^*) = Im(\zeta_X - \mathrm{Hom}_\Gamma[M, f^*])$, *where* $f^* : X \to \mathrm{Hom}_\Gamma[M, X]$ *is defined by* $f^*(x)(m) = f(x \otimes m)$ *and* $\zeta_X : \mathrm{Hom}_\Gamma[M, X] \to \mathrm{Hom}_\Gamma[M, \mathrm{Hom}_\Gamma[M, X]]$ *is defined by* $\zeta_X(\phi)(m)(n) = \phi(\vartheta(m \otimes n))$, $\forall m, n \in M$, $\forall x \in X$ *and* $\forall \phi \in \mathrm{Hom}_\Gamma[M, X]$.

(c) $\mathrm{Hom}_\Lambda[{}_\Gamma\Gamma_\Lambda, (X, f)]$ *is an injective right* Γ-*module and* $(X, f) \cong \mathrm{Hom}_\Gamma[{}_\Lambda\Lambda_\Gamma, \mathrm{Hom}_\Lambda[{}_\Gamma\Gamma_\Lambda, (X, f)]]$.

(d) $\mathscr{E}xt^1_\Lambda[\mathrm{Hom}_\Gamma[{}_\Lambda\Gamma_\Gamma, -], (X, f)] = 0$.

Remark 5.5. Suppose that $F \neq 0$. If $\mathrm{Ker}\,\mathbf{C} = 0$, then $\mathbf{C}$ is never exact, and always $\Sigma, \mathfrak{F}\text{-gl.dim}\,\mathscr{D}_F(\eta) \geq 1$. Indeed otherwise applying $\mathbf{C}$ to the short exact sequence $0 \to \Phi\mathbf{Z}(X) \to \mathbf{T}(X) \to \mathbf{Z}(X) \to 0$, we get directly that $\mathbf{C}\Phi\mathbf{Z}(X) = 0$, thus $\Phi\mathbf{Z}(X) = 0$ since $\mathrm{Ker}\,\mathbf{C} = 0$. But then $\mathbf{U}\Phi\mathbf{Z}(X) = FX = 0$. Since this happens for any $X \in \mathscr{D}$, we get that $F = 0$.

5.1. Cleft extensions of finite global dimension

The next result gives necessary and sufficient conditions for the $\mathfrak{F}$-global dimension of $\mathscr{D}_F(\eta)$ to be finite.

Theorem 5.6. (1) *If* $\mathrm{Ker}\,\mathbf{C} = 0$, *then the following are equivalent*:

(i) $\mathfrak{F}\text{-gl.dim}\,\mathscr{D}_F(\eta) < \infty$.

(ii) $\mathfrak{R}\text{-gl.dim}\,\mathscr{D} < \infty$ *and* $\exists i \geq 0$: $\mathscr{L}^{\mathfrak{F}}_{i+1}\mathbf{C} = 0$.

In any case we have the following bounds:

$$\mathfrak{R}\text{-gl.dim}\,\mathscr{D} \leq \mathfrak{F}\text{-gl.dim}\,\mathscr{D}_F(\eta) \leq min\{n \geq 0\colon \mathscr{L}^{\mathfrak{F}}_{n+1}\mathbf{C} = 0\} + \mathfrak{R}\text{-gl.dim}\,\mathscr{D}.$$

(2) *Suppose that* $\mathscr{D}_F(\eta)$ *is admissible. Then the following are equivalent*:

(i) $\mathfrak{F}\text{-gl.dim}\,\mathscr{D}_F(\eta) < \infty$.

(ii) $\mathfrak{R}\text{-gl.dim}\,\mathscr{D} < \infty$ *and* $\exists i \geq 0 : \mathscr{L}^{\mathfrak{F}}_{i+1}\mathbf{CZ} = 0$.

In any case we have the following bounds:

$$\mathfrak{R}\text{-gl.dim}\,\mathscr{D} \leq \mathfrak{F}\text{-gl.dim}\,\mathscr{D}_F(\eta) \leq min\{n \geq 0\colon \mathscr{L}^{\mathfrak{F}}_{n+1}\mathbf{CZ} = 0\} + \mathfrak{R}\text{-gl.dim}\,\mathscr{D}.$$

Proof. (1) ($\Rightarrow$) If $\mathfrak{F}\text{-gl.dim}\,\mathscr{D}_F(\eta) = m < \infty$, then $\mathscr{L}^{\mathfrak{F}}_{m+1}\mathbf{C} = 0$, and by Corollary 4.5 we have $\mathfrak{R}\text{-gl.dim}\,\mathscr{D} \leq m < \infty$.

($\Leftarrow$) Suppose that $\mathfrak{R}\text{-gl.dim}\,\mathscr{D} = k < \infty$ and let $n = min\{i \geq 0\colon \mathscr{L}^{\mathfrak{F}}_{i+1}\mathbf{C} = 0\}$. Let $(X, f) \in \mathscr{D}_F(\eta)$ be an arbitrary object and let $\mathfrak{F}^\bullet_{(X,f)}$ be the $\mathfrak{F}$-associated complex of

(X, f). From the construction of $\mathfrak{F}^{\bullet}_{(X,f)}$, the following complex

$$\cdots \to P_{n+t+1} \xrightarrow{\omega_{n+t}} P_{n+t} \to \cdots \to P_{n+1} \xrightarrow{\omega_n} P_n \xrightarrow{coker(\omega_n)} Coker(\omega_n) \to 0$$

is an acyclic complex consisting of $\mathfrak{R}$-projective objects. Since by Remark 3.2 the above complex is $\mathfrak{R}$-proper, it is an $\mathfrak{R}$-projective resolution of $Coker(\omega_n) \cong \mathbf{C}(Q_{n-1}, g_{n-1})$. Since $\mathfrak{R}$-gl.dim $\mathscr{D} = k$, $Coker(\omega_{n+k})$ is in $\mathscr{P}(\mathfrak{R})$ as a kth-syzygy of $Coker(\omega_n)$ with respect to the proper class $\mathfrak{R}$. Since by construction the object (Q_j, g_j) is a $(j+1)$th-syzygy of (X, f) with respect to the proper class, $\mathfrak{F}$, $\forall j \geq 0$, the hypothesis implies that $\mathscr{L}_1^{\mathfrak{F}}\mathbf{C}(Q_{n+k-1}, g_{n+k-1}) = \mathscr{L}_{n+k+1}^{\mathfrak{F}}\mathbf{C}(X, f) = 0$. Hence by Lemma 5.3 we have $(Q_{n+k-1}, g_{n+k-1}) \in \mathscr{P}(\mathfrak{F})$ and consequently $\mathfrak{F}$-p.d $(X, f) \leq n + k$. We conclude: $\mathfrak{F}$-gl.dim $\mathscr{D}_F(\eta) \leq n + k < \infty$.

(2) Since $\mathscr{D}_F(\eta)$ is admissible, by Remark 5.2(2) we have that $\operatorname{Ker}\mathbf{C} = 0$, and the hypotheses of (1) are satisfied. Hence it suffices to show that: $\mathscr{L}_{n+1}^{\mathfrak{F}}\mathbf{C} = 0 \Leftrightarrow \mathscr{L}_{n+1}^{\mathfrak{F}}\mathbf{CZ} = 0$. The ($\Rightarrow$) direction is trivial and the ($\Leftarrow$) direction follows directly from the fact that $\mathscr{D}_F(\eta)$ is admissible. $\square$

Theorem 5.7. (1) *If the functor F is $\mathfrak{R}$-exact, then* $\forall (X, f) \in \mathscr{D}_F(\eta)$:

$$(*) \qquad \mathfrak{F}\text{-p.d}\,(X, f) \leq \Sigma\text{-p.d}\,(X, f) + \mathfrak{R}\text{-gl.dim}\,\mathscr{D}.$$

(2) *If the functor F is $\mathfrak{R}$-exact, and $F(\mathscr{P}(\mathfrak{R})) \subseteq \mathscr{P}(\mathfrak{R})$, then* $\forall (X, f) \in \mathscr{D}_F(\eta)$:

$$(**) \qquad \mathfrak{F}\text{-p.d}\,(X, f) \leq \Sigma\text{-p.d}\,(X, f) + \mathfrak{R}\text{-p.d}\,X.$$

In any of the above cases we have the bounds:

$$\mathfrak{R}\text{-gl.dim}\,\mathscr{D} \leq \mathfrak{F}\text{-gl.dim}\,\mathscr{D}_F(\eta) \leq \mathfrak{R}\text{-gl.dim}\,\mathscr{D} + \Sigma\text{-gl.dim}\,\mathscr{D}_F(\eta).$$

Proof. If Σ-p.d $(X, f) = \infty$ or $\mathfrak{R}$-gl.dim $\mathscr{D} = \infty$ then relation $(*)$ is true trivially. Suppose that $\mathfrak{R}$-gl.dim $\mathscr{D} = t < \infty$, Σ-p.d $(X, f) < \infty$ and we apply induction on Σ-p.d (X, f). If Σ-p.d $(X, f) = 0$, then $(X, f) \in \mathscr{P}(\Sigma)$ and the Σ-exact sequence: $(1) : 0 \to \Phi(X, f) \to \mathbf{T}(X) \xrightarrow{\varepsilon(X,f)} (X, f) \to 0$ splits. Hence we have $\mathbf{T}(X) \cong (X, f) \oplus \Phi(X, f)$, and

$$\mathfrak{F}\text{-p.d}\,\mathbf{T}(X) = max\{\mathfrak{F}\text{-p.d}\,(X, f), \mathfrak{F}\text{-p.d}\,\Phi(X, f)\}.$$

Since F is $\mathfrak{R}$-exact, by Corollary 4.2 we have $\mathfrak{F}$-p.d $\mathbf{T}(X) = \mathfrak{R}$-p.d X, thus $\mathfrak{F}$-p.d $(X, f) \leq \mathfrak{R}$-p.d $X \leq \mathfrak{R}$-gl.dim $\mathscr{D}$ and $(*)$ is true. Let Σ-p.d $(X, f) > 0$ and assume $(*)$ is true for all objects (Y, g) with Σ-p.d $(Y, g) < \Sigma$-p.d (X, f). Since $\Sigma \subseteq \mathfrak{F}$, the Σ-exact sequence (1) is also $\mathfrak{F}$-exact, hence for any $(W, h) \in \mathscr{D}_F(\eta)$ we have a long exact sequence:

$$\cdots \to \mathscr{E}xt_{\mathfrak{F}}^{n-1}[\mathbf{T}(X), (W, h)] \to \mathscr{E}xt_{\mathfrak{F}}^{n-1}[\Phi(X, f), (W, h)] \to \mathscr{E}xt_{\mathfrak{F}}^{n}[(X, f), (W, h)]$$

$$\to \mathscr{E}xt_{\mathfrak{F}}^{n}[\mathbf{T}(X), (W, h)] \to \mathscr{E}xt_{\mathfrak{F}}^{n}[\Phi(X, f), (W, h)] \to \cdots$$

We set $s := \Sigma$-p.d $(X, f) + t + 1$. Since F is $\mathfrak{R}$-exact from Corollary 4.2, we have an isomorphism $\mathscr{E}xt_{\mathfrak{F}}^{s-1}[\mathbf{T}(X), (W, h)] \cong \mathscr{E}xt_{\mathfrak{R}}^{s-1}[X, W] = 0$, since $s - 1 = \Sigma$-p.d $(X, f) + t > 0$

and $t = \mathfrak{R}$-gl.dim$\,\mathscr{D}$. Hence the above long exact sequence induces an isomorphism $\mathscr{E}xt_{\mathfrak{F}}^{s-1}[\Phi(X,f),(W,h)] \cong \mathscr{E}xt_{\mathfrak{F}}^{s}[(X,f),(W,h)]$. By induction hypothesis, since Σ-p.d $(X,f)-1 = \Sigma$-p.d$\,\Phi(X,f) < \Sigma$-p.d$\,(X,f)$, we have $\mathfrak{F}$-p.d$\,\Phi(X,f) \leq \Sigma$-p.d$\,\Phi(X,f)+t = s-2$. But then $\mathscr{E}xt_{\mathfrak{F}}^{s-1}[\Phi(X,f),(W,h)] \cong \mathscr{E}xt_{\mathfrak{F}}^{s}[(X,f),(W,h)] = 0$, and consequently $\mathfrak{F}$-p.d$\,(X,f) \leq s-1 = \Sigma$-p.d$\,(X,f)+t$, because (W,h) was arbitrary. We conclude that $(*)$ is true.

If we have further $F(\mathscr{P}(\mathfrak{R})) \subseteq \mathscr{P}(\mathfrak{R})$, then since F is $\mathfrak{R}$-exact, obviously $\forall X \in \mathscr{D}$: $\mathfrak{R}$-p.d$\,FX \leq \mathfrak{R}$-p.d$\,X$. Applying the above induction hypothesis with $t = \mathfrak{R}$-p.d$\,X$, we will arrive at the relation $\mathfrak{F}$-p.d$\,\Phi(X,f) \leq \mathfrak{R}$-p.d$\,FX + t$, because $\Phi(X,f) = (FX, \eta_X - Ff)$. Then $\mathfrak{F}$-p.d$\,\Phi(X,f) \leq \mathfrak{R}$-p.d$\,X + t$ and we can continue the induction as above, concluding that $(**)$ is true. The last relation follows from $(*)$ and Corollary 4.5. $\square$

It is interesting to note that the conditions in (2) of Theorem 5.7, forces Σ to be central in $\mathfrak{F}$, in the sense of [7]. Combining the results of Section 4 and Proposition 5.8(iv) of [6] (which also can be deduced easily from our previous results), we have the following corollary, in which we denote by $c(F) = \infty$ or $c(F) = min\{n \geq 0 : F^{n+1} = 0\}$ the *nilpotency class* of F.

Corollary 5.8. (1) Σ-gl.dim$\,\mathscr{D}_F(\eta) \leq c(F)$. *If* $\eta = 0$ *then* Σ-gl.dim$\,\mathscr{D}_F(0) < \infty \Leftrightarrow$ *the functor* F *is nilpotent*; *in the last case we have* Σ-gl.dim$\,\mathscr{D}_F(0) = c(F)$.

(2) *If* F *is* $\mathfrak{R}$-*exact*, *then we have the following*:

(i) *If the functor* F *is nilpotent*, *then*

$$\mathfrak{R}\text{-gl.dim}\,\mathscr{D} \leq \mathfrak{F}\text{-gl.dim}\,\mathscr{D}_F(\eta) \leq c(F) + \mathfrak{R}\text{-gl.dim}\,\mathscr{D}.$$

(ii) *If* Σ-gl.dim$\,\mathscr{D}_F(\eta) < \infty$: $\mathfrak{F}$-gl.dim$\,\mathscr{D}_F(\eta) < \infty \Leftrightarrow \mathfrak{R}$-gl.dim$\,\mathscr{D} < \infty$.

(iii) *If* Σ-gl.dim$\,\mathscr{D}_F(\eta) = 0$ *then* $\mathfrak{F}$-gl.dim$\,\mathscr{D}_F(\eta) = c(F) + \mathfrak{R}$-gl.dim$\,\mathscr{D}$.

Corollary for Rings 5.9. *Let* $\Lambda = \Gamma \ltimes_{\vartheta} M$ *be a cleft extension of rings.*

(i) (Λ,Γ)-r.gl.dim$\,\Lambda \leq min\{m \in \mathbb{N}\colon \otimes_{\Gamma}^{n+1} M = 0\}$, with equality if $\vartheta = 0$.

(ii) *If* M *is left* Γ-*flat then*

$$\text{r.gl.dim}\,\Gamma \leq \text{r.gl.dim}\,\Lambda \leq (\Lambda,\Gamma)\text{-r.gl.dim}\,\Lambda + \text{r.gl.dim}\,\Gamma.$$

In particular: r.gl.dim$\,\Gamma \leq$ r.gl.dim$\,\Lambda \leq n +$ r.gl.dim$\,\Gamma$, *if* $\otimes_{\Gamma}^{n+1} M = 0$.

(iii) *If* M *is left* Γ-*flat and right* Γ-*projective*, *then* $\forall (X,f) \in \text{Mod}\,(\Lambda)$:

$$\text{p.d}\,(X,f) \leq (\Lambda,\Gamma)\text{-p.d}\,(X,f) + \text{p.d}\,X.$$

(iv) *If* ϑ *is right T-nilpotent*, *then* r.gl.dim$\,\Lambda < \infty \Leftrightarrow$ r.gl.dim$\,\Gamma < \infty$ *and* w.d$_{\Lambda}\Gamma < \infty$. *In any case we have the bounds*

$$\text{r.gl.dim}\,\Gamma \leq \text{r.gl.dim}\,\Lambda \leq \text{w.d}_{\Lambda}\Gamma + \text{r.gl.dim}\,\Gamma.$$

In particular if Γ *is semisimple*, *then* r.gl.dim$\,\Lambda = $ w.d$_{\Lambda}\Gamma$.

For a proper class $\mathfrak{R}$ in an abelian category $\mathscr{C}$, we denote by $\mathscr{P}^{\infty}(\mathfrak{R})$ the full subcategory of $\mathscr{C}$ consisting of objects with finite $\mathfrak{R}$-projective dimension. We recall that the *finitistic* $\mathfrak{R}$*-global dimension* $\mathfrak{R}$-f.gl.dim $\mathscr{C}$ of $\mathscr{C}$ is defined as $\mathfrak{R}\text{-f.gl.dim}\,\mathscr{C} = sup\{\mathfrak{R}\text{-p.d}\,X : X \in \mathscr{P}^{\infty}(\mathfrak{R})\}$. The next result generalizes Lemma 5.3(iii).

Lemma 5.10. *Suppose that* $\operatorname{Ker}\mathbf{C} = 0$.
(1) *If* $(X, f) \in \mathscr{P}^{\infty}(\mathfrak{F})$ *and* $\mathscr{L}_n^{\mathfrak{F}}\mathbf{C}(X, f) = 0$, $\forall n \geq 1$, *then*:

$$\mathfrak{F}\text{-p.d}\,(X, f) = \mathfrak{R}\text{-p.d}\,\mathbf{C}(X, f).$$

(2) *The following are equivalent for an object* $(X, f) \in \mathscr{D}_F(\eta)$:
(i) $(X, f) \in \mathscr{P}^{\infty}(\mathfrak{F})$.
(ii) $\exists n \geq 0$: $\mathscr{E}xt_{\mathfrak{F}}^{n+1}[(X, f), \mathbf{Z}(-)] = 0$.
(iii) $\exists n \geq 0$: *Coker*$(\omega_n^{(X,f)}) \in \mathscr{P}^{\infty}(\mathfrak{R})$, *and* $\mathscr{L}_{n+k}^{\mathfrak{F}}\mathbf{C}(X, f) = 0$, $\forall k \geq 1$, *where* $\omega_n^{(X,f)}$ *is the nth differential of the* $\mathfrak{F}$*-associated complex* $\mathfrak{F}^{\bullet}_{(X,f)}$ *of* (X, f).

If one of the above equivalent statements is true, then setting $n = min\{m \geq 0$: $\mathscr{L}_{m+k}^{\mathfrak{F}}\mathbf{C}(X, f) = 0$, $\forall k \geq 1\}$, *we have*

$$\mathfrak{F}\text{-p.d}\,(X, f) \leq n + \mathfrak{R}\text{-p.d}\,Coker(\omega_n^{(X,f)}).$$

Proof. (1) By induction on $\mathfrak{F}$-p.d$\,(X, f)$. If $\mathfrak{F}$-p.d$\,(X, f)$=0, obviously $\mathfrak{R}$-p.d$\,\mathbf{C}(X, f)$= 0. Suppose that the result is true for all (Y, g) with $\mathfrak{F}\text{-p.d}\,(Y, g) < \mathfrak{F}\text{-p.d}\,(X, f)$ and $\mathscr{L}_t^{\mathfrak{F}}\mathbf{C}(Y, g) = 0$, $\forall t \geq 1$. If $\mathfrak{F}\text{-p.d}\,(X, f) > 0$, consider the $\mathfrak{F}$-projective presentation $0 \to (Q_0, g_0) \xrightarrow{\kappa_0} \mathbf{T}(P_0) \xrightarrow{\alpha_0} (X, f) \to 0$. Trivially (Q_0, g_0) satisfies the induction hypothesis, so we have $\mathfrak{F}\text{-p.d}\,(Q_0, g_0) = \mathfrak{R}\text{-p.d}\,\mathbf{C}(Q_0, g_0)$. Since $\mathscr{L}_t^{\mathfrak{F}}\mathbf{C}(X, f) = 0$, $\forall t \geq 1$, the sequence $0 \to \mathbf{C}(Q_0, g_0) \xrightarrow{\mathbf{C}(\kappa_0)} P_0 \xrightarrow{\mathbf{C}(\alpha_0)} \mathbf{C}(X, f) \to 0$ is an $\mathfrak{R}$-projective presentation of $\mathbf{C}(X, f)$. So: $\mathfrak{F}\text{-p.d}\,(X, f) = 1 - \mathfrak{F}\text{-p.d}\,(Q_0, g_0) = 1 - \mathfrak{R}\text{-p.d}\,\mathbf{C}(Q_0, g_0) = \mathfrak{R}\text{-p.d}\,\mathbf{C}(X, f)$.

(2) (i) $\Leftrightarrow$ (ii) Follows from Lemma 5.3(iii), and an easy induction argument.

(i) $\Rightarrow$ (iii) If $\mathfrak{F}\text{-p.d}\,(X, f) = m < \infty$, then the mth-syzygy (Q_{m-1}, g_{m-1}) of (X, f) is $\mathfrak{F}$-projective. By Lemma 5.3(iii), we have $\mathbf{C}(Q_{m-1}, g_{m-1}) \cong Coker(\omega_m) \in \mathscr{P}(\mathfrak{R})$ and $\mathscr{L}_1^{\mathfrak{F}}\mathbf{C}(Q_{m-1}, g_{m-1}) = \mathscr{L}_{m+1}^{\mathfrak{F}}\mathbf{C}(X, f) = 0$.

(iii) $\Rightarrow$ (i) If $\mathfrak{R}\text{-p.d}\,Coker(\omega_n) = t < \infty$, and $\mathscr{L}_{n+k}^{\mathfrak{F}}\mathbf{C}(X, f) = 0$, $\forall k \geq 1$, then the tth-syzygy of $Coker(\omega_n)$, (which is $Coker(\omega_{n+t})$ because $\mathscr{L}_{n+k}^{\mathfrak{F}}\mathbf{C}(X, f)$=0, $\forall k \geq 1$), is $\mathfrak{R}$-projective. But $Coker(\omega_{n+t}) \cong \mathbf{C}(Q_{n+t-1}, g_{n+t-1})$. Now since $\mathscr{L}_1^{\mathfrak{F}}\mathbf{C}(Q_{n+t-1}, g_{n+t-1})$= $\mathscr{L}_{n+t+1}^{\mathfrak{F}}\mathbf{C}(X, f) = 0$, we have by Lemma 5.3(iii), that $(Q_{n+t-1}, g_{n+t-1}) \in \mathscr{P}(\mathfrak{F})$. Since (Q_{n+t-1}, g_{n+t-1}) is a $(n + t)$th-syzygy of (X, f), we have $\mathfrak{F}\text{-p.d}\,(X, f) \leq n + t$, and $(X, f) \in \mathscr{P}^{\infty}(\mathfrak{F})$. The last assertion follows from the above proof. □

Theorem 5.11. (1) *If* $\operatorname{Ker}\mathbf{C} = 0$, *then*

$$\mathfrak{F}\text{-f.gl.dim}\,\mathscr{D}_F(\eta) \leq \aleph + \mathfrak{R}\text{-f.gl.dim}\,\mathscr{D},$$

where $\aleph = sup\{m \geq 0$: $\mathscr{L}_{m+k}^{\mathfrak{F}}\mathbf{C}(X, f) = 0$, $\forall k \geq 1$, $\forall (X, f) \in \mathscr{P}^{\infty}(\mathfrak{F})\}$.

(2) *If the functor* $\mathscr{L}^{\mathfrak{F}}_{n+1}\mathbf{C}$ *vanishes on* $\mathscr{P}^{\infty}(\mathfrak{F})$, *in particular if* $\mathscr{L}^{\mathfrak{F}}_{n+1}\mathbf{C}=0$, *then* $\mathfrak{F}$-f.gl.dim $\mathscr{D}_F(\eta) \leq n + \mathfrak{R}$-f.gl.dim $\mathscr{D}$. *In particular if* $\mathscr{L}^{\mathfrak{F}}_{t}\mathbf{C}|_{\mathscr{P}^{\infty}(\mathfrak{F})}=0$, $\forall t \geq 1$, *then* $\mathfrak{F}$-f.gl.dim $\mathscr{D}_F(\eta) \leq \mathfrak{R}$-f.gl.dim $\mathscr{D}$.

(3) *If the functor* F *is* $\mathfrak{R}$*-exact, then* $\mathfrak{R}$-f.gl.dim $\mathscr{D} \leq \mathfrak{F}$-f.gl.dim $\mathscr{D}_F(\eta)$, *and the following are equivalent*:

(i) $\mathfrak{F}$-f.gl.dim $\mathscr{D}_F(\eta) < \infty$.

(ii) (a) $\aleph = sup\{m \geq 0\colon \mathscr{L}^{\mathfrak{F}}_{m+k}\mathbf{C}(X,f)=0,\ \forall k \geq 1,\ \forall (X,f) \in \mathscr{P}^{\infty}(\mathfrak{F})\} < \infty$.

(b) $\mathfrak{R}$-f.gl.dim $\mathscr{D} < \infty$.

If one of the above equivalent statements is true then

$$\mathfrak{R}\text{-f.gl.dim}\,\mathscr{D} \leq \mathfrak{F}\text{-f.gl.dim}\,\mathscr{D}_F(\eta) \leq \aleph + \mathfrak{R}\text{-f.gl.dim}\,\mathscr{D}.$$

Finally if in addition the functor $\mathscr{L}^{\mathfrak{F}}_{t}\mathbf{C}$ *vanishes on* $\mathscr{P}^{\infty}(\mathfrak{F})$, $\forall t \geq 1$, *then*

$$\mathfrak{F}\text{-f.gl.dim}\,\mathscr{D}_F(\eta) = \mathfrak{R}\text{-f.gl.dim}\,\mathscr{D}.$$

(4) *If* F *is* $\mathfrak{R}$*-exact, then*

$$\mathfrak{R}\text{-f.gl.dim}\,\mathscr{D} \leq \mathfrak{F}\text{-f.gl.dim}\,\mathscr{D}_F(\eta) \leq \Sigma\text{-f.gl.dim}\,\mathscr{D}_F(\eta) + \mathfrak{R}\text{-gl.dim}\,\mathscr{D}.$$

If F *is* $\mathfrak{R}$*-exact and* $F(\mathscr{P}(\mathfrak{R})) \subseteq \mathscr{P}(\mathfrak{R})$, *then*

$$\mathfrak{R}\text{-f.gl.dim}\,\mathscr{D} \leq \mathfrak{F}\text{-f.gl.dim}\,\mathscr{D}_F(\eta) \leq \Sigma\text{-f.gl.dim}\,\mathscr{D}_F(\eta) + \mathfrak{R}\text{-f.gl.dim}\,\mathscr{D}.$$

Proof. (1) and (2) are consequences of Lemma 5.10. If F is exact, then by Corollary 4.2, we have $\forall X \in \mathscr{D}$: $\mathfrak{R}$-p.d $X = \mathfrak{F}$-p.d $\mathbf{T}(X)$. Hence $\mathfrak{R}$-f.gl.dim $\mathscr{D} \leq \mathfrak{F}$-f.gl.dim $\mathscr{D}_F(\eta)$, and (3) follows from (2). Part (4) follows from Theorem 5.7. □

Let $\Lambda = \Gamma \ltimes_{\vartheta} M$ be a cleft extension of rings. In the following consequence of Theorem 5.11, we denote by $\mathscr{P}^{\infty}(\Lambda)$, the full subcategory of Mod(Λ) consisting of all modules of finite projective dimension.

Corollary for Rings 5.12. (1) *If* ϑ *is right* T*-nilpotent, then*

$$\text{f.gl.dim}\,\Lambda \leq \aleph + \text{f.gl.dim}\,\Gamma,$$

where

$$\aleph = sup\{m \geq 0\colon \mathscr{T}or^{\Lambda}_{m+k}[(X,f),\Gamma]=0,\ \forall k \geq 1, \forall (X,f) \in \mathscr{P}^{\infty}(\Lambda)\}.$$

(2) *If the functor* $\mathscr{T}or^{\Lambda}_{n+1}[-,_{\Lambda}\Gamma]$ *vanishes on* $\mathscr{P}^{\infty}(\Lambda)$, *in particular if* w.d$_{\Lambda}\Gamma \leq n$, *then* r.f.gl.dim $\Lambda \leq n +$ r.f.gl.dim Γ.

(3) *If* M *is left* Γ*-flat, then* r.f.gl.dim $\Gamma \leq$ r.f.gl.dim Λ, *and the following are equivalent*:

(i) f.gl.dim $\Lambda < \infty$.

(ii) (a) *The number* $\aleph$ *in* (1) *is finite.*

(b) r.f.gl.dim $\Gamma < \infty$.

Finally if in addition the functor $\mathscr{T}or^{\Lambda}_{t}[-,_{\Lambda}\Gamma]$ *vanishes on* $\mathscr{P}^{\infty}(\Lambda)$, $\forall t \geq 1$, *then*

$$\text{r.f.gl.dim}\,\Lambda = \text{r.f.gl.dim}\,\Gamma.$$

(4) *If M is left Γ-flat, then*

$$\text{r.f.gl.dim}\,\Gamma \le \text{r.f.gl.dim}\,\Lambda \le (\Lambda,\Gamma)\text{-r.f.gl.dim}\,\Lambda + \text{r.gl.dim}\,\Gamma.$$

If M is left Γ-flat and right Γ-projective, then

$$\text{r.f.gl.dim}\,\Gamma \le \text{r.f.gl.dim}\,\Lambda \le (\Lambda,\Gamma)\text{-r.f.gl.dim}\,\Lambda + \text{r.f.gl.dim}\,\Gamma.$$

5.2. Applications to free, symmetric, polynomial and exterior categories

We apply the results obtained so far to the above mentioned categories. We follow the notation introduced in Section 2.5.

Let $\mathscr{D}\langle F_i,\ i\in I\rangle$ be the free category of the abelian category $\mathscr{D}$, with respect to the family of right exact endofunctors $\{F_i\}_{i\in I}$ of $\mathscr{D}$, $F_i\neq 0$, $\forall i$. Let $J := max\{\aleph_0, |I|\}$ and assume that $\mathscr{D}$ is an $AB4^{(J)}$-category, i.e. $\mathscr{D}$ has exact coproducts indexed by J. The following generalizes a result of Mitchell [28].

Theorem 5.13. *If F_i is $\mathfrak{R}$-exact $\forall i\in I$, then $\Sigma\text{-gl.dim}\,\mathscr{D}\langle F_i,\ i\in I\rangle = 1$ and*

$$\mathfrak{R}\text{-gl.dim}\,\mathscr{D} \le \mathfrak{F}\text{-gl.dim}\,\mathscr{D}\langle F_i,\ i\in I\rangle \le 1 + \mathfrak{R}\text{-gl.dim}\,\mathscr{D}.$$

Moreover, the following are equivalent:

(a) $\mathfrak{F}\text{-gl.dim}\,\mathscr{D}\langle F_i,\ i\in I\rangle = 1 + \mathfrak{R}\text{-gl.dim}\,\mathscr{D}$.

(b) $\exists X\in\mathscr{D}$: $sup_{i\in I}\{\mathfrak{R}\text{-p.d}\,F_iX\} = \mathfrak{R}\text{-gl.dim}\,\mathscr{D}$.

In particular $\mathfrak{F}\text{-gl.dim}\,\mathscr{D}\langle I\rangle = 1 + \mathfrak{R}\text{-gl.dim}\,\mathscr{D}$ *and* $\forall n\ge 0$:

$$\mathscr{E}xt^n_{\mathfrak{F}}[\mathbf{Z}(-),\mathbf{Z}(-)] \cong \mathscr{E}xt^n_{\mathfrak{R}}[-,-] \oplus \prod_{i\in I} \mathscr{E}xt^{n-1}_{\mathfrak{R}}[F_i-,-].$$

Proof. (i) We view $\mathscr{D}\langle F_i,\ i\in I\rangle$ as $\mathscr{D}(F)$, which is a cleft extension of $\mathscr{D}$. Then F is $\mathfrak{R}$-exact and moreover from the shape of the left adjoint $\mathbf{T}$ of $\mathbf{U}$ (see Section 2.5.1 or [6]), it follows that $\forall (X,f)\in\mathscr{D}(F)$: $\Phi(X,f)\cong \mathbf{T}(FX)$ $(*)$. From $(*)$ we see that $\Sigma\text{-gl.dim}\,\mathscr{D}(F)\le 1$. In addition if $F_i\neq 0$ for some $i\in I$, then $\Sigma\text{-gl.dim}\,\mathscr{D}(F) = 1$, since otherwise $\forall X\in\mathscr{D}$, the Σ-epic $\varepsilon_{\mathbf{Z}(X)}$ splits, and this implies trivially that $F(X)=0$. Then by Theorem 5.7: $\mathfrak{R}\text{-gl.dim}\,\mathscr{D} \le \mathfrak{F}\text{-gl.dim}\,\mathscr{D}(F) \le 1 + \mathfrak{R}\text{-gl.dim}\,\mathscr{D}$. From the functorial Σ-projective presentation of (X,f) we deduce the following long exact sequence, $\forall (Y,g)\in\mathscr{D}(F)$:

$$\cdots \to \mathscr{E}xt^{n-1}_{\mathfrak{F}}[\Phi(X,f),(Y,g)] \to \mathscr{E}xt^n_{\mathfrak{F}}[(X,f),(Y,g)] \to \mathscr{E}xt^n_{\mathfrak{F}}[\mathbf{T}(X),(Y,g)]$$

$$\to \mathscr{E}xt^n_{\mathfrak{F}}[\Phi(X,f),(Y,g)] \to \cdots$$

(a) $\Rightarrow$ (b) Suppose $\mathfrak{R}\text{-gl.dim}\,\mathscr{D} = m < \infty$ and $\mathfrak{F}\text{-gl.dim}\,\mathscr{D}(F) = 1 + m$. Since F is $\mathfrak{R}$-exact by $(*)$ and Corollary 4.2(3), the above long exact sequence induces an epic: $\mathscr{E}xt^m_{\mathfrak{R}}[FX,Y] \to \mathscr{E}xt^{m+1}_{\mathfrak{F}}[(X,f),(Y,g)] \to 0$. Hence $\mathfrak{R}\text{-p.d}\,FX = m$, since otherwise $\mathfrak{F}\text{-gl.dim}\,\mathscr{D}(F) < 1 + m$ contrary to our assumption. (b) $\Rightarrow$ (a) Since F is $\mathfrak{R}$-exact, by Corollary 4.2(3) we have

$$\begin{aligned}\mathscr{E}xt^{m+1}_{\mathfrak{F}}[\mathbf{Z}(X),\mathbf{Z}(Y)] &\cong \mathscr{E}xt^{m}_{\mathfrak{F}}[\Phi\mathbf{Z}(X),\mathbf{Z}(Y)] \cong \mathscr{E}xt^{m}_{\mathfrak{F}}[\mathbf{T}(FX),\mathbf{Z}(Y)]\\ &\cong \mathscr{E}xt^{m}_{\mathfrak{R}}[FX,Y].\end{aligned}$$

Since by hypothesis the last group is $\neq 0$, we have $\mathfrak{F}$-gl.dim $\mathscr{D}(F)=1+m$. The proof of the last assertion follows from Corollary 4.2(3). □

As a corollary of the above theorem and its dual we have the following result (part (1) is classical and part (2) is a result of Roganov, see [9]).

Corollary for Rings 5.14. (1) r.gl.dim $\Gamma\langle S\rangle = 1 + \text{r.gl.dim}\,\Gamma$.

(2) *Let M be a Γ–Γ-bimodule, flat as a left Γ-module. Then*

$$\text{r.gl.dim}\,\Gamma \leq \text{r.gl.dim}\,\mathrm{T}_{\Gamma}(M) \leq 1 + \text{r.gl.dim}\,\Gamma$$

and the following are equivalent:

(i) r.gl.dim $\mathrm{T}_{\Gamma}(M) = 1 + \text{r.gl.dim}\,\Gamma$.

(ii) $\exists X \in \mathrm{Mod}(\Gamma)$: p.d $X\otimes_{\Gamma} M = \text{r.gl.dim}\,\Gamma$.

(iii) $\exists X \in \mathrm{Mod}(\Gamma)$: i.d $\mathrm{Hom}_{\Gamma}[M,X] = \text{r.gl.dim}\,\Gamma$.

The following are equivalent:

(i) r.gl.dim $\mathrm{T}_{\Gamma}(M) \leq 1$.

(ii) r.gl.dim $\Gamma \leq 1$ *and* $\forall X \in \mathrm{Mod}(\Gamma)$, *the Γ-module $X\otimes_{\Gamma} M$ is projective.*

(iii) r.gl.dim $\Gamma \leq 1$ *and* $\forall X \in \mathrm{Mod}(\Gamma)$, *the Γ-module* $\mathrm{Hom}_{\Gamma}[M,X]$ *is injective.*

Corollary 5.15. *If $\mathscr{Q}$ is a quiver with finitely many vertices and Γ is semisimple algebra, then the quiver-algebra $\Gamma\mathscr{Q}$ is hereditary.*

Consider the symmetric category $\mathscr{D}[F_1,F_2,\dots,F_m]$ of $\mathscr{D}$ with respect to a set $\{F_i\}_{i=1}^n$ of commuting right exact endofunctors of $\mathscr{D}$ and let $\mathscr{D}[m]$ be the polynomial category of $\mathscr{D}$ in m-variables. Assume that $\mathscr{D}$ is an $AB4^{(\aleph_0)}$-category. The following includes a result of Mitchell [28].

Theorem 5.16. *Assume that F_i is $\mathfrak{R}$-exact, $\forall i=1,\dots,m$. Then*:

$$\mathfrak{R}\text{-gl.dim}\,\mathscr{D} \leq \mathfrak{F}\text{-gl.dim}\,\mathscr{D}[F_1,F_2,\dots,F_m] \leq m + \mathfrak{R}\text{-gl.dim}\,\mathscr{D},$$

$$\Sigma\text{-gl.dim}\,\mathscr{D}[F_1,F_2,\dots,F_m] = m, \quad \mathfrak{F}\text{-gl.dim}\,\mathscr{D}[m] = m + \mathfrak{R}\text{-gl.dim}\,\mathscr{D},$$

$$\begin{aligned}\mathscr{E}xt^{n}_{\mathfrak{F}}[\mathbf{Z}(-),\mathbf{Z}(-)] &\cong \mathscr{E}xt^{n}_{\mathfrak{R}}[-,-] \oplus \bigoplus_{i=1}^{m} \mathscr{E}xt^{n-1}_{\mathfrak{R}}[F_i-,-] \oplus \bigoplus_{i<j} \mathscr{E}xt^{n-2}_{\mathfrak{R}}[F_iF_j-,-]\\ &\oplus \bigoplus_{i<j<k} \mathscr{E}xt^{n-3}_{\mathfrak{R}}[F_iF_jF_k-,-] \oplus \cdots \oplus \mathscr{E}xt^{n-m}_{\mathfrak{R}}[F_1F_2\cdots F_m-,-].\end{aligned}$$

$\forall X \in \mathscr{D}$, *the $\mathfrak{F}$-cohomology ring $\mathscr{E}xt^{*}_{\mathfrak{F}}[\mathbf{Z}(X),\mathbf{Z}(X)]$ of $\mathbf{Z}(X)$ in the polynomial category $\mathscr{D}[m]$, is the exterior ring of* $\mathrm{End}_{\mathscr{D}}(X)$ *in m-variables, and is a cleft extension of the $\mathfrak{R}$-cohomology ring $\mathscr{E}xt^{*}_{\mathfrak{R}}[X,X]$* of X in $\mathscr{D}$.

Proof. Follows directly from Theorem 5.13 and induction. □

Corollary for Rings 5.17. r.gl.dim $\Gamma[\mathbf{x}_1,\mathbf{x}_2,\ldots,\mathbf{x}_m] = m +$ r.gl.dim Γ, *and*

$$\forall n \geq 0\colon\ \mathscr{E}xt^n_{\Gamma[\mathbf{x}_1,\mathbf{x}_2,\ldots,\mathbf{x}_m]}[\Gamma,\Gamma] \cong \Gamma^{(m!/n!(m-n)!)}.$$

Finally, consider the exterior category $\bigwedge(\mathscr{D})(F_1,F_2,\ldots,F_m)$ of $\mathscr{D}$ with respect to a set of commuting right exact endofunctors $\{F_i\}_{i=1}^m$ of $\mathscr{D}$ and the exterior category $\bigwedge^m(\mathscr{D})$ of $\mathscr{D}$ in m-variables. The next theorem which includes a result of Mitchell [28], follows by induction directly from the results of this section.

Theorem 5.18. *Assume that each F_i is $\mathfrak{R}$-exact.*

(1) *The following are equivalent*:

(i) $\mathfrak{F}$-gl.dim $\bigwedge(\mathscr{D})(F_1,F_2,\ldots,F_m) < \infty$.

(ii) $\mathfrak{R}$-gl.dim $\mathscr{D} < \infty$ *and F_i is nilpotent*, $\forall i = 1,2,\ldots,m$.

(2) $\mathfrak{F}$-gl.dim $\bigwedge^m(\mathscr{D}) = \infty$, *and* $\forall n \geq 0$:

$$\mathscr{E}xt^n_{\mathfrak{F}}[\mathbf{Z}(-),\mathbf{Z}(-)] \cong \bigoplus_{k_{m+1}+k_m+\cdots+k_1=n} \mathscr{E}xt^{k_{m+1}}_{\mathfrak{R}}[F_1^{k_m}F_2^{k_{m-1}}\cdots F_{m-1}^{k_2}F_m^{k_1}-,-],$$

$$\mathfrak{R}\text{-gl.dim}\,\mathscr{D} \leq \mathfrak{F}\text{-gl.dim} \bigwedge(\mathscr{D})(F_1,F_2,\ldots,F_m) \leq \sum_{i=1}^m c(F_i) + \mathfrak{R}\text{-gl.dim}\,\mathscr{D}.$$

Corollary for Rings 5.19. r.gl.dim $\bigwedge^n_{\Gamma}(\Gamma) = \infty$.

5.3. The Hilbert basis theorem

We seek conditions for the cleft extension $\mathscr{D}_F(\eta)$ to be (locally) Noetherian. The results of this subsection are independent of the results of the previous sections.

Lemma 5.20. *If X, FX are Noetherian objects in $\mathscr{D}$, then so is $\mathbf{T}(X)$ in $\mathscr{D}_F(\eta)$. Hence $\mathscr{D}_F(\eta)$ is Noetherian iff $\mathscr{D}$ is Noetherian.*

Proof. Let $(X_0,f_0) \subseteq \cdots \subseteq (X_n,f_n) \subseteq \cdots \subseteq \mathbf{T}(X)$ be a chain of subobjects. Applying $\mathbf{U}$ we have a chain of subobjects $X_0 \subseteq \cdots \subseteq X_n \subseteq \cdots \subseteq X \oplus FX$. Since $X \oplus FX$ is Noetherian, there exists n_0 such that $X_n = X_{n_0}, \forall n \geq n_0$. Since $\mathbf{U}$ is faithful, we have $(X_n,f_n) = (X_{n_0},f_{n_0})$, $\forall n \geq n_0$. Hence $\mathbf{T}(X)$ is Noetherian. If $\mathscr{D}$ is Noetherian then $\mathscr{D}_F(\eta)$ is Noetherian, since any (X,f) is a quotient of $\mathbf{T}(X)$. The converse is easy since $\mathscr{D} = \mathbf{Z}(\mathscr{D})$ is closed under subobjects in $\mathscr{D}_F(\eta)$. □

From now on we assume that $\mathscr{D}$ has injectives and exact coproducts and the functor F has a right adjoint G. Then it is easy to see that $\mathscr{D}_F(\eta)$ has coproducts and $\mathscr{D}_F(\eta)$ is also a cleft coextension. We view the isomorphism $\mathscr{D}^G(\zeta) \cong \mathscr{D}_F(\eta)$ of Section 2, as an identification. Then we have the adjoint pairs $(\mathbf{T},\mathbf{U})$, $(\mathbf{U},\mathbf{H})$, $(\mathbf{C},\mathbf{Z})$ and $(\mathbf{Z},\mathbf{K})$.

The adjoint pair $(\mathbf{U},\mathbf{H})$ induces an exact sequence of functors

$$0 \to \mathrm{Id}_{\mathscr{D}_F(\eta)} \xrightarrow{\mu} \mathbf{HU} \xrightarrow{\nu} \Psi \to 0,$$

where μ is the unit of the adjoint pair $(\mathbf{U},\mathbf{H})$ defined by $\mu_{(X,f)} = (f, 1_X)$, $\nu_{(X,f)} = {}^t(1_{GX}, -f)$ and the functor Ψ is defined in $\mathscr{D}^G(\zeta)$ by $\Psi(X,f) = (GX, \zeta_X - Gf)$. Obviously, the functor Ψ is a right adjoint of the functor Φ introduced in Section 3.2. Recall that Φ is defined in $\mathscr{D}_F(\eta)$ by $\Phi(X,f) = (FX, \eta_X - Ff)$.

Theorem 5.21. *Assume that F is exact and its right adjoint G preserves coproducts. Any coproduct of injectives in $\mathscr{D}$ is injective iff the same is true in $\mathscr{D}_F(\eta)$.*

Proof. Since F is exact it follows trivially that the functor Φ is exact. Hence its right adjoint Ψ preserves injectives. Similarly, since $\mathbf{U}$ is exact, its right adjoint $\mathbf{H}$ preserves injectives. Let $(X_i, f_i)_{i\in I}$ be a set of injective objects in $\mathscr{D}^G(\zeta)$ and consider the object $(X,f) := \bigoplus_{i\in I}(X_i, f_i)$. Then we have short exact sequences

$$(*)\colon\ 0 \to (X_i, f_i) \xrightarrow{\mu_{(X_i,f_i)}} \mathbf{H}(X_i) \xrightarrow{\nu_{(X_i,f_i)}} \Psi(X_i, f_i) \to 0, \quad \forall i \in I.$$

Since (X_i, f_i) is an injective object, $\forall i \in I$, we have that $\mathbf{H}(X_i) \cong (X_i, f_i) \oplus \Psi(X_i, f_i)$. Since (X_i, f_i) is injective and Ψ preseves injectives, it follows that $\mathbf{H}(X_i)$ is an injective object $\forall i \in I$. Since $\mathbf{Z}$ is exact, it follows that $\mathbf{K}$ preserves injectives. Since $\mathbf{KH} = \mathrm{Id}_{\mathscr{D}^G(\zeta)}$, we have that X_i is injective, $\forall i \in I$. Since any coproduct of injectives $\mathscr{D}$ is injective, we have that $\bigoplus_{i\in I} X_i$ is injective in $\mathscr{D}$. Since $\mathbf{H}$ preserves injectives, we have that $\mathbf{H}(\bigoplus_{i\in I} X_i)$ is injective in $\mathscr{D}^G(\zeta)$. But since G preserves coproducts, it follows easily that the functors $\mathbf{H}$, Ψ preserve coproducts. Hence $\bigoplus_{i\in I}\mathbf{H}(X_i)$ is injective. The coproduct of the split short exact sequences $(*)$, is the split short exact sequence:

$$0 \to \bigoplus_{i\in I}(X_i, f_i) \to \bigoplus_{i\in I}\mathbf{H}(X_i) \to \bigoplus_{i\in I}\Psi(X_i, f_i) \to 0.$$

Since $\bigoplus_{i\in I}(X_i, f_i)$ is a direct summand of $\bigoplus_{i\in I}\mathbf{H}(X_i)$, it is injective. The converse is easy and is left to the reader. □

From now on we assume that $\mathscr{D}$ is a Grothendieck category and $F : \mathscr{D} \to \mathscr{D}$ preserves colimits. By [6], $\mathscr{D}_F(\eta)$ is a Grothendieck category. Since F preserves colimits and $\mathscr{D}$ is Grothendieck, by the Special Adjoint Theorem it follows that F has a right adjoint $G\colon \mathscr{D} \to \mathscr{D}$. We recall that an object $X \in \mathscr{D}$ is called *finitely presented*, if the functor $\mathscr{D}(X, -)$ preserves filtered colimits. We recall that $\mathscr{D}$ is called *locally finitely presented* if $\mathscr{D}$ has a set of finitely presented generators. The category $\mathscr{D}$ is called *locally Noetherian* if $\mathscr{D}$ has a set of Noetherian generators. Note that in this case by Gabriel's theory [14], the category of finitely presented objects of $\mathscr{D}$ is identified with the category of Noetherian objects of $\mathscr{D}$.

Theorem 5.22. *Assume that $\mathscr{D}$ is a locally finitely presented Grothendieck category and the functor F preserves colimits. If F preserves finitely presented objects* (*equivalently its right adjoint* G *preserves filtered colimits*), *then* $\mathscr{D}_F(\eta)$ *is locally Noetherian iff so is* $\mathscr{D}$.

Proof. From the adjoint pair $(\mathbf{T},\mathbf{U})$ and the fact that $\mathbf{U}$ is faithful and preserves colimits, it follows directly that $\mathbf{T}$ preserves generators and finitely presented objects. Hence $\mathscr{D}_F(\eta)$ is locally finitely presented. If $\mathscr{D}$ is locally Noetherian by Lemma 5.20, it suffices to show that F preserves Noetherian or equivalently finitely presented objects, and this is true by hypothesis. The converse and the parenthetical assertion are easy and are left to the reader. $\square$

Corollary for Rings 5.23. *Let* $\Lambda = \Gamma \ltimes_\vartheta M$ *be a cleft extension of rings. If* Γ *is right Noetherian and* M_Γ *is finitely generated, then* Λ *is right Noetherian.*

Applying Theorems 5.21 and 5.22 to the free cleft extension $\mathscr{D}(F)$ and using induction we have the following.

Corollary 5.24. *Let* $\{F_1, F_2, \ldots, F_m\}$ *be a set of commuting endofunctors of* $\mathscr{D}$. *Assume that* $\mathscr{D}$ *has exact coproducts and each* F_i *has a right adjoint* G_i.

(i) *Suppose that each* F_i *is exact and its right adjoint* G_i, *preserves coproducts. Then any coproduct of injectives objects in* $\mathscr{D}$ *is an injective object iff the same is true for the symmetric category* $\mathscr{D}[F_1, F_2, \ldots, F_m]$.

(ii) *If* $\mathscr{D}$ *is locally finitely presented and if each* G_i *preserves filtered colimits, then* $\mathscr{D}$ *is locally Noetherian iff so is* $\mathscr{D}[F_1, F_2, \ldots, F_m]$.

As a corollary of the above result, we obtain the classical Hilbert Basis Theorem:

Corollary for Rings 5.25. *If the ring* Γ *is right Noetherian, then so is the polynomial ring* $\Gamma[\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_m]$.

6. Cleft extensions of small global dimension and Frobenius cleft extensions

6.1. Cleft extensions of global dimension ≤ 2

We characterize when $\mathscr{D}_F(\eta)$ has relative $\mathfrak{F}$-global dimension (smaller or equal to) 1 or 2.

Proposition 6.1. (1) *If* $\operatorname{Ker}\mathbf{C} = 0$, *then the following are equivalent*:

(i) $\mathfrak{F}$-gl.dim $\mathscr{D}_F(\eta) \leq 1$.

(ii) (a) $\mathfrak{R}$-gl.dim $\mathscr{D} \leq 1$.

(b) *The functor* F *is* $\mathfrak{R}$-*exact.*

(c) $\forall (X, f) \in \mathscr{D}_F(\eta)$: *Ker*$(f)$/*Im*$(\eta_X - Ff) \in \mathscr{P}(\mathfrak{R})$ *and Ker*$(\eta_X - Ff)$=*Im*$(\eta_{FX} - F\eta_X + F^2 f)$.

(2) *If* $\mathscr{D}_F(\eta)$ *is admissible, then the following are equivalent*:

(i) $\mathfrak{F}$-gl.dim $\mathscr{D}_F(\eta) \leq 1$.

(ii) (a) $\mathfrak{R}$-gl.dim $\mathscr{D} \leq 1$.

(b) *The functor* F *is* $\mathfrak{R}$*-exact*.

(c) $\forall X \in \mathscr{D}$: $Coker(\eta_X) \in \mathscr{P}(\mathfrak{R})$ $(FX \in \mathscr{P}(\mathfrak{R})$, *if* $\eta = 0)$.

(d) $Ker(\eta) = Im(\eta F - F\eta)$ $(F^2 = 0$, *if* $\eta = 0)$.

(3) *In* (1) and (2) *if* $F \neq 0$, *then* $\mathfrak{F}$-gl.dim $\mathscr{D}_F(\eta) = 1$.

Proof. (1) Suppose that $\mathfrak{F}$-gl.dim $\mathscr{D}_F(\eta) \leq 1$. By Corollary 4.5, we have that $\mathfrak{R}$-gl.dim $\mathscr{D} \leq 1$. Since $\mathscr{L}_2^{\mathfrak{F}}\mathbf{C} = 0$, by Theorem 4.7 we have $\mathscr{L}_2^{\Sigma}\mathbf{C}(X, f) = Ker(\eta_X - Ff)/Im(\eta_{FX} - F\eta_X + F^2 f) = 0$, and by Corollary 4.2 we have $\mathscr{L}_1^{\Sigma}\mathbf{C}(X, f) = Ker(f)/Im(\eta_X - Ff) \in \mathscr{P}(\mathfrak{R})$. Since $\mathscr{L}_2^{\mathfrak{F}}\mathbf{CT} = 0$, by Corollary 4.2 we have $\mathbf{C}\mathscr{L}_1^{\mathfrak{R}}\mathbf{T} = 0$ and then $\mathscr{L}_1^{\mathfrak{R}}\mathbf{T} = 0$, since Ker $\mathbf{C} = 0$. Hence $\mathscr{L}_1^{\mathfrak{R}}F = 0$ and F is $\mathfrak{R}$-exact. Suppose now that (a)–(c) in (ii) are true. Since F is $\mathfrak{R}$-exact, by Theorem 4.7 we have $\mathscr{L}_i^{\mathfrak{F}}\mathbf{C}(X, f) \cong \mathscr{L}_i^{\Sigma}\mathbf{C}(X, f)$, $\forall i \geq 0$. Hence by (c) we have $\mathscr{L}_2^{\mathfrak{F}}\mathbf{C}(X, f) = 0$ and $\mathscr{L}_1^{\mathfrak{F}}\mathbf{C}(X, f) \in \mathscr{P}(\mathfrak{R})$. If $0 \to (Q_0, g_0) \to \mathbf{T}(P_0) \to (X, f) \to 0$ is an $\mathfrak{F}$-projective presentation of (X, f) then we have the induced $\mathfrak{R}$-exact sequence: $0 \to \mathscr{L}_1^{\mathfrak{F}}\mathbf{C}(X, f) \to \mathbf{C}(Q_0, g_0) \to P_0 \to \mathbf{C}(X, f) \to 0$. Since $\mathfrak{R}$-gl.dim$\mathscr{D}$ and $\mathscr{L}_1^{\mathfrak{F}}\mathbf{C}(X, f) \in \mathscr{P}(\mathfrak{R})$, we see directly that $\mathbf{C}(Q_0, g_0) \in \mathscr{P}(\mathfrak{R})$. But $\mathscr{L}_1^{\mathfrak{F}}\mathbf{C}(Q_0, g_0) \cong \mathscr{L}_2^{\mathfrak{F}}\mathbf{C}(X, f) = 0$, so by Lemma 5.2, $(Q_0, g_0) \in \mathscr{P}(\mathfrak{F})$. Hence $\mathfrak{F}$-gl.dim $\mathscr{D}_F(\eta) \leq 1$. Parts (2) and (3) follow from (1). □

The case $\vartheta = 0$ of the next result is due to Reiten [34].

Corollary for Rings 6.2. *Let* $\Lambda = \Gamma \ltimes_{\vartheta} M$ *be a cleft extension of rings, and suppose that the morphism* ϑ *is nilpotent. Then the following are equivalent*:

(i) Λ *is right hereditary*.

(ii) (a) Γ *is right hereditary, and* ${}_{\Gamma}M$ *is flat*.

(b) $Ker(\vartheta) = Im(\vartheta \otimes 1_M - 1_M \otimes \vartheta)$ $(M \otimes_{\Gamma} M = 0$, *if* $\vartheta = 0)$.

(c) $\forall X \in \mathrm{Mod}(\Gamma)$: $X \otimes_{\Gamma} Coker(\vartheta)$ $(X \otimes_{\Gamma} M$, *if* $\vartheta = 0)$ *is right* Γ*-projective*.

If $M \neq 0$ *and* (ii) *is true*, *then* r.gl.dim $\Lambda = 1$.

The characterization of when $\mathfrak{F}$-gl.dim $\mathscr{D}_F(\eta) \leq 2$, is much more difficult, and is more precise if $\eta = 0$ or more generally if $\mathscr{D}_F(\eta)$ is a truncated extension, see Section 8. First we need to construct for any $X \in \mathscr{D}$, a specific natural morphism

$$\varphi^X : \mathscr{D}[\mathscr{L}_2^{\mathfrak{F}}\mathbf{CZ}(X), -] \to \mathscr{E}xt^3_{\mathfrak{R}}[X, -] \oplus \mathscr{E}xt^2_{\mathfrak{R}}[Coker(\eta_X), -].$$

Let $X \in \mathscr{D}$; since $\mathbf{Z}(X), \Omega_{\mathfrak{F}}\mathbf{Z}(X)$ are primitive objects, we have as in Theorem 3.4, an $\mathfrak{F}$-projective presentation of $\Omega_{\mathfrak{F}}\mathbf{Z}(X)$ of the form: (1) : $0 \to \Omega^2_{\mathfrak{F}}\mathbf{Z}(X) \xrightarrow{\kappa_1} \mathbf{T}(P_1) \oplus \mathbf{T}(Q) \xrightarrow{\beta_1} \Omega_{\mathfrak{F}}\mathbf{Z}(X) \to 0$. It is easy to see that $\mathbf{C}(\Omega_{\mathfrak{F}}\mathbf{Z}(X)) \cong K_0 \oplus Coker(\eta_X)$. Hence applying to (1) the functor $\mathbf{C}$ we get the short exact sequence: $0 \to \mathscr{L}_2^{\mathfrak{F}}\mathbf{CZ}(X) \to \mathbf{C}(\Omega^2_{\mathfrak{F}}\mathbf{Z}(X)) \to \Omega^2_{\mathfrak{R}}(X) \oplus \Omega^1_{\mathfrak{R}}(Coker(\eta_X)) \to 0$. From this we get the following exact sequence and the desired morphism φ^X:

$$\cdots \to \mathscr{D}[\mathbf{C}(\Omega^2_{\mathfrak{F}}\mathbf{Z}(X)),-] \to \mathscr{D}[\mathscr{L}_2^{\mathfrak{F}}\mathbf{CZ}(X),-] \xrightarrow{\phi^X} \mathscr{E}xt^3_{\mathfrak{R}}[X,-] \oplus \mathscr{E}xt^2_{\mathfrak{R}}[Coker(\eta_X),-] \to \mathscr{E}xt^1_{\mathfrak{R}}[\mathbf{C}(\Omega^2_{\mathfrak{F}}\mathbf{Z}(X)),-] \to \mathscr{E}xt^1_{\mathfrak{R}}[\mathscr{L}_2^{\mathfrak{F}}\mathbf{C}(\mathbf{Z}(X)),-] \to \cdots$$

Now we are ready to prove the following result.

Theorem 6.3. *If $\mathscr{D}_F(\eta)$ is admissible, then the following are equivalent*:

(i) $\mathfrak{F}$-gl.dim $\mathscr{D}_F(\eta) \le 2$.

(ii) (a) $\forall X \in \mathscr{D}$: $\mathscr{L}_2^{\Sigma}\mathbf{C}(\Omega_{\mathfrak{F}}\mathbf{Z}(X)) = 0$.

(b) $\forall P \in \mathscr{P}(\mathfrak{R})$: $\mathscr{L}_2^{\mathfrak{F}}\mathbf{C}(\Phi\mathbf{Z}(P)) = 0$.

(c) $\forall X \in \mathscr{D}$: $\mathscr{L}_2^{\mathfrak{F}}\mathbf{C}(\mathbf{Z}(X)) \in \mathscr{P}(\mathfrak{R})$.

(d) $\mathscr{L}_2^{\mathfrak{R}}F = 0$.

(e) $\forall X \in \mathscr{D}$, *the natural morphism φ^X constructed above, is epic.*

If (ii) *is true and $F^2 \neq 0$, then* $\mathfrak{F}$-gl.dim $\mathscr{D}_F(\eta) = 2$.

Proof. (i) $\Rightarrow$ (ii) By Corollary 4.5, $\mathfrak{R}$-gl.dim $\mathscr{D} \le \mathfrak{F}$-gl.dim $\mathscr{D}_F(\eta) \le 2$. Let $X \in \mathscr{D}$; since $\mathfrak{F}$-gl.dim $\mathscr{D}_F(\eta) \le 2, \Omega^2_{\mathfrak{F}}\mathbf{Z}(X) \in \mathscr{P}(\mathfrak{F})$, thus $\mathbf{C}\Omega^2_{\mathfrak{F}}\mathbf{Z}(X)) \in \mathscr{P}(\mathfrak{R})$. Applying $\mathbf{C}$ to the $\mathfrak{F}$-projective presentation (1): $0 \to \Omega^2_{\mathfrak{F}}\mathbf{Z}(X) \xrightarrow{\kappa_1} \mathbf{T}(P_1) \oplus \mathbf{T}(Q) \xrightarrow{\beta_1} \Omega_{\mathfrak{F}}\mathbf{Z}(X) \to 0$ above, we get the exact sequence $0 \to \mathscr{L}_2^{\mathfrak{F}}\mathbf{CZ}(X) \to \mathbf{C}(\Omega^2_{\mathfrak{F}}\mathbf{Z}(X)) \to P_1 \oplus Q \to \mathbf{C}(\Omega_{\mathfrak{F}}\mathbf{Z}(X)) \to 0$, which is therefore an $\mathfrak{R}$-projective resolution of $\mathbf{C}(\Omega_{\mathfrak{F}}\mathbf{Z}(X)) \cong \Omega_{\mathfrak{R}}(X) \oplus Coker(\eta_X)$. Hence $\mathscr{L}_2^{\mathfrak{F}}\mathbf{CZ}(X) \in \mathscr{P}(\mathfrak{R})$. Further since $\mathbf{C}(\Omega^2_{\mathfrak{F}}\mathbf{Z}(X)) \in \mathscr{P}(\mathfrak{R})$, the natural morphism φ^X is an epimorphism. Since $\mathfrak{F}$-gl.dim $\mathscr{D}_F(\eta) \le 2, \mathscr{L}_3^{\mathfrak{F}}\mathbf{CZ}(X)=0$. But $\mathscr{L}_3^{\mathfrak{F}}\mathbf{CZ}(X)$ is isomorphic to $\mathscr{L}_2^{\mathfrak{F}}\mathbf{C}(\Omega_{\mathfrak{F}}\mathbf{Z}(X))$ and $\mathscr{L}_2^{\Sigma}\mathbf{C}(\Omega_{\mathfrak{F}}\mathbf{Z}(X))$ is an epimorphic image of $\mathscr{L}_2^{\mathfrak{F}}\mathbf{C}(\Omega_{\mathfrak{F}}\mathbf{Z}(X))$, and this proves (a). If $P \in \mathscr{P}(\mathfrak{R})$, then obviously $\Phi\mathbf{Z}(P)$ is a first $\mathfrak{F}$-syzygy of $\mathbf{Z}(P)$, and this implies that $\mathscr{L}_2^{\mathfrak{F}}\mathbf{C}(\Phi\mathbf{Z}(P)) = \mathscr{L}_3^{\mathfrak{F}}\mathbf{C}(\mathbf{Z}(P)) = 0$. It remains to prove (d). Consider the primitive object $\Omega^2_{\mathfrak{F}}\mathbf{Z}(X)$. We know from (the proof of) Theorem 3.4 that if $0 \to K_1 \to P_1 \to P_0 \to X \to 0$ is an $\mathfrak{R}$-projective resolution of X, then

$$\Omega^2_{\mathfrak{F}}\mathbf{Z}(X) = \left(K_1 \oplus L_1, \begin{pmatrix} 0 & x_1 \\ 0 & y_1 \end{pmatrix}\right),$$

and there exists a push-out diagram:

$$\begin{tikzcd}
0 \arrow[r] & \Phi\mathbf{Z}(K_1) \arrow[r] \arrow[d, "x_1"] & \mathbf{T}(K_1) \arrow[r, "{}^t(1_{K_1},0)"] \arrow[d, "x_1^!"] & \mathbf{Z}(K_1) \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & (L_1, y_1) \arrow[r] & \Omega^2_{\mathfrak{F}}\mathbf{Z}(X) \arrow[r, "{}^t(1_{K_1},0)"] & \mathbf{Z}(K_1) \arrow[r] & 0
\end{tikzcd}$$

where

$$x_1^! = \begin{pmatrix} 1_{K_1} & 0 \\ 0 & x_1 \end{pmatrix}.$$

Since $K_1 = \Omega^2_{\mathfrak{R}}(X)$, we have $K_1 \in \mathscr{P}(\mathfrak{R})$ and $\mathbf{T}(K_1) \in \mathscr{P}(\mathfrak{F})$. Since $\mathscr{D}_F(\eta)$ is admissible, Ker $\mathbf{C} = 0$, hence the unit of the adjoint pair $(\mathbf{C}, \mathbf{Z})$ is a minimal epic.

But $\lambda_{\mathbf{T}(K_1)} = {}^t(1_{K_1},0)$. Thus ${}^t(1_{K_1},0): \mathbf{T}(K_1) \to \mathbf{Z}(K_1)$ is an $\mathfrak{F}$-projective cover. This implies that $x_1^!$ is split monic since $\Omega^2_{\mathfrak{F}}\mathbf{Z}(X) \in \mathscr{P}(\mathfrak{F})$. But then x_1 is split monic. From the proof of Theorem 3.4 we have that x_1 is monic iff $\mathscr{L}_1^{\mathfrak{R}}FK_0 = \mathscr{L}_2^{\mathfrak{R}}FX = 0$. This proves (d).

(ii) $\Rightarrow$ (i) Since $\mathscr{D}_F(\eta)$ is admissible it suffices to show that $\Omega^2\mathbf{Z}(X) \in \mathscr{P}(\mathfrak{F})$, $\forall X \in \mathscr{D}$. This happens iff $\mathbf{C}(\Omega^2\mathbf{Z}(X)) \in \mathscr{P}(\mathfrak{R})$ and $\mathscr{L}_1^{\mathfrak{F}}\mathbf{C}(\Omega^2\mathbf{Z}(X)) = \mathscr{L}_3^{\mathfrak{F}}\mathbf{C}(\mathbf{Z}(X)) = 0$. By (e), (c) and the long exact sequence above which defines the morphism ϕ^X, we have that $\mathscr{E}xt^1_{\mathfrak{R}}[\mathbf{C}(\Omega^2\mathbf{Z}(X)), -] = 0$, thus $\mathbf{C}(\Omega^2\mathbf{Z}(X)) \in \mathscr{P}(\mathfrak{R})$. Now consider the functorial Σ-projective presentation of $\Omega_{\mathfrak{F}}\mathbf{Z}(X)$: $0 \to \Phi(\Omega_{\mathfrak{F}}\mathbf{Z}(X)) \to \mathbf{TU}(\Omega_{\mathfrak{F}}\mathbf{Z}(X)) \to \Omega_{\mathfrak{F}}\mathbf{Z}(X) \to 0$, and apply to it the functor $\mathbf{C}$. Then we get the following exact sequence:

$$\begin{aligned}\cdots &\to \mathscr{L}_2^{\mathfrak{F}}\mathbf{C}(\Phi(\Omega_{\mathfrak{F}}\mathbf{Z}(X))) \to \mathscr{L}_2^{\mathfrak{F}}\mathbf{C}(\mathbf{TU}(\Omega_{\mathfrak{F}}\mathbf{Z}(X))) \to \mathscr{L}_3^{\mathfrak{F}}\mathbf{C}(\mathbf{Z}(X))\\ &\to \mathscr{L}_1^{\mathfrak{F}}\mathbf{C}(\Phi(\Omega_{\mathfrak{F}}\mathbf{Z}(X))) \to \mathscr{L}_1^{\mathfrak{F}}\mathbf{C}(\mathbf{TU}(\Omega_{\mathfrak{F}}\mathbf{Z}(X))) \to \cdots\end{aligned}$$

Since $\mathscr{L}_1^{\mathfrak{F}}\mathbf{C}(\Phi(\Omega_{\mathfrak{F}}\mathbf{Z}(X))) \cong \mathscr{L}_2^{\Sigma}\mathbf{C}(\Omega_{\mathfrak{F}}\mathbf{Z}(X)) = 0$ by (a), we have that $\mathscr{L}_3^{\mathfrak{F}}\mathbf{C}(\mathbf{Z}(X))$ is an epimorphic image of $\mathscr{L}_2^{\mathfrak{F}}\mathbf{C}(\mathbf{TU}(\Omega_{\mathfrak{F}}\mathbf{Z}(X)))$. Thus it suffices to show that $\mathscr{L}_2^{\mathfrak{F}}\mathbf{C}(\mathbf{TU}(\Omega_{\mathfrak{F}}\mathbf{Z}(X))) = 0$. Now since $\mathscr{L}_2^{\mathfrak{F}}\mathbf{CT} \cong \mathbf{C}\mathscr{L}_1^{\mathfrak{R}}\mathbf{T}$ and $\operatorname{Ker}\mathbf{C} = 0$, we have that $\mathscr{L}_2^{\mathfrak{F}}\mathbf{C}(\mathbf{TU}(\Omega_{\mathfrak{F}}\mathbf{Z}(X))) = 0 \Leftrightarrow \mathscr{L}_1^{\mathfrak{R}}\mathbf{T}(\mathbf{U}(\Omega_{\mathfrak{F}}\mathbf{Z}(X))) = 0$. But $\mathbf{U}(\Omega_{\mathfrak{F}}\mathbf{Z}(X)) \cong \Omega_{\mathfrak{R}}X \oplus FP_0$. Hence $\mathscr{L}_1^{\mathfrak{R}}\mathbf{T}(\mathbf{U}(\Omega_{\mathfrak{F}}\mathbf{Z}(X))) = 0 \Leftrightarrow \mathscr{L}_1^{\mathfrak{R}}\mathbf{T}(\Omega_{\mathfrak{R}}X \oplus FP_0) = \mathscr{L}_1^{\mathfrak{R}}\mathbf{T}(X) \oplus \mathscr{L}_1^{\mathfrak{R}}\mathbf{T}(FP_0) = 0$. Since $\mathbf{U}(\mathscr{L}_i^{\mathfrak{R}}\mathbf{T}) = \mathscr{L}_i^{\mathfrak{R}}F$, by (d) we have that $\mathscr{L}_2^{\mathfrak{R}}\mathbf{T}(X) = 0$, and it remains to show that $\mathscr{L}_1^{\mathfrak{R}}F(FP) = 0$, $\forall P \in \mathscr{P}(\mathfrak{R})$. Consider the Σ-projective presentation of $\mathbf{Z}(P)$: $0 \to \Phi(\mathbf{Z}(P)) \to \mathbf{T}(FP) \to \mathbf{Z}(P) \to 0$, and apply to it the functor $\mathbf{C}$. Since $\mathscr{L}_1^{\mathfrak{F}}\mathbf{C}(\mathbf{T}(FP)) = 0$, in the induced long exact sequence:

$$\cdots \to \mathscr{L}_2^{\mathfrak{F}}\mathbf{C}(\Phi\mathbf{Z}(P)) \to \mathscr{L}_2^{\mathfrak{F}}\mathbf{C}(\mathbf{T}(FP)) \to \mathscr{L}_2^{\mathfrak{F}}\mathbf{C}(\mathbf{Z}(P)) \xrightarrow{m} \mathscr{L}_1^{\mathfrak{F}}\mathbf{C}(\Phi\mathbf{Z}(P)) \to \cdots$$

the morphism m is an epimorphism and by (b), $\mathscr{L}_2^{\mathfrak{F}}\mathbf{C}(\Phi\mathbf{Z}(P)) = 0$. Since $\Phi\mathbf{Z}(P) = \Omega_{\mathfrak{F}}\mathbf{Z}(P)$ we have that $\mathscr{L}_1^{\mathfrak{F}}\mathbf{C}(\Phi\mathbf{Z}(P)) \cong \mathscr{L}_2^{\mathfrak{F}}\mathbf{C}(\mathbf{Z}(P))$ and the morphism m is an isomorphism. Hence $\mathscr{L}_2^{\mathfrak{F}}\mathbf{C}(\mathbf{T}(FP)) = 0$. But $\mathscr{L}_2^{\mathfrak{F}}\mathbf{C}(\mathbf{T}(FP)) = \mathbf{C}\mathscr{L}_1^{\mathfrak{R}}\mathbf{T}(FP))$ and this is zero iff $\mathscr{L}_1^{\mathfrak{R}}\mathbf{T}(FP)) = 0$ iff $\mathscr{L}_1^{\mathfrak{R}}F(FP)) = 0$.

The last assertion follows from Proposition 6.1. □

Corollary 6.4. *The following are equivalent*:

(i) $\mathfrak{F}$-gl.dim $\mathscr{D}_F(0) \leq 2$.

(ii) (a) $F^3 = \mathscr{L}_1^{\mathfrak{R}}F^2 = \mathscr{L}_2^{\mathfrak{R}}F = 0$.

(b) $\mathscr{L}_1^{\mathfrak{R}}F(FP) = 0$, $\forall P \in \mathscr{P}(\mathfrak{R})$.

(c) $\forall X \in \mathscr{D}$: F^2X, $\mathscr{L}_1^{\mathfrak{R}}F(X) \in \mathscr{P}(\mathfrak{R})$.

(d) $\forall X \in \mathscr{D}$, *the natural morphism* ϕ^X *constructed above is epic.*

Proof. We translate the conditions of Theorem 6.3, when necessary, to the case $\eta = 0$, and we use the formula $\mathscr{L}_2^{\mathfrak{F}}\mathbf{CZ} \cong F^2 \oplus \mathscr{L}_1^{\mathfrak{R}}F$ of Corollary 4.4.

(i) $\Rightarrow$ (ii) $\forall P \in \mathscr{P}(\mathfrak{R})$: $0 = \mathscr{L}_3^{\mathfrak{F}}\mathbf{CZ}(P) \cong \mathscr{L}_2^{\mathfrak{F}}\mathbf{C}\Omega_{\mathfrak{F}}\mathbf{Z}(P) = \mathscr{L}_2^{\mathfrak{F}}\mathbf{CZ}(FP) \cong \mathscr{L}_1^{\mathfrak{R}}F(FP) \oplus F^3P$. Hence $\forall P \in \mathscr{P}(\mathfrak{R})$: $\mathscr{L}_1^{\mathfrak{R}}F(FP) = 0$ and $F^3P = 0$, so $F^3 = 0$. Since $\mathscr{L}_2^{\Sigma}\mathbf{C}\Omega_{\mathfrak{F}}\mathbf{Z}(X) = 0$ and $F^3 = 0$, the Σ-associated complex of $\Omega_{\mathfrak{F}}\mathbf{Z}(X)$ is of the form:

$\cdots \to 0 \to F^2K \stackrel{(0,F^2k)}{\to} FK \oplus F^2P \to K \oplus FP \to 0$, see the proof of Corollary 4.4. Hence $0 = \mathscr{L}_2^{\Sigma}\mathbf{C}\Omega_{\mathfrak{F}}\mathbf{Z}(X) = Ker(F^2k) = \mathscr{L}_1^{\mathfrak{R}}F^2X$.

(ii) $\Rightarrow$ (i) By (a) we have $\mathscr{L}_2^{\Sigma}\mathbf{C}\Omega_{\mathfrak{F}}\mathbf{Z}(X) \cong \mathscr{L}_1^{\mathfrak{R}}F^2X = 0$ and $\mathscr{L}_2^{\mathfrak{F}}\mathbf{C}\Omega_{\mathfrak{F}}\Phi\mathbf{Z}(P) = \mathscr{L}_2^{\mathfrak{F}}\mathbf{C}\Omega_{\mathfrak{F}}\mathbf{Z}(FP) = \mathscr{L}_1^{\mathfrak{R}}F(FP) = 0$. $\square$

The next corollary of the above result, improves a Theorem of Palmer [30].

Corollary for Rings 6.5. *If $\Lambda = \Gamma \ltimes_0 M$ is trivial cleft extension of Γ, then the following are equivalent*:

(i) r.gl.dim $\Lambda \leq 2$.

(ii) (a) $\otimes_{\Gamma}^3 M = \mathscr{T}or_1^{\Gamma}(M,M) = 0$ *and* w.d $\otimes_{\Gamma}^2 M = 0$, w.d$_{\Gamma}M \leq 1$.

(b) $\forall X \in \mathrm{Mod}(\Gamma)$: $X \otimes_{\Gamma}^2 M, \mathscr{T}or_1^{\Gamma}(X,M)$ *are projective right Γ-modules.*

(c) $\forall X \in \mathrm{Mod}(\Gamma)$, *the naturally induced morphism*:
$\mathrm{Hom}_{\Gamma}[\mathscr{T}or_1^{\Gamma}(X,M) \oplus X \otimes_{\Gamma} (\otimes_{\Gamma}^2 M), -] \to \mathscr{E}xt_{\Gamma}^2[X \otimes_{\Gamma} M, -] \oplus \mathscr{E}xt_{\Gamma}^3[X,-]$
is an epimorphism.

If $M \otimes_{\Gamma} M \neq 0$ and condition (ii) *is true*, *then* r.gl.dim $\Lambda = 2$.

6.2. Frobenius cleft extensions

We close this section by studing when a cleft extension $\mathscr{D}_F(\eta)$ is $\mathfrak{F}$-Frobenius. We recall that an abelian category $\mathscr{C}$ equipped with a proper class of short exact sequences $\mathfrak{F}$ is called *$\mathfrak{F}$-Frobenius*, if $\mathscr{C}$ has enough $\mathfrak{F}$-projectives, enough $\mathfrak{F}$-injectives and the class of $\mathfrak{F}$-projectives $\mathscr{P}(\mathfrak{F})$ coincides with the class $\mathscr{I}(\mathfrak{F})$ of $\mathfrak{F}$-injectives.

We suppose now that the functor F has a right adjoint G with counit $\rho : FG \to \mathrm{Id}_{\mathscr{D}}$ and unit $\sigma : \mathrm{Id}_{\mathscr{D}} \to GF$. We denote by $\mathbf{H} : \mathscr{D} \to \mathscr{D}_F(\eta)$ the right adjoint of $\mathbf{U}$ and by $\mathbf{K} : \mathscr{D}_F(\eta) \to \mathscr{D}$ the right adjoint of $\mathbf{Z}$, using the isomorphism $\mathbb{D} : \mathscr{D}_F(\eta) \cong \mathscr{D}^G(\zeta)$ from Section 2. We assume that $\mathscr{D}$, in addition to our default assumption that has enough $\mathfrak{R}$-projectives, it has also enough $\mathfrak{R}$-injectives. Then by the dual of the Adjoint Theorem [12], $\mathscr{D}^G(\zeta) \cong \mathscr{D}_F(\eta)$ has also enough $\mathfrak{F}$-injectives.

Proposition 6.6. *If* $\mathrm{Ker}\,\mathbf{C} = 0$, $\mathrm{Ker}\,\mathbf{K} = 0$, *then the following are equivalent for the cleft extension $\mathscr{D}_F(\eta)$* of $\mathscr{D}$:

(i) *$\mathscr{D}_F(\eta)$ is $\mathfrak{F}$-Frobenius.*

(ii) (a) $\forall P \in \mathscr{P}(\mathfrak{R}) : \mathbf{KT}(P) \in \mathscr{I}(\mathfrak{R})$ *and* $\mathscr{R}_{\mathfrak{F}}^1\mathbf{KT}(P) = 0$.

(b) $\forall I \in \mathscr{I}(\mathfrak{R}) : \mathbf{CH}(I) \in \mathscr{P}(\mathfrak{R})$ *and* $\mathscr{L}_1^{\mathfrak{F}}\mathbf{CH}(I) = 0$.

Proof. If $\mathrm{Ker}\,\mathbf{C} = 0$, then $(X,f) \in \mathscr{P}(\mathfrak{F}) \Leftrightarrow \mathbf{C}(X,f) \in \mathscr{P}(\mathfrak{R})$ and $\mathscr{L}_1^{\mathfrak{F}}\mathbf{C}(X,f) = 0 \Leftrightarrow (X,f) \cong \mathbf{T}(P)$ for $P \in \mathscr{P}(\mathfrak{R})$. Dually if $\mathrm{Ker}\,\mathbf{K} = 0$, then $(X,f) \in \mathscr{I}(\mathfrak{F}) \Leftrightarrow \mathbf{K}(X,f) \in \mathscr{I}(\mathfrak{R})$ and $\mathscr{R}_{\mathfrak{F}}^1\mathbf{K}(X,f) = 0 \Leftrightarrow (X,f) \cong \mathbf{H}(I)$ for $I \in \mathscr{I}(\mathfrak{R})$. Hence $\mathscr{P}(\mathfrak{F}) = \mathscr{I}(\mathfrak{F}) \Leftrightarrow \forall P \in \mathscr{P}(\mathfrak{R})$, $\forall I \in \mathscr{I}(\mathfrak{R})$: $\mathbf{KT}(P) \in \mathscr{I}(\mathfrak{R})$, $\mathscr{R}_{\mathfrak{F}}^1\mathbf{KT}(P) = 0$, $\mathbf{CH}(I) \in \mathscr{P}(\mathfrak{R})$ and $\mathscr{L}_1^{\mathfrak{F}}\mathbf{CH}(I) = 0$. $\square$

Corollary 6.7. *The following are equivalent for the cleft extension $\mathscr{D}_F(0)$*:

(i) *$\mathscr{D}_F(0)$ is $\mathfrak{F}$-Frobenius.*

(ii) *The following are true*, $\forall I \in \mathscr{I}(\mathfrak{R})$, $\forall P \in \mathscr{P}(\mathfrak{R})$:

(a) $FP, Ker(\sigma_P) \in \mathscr{I}(\mathfrak{R})$, σ_P *is epic and* $GKer(\sigma_P) = 0$.
(b) GI, $Coker(\rho_I) \in \mathscr{P}(\mathfrak{R})$, ρ_I *is monic and* $FCoker(\rho_I) = 0$.

Proof. Obviously $\operatorname{Ker}\mathbf{C} = 0 = \operatorname{Ker}\mathbf{K}$. We observe that $\mathbf{KT}(P) \cong Ker(\sigma_P) \oplus FP$, and $\mathbf{CH}(I) \cong Coker(\rho_I) \oplus GI$. Also using Theorem 4.7(i) and its dual, we see that $\mathscr{R}^1_{\mathfrak{F}}\mathbf{KT}(P) \cong GKer(\sigma_P) \oplus Coker(\sigma_P)$, and $\mathscr{L}_1^{\mathfrak{F}}\mathbf{CH}(I) \cong FCoker(\rho_I) \oplus Ker(\rho_I)$. Then the result follows from Proposition 6.6. $\square$

In the following let $\Lambda = \Gamma \ltimes_\vartheta M$ be a cleft extension of rings. We denote by $\operatorname{Lann}_\Gamma(M)$ the left annihilator of M in Γ.

Corollary for Rings 6.8. (1) *Let* $\Lambda = \Gamma \ltimes_\vartheta M$ *be a cleft extension of rings and suppose that* ϑ *is right T-nilpotent. Then the following are equivalent*:
(i) Λ *is a right selfinjective ring* (*QF-ring*).
(ii) $\mathscr{E}xt^1_\Lambda[\Gamma, \Lambda] = 0$ *and* $\operatorname{Hom}_\Gamma[\Gamma, \Lambda]$ *is an injective right* Γ*-module* (*and* Γ *is right Noetherian and* M_Γ *is finitely generated*).
(2) *If* $\vartheta = 0$, *then the following are equivalent*:
(i) Λ *is a right selfinjective ring* (*QF-ring*):
(ii) (a) ${}_\Gamma M, {}_\Gamma\operatorname{Lann}_\Gamma(M)$ *are injective, and* $\operatorname{Hom}_\Gamma[M, \operatorname{Lann}_\Gamma(M)] = 0$.
(b) *The natural morphism* $\Gamma \to \operatorname{End}_\Gamma(M_\Gamma)$ *is an epimorphism* (*and* Γ *is right Noetherian and* M_Γ *is finitely generated*).

A direct consequence of the above results is the following.

Corollary 6.9. *Let* Γ *be a ring and* M *be a* Γ–Γ*-bimodule such that the functor* $-\otimes_\Gamma M : \operatorname{Mod}(\Gamma) \to \operatorname{Mod}(\Gamma)$ *induces an equivalence* $-\otimes_\Gamma M : \mathbf{P}_\Gamma \xrightarrow{\approx} \mathbf{I}_\Gamma$ *between the category* $\mathbf{P}_\Gamma$ *of projective right* Γ*-modules and the category* $\mathbf{I}_\Gamma$ *of injective* Γ*-modules. Then the trivial extension* $\Lambda = \Gamma \ltimes_0 M$ *is QF.*

Examples of rings satisfying the assumptions of the above corollary are rings with a (Morita) self-duality, for instance Artin algebras.

7. Formulas for $\mathscr{L}_n^{\mathfrak{F}}\mathbf{CZ}$, $\mathscr{E}xt^n_{\mathfrak{F}}[\mathbf{Z}(-), \mathbf{Z}(-)]$ and spectral sequences

In this section we use spectral sequences to study further the relative homological behaviour of $\mathscr{D}_F(\eta)$, with respect the proper class $\mathfrak{F} = \mathbf{U}^{-1}(\mathfrak{R})$. We pose a condition on the functor F for a nice behaviour of the derived spectral sequences. In general, we get better estimates for $\mathfrak{F}$-gl.dim $\mathscr{D}_F(\eta)$, and more precise formulas for the functors $\mathscr{L}_n^{\mathfrak{F}}\mathbf{CZ}$, $\mathscr{E}xt^n_{\mathfrak{F}}[\mathbf{Z}(-), \mathbf{Z}(-)]$, when we impose on the functor F the following *vanishing condition* (see also [13]):

$$(\dagger) \qquad \mathscr{L}_i^{\mathfrak{R}} F^j(FP) = 0, \quad \forall i, j \geq 1, \ \forall P \in \mathscr{P}(\mathfrak{R}).$$

In case the cleft extension is induced by a cleft extension of rings $\Lambda = \Gamma \ltimes_{\vartheta} M$, the vanishing condition (for the absolute theory) is the following:

$$\mathscr{T}or_i^{\Gamma}[M, \otimes_{\Gamma}^{j} M] = 0, \quad \forall i, j \geq 1.$$

These estimates and formulas become exact, when $\eta = 0$. Condition (†) will be very useful in the study of the Butler–Horrocks spectral sequence. We begin with the following preliminary result:

Proposition 7.1. *There is a functorial spectral sequence*:

$$E_2^{p,q} = \mathscr{E}xt_{\mathfrak{R}}^{p}[\mathscr{L}_q^{\mathfrak{F}}\mathbf{C}-, ?] \Rightarrow \mathscr{E}xt_{\mathfrak{F}}^{n}[-, \mathbf{Z}(?)].$$

Proof. Let $(X, f) \in \mathscr{D}_F(\eta)$ and $Y \in \mathscr{D}$, and consider the functor $h_Y := \mathscr{D}[-, Y] : \mathscr{D} \to \mathscr{A}b$. Obviously, the right h_Y-acyclic objects of $\mathscr{D}$ with respect to $\mathfrak{R}$ are precisely the objects of $\mathscr{P}(\mathfrak{R})$. Since $\mathbf{C}(\mathscr{P}(\mathfrak{F})) \subseteq \mathscr{P}(\mathfrak{R})$, the functor $\mathbf{C}$ sends objects of $\mathscr{P}(\mathfrak{F})$ to h_Y-acyclic objects with respect to $\mathfrak{R}$. From the adjoint pair $(\mathbf{C}, \mathbf{Z})$ we have an isomorphism $h_Y\mathbf{C} \cong \mathscr{D}_F(\eta)[-, \mathbf{Z}(Y)]$. Since $\mathscr{R}_{\mathfrak{R}}^{n} h_Y = \mathscr{E}xt_{\mathfrak{R}}^{n}[-, Y]$, and $\mathscr{R}_{\mathfrak{F}}^{n} h_Y\mathbf{C} \cong \mathscr{R}_{\mathfrak{F}}^{n}\mathscr{D}_F(\eta)[-, \mathbf{Z}(Y)] = \mathscr{E}xt_{\mathfrak{F}}^{n}[-, \mathbf{Z}(Y)]$, we have a third quadrant spectral sequence of Grothendieck type $E_2^{p,q} = \mathscr{E}xt_{\mathfrak{R}}^{p}[\mathscr{L}_q^{\mathfrak{F}}\mathbf{C}(-), Y] \Rightarrow \mathscr{E}xt_{\mathfrak{F}}^{n}[-, \mathbf{Z}(Y)]$, if the $\mathfrak{F}$-associated complex $\mathfrak{F}^{\bullet}_{(X,f)}$ of any $(X, f) \in \mathscr{D}_F(\eta)$ has an $\mathfrak{R}$-proper $\mathfrak{R}$-projective resolution in the sense of Cartan–Eilenberg. This happens in our case, as follows from our conventions and the classical construction of the Grothendieck's spectral sequence, see [18,35] or [38]. □

As a corollary of the above proposition we have the following:

Corollary 7.2. *Let* $(X, f) \in \mathscr{D}_F(\eta)$ *be an object with* $\mathscr{L}_n^{\mathfrak{F}}\mathbf{C}(X, f) = 0$, $\forall n \geq 1$. *Then* $\forall Y \in \mathscr{D}$, $\forall n \geq 1$, *there is an isomorphism*:

$$\mathscr{E}xt_{\mathfrak{F}}^{n}[(X, f), \mathbf{Z}(Y)] \cong \mathscr{E}xt_{\mathfrak{R}}^{n}[\mathbf{C}(X, f), Y].$$

If in addition $\operatorname{Ker}\mathbf{C} = 0$, *then* $\mathfrak{F}$-p.d$(X, f) = \mathfrak{R}$-p.d $\mathbf{C}(X, f)$.

Proof. Trivial since the spectral sequence of Proposition 7.1 collapses. If $\operatorname{Ker}\mathbf{C} = 0$, the result follows from Lemma 5.3. □

Corollary 7.3. *The following are equivalent for the cleft extension* $\mathscr{D}_F(\eta)$:

(i) *The natural morphism* $\mathbf{Z}^i : \mathscr{E}xt_{\mathfrak{R}}^{i}[-,-] \to \mathscr{E}xt_{\mathfrak{F}}^{i}[\mathbf{Z}(-), \mathbf{Z}(-)]$ *induced by the exact functor* $\mathbf{Z}$ *is an isomorphism*, $\forall i \geq 1$.

(ii) *The natural morphism* $\mathbf{U}^i : \mathscr{E}xt_{\mathfrak{F}}^{i}[\mathbf{Z}(-), \mathbf{Z}(-)] \to \mathscr{E}xt_{\mathfrak{R}}^{i}[-,-]$ *induced by the exact functor* $\mathbf{U}$ *is an isomorphism*, $\forall i \geq 1$.

(iii) $\mathscr{L}_i^{\mathfrak{F}}\mathbf{CZ}(P) = 0$, $\forall i \geq 1$, $\forall P \in \mathscr{P}(\mathfrak{R})$.

(iv) $\mathscr{L}_i^{\mathfrak{F}}\mathbf{CZ} = 0$, $\forall i \geq 1$.

(v) $\mathbf{Z}(\mathscr{D})$ *is extension-closed in* $\mathscr{D}_F(\eta)$ *and* $\mathscr{L}_i^{\mathfrak{F}}\mathbf{CZ}(P) = 0$, $\forall i \geq 2$, $\forall P \in \mathscr{P}(\mathfrak{R})$.

(vi) η *is epic and* $\mathscr{L}_i^{\mathfrak{F}}\mathbf{CZ}(P) = 0$, $\forall i \geq 2$, $\forall P \in \mathscr{P}(\mathfrak{R})$.

Proof. (i) $\Leftrightarrow$ (ii) Trivial since $\mathbf{U}^i \circ \mathbf{Z}^i = \mathrm{Id}_{\mathscr{E}xt^i_{\mathfrak{R}}[-,-]}$, $\forall i \geq 1$. (iii) $\Leftrightarrow$ (iv) Assuming (iii), let $X \in \mathscr{D}$, and $P^\bullet \to X$ be an $\mathfrak{R}$-projective resolution of X. Then $\mathbf{Z}(P^\bullet) \to \mathbf{Z}(X)$ is an $\mathfrak{F}$-exact complex consisting of $\mathbf{C}$-acyclic objects with respect to $\mathfrak{R}$. Hence $\mathscr{L}_i^{\mathfrak{F}}\mathbf{CZ}(X) = H_i[\mathbf{CZ}(P^\bullet)] = 0$, $\forall i \geq 1$, since $\mathbf{CZ} = \mathrm{Id}_{\mathscr{D}}$. The other direction is trivial. (iv) $\Rightarrow$ (v) It suffices to show that $\mathscr{L}_1^{\mathfrak{F}}\mathbf{CZ}=0$ iff η is epic. But $\forall X \in \mathscr{D}$: $\mathscr{L}_1^{\mathfrak{F}}\mathbf{CZ}(X)= \mathit{Coker}(\eta_X)$. (v) $\Leftrightarrow$ (vi) Obviously $\mathbf{Z}(\mathscr{D})$ is extension closed iff the morphism $\mathbf{Z}^1$ is an isomorphism. Since $\mathbf{Z}^1$ is always split monic and

$$\mathit{Coker}(\mathbf{Z}^1) \cong \mathscr{E}xt^1_{\Sigma}[\mathbf{Z}(X), \mathbf{Z}(Y)] \cong \mathscr{D}[\Phi\mathbf{Z}(X), \mathbf{Z}(Y)] \cong \mathscr{D}[\mathit{Coker}(\eta_X), Y],$$

we have that $\mathbf{Z}(\mathscr{D})$ is extension closed iff η is epic. (iv) $\Rightarrow$ (i) Assuming (iv), the spectral sequence of Proposition 7.1 for $f = 0$, collapses, so the horizontal edge morphism $e^i : \mathscr{E}xt^i_{\mathfrak{R}}[X, Y] \to \mathscr{E}xt^i_{\mathfrak{F}}[\mathbf{Z}(X), \mathbf{Z}(Y)]$, $\forall X, Y \in \mathscr{D}$, $\forall i \geq 1$ is an isomorphism. It is easy to see that e^i coincides with the natural morphism $\mathbf{Z}^i$ induced by the exact functor $\mathbf{Z}$. (i) $\Rightarrow$ (iv) The proof consists of a simple induction argument and Theorem 5.12, p. 328 of [8]. $\square$

7.1. The vanishing condition $(\dagger) : \mathscr{L}_i^{\mathfrak{R}} F^j(FP) = 0, \ \forall i, j \geq 1, \ \forall P \in \mathscr{P}(\mathfrak{R})$ *and its consequences*

Throughout this subsection we assume that condition $(\dagger)$ cited above holds. The next lemma describes an equivalent formulation of $(\dagger)$ and indicates some useful consequences.

Lemma 7.4. (α) *Consider the following conditions*:

(i) $\mathscr{L}_i^{\mathfrak{R}} F(F^jP) = 0$, $\forall i, j \geq 1$, $\forall P \in \mathscr{P}(\mathfrak{R})$.

(ii) $\mathscr{L}_i^{\mathfrak{R}} F^j(FP) = 0$, $\forall i, j \geq 1$, $\forall P \in \mathscr{P}(\mathfrak{R})$.

(iii) $\mathscr{L}_i^{\mathfrak{F}} \mathbf{CT}(F^jP) = 0$, $\forall i, j \geq 1$, $\forall P \in \mathscr{P}(\mathfrak{R})$.

Then (i) $\Leftrightarrow$ (ii) $\Rightarrow$ (iii) *and if* $\mathrm{Ker}\,\mathbf{C} = 0$, *then these conditions are equivalent.*

(β) *If one of the above conditions is true, then we have natural isomorphisms*:

$$\mathscr{L}_n^{\mathfrak{F}}\mathbf{CZ} \cong \mathscr{L}_n^{\Sigma}\mathbf{CZ}, \quad \mathscr{L}_{n+m}^{\mathfrak{F}}\mathbf{CZ} \cong \mathscr{L}_n^{\mathfrak{F}}\mathbf{C}\Phi^m\mathbf{Z} : \mathscr{P}(\mathfrak{R}) \to \mathscr{D}, \quad \forall n, m \geq 0.$$

Proof. (α) (i) $\Rightarrow$ (ii) Assuming (i), the functor F^r, $\forall r \geq 1$, sends $\mathfrak{R}$-projectives to F-acyclic objects with respect to $\mathfrak{R}$. Hence there exists a family of spectral sequences of Grothendieck type:

$${}_rE^2_{p,q} = \mathscr{L}_p^{\mathfrak{R}} F[\mathscr{L}_q^{\mathfrak{R}} F^r(X)] \Rightarrow \mathscr{L}_n^{\mathfrak{R}} F^{r+1}(X), \quad \forall X \in \mathscr{D}, \ \forall r \geq 1.$$

We apply induction to $j \geq 0$; the hypothesis ensures that (ii) is true for $j = 1$, and we assume that is true for $j - 1$. We set $X = FP$, for $P \in \mathscr{P}(\mathfrak{R})$ and consider the spectral sequence ${}_{j-1}E^2_{p,q}$. From the induction hypothesis this spectral sequence collapses, thus $\mathscr{L}_p^{\mathfrak{R}} F[F^{j-1}(FP)] = \mathscr{L}_p^{\mathfrak{R}} F[F^jP] \cong \mathscr{L}_p^{\mathfrak{R}} F^j(FP)$, $\forall p \geq 1$. Hence since by our hypothesis $\mathscr{L}_p^{\mathfrak{R}} F[F^jP] = 0$, $\forall j \geq 1$, we have that $\mathscr{L}_p^{\mathfrak{R}} F^j(FP) = 0$, $\forall p \geq 1$, which completes the proof. Part (ii) $\Rightarrow$ (i) is similar to (i) $\Rightarrow$ (ii) and the rest follows from Theorem 4.7. Part (β) is easy and is left to the reader. $\square$

Theorem 7.5. *If $\mathscr{D}_F(\eta)$ is admissible, then the following are equivalent*:

(i) $\mathfrak{F}$-gl.dim $\mathscr{D}_F(\eta) < \infty$.

(ii) (a) $\mathfrak{R}$-gl.dim $\mathscr{D} < \infty$.

(b) $\exists m \geq 0$: $\mathscr{L}_n^{\Sigma}\mathbf{CZ}(P) = 0$, $\forall n \geq m+1$, $\forall P \in \mathscr{P}(\mathfrak{R})$.

Setting $m_0 = min\{m \geq 0 : \mathscr{L}_n^{\Sigma}\mathbf{CZ}(P) = 0,\ \forall n \geq m+1,\ \forall P \in \mathscr{P}(\mathfrak{R})\}$, *we have*

$$\mathfrak{R}\text{-gl.dim}\,\mathscr{D} \leq \mathfrak{F}\text{-gl.dim}\,\mathscr{D}_F(\eta) \leq m_0 + 2\cdot\mathfrak{R}\text{-gl.dim}\,\mathscr{D}.$$

Proof. (i) $\Rightarrow$ (ii) Follows from Theorem 5.6, and Lemma 7.4(β). (ii) $\Rightarrow$ (i) By Theorem 5.6(2), it is enough to show that if $\mathfrak{R}$-gl.dim $\mathscr{D} < \infty$, and m_0 is as in the statement of the Theorem, then $\mathfrak{F}$-gl.dim $\mathscr{D}_F(\eta) \leq m_0 + 2\cdot\mathfrak{R}$-gl.dim $\mathscr{D}$. Suppose that $\mathfrak{R}$-gl.dim $\mathscr{D} = n < \infty$, and for an arbitrary $X \in \mathscr{D}$, choose an $\mathfrak{R}$-projective resolution $0 \to P_n \to P_{n-1} \to \cdots \to P_0 \to X \to 0$, with corresponding $\mathfrak{R}$-syzygies $K_0, K_1, \ldots, K_{n-1} = P_n$. From the long exact sequence induced by the $\mathfrak{F}$-exact sequence $0 \to \mathbf{Z}(K_0) \to \mathbf{Z}(P_0) \to \mathbf{Z}(X) \to 0$, using our hypothesis we have $\mathscr{L}^{\mathfrak{F}}_{m_0+k+1}\mathbf{CZ}(X) \cong \mathscr{L}^{\mathfrak{F}}_{m_0+k}\mathbf{CZ}(K_0)$, $\forall k \geq 1$. In the same way from the long exact sequence induced by the $\mathfrak{F}$-exact sequence $0 \to \mathbf{Z}(K_1) \to \mathbf{Z}(P_1) \to \mathbf{Z}(K_0) \to 0$, using our hypothesis we have $\mathscr{L}^{\mathfrak{F}}_{m_0+k+1}\mathbf{CZ}(K_0) \cong \mathscr{L}^{\mathfrak{F}}_{m_0+k}\mathbf{CZ}(K_1)$, $\forall k \geq 1$. Similarly $\mathscr{L}^{\mathfrak{F}}_{m_0+k+1}\mathbf{CZ}(K_i) \cong \mathscr{L}^{\mathfrak{F}}_{m_0+k}\mathbf{CZ}(K_{i-1})$, $\forall k \geq 1$, $\forall i = 0, 1, \ldots, n-1$. Hence

$$\mathscr{L}^{\mathfrak{F}}_{m_0+n+1}\mathbf{CZ}(X) \cong \mathscr{L}^{\mathfrak{F}}_{m_0+n}\mathbf{CZ}(K_0) \cong \mathscr{L}^{\mathfrak{F}}_{m_0+n-1}\mathbf{CZ}(K_1) \cong \cdots$$

$$\cdots \cong \mathscr{L}^{\mathfrak{F}}_{m_0+1}\mathbf{CZ}(K_{n-1}) \cong \mathscr{L}^{\mathfrak{F}}_{m_0+1}\mathbf{CZ}(P_n) \cong \mathscr{L}^{\Sigma}_{m_0+1}\mathbf{CZ}(P_n) = 0.$$

We conclude that $\mathscr{L}^{\mathfrak{F}}_{m_0+n+1}\mathbf{CZ}(X) = 0$, $\forall X \in \mathscr{D}$. But since $\mathscr{D}_F(\eta)$ is $\mathfrak{F}$-admissible, we have $\mathscr{L}^{\mathfrak{F}}_{m_0+n+1}\mathbf{C} = 0$. Finally by Theorem 5.6, we have the formula $\mathfrak{F}$-gl.dim $\mathscr{D}_F(\eta) \leq m_0 + n + \mathfrak{R}$-gl.dim $\mathscr{D} = m_0 + 2\cdot\mathfrak{R}$-gl.dim $\mathscr{D}$. □

As a corollary we get a generalization of the main result of [13] about the global dimension of the trivial extension $\mathscr{D}_F(0)$.

Corollary 7.6. *If* $\eta = 0$, *then the following are equivalent*:

(i) $\mathfrak{F}$-gl.dim $\mathscr{D}_F(0) < \infty$.

(ii) (a) $\mathfrak{R}$-gl.dim $\mathscr{D} < \infty$.

(b) *The functor F is nilpotent.*

If $c(F)$ is the nilpotency class of F, we have the bounds:

$$\mathfrak{R}\text{-gl.dim}\,\mathscr{D} \leq \mathfrak{F}\text{-gl.dim}\,\mathscr{D}_F(0) \leq c(F) + 2\cdot\mathfrak{R}\text{-gl.dim}\,\mathscr{D}.$$

Proof. If $\eta=0$ then $\mathscr{D}_F(0)$ is admissible and of course $\mathscr{L}_n^{\Sigma}\mathbf{CZ} = F^n$. Hence $\mathscr{L}_n^{\Sigma}\mathbf{CZ}(P) = 0$, $\forall P \in \mathscr{P}(\mathfrak{R}) \Leftrightarrow F^n(P) = 0$, $\forall P \in \mathscr{P}(\mathfrak{R}) \Leftrightarrow F^n = 0$. The rest follows from the previous theorem. □

Remark 7.7. Fix the abelian category $\mathscr{D}$ and the functor F. Consider the class $\mathscr{H}$ of all nilpotent and associative "multiplications" $\eta : F^2 \to F$ in $\mathscr{D}$. Set

$$\forall \eta \in \mathscr{H} : \mathfrak{m}(\eta) := min\{m \geq 0: \mathscr{L}_n^{\Sigma}\mathbf{CZ}(P) = 0,\ \forall n \geq m+1,\ \forall P \in \mathscr{P}(\mathfrak{R})\},$$

$$\mathfrak{m}(\mathscr{D}, F, \eta) := sup\{\mathfrak{m}(\eta); \eta \in \mathscr{H}\}, \quad \aleph(\mathscr{D}, F, \eta) := \sup_{\eta : F^2 \to F \in \mathscr{H}} \{\mathfrak{F}\text{-gl.dim}\, \mathscr{D}_F(\eta)\}.$$

It is clear that $\mathfrak{m}(\mathscr{D}, F, \eta) = \mathfrak{m}(0) = c(F)$, hence $\mathfrak{m}(\mathscr{D}, F, \eta)$ is independent of η. We `conjecture` that $\aleph(\mathscr{D}, F, \eta)$ is also independent of η and

$$\aleph(\mathscr{D}, F, \eta) = \mathfrak{F}\text{-gl.dim}\, \mathscr{D}_F(0).$$

Clearly $\mathfrak{F}\text{-gl.dim}\, \mathscr{D}_F(0) \leq \aleph(\mathscr{D}, F, \eta)$. Assuming the vanishing condition (†), by the above results:

$$\mathfrak{R}\text{-gl.dim}\, \mathscr{D} \leq \aleph(\mathscr{D}, F, \eta) \leq c(F) + 2 \cdot \mathfrak{R}\text{-gl.dim}\, \mathscr{D}.$$

If $\mathscr{D}$ is semisimple, so the homological theory is the absolute one, then for any admissible $\mathscr{D}_F(\eta)$, we have that $\mathfrak{m}(\eta) = \text{gl.dim}\, \mathscr{D}_F(\eta)$, as follows from the isomorphism $\mathscr{E}xt^n_{\mathscr{D}_F(\eta)}[\mathbf{Z}(X), \mathbf{Z}(Y)] \cong \mathscr{D}[\mathscr{L}_n^{\Sigma}\mathbf{CZ}(X), \mathbf{Z}(Y)] \cong \mathscr{D}[\mathrm{H}_n \Sigma^{\bullet}_{\mathbf{Z}(X)}, Y]$, where $\Sigma^{\bullet}_{\mathbf{Z}(X)}$ is the Σ-associated complex of $\mathbf{Z}(X)$. It follows in this case that $\aleph(\mathscr{D}, F, \eta) = c(F) = \text{gl.dim}\, \mathscr{D}_F(0)$ and the above conjecture is true. In particular if $\text{gl.dim}\, \mathscr{D}_F(0) < \infty$, or equivalently if F is nilpotent, then

$$\forall \eta : \text{gl.dim}\, \mathscr{D}_F(\eta) \leq \text{gl.dim}\, \mathscr{D}_F(0) = c(F) < \infty.$$

Let $\Lambda = \Gamma \ltimes_{\vartheta} M$ be a cleft extension of rings. The *tensor-nilpotency class* $c(M)$ of M is the nilpotency class of the functor $- \otimes_{\Gamma} M$, i.e. $c(M) = \infty$ or $c(M) = min\{k \in \mathbb{N}: \otimes_{\Gamma}^{k+1} M = 0\}$. We leave to the reader to interpret the above Remark in the case of rings, noting that by [6] the class of algebras $\Lambda = \Gamma \ltimes_{\vartheta} M$ we obtain using tensor nilpotent Γ–Γ-bimodules M over semisimple k-algebras Γ over a field k, coincides with the class of triangular algebras, i.e. algebras without oriented cycles in their ordinary quiver [3].

Corollary for Rings 7.8. *Let $\Lambda = \Gamma \ltimes_{\vartheta} M$ be a cleft extension of rings and assume that ϑ is nilpotent and $\mathscr{T}or_i^{\Gamma}[M, \otimes_{\Gamma}^{j} M] = 0,\ \forall i, j \geq 1$. Then*

(i) r.gl.dim $\Lambda < \infty$.

(ii) r.gl.dim $\Gamma < \infty$ *and* $\exists m \geq 0 : \mathscr{T}or_n^{(\Lambda,\Gamma)}[\Gamma, \Gamma] = 0,\ \forall n \geq m+1$.

If $m_0 = min\{m \geq 0: \mathscr{T}or_n^{(\Lambda,\Gamma)}[\Gamma, \Gamma] = 0,\ \forall n \geq m+1\}$, then

$$\text{r.gl.dim}\, \Gamma \leq \text{r.gl.dim}\, \Lambda \leq m_0 + 2 \cdot \text{r.gl.dim}\, \Gamma.$$

In particular if $\vartheta = 0$, then

$$\text{r.gl.dim}\, \Gamma \leq \text{r.gl.dim}\, \Lambda \leq c(M) + 2 \cdot \text{r.gl.dim}\, \Gamma.$$

Our purpose now is to construct, $\forall X \in \mathscr{D}$, an $\mathfrak{F}$-exact resolution of $\mathbf{Z}(X)$ consisting of $\mathbf{C}$-acyclic objects with respect to $\mathfrak{F}$. We will use this resolution to compute the derived functors $\mathscr{L}_i^{\mathfrak{F}}\mathbf{C}\mathbf{Z}$, $\mathscr{E}xt^i_{\mathfrak{F}}[\mathbf{Z}(-),\mathbf{Z}(-)]$. We begin by considering an arbitrary $\mathfrak{R}$-projective resolution $\mathscr{P}^{\bullet}_{\mathfrak{R}}(X)$ of X:

$$\cdots \to P_{n+1} \xrightarrow{a_{n+1}} P_n \xrightarrow{a_n} P_{n-1} \to \cdots \to P_1 \xrightarrow{a_1} P_0 \xrightarrow{a_0} X \to 0.$$

Let $a_{n+1} = l_{n+1} \circ k_n$ be the canonical factorization of a_{n+1}, and let $K_n = Im(a_{n+1})$, $\forall n \geq 0$ be the corresponding $\mathfrak{R}$-syzygies of X.

Lemma 7.9. *There exists an $\mathfrak{F}$-exact resolution of $\mathbf{Z}(X)$ consisting of $\mathbf{C}$-acyclic objects with respect to $\mathfrak{F}$, of the form $\mathfrak{X}^{\bullet}_{\mathfrak{F}}\mathbf{Z}(X)$:*

$$\cdots \to \mathbf{T}\left(\bigoplus_{p+q=n+1} F^pP_q\right) \xrightarrow{\varpi_n} \mathbf{T}\left(\bigoplus_{p+q=n} F^pP_q\right) \xrightarrow{\varpi_{n-1}} \mathbf{T}\left(\bigoplus_{p+q=n-1} F^pP_q\right) \to$$

$$\cdots \to \mathbf{T}(P_2 \oplus FP_1 \oplus F^2P_0) \xrightarrow{\varpi_1} \mathbf{T}(P_1 \oplus FP_0) \xrightarrow{\varpi_0} \mathbf{T}(P_0) \xrightarrow{\alpha} \mathbf{Z}(X) \to 0.$$

Proof. As in the proof of Theorem 3.4, we have the following $\mathfrak{F}$-projective presentation of $\mathbf{Z}(X)$: $0 \to (Q_0, g_0) \xrightarrow{\kappa_0} \mathbf{T}(P_0) \xrightarrow{\alpha_0} \mathbf{Z}(X) \to 0$, where $Q_0 = K_0 \oplus FP_0$,

$$g_0 = \begin{pmatrix} 0 & Fk_0 \\ 0 & \eta_{P_0} \end{pmatrix}, \quad \kappa_0 = \begin{pmatrix} k_0 & 0 \\ 0 & 1_{FP_0} \end{pmatrix}, \quad \alpha_0 = {}^t(a_0, 0).$$

It is easy to see that we have an $\mathfrak{F}$-exact sequence $0 \to (Q_1, g_1) \xrightarrow{\kappa_1} \mathbf{T}(P_1 \oplus FP_0) \xrightarrow{\alpha_1} (Q_0, g_0) \to 0$. By induction we have $\mathfrak{F}$-exact sequences

$$0 \to (Q_{n-1}, g_{n-1}) \xrightarrow{\kappa_{n-1}} \mathbf{T}\left(\bigoplus_{p+q=n-1} F^pP_q\right) \xrightarrow{\alpha_{n-1}} (Q_{n-2}, g_{n-2}) \to 0,$$

where $Q_{n-1} = K_{n-1} \oplus \bigoplus_{p+q=n-1} F^pP_q$. The Yoneda composition of the above $\mathfrak{F}$-exact sequences, is the desired $\mathfrak{F}$-exact resolution of $\mathbf{Z}(X)$, which by condition (†), consists of $\mathbf{C}$-acyclic objects with respect to $\mathfrak{F}$. $\square$

From the proof of the above lemma, an easy calculation shows that

$$\mathbf{C}(\varpi_n) = \begin{pmatrix} a_{n+1} & 0 & 0 & \cdots & 0 & 0 \\ 0 & (-1)^nFa_n & 0 & \cdots & 0 & 0 \\ 0 & \eta^1_{P_{n-1}} & (-1)^nF^2a_{n-1} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \eta^{n-1}_{P_1} & (-1)^nF^na_1 \\ 0 & 0 & 0 & \cdots & 0 & \eta^n_{P_0} \end{pmatrix},$$

where $\eta^j_{P_i} = \sum_{k=0}^{j-1}(-1)^kF^k\eta_{F^{j-k-1}P_j} : F^{j+1}P_i \to F^jP_i$.

Theorem 7.10. *The $\mathfrak{F}$-associated complex $\mathfrak{F}^{\bullet}_{\mathbf{Z}(X)}$ of $\mathbf{Z}(X)$ is quasi-isomorphic to the complex $\mathbf{C}\mathfrak{X}^{\bullet}_{\mathfrak{F}}\mathbf{Z}(X)$, thus*

$$\mathscr{L}_n^{\mathfrak{F}}\mathbf{C}\mathbf{Z}(X) \cong H_n[\mathbf{C}\mathfrak{X}^{\bullet}_{\mathfrak{F}}\mathbf{Z}(X)], \quad \forall n \geq 0.$$

In particular if $\eta = 0$, then

$$\mathscr{L}_n^{\mathfrak{F}} \mathbf{CZ} \cong \bigoplus_{p+q=n} \mathscr{L}_q^{\mathfrak{R}} F^p, \quad \forall n \geq 0.$$

Proof. By Lemma 7.9, the $\mathfrak{F}$-exact resolution $\mathfrak{X}_{\mathfrak{F}}^{\bullet}\mathbf{Z}(X)$ of $\mathbf{Z}(X)$, consists of $\mathbf{C}$-acyclic objects. So $\mathscr{L}_i^{\mathfrak{F}}\mathbf{CZ}(X) \cong H_i[\mathbf{C}\mathfrak{X}_{\mathfrak{F}}^{\bullet}\mathbf{Z}(X)]$, $\forall i \geq 0$. If $\eta = 0$, then $\eta_{P_i}^j = 0$, $\forall i, j \geq 0$. Then the assertion follows from the shape of the morphism $\mathbf{C}(\varpi_n)$. $\square$

Corollary 7.11. *If $\eta = 0$, then the following are equivalent*:
(i) $\mathfrak{F}$-gl.dim $\mathscr{D}_F(0) < \infty$.
(ii) (a) $\mathfrak{R}$-gl.dim $\mathscr{D} < \infty$.
(b) $\exists n \geq 0$: $\bigoplus_{p+q=n} \mathscr{L}_q^{\mathfrak{R}} F^p = 0$.
If $m = min\{n \geq 0$: $\bigoplus_{p+q=n+1} \mathscr{L}_q^{\mathfrak{R}} F^p = 0\}$, we have the bounds:

$$\mathfrak{R}\text{-gl.dim}\,\mathscr{D} \leq \mathfrak{F}\text{-gl.dim}\,\mathscr{D}_F(0) \leq m + \mathfrak{R}\text{-gl.dim}\,\mathscr{D}.$$

Corollary for Rings 7.12. *Let $\Lambda = \Gamma \ltimes_0 M$ be a trivial cleft extension of rings and assume that $\mathscr{T}or_i^{\Gamma}[M, \otimes_{\Gamma}^j M] = 0$, $\forall i, j \geq 1$. Then*

$$\forall X \in \mathrm{Mod}(\Gamma)\colon \mathscr{T}or_n^{\Lambda}[X_{\Lambda}, {}_{\Lambda}\Gamma] \cong \bigotimes_{p+q=n} \mathscr{T}or_q^{\Gamma}[X, \otimes_{\Gamma}^p M], \quad \forall n \geq 0,$$

$$\text{gl.dim}\,\Gamma \leq \text{gl.dim}\,\Lambda \leq min\,\{\, n \geq 0\colon \mathscr{T}or_q^{\Gamma}[-, \otimes_{\Gamma}^p M] = 0, \ \forall p + q = n + 1\} + \text{gl.dim}\,\Gamma.$$

From now on we assume that $\mathscr{D}$ has enough $\mathfrak{R}$-injectives.

Theorem 7.13. *$\forall X, Y \in \mathscr{D}$, there exists an isomorphism*:

$$\mathscr{E}xt_{\mathfrak{F}}^n[\mathbf{Z}(X), \mathbf{Z}(Y)] \cong H^n[\mathrm{Tot}\, M^{\bullet\bullet}], \quad \forall n \geq 0$$

where $M^{\bullet\bullet} = \mathscr{D}[\mathbf{C}\mathfrak{X}_{\mathfrak{F}}^{\bullet}\mathbf{Z}(X), I_{\mathfrak{R}}^{\bullet}(Y)]$ is the double complex arising from an arbitrary $\mathfrak{R}$-injective resolution $I_{\mathfrak{R}}^{\bullet}(Y)$ of Y in $\mathscr{D}$. In particular if $\eta = 0$, then

$$\mathscr{E}xt_{\mathfrak{F}}^n[\mathbf{Z}(X), \mathbf{Z}(Y)] \cong \bigoplus_{p+q=n} \mathscr{R}_{\mathfrak{R}}^p \mathscr{D}[F^q X, Y], \quad \forall n \geq 0.$$

Proof. For any $X \in \mathscr{D}$, consider the $\mathfrak{F}$-projective resolution $\mathscr{P}_{\mathfrak{F}}^{\mathbf{Z}(X)}$, constructed in Section 3, and the $\mathfrak{F}$-exact $\mathbf{C}$-acyclic resolution $\mathfrak{X}_{\mathfrak{F}}^{\bullet}\mathbf{Z}(X)$ constructed above. Obviously, the identity morphism of $\mathbf{Z}(X)$ induces a morphism of resolutions $\xi^{\circ}: \mathscr{P}_{\mathfrak{F}}^{\mathbf{Z}(X)} \to \mathfrak{X}_{\mathfrak{F}}^{\bullet}\mathbf{Z}(X)$. By Theorem 7.10, the morphism $\mathbf{C}(\xi^{\circ})$ is a quasi-isomorphism, i.e. $\mathbf{C}(\xi^{\circ})$ induces an isomorphism $H_n[\mathbf{C}(\xi^{\circ})] : H_n[\mathbf{C}(\mathscr{P}_{\mathfrak{F}}^{\mathbf{Z}(X)})] \to H_n[\mathbf{C}(\mathfrak{X}_{\mathfrak{F}}^{\bullet}\mathbf{Z}(X))] \cong \mathscr{L}_n^{\mathfrak{F}}\mathbf{CZ}(X)$, $\forall n \geq 0$. For any $Y \in \mathscr{D}$, the morphism $\mathbf{C}(\xi^{\circ})$, induces a morphism of complexes: $\zeta^{\circ} = \mathscr{D}[\mathbf{C}(\xi^{\circ}), Y] : \mathscr{D}[\mathbf{C}(\mathfrak{X}_{\mathfrak{F}}^{\bullet}\mathbf{Z}(X)), Y] \to \mathscr{D}[\mathbf{C}(\mathscr{P}_{\mathfrak{F}}^{\mathbf{Z}(X)}), Y]$, and in turn ζ° induces a morphism in cohomology $H^n(\zeta^{\circ}) : H^n[\mathscr{D}[\mathbf{C}(\mathfrak{X}_{\mathfrak{F}}^{\bullet}\mathbf{Z}(X)), Y]] \to H^n[\mathscr{D}[\mathbf{C}(\mathscr{P}_{\mathfrak{F}}^{\mathbf{Z}(X)}), Y]]$. By Section 3, we have an isomorphism: $H^n[\mathscr{D}[\mathbf{C}(\mathscr{P}_{\mathfrak{F}}^{\mathbf{Z}(X)}), Y]] \cong \mathscr{E}xt_{\mathfrak{F}}^n[\mathbf{Z}(X), \mathbf{Z}(Y)]$. Hence we have a well-defined morphism

$$H^n(\zeta^{\circ}) : H^n[\mathscr{D}[\mathbf{C}(\mathfrak{X}_{\mathfrak{F}}^{\bullet}\mathbf{Z}(X)), Y]] \to \mathscr{E}xt_{\mathfrak{F}}^n[\mathbf{Z}(X), \mathbf{Z}(Y)], \quad \forall n \geq 0.$$

Consider now an $\mathfrak{R}$-injective resolution $I^\bullet_{\mathfrak{R}}(Y)$ of Y in $\mathscr{D}$, and denote by $M^{\bullet\bullet} = \mathscr{D}[\mathbf{C}\mathfrak{X}^\bullet_{\mathfrak{F}}\mathbf{Z}(X), I^\bullet_{\mathfrak{R}}(Y)]$, $N^{\bullet\bullet} = \mathscr{D}[\mathbf{C}\mathscr{P}^{\mathbf{Z}(X)}_{\mathfrak{F}}, I^\bullet_{\mathfrak{R}}(Y)]$ the induced double complexes. The morphism ζ°, extends to a morphism of double complexes $\zeta^{\circ\circ} : M^{\bullet\bullet} \to N^{\bullet\bullet}$ and we have the following usual spectral sequences arising from $M^{\bullet\bullet}, N^{\bullet\bullet}$:

(i) ${}_{\mathrm{I}}{}^1E_2^{p,q} = {}_{\mathrm{I}}H^q[{}_{\mathrm{II}}H^p(M^{\bullet\bullet})] \Rightarrow H^{p+q}[\operatorname{Tot} M^{\bullet\bullet}]$,

(ii) ${}_{\mathrm{II}}{}^1E_2^{p,q} = {}_{\mathrm{II}}H^p[{}_{\mathrm{I}}H^q(M^{\bullet\bullet})] \Rightarrow H^{p+q}[\operatorname{Tot} M^{\bullet\bullet}]$,

(iii) ${}_{\mathrm{I}}{}^2E_2^{p,q} = {}_{\mathrm{I}}H^q[{}_{\mathrm{II}}H^p(N^{\bullet\bullet})] \Rightarrow H^{p+q}[\operatorname{Tot} N^{\bullet\bullet}]$,

(iii) ${}_{\mathrm{II}}{}^2E_2^{p,q} = {}_{\mathrm{II}}H^p[{}_{\mathrm{I}}H^q(N^{\bullet\bullet})] \Rightarrow H^{p+q}[\operatorname{Tot} N^{\bullet\bullet}]$.

Calculating the spectral sequence (ii), we have

$$ {}_{\mathrm{I}}H^q(M^{\bullet\bullet}) \cong H^q[\mathscr{D}[\mathbf{C}\mathfrak{X}^\bullet_{\mathfrak{F}}\mathbf{Z}(X), I^p]] \cong \mathscr{D}[\mathscr{L}^{\mathfrak{F}}_q\mathbf{C}\mathbf{Z}(X), I^p]. $$

Hence ${}_{\mathrm{II}}{}^1E_2^{p,q} = \mathscr{E}xt^p_{\mathfrak{R}}[\mathscr{L}^{\mathfrak{F}}_q\mathbf{C}\mathbf{Z}(X), Y]$. Similarly calculating the spectral sequence (iii), we see easily that (iii) collapses giving isomorphisms:

$$ {}_{\mathrm{I}}{}^2E_2^{n,0} \cong \mathscr{E}xt^n_{\mathfrak{F}}[\mathbf{Z}(X), \mathbf{Z}(Y)] \cong H^n[\operatorname{Tot} N^{\bullet\bullet}], \quad \forall n \geq 0. $$

Finally calculating the spectral sequence (iv), we get as in the case of (ii) that

$$ {}_{\mathrm{II}}{}^2E_2^{p,q} \cong \mathscr{E}xt^p_{\mathfrak{R}}[\mathscr{L}^{\mathfrak{F}}_q\mathbf{C}\mathbf{Z}(X), Y]. $$

Now the morphism $\zeta^{\circ\circ} : M^{\bullet\bullet} \to N^{\bullet\bullet}$, induces a morphism of spectral sequences (ii) $\to$ (iv), such that $\zeta^{\circ\circ}$ induces isomorphisms between the terms ${}_{\mathrm{II}}{}^1E_2^{p,q}$, ${}_{\mathrm{II}}{}^2E_2^{p,q}$. From the Comparison Theorem of spectral sequences, see [8], the morphism $\zeta^{\circ\circ} : M^{\bullet\bullet} \to N^{\bullet\bullet}$, induces isomorphisms in cohomology: $H^n(\zeta^{\circ\circ}) : H^n(\operatorname{Tot} M^{\bullet\bullet}) \to H^n(\operatorname{Tot} N^{\bullet\bullet})$, $\forall n \geq 0$. Since $H^n(\operatorname{Tot} N^{\bullet\bullet})$ is isomorphic to $\mathscr{E}xt^n_{\mathfrak{F}}[\mathbf{Z}(X), \mathbf{Z}(Y)]$, we get finally that $\zeta^{\circ\circ}$ induces isomorphisms:

$$ \mathscr{E}xt^n_{\mathfrak{F}}[\mathbf{Z}(X), \mathbf{Z}(Y)] \cong H^n(\operatorname{Tot} M^{\bullet\bullet}), \quad \forall X, Y \in \mathscr{D},\ \forall n \geq 0. $$

Suppose now $\eta = 0$. Then from the shape of the complex $\mathbf{C}\mathfrak{X}^\bullet_{\mathfrak{F}}\mathbf{Z}(X)$, we have

$$ \begin{aligned} (\operatorname{Tot} M^{\bullet\bullet})^n &= \bigoplus_{p+q=n} \mathscr{D}[\mathbf{C}\mathfrak{X}^\bullet_{\mathfrak{F}}\mathbf{Z}(X)_p, I^q] = \bigoplus_{p+q=n} \mathscr{D}\left[\bigoplus_{r+s=p} F^r P_s, I^q\right] \\ &= \bigoplus_{p+q=n}\bigoplus_{r+s=p} \mathscr{D}[F^r P_s, I^q]. \end{aligned} $$

Hence $(\operatorname{Tot} M^{\bullet\bullet})^n = \bigoplus_{p+q=n}(\operatorname{Tot} K_q^{\bullet\bullet})^p$, where $\operatorname{Tot} K_q^{\bullet\bullet}$ is the total complex of the double complex $\mathscr{D}[F^q(\mathscr{P}^\bullet_{\mathfrak{R}}(X)), I^\bullet_{\mathfrak{R}}(Y)]$ and $\mathscr{P}^\bullet_{\mathfrak{R}}(X)$ is an $\mathfrak{R}$-projective resolution of X. Consequently $H^n[\operatorname{Tot} M^{\bullet\bullet}] = \bigoplus_{p+q=n} H^p[\operatorname{Tot} K_q^{\bullet\bullet}]$, $\forall n \geq 0$. From the complex $\operatorname{Tot} K_q^{\bullet\bullet}$, we see directly that $H^p[\operatorname{Tot} K_q^{\bullet\bullet}] = \mathscr{R}^p_{\mathfrak{R}}\mathscr{D}[F^q X, Y]$ is the pth right derived functor of the nonbalanced functor $\mathscr{D}[F^q-,-] : \mathscr{D}^{op} \times \mathscr{D} \to \mathscr{A}b$. We conclude that

$$ \mathscr{E}xt^n_{\mathfrak{F}}[\mathbf{Z}(X), \mathbf{Z}(Y)] \cong \bigoplus_{p+q=n} H^p[\operatorname{Tot} K_q^{\bullet\bullet}] \cong \bigoplus_{p+q=n} \mathscr{R}^p_{\mathfrak{R}}\mathscr{D}[F^q X, Y], \quad \forall n \geq 0. \qquad \square $$

The case $\eta = 0$ in the previous theorem as well as in the following corollaries generalizes, simplifies and completes all the analogous results of [31].

Corollary 7.14. *If* $\eta = 0$, *then*

$$\mathfrak{F}\text{-gl.dim}\,\mathscr{D}_F(0) = sup\left\{ n \geq 0\colon \bigoplus_{p+q=n+1} \mathscr{R}^p_{\mathfrak{R}}\,\mathscr{D}[F^q-,-] = 0 \right\}.$$

Corollary for Rings 7.15. $\Lambda = \Gamma \ltimes_0 M$ *be a trivial cleft extension of rings and assume that* $\mathscr{T}or_i^{\Gamma}[M, \otimes^j_{\Gamma} M] = 0$, $\forall i, j \geq 1$. *Then*

$$\text{gl.dim}\,\Lambda = sup\left\{ n \geq 0\colon \bigoplus_{p+q=n+1} \mathscr{R}^p\,\text{Hom}_{\Gamma}[-\otimes_{\Gamma}(\otimes^q_{\Gamma} M), -] = 0 \right\}.$$

7.2. Morphism and comma categories, categories of complexes

Corollary 7.16. *If* $\eta = 0$, *and the following vanishing condition* (††) *below is true*:

(††) $\quad \mathscr{L}^{\mathfrak{R}}_i F(FP) = 0, \quad \forall i \geq 0, \ \forall P \in \mathscr{P}(\mathfrak{R})$

then we have the following:

$$\mathscr{L}^{\mathfrak{F}}_n \mathbf{CZ} = \mathscr{L}^{\mathfrak{R}}_{n-1} F, \ \mathscr{E}xt^n_{\mathfrak{F}}[\mathbf{Z}(X), \mathbf{Z}(Y)] \cong \mathscr{E}xt^n_{\mathfrak{F}}[X, Y] \oplus \mathscr{R}^{n-1}_{\mathfrak{R}}\mathscr{D}[FX, Y], \quad \forall n \geq 1,$$

$$\mathfrak{F}\text{-gl.dim}\,\mathscr{D}_F(0) = max\{\mathfrak{R}\text{-gl.dim}\,\mathscr{D}, \{n \geq 0\colon \mathscr{R}^{n-1}_{\mathfrak{R}}\mathscr{D}[F-,-] = 0\}\}.$$

Proof. Condition (††) ensures that $F^2 = 0$, and that the vanishing condition (†) is true for F. The rest follows from Corollary 7.14. □

Corollary for Rings 7.17. Let $\Lambda = \Gamma \ltimes_0 M$ *be a trivial cleft extension of rings and assume that* $\mathscr{T}or_i^{\Gamma}[M, M] = 0$, $\forall i \geq 0$. *Then* $\forall X, Y \in \text{Mod}(\Gamma)$, $\forall n \geq 1$:

$$\mathscr{T}or_n^{\Lambda}[X_{\Lambda}, {}_{\Lambda}\Gamma] \cong \mathscr{T}or^{\Gamma}_{n-1}[X_{\Gamma}, {}_{\Gamma}M],$$

$$\mathscr{E}xt^n_{\Lambda}[X_{\Lambda}, Y_{\Lambda}] \cong \mathscr{E}xt^n_{\Lambda}[X_{\Gamma}, Y_{\Gamma}] \oplus \mathscr{R}^{n-1}\,\text{Hom}_{\Gamma}[X \otimes_{\Gamma} M, Y],$$

$$\text{gl.dim}\,\Lambda = max\{\text{gl.dim}\,\Gamma, sup\{n \geq 0\colon \mathscr{R}^{n-1}\,\text{Hom}_{\Gamma}[-\otimes_{\Gamma} M, -] = 0\}.$$

Let $G : \mathscr{D} \to \mathscr{E}$ be a right exact functor between abelian categories, and let $\mathscr{C} = (\mathscr{D}, G, \mathscr{E})$ be the induced comma-category. As in Section 2, $\mathscr{C}$ can be considered as a trivial extension of $\mathscr{D} \times \mathscr{E}$ by the right exact functor $F : \mathscr{D} \times \mathscr{E} \to \mathscr{D} \times \mathscr{E}$ given by $F(A, B) = (0, GA)$. It is clear that the functor F satisfies the condition (††) of Corollary 7.16. Hence we have the following.

Corollary 7.18. $\forall (X_1, Y_1), (X_2, Y_2) \in \mathscr{C}$, $\forall n \geq 0$:

$$\mathscr{E}xt^n_{\mathscr{C}}[\mathbf{Z}(X_1, Y_1), \mathbf{Z}(X_2, Y_2)] \cong \mathscr{E}xt^n_{\mathscr{D}}[X_1, X_2] \oplus \mathscr{E}xt^n_{\mathscr{E}}[Y_1, Y_2] \oplus \mathscr{R}^{n-1}\mathscr{D} \times \mathscr{E}[GX_1, Y_2].$$

$$\text{gl.dim}\,\mathscr{C} = max\{\text{gl.dim}\,\mathscr{D}, \ \text{gl.dim}\,\mathscr{E}, \ sup\{n\colon \mathscr{R}^n\,\text{Hom}_{\mathscr{D}\times\mathscr{E}}[G-,-] = 0\}\}.$$

Let ${}_RM_S$ be an R-S-bimodule, and let $\left(\begin{smallmatrix} R & {}_RM_S \\ 0 & S \end{smallmatrix}\right)$ be the induced triangular matrix ring. A direct consequence of the above result is the following.

Corollary for Rings 7.19.

$$\text{r.gl.dim}\begin{pmatrix} R & {}_RM_S \\ 0 & S \end{pmatrix} = max\,\{\text{r.gl.dim.}\,R,\ \text{r.gl.dim}\,S,$$
$$sup\{n\colon \mathscr{R}^{n-1}\text{Hom}_{R\times S}[-\otimes_R M_S, -] = 0\}\}.$$

Let $\mathscr{D}^n$ be the category of composable morphisms of length $n-1$, over the abelian category $\mathscr{D}$. Then $\mathscr{D}^n$ is a trivial extension (of $\mathscr{D}^{n-1}\times\mathscr{D}$), as is the category $\mathscr{C}^{[0,n]}(\mathscr{D})$ of complexes of length n over $\mathscr{D}$. Let $\mathscr{C}^{\emptyset}(\mathscr{D})$ be the category of complexes over $\mathscr{D}$ and let $\mathscr{C}^b(\mathscr{D})$, $\mathscr{C}^-(\mathscr{D})$ and $\mathscr{C}^+(\mathscr{D})$, be the full subcategories of bounded, bounded below and bounded above complexes, respectively. The following is a trivial consequence of the above results.

Corollary 7.20. $\forall n \geq 0$: $\text{gl.dim}\,\mathscr{D}^n = \text{gl.dim}\,\mathscr{C}^{[0,n]}(\mathscr{D}) = n + \text{gl.dim}\,\mathscr{D}$,

$$\text{gl.dim}\,\mathscr{C}^*(\mathscr{D}) = \infty \quad \text{for } * = b, -, +, \emptyset.$$

7.3. The Butler–Horrocks spectral sequence

For any object $(X, f) \in \mathscr{D}_F(\eta)$ we consider the $\mathfrak{F}$-projective resolution of (X, f):

$$\mathscr{P}_{\mathfrak{F}}^{(X,f)} : \cdots \to \mathbf{T}(P_p) \xrightarrow{\phi_p^{(X,f)}} \mathbf{T}(P_{p-1}) \to \cdots \to \mathbf{T}(P_0) \xrightarrow{\alpha_0^{(X,f)}} (X, f) \to 0$$

and the functorial Σ-projective resolutions of the objects $(X, f), \mathbf{T}(P_p)$, $\forall p \geq 0$:

$$\mathscr{P}_{\Sigma}^{(X,f)} : \cdots \to \mathbf{T}(F^qX) \xrightarrow{\theta_q^{(X,f)}} \mathbf{T}(F^{q-1}X) \to \cdots \to \mathbf{T}(X) \xrightarrow{\varepsilon_{(X,f)}} (X, f) \to 0,$$

$$\mathscr{P}_{\Sigma}^{\mathbf{T}(P_p)} : \cdots \to \mathbf{T}(F^qP_p \oplus F^{q+1}P_p) \xrightarrow{\theta_q^{\mathbf{T}(P_p)}} \mathbf{T}(F^{q-1}P_p \oplus F^qP_p) \to$$
$$\cdots \to \mathbf{T}(FP_p \oplus F^2P_p) \xrightarrow{\theta_1^{\mathbf{T}(P_p)}} \mathbf{T}(P_p \oplus FP_p) \xrightarrow{\varepsilon_{\mathbf{T}(P_p)}} \mathbf{T}(P_p) \to 0$$

constructed in Section 3. All these resolutions are imbedded in the following complex of Σ-projective resolutions:

$$\cdots \to \mathscr{P}_{\Sigma}^{\mathbf{T}(P_p)} \to \mathscr{P}_{\Sigma}^{\mathbf{T}(P_{p-1})} \to \cdots \to \mathscr{P}_{\Sigma}^{\mathbf{T}(P_1)} \to \mathscr{P}_{\Sigma}^{\mathbf{T}(P_0)} \to \mathscr{P}_{\Sigma}^{(X,f)} \to 0. \tag{1}$$

The differentials in (1) are $\mathbf{T}F^q\mathbf{U}(\phi_p^{(X,f)}) : \mathscr{P}_{\Sigma^p}^{\mathbf{T}(P_p)} \to \mathscr{P}_{\Sigma^p}^{\mathbf{T}(P_{p-1})}$. Now applying to this sequence the functor $\mathbf{C}$, we obtain the following sequence of the Σ-associated complexes of the objects $(X, f), \mathbf{T}(P_p)$, $\forall p \geq 0$:

$$\cdots \to \Sigma^{\bullet}_{\mathbf{T}(P_p)} \to \Sigma^{\bullet}_{\mathbf{T}(P_{p-1})} \to \cdots \to \Sigma^{\bullet}_{\mathbf{T}(P_1)} \to \Sigma^{\bullet}_{\mathbf{T}(P_0)} \to \Sigma^{\bullet}_{(X,f)} \to 0. \tag{2}$$

The differentials in (2) are $F^q\mathbf{U}(\phi_p^{(X,f)}) : \Sigma^p_{\mathbf{T}(P_p)} \to \Sigma^p_{\mathbf{T}(P_{p-1})}$. Using the sign lemma [35], we can view sequences (1) and (2) as first-quadrant bicomplexes ${}_iM_{\bullet\bullet}^{(X,f)}$, $i =$

1, 2, in which we have deleted the first row and column. Then ${}_2M_{p,q}^{(X,f)} = F^qP_p \oplus F^{q+1}P_p$, $\forall p,q \geq 0$ and the differentials are given by $\delta^1_{p,q} = F^q\mathbf{U}(\phi_p^{(X,f)}) : M_{p,q}^{(X,f)} \to M_{p-1,q}^{(X,f)}$, $\delta^2_{p,q} = (-1)^p\mathbf{C}(\theta_q^{\mathbf{T}(P_p)}) : M_{p,q}^{(X,f)} \to M_{p,q-1}^{(X,f)}$. Since the columns are the Σ-associated complexes of the objects $\mathbf{T}(P_p)$, all columns are contractible and the rows are obtained from the deleted $\mathfrak{F}$-projective resolution of (X, f) by applying succesively the functors $F^q\mathbf{U}$, $\forall\, q \geq 0$. We observe that we have an augmentation $\mathrm{Tot}_1\, M_{\bullet\bullet}^{(X,f)} \to (X, f) \to 0$.

Lemma 7.21. *The total complex* $\mathrm{Tot}_1\, M_{\bullet\bullet}^{(X,f)}$ *is an* $\mathfrak{F}$*-exact resolution of* (X, f), *which consists of* $\mathbf{C}$*-acyclic objects with respect to* $\mathfrak{F}$ *if the functor F satisfies the vanishing condition* (†), *in which case*

$$\mathscr{L}_n^{\mathfrak{F}}\mathbf{C}(X, f) \cong H_n[\mathrm{Tot}_1\, M_{\bullet\bullet}^{(X,f)}], \quad \forall n \geq 0.$$

Proof. The first part follows from the spectral sequence induced by the bicomplex ${}_1M_{\bullet\bullet}^{(X,f)}$, and the second follows from the first. □

Now we define the *Σ-projective resolution functor* $\mathscr{S} : \mathscr{D}_F(\eta) \to \mathscr{C}^{0]}(\mathscr{D}_F(\eta))$ to the abelian category $\mathscr{C}^{0]}(\mathscr{D}_F(\eta))$ of positive complexes of $\mathscr{D}_F(\eta)$, as follows: $\mathscr{S}(X, f) = \mathscr{P}_{\Sigma}^{(X,f)}$, and if $a : (X, f) \to (Y, g)$ is a morphism in $\mathscr{D}_F(\eta)$, then $\mathscr{S}(a)^i = \mathbf{T}(F^ia)$, $\forall i \geq 0$. Let $\mathbf{C}^{0]} : \mathscr{C}^{0]}(\mathscr{D}_F(\eta)) \to \mathscr{C}^{0]}(\mathscr{D})$ be the natural extension of the functor $\mathbf{C} : \mathscr{D}_F(\eta) \to \mathscr{D}$, and define:

$$\Sigma^\bullet = \mathbf{C}^{0]}\mathscr{S} : \mathscr{D}_F(\eta) \to \mathscr{C}^{0]}(\mathscr{D}).$$

Then $\Sigma^\bullet(X, f) = \Sigma^\bullet_{(X,f)} =$ the Σ-associated complex of (X, f), and if $a : (X, f) \to (Y, g)$ is a morphism in $\mathscr{D}_F(\eta)$, then $\Sigma^\bullet(a)^i = F^ia$, $\forall i \geq 0$. Complexes (1) and (2) are therefore the images of $\mathscr{P}_{\mathfrak{F}}^{(X,f)}$ under the functors $\mathscr{S}, \Sigma^\bullet$. Obviously $\Sigma^\bullet$ is a right $\mathfrak{F}$-exact and Σ-exact embedding. So it is interesting to consider the functors $\mathscr{L}_q^{\mathfrak{F}}\Sigma^\bullet$. The objects $\mathscr{L}_q^{\mathfrak{F}}\Sigma^\bullet(X, f)$ are by construction positive complexes in $\mathscr{D}$. The following describes the Butler–Horrocks spectral sequence for the functor $\mathbf{C}$, induced by the inclusion of the proper classes $\Sigma \subseteq \mathfrak{F}$, see [7,12].

Theorem 7.22. $\forall (X, f) \in \mathscr{D}_F(\eta)$, *there is a spectral sequence of the form*:

$$(!) \qquad E^2_{p,q} = \mathscr{L}_p^{\Sigma}[\mathscr{L}_q^{\mathfrak{F}}\mathbf{C}](X, f) = H_p\mathscr{L}_q^{\mathfrak{F}}\Sigma^\bullet(X, f) \Rightarrow \mathscr{L}_n^{\mathfrak{F}}\mathbf{C}(X, f).$$

If the vanishing condition (†) *is true for F, then the following are equivalent*:

(i) $\forall p,q \geq 1$ such that $p + q \geq n + 1$: $\mathscr{L}_p^{\mathfrak{R}}F^q = 0$.

(ii) $\mathscr{L}_m^{\mathfrak{F}}\mathbf{C} \cong \mathscr{L}_m^{\Sigma}\mathbf{C}$, $\forall m \geq n + 1$.

In particular if $\eta = 0$, *the vanishing condition* (††) *of Corollary* 7.16 *is true for F, and* (i) *above holds, then* $\mathscr{L}_m^{\mathfrak{F}}\mathbf{C} = 0$, $\forall m \geq n + 1$.

Proof. We compute the spectral sequences arising from the bicomplex ${}_2M_{\bullet\bullet}^{(X,f)}$. For the first filtration, since the columns are contractible, we have ${}^{\mathrm{I}}E^2_{p,q} = 0$, $\forall q \geq 1$, and ${}^{\mathrm{I}}E^2_{p,0} = H_p[\mathfrak{F}^\bullet_{(X,f)}]$. Thus the spectral sequence arising from the first filtration collapses, and ${}^{\mathrm{I}}E^2_{n,0} \cong \mathscr{L}_n^{\mathfrak{F}}\mathbf{C}(X, f) \cong H_n[\mathrm{Tot}\,({}_2M_{\bullet\bullet}^{(X,f)})]$, $\forall n \geq 0$. For filtration II, we have $H^1_{p,q}[{}_2M^{(X,f)}_{\bullet\bullet}] = Ker(F^p\mathbf{U}(\phi^q))/Im(F^p\mathbf{U}(\phi^{q+1})) = (\mathscr{L}^{\mathfrak{F}}_q\Sigma^{\bullet}(X,f))^p$. Hence ${}^{\mathrm{II}}E^2_{p,q} = H_p(\mathscr{L}^{\mathfrak{F}}_q\Sigma^{\bullet}(X,f))$.

Suppose now that condition (†) is true for F. Then the underline graded object of the complex $\mathscr{L}^{\mathfrak{F}}_q\Sigma^{\bullet}(X,f)$ is $\mathscr{L}^{\mathfrak{R}}_p F^q X$, $\forall p,q \geq 1$. Indeed, as we have seen $(\mathscr{L}^{\mathfrak{F}}_q\Sigma^{\bullet}(X,f))^p = Ker(F^p\mathbf{U}(\phi^q))/Im(F^p\mathbf{U}(\phi^{q+1}))$. But this homology is obtained applying the $\mathfrak{F}$-exact functor $\mathbf{U}$ to the $\mathfrak{F}$-projective resolution $\mathscr{P}^{(X,f)}_{\mathfrak{F}}$ of (X,f) and then applying to the resulting $\mathfrak{R}$-exact sequence $\mathbf{U}[\mathscr{P}^{(X,f)}_{\mathfrak{F}}]$ the functor F^p. By (†), the $\mathfrak{R}$-exact resolution $\mathbf{U}[\mathscr{P}^{(X,f)}_{\mathfrak{F}}]$ of X consists of F^p-acyclic objects with respect to $\mathfrak{R}$. So $Ker(F^p\mathbf{U}(\phi^q))/Im(F^p\mathbf{U}(\phi^{q+1})) = (\mathscr{L}^{\mathfrak{F}}_q\Sigma^{\bullet}(X,f))^p \cong \mathscr{L}^{\mathfrak{R}}_p F^q X$, $\forall p,q \geq 1$. Observe that ${}^{\mathrm{II}}E^2_{p,0} = H_p(\Sigma^{\bullet}(X,f)) = H_p[\Sigma^{\bullet}_{(X,f)}] \cong \mathscr{L}^{\Sigma}_p\mathbf{C}(X,f)$.

That (i) $\Rightarrow$ (ii) follows easily from the theory of spectral sequences, and the proof of (ii) $\Rightarrow$ (i), consists of a simple induction argument and is left to the reader. □

Combining Theorems 5.11(2) and 7.22, we have the following:

Corollary 7.23. *Suppose that the vanishing condition* (†) *is true for* F, *and* $\forall p \geq 0$, $q \geq 1$: $p+q = n+1$, $\mathscr{L}^{\mathfrak{R}}_p F^q = 0$. *Then*

$$\mathfrak{F}\text{-f.gl.dim}\,\mathscr{D}_F(\eta) \leq n + \mathfrak{R}\text{-f.gl.dim}\,\mathscr{D}.$$

The sequence of complexes (2), viewed as a bicomplex, has been constructed in an ad hoc manner in case $\eta = 0$ in [13]. The above theorem and corollary generalizes and improves all the corresponding results of [13].

Corollary for Rings 7.24. *Let* $\Lambda = \Gamma \ltimes_{\vartheta} M$ *be a cleft extension of rings. and suppose that* $\mathscr{T}or^{\Gamma}_i[M, \otimes^j_{\Gamma} M] = 0$, $\forall i,j \geq 1$. *If* $\mathscr{T}or^{\Gamma}_q[-, \otimes^p_{\Gamma} M] = 0$, $\forall p \geq 1$, $q \geq 1$: $p+q = n+1$, *then* $\mathscr{T}or^{\Lambda}_i[-,\Gamma] \cong \mathscr{T}or^{(\Lambda,\Gamma)}_i[-,\Gamma]$, $\forall i \geq n+1$. *If* $\mathscr{T}or^{\Gamma}_q[-, \otimes^p_{\Gamma} M] = 0$, $\forall p \geq 0$, $q \geq 1$: $p+q = n+1$, *then*

$$\text{r.f.gl.dim}\,\Lambda \leq n + \text{r.f.gl.dim}\,\Gamma.$$

8. A homological quide to the truncated extensions of rings and abelian categories

Throughout this section we consider a t-truncated extension $\mathscr{D} \ltimes_t F$ of the abelian category $\mathscr{D}$ by the right exact functor F, where $t \geq 0$. We recall that the objects of $\mathscr{D} \ltimes_t F$ are of the form (X,f) where $f: FX \to X$ is a morphism in $\mathscr{D}$ such that $f_{t+1} = 0$, where $f_i := F^i f \circ F^{i-1} f \circ \cdots \circ F^2 f \circ Ff \circ f$: $F^{i+1}X \to X$. From Section 2 we have an isomorphism of categories $\mathbb{A}: \mathscr{D} \ltimes_t F \to \mathscr{D}_{\tilde{F}}(\tilde{\eta})$, for an $\tilde{\eta}$-extension, where $\tilde{F} := \bigoplus_{i=1}^{t+1} F^i$ and $\tilde{\eta}: \tilde{F}^2 \to \tilde{F}$ is a suitable associative morphism. Hence we can apply the results of the previous sections, using the isomorphism $\mathbb{A}$. We point out that the absolute homology of $\mathscr{D} \ltimes_t F$ in the case F is exact, is studied in [27] using triple complexes. We begin the study of the $\mathfrak{F}, \Sigma$-homology of $\mathscr{D} \ltimes_t F$ by determining the Σ-associated complex of an arbitrary object (X,f) of $\mathscr{D} \ltimes_t F$. We denote by $\mathfrak{F}, \Sigma$ the usual proper classes in $\mathscr{D} \ltimes_t F$ and in $\mathscr{D}_{\tilde{F}}(\tilde{\eta})$. From the isomorphism $\mathbb{A}$ above and the shape of the Σ-associated complex $\Sigma^{\bullet}_{(X,f)}$ of an object $(X,f) \in \mathscr{D}_{\tilde{F}}(\tilde{\eta})$, it follows that $\Sigma^{\bullet}_{(X,f)}$ is too big for calculations. However we have the following:

Lemma 8.1. *$\forall (X,f) \in \mathscr{D} \ltimes_t F \cong \mathscr{D}_{\tilde{F}}(\tilde{\eta})$, the Σ-associated complex $\Sigma^{\bullet}_{(X,f)}$ of (X,f) is homotopy equivalent to the following complex*:

$$\begin{aligned} {}^t\Sigma^{\bullet}_{(X,f)}\colon \cdots \to F^{\phi(2n+1)}X &\xrightarrow{F^{\phi(2n)}f} F^{\phi(2n)}X \xrightarrow{-F^{\phi(2n-1)}f_t} F^{\phi(2n-1)}X \to \\ \cdots \to F^{\phi(3)}X &\xrightarrow{-F^{\phi(2)}f} F^{\phi(2)}X \xrightarrow{F^{\phi(1)}f_t} F^{\phi(1)}X \xrightarrow{-F^{\phi(0)}f} F^{\phi(0)}X \to 0, \end{aligned}$$

where $\phi:\mathbb{N}\to\mathbb{N}$ is the function defined by

$$\phi(n) = \begin{cases} m\cdot(t+2) & \text{if } n = 2\cdot m, \\ m\cdot(t+2)+1 & \text{if } n = 2\cdot m+1. \end{cases}$$

Proof. Let $(X,f) \in \mathscr{D} \ltimes_t F$. Then we have a short exact sequence:

$$0 \to \left(\bigoplus_{i=1}^{t+1} F^iX, \sigma_0\right) \xrightarrow{\kappa_0} \mathbf{T}(X) \xrightarrow{\varpi_{(X,f)}} (X,f) \to 0,$$

where ϖ is the counit of the adjunction $(\mathbf{T},\mathbf{U})$, defined by $\varpi_{(X,f)} = {}^t(1_X, f_0, f_1, \ldots, f_t)$, where σ_0: $\bigoplus_{i=2}^{t+2} F^iX \to \bigoplus_{i=1}^{t+1} F^iX$ is represented by a $(t+1)\times(t+1)$ matrix $(a_{i,j})$, where $a_{i,1} = -Ff_i$, $i = 0,1,\ldots,t$, $a_{i,i+1} = 1_{F^{i+1}X}$, $i = 0,1,\ldots,t$, and all other entries are zero, and finally $\kappa_0 = ker(\varpi_{(X,f)})$. Next we have the following exact sequence in $\mathscr{D} \ltimes_t F$:

$$0 \to (F^{t+2}X, F^{t+2}f_0) \xrightarrow{\kappa_1} \mathbf{T}(FX) \xrightarrow{\pi_1} \left(\bigoplus_{i=1}^{t+1} F^iX, \sigma_0\right) \to 0,$$

where π_1 is represented by a $(t+1)\times(t+2)$ matrix $(a_{i,j})$, where $a_{i,i-1} = -F^{i-1}f_0$, $i = 2,\ldots,t+1$, $a_{i,i} = 1_{F^iX}$, $i = 1,\ldots,t+1$, and all other entries are zero, and finally $\kappa_1 = ker(\pi_1)$. The Yoneda composition of the above sequences gives the following exact sequence:

$$0 \to (F^{t+2}X, F^{t+2}f_0) \to \mathbf{T}(FX) \to \mathbf{T}(X) \to (X,f) \to 0.$$

Continuing the above procedure starting now with $(F^{t+2}X, F^{t+2}f_0)$, and using the function ϕ, we have a resolution of (X,f):

$$\cdots \to \mathbf{T}(F^{\phi(n)}X) \to \mathbf{T}(F^{\phi(n-1)}X) \to \cdots \to \mathbf{T}(F^{\phi(0)}X) \to (X,f) \to 0 \tag{1}$$

Obviously (1) is a functorial Σ-projective resolution of (X,f) in $\mathscr{D} \ltimes_t F$. We denote (1) by ${}^t\mathscr{P}_{\Sigma}^{(X,f)}$. Now using the isomorphism $\mathbb{A}$, $\mathbb{A}({}^t\mathscr{P}_{\Sigma}^{(X,f)})$ is a Σ-projective resolution of $\mathbb{A}(X,f)$ in $\mathscr{D}_{\tilde{F}}(\tilde{\eta})$. Hence $\mathbb{A}({}^t\mathscr{P}_{\Sigma}^{(X,f)})$ is homotopy equivalent to $\mathscr{P}_{\Sigma}^{\mathbb{A}(X,f)}$, and this implies that $\Sigma^{\bullet}_{\mathbb{A}(X,f)}$ is homotopy equivalent to ${}^t\Sigma^{\bullet}_{(X,f)}$. $\square$

We note the function ϕ is defined in [27]. From now on we view the isomorphism $\mathbb{A}$ as an identification, and we call the complex ${}^t\Sigma^{\bullet}_{(X,f)}$, the *t-truncated Σ-associated complex* of (X,f). In case $t=0$, then (and only then) $\mathscr{D}_F(0)\cong\mathscr{D}\ltimes_0 F$ is the trivial extension, $\phi=\mathrm{Id}_{\mathbb{N}}$ and ${}^0\Sigma^{\bullet}_{(X,f)}=\Sigma^{\bullet}_{(X,f)}$. Having established a convenient description of $\Sigma^{\bullet}_{(X,f)}$ as ${}^t\Sigma^{\bullet}_{(X,f)}$, we can apply the results of the previous sections, observing that in $\mathscr{D}\ltimes_t F$ we have $\mathrm{Ker}\,\mathbf{C}=0$, since by Remark 5.2(3), $\mathscr{D}\ltimes_t F$ is $\mathfrak{F}$-admissible. Many of these results apply directly to the truncated case. Namely the results that are independent of the multiplication $\tilde{\eta}$. This is true, for example, for Theorem 4.11 concerning lifting of tilting objects. This theorem is true as stated replacing everywhere F with $\tilde{F}=\bigoplus_{i=1}^{t+1}F^i$. Now we state some results which depend on the index t in $\mathscr{D}\ltimes_t F$.

Corollary 8.2. *The following are equivalent for the cleft extension $\mathscr{D}\ltimes_t F$.*

(i) $\mathfrak{F}$-gl.dim $\mathscr{D}\ltimes_t F\leq 1$.

(ii) (a) $\mathfrak{R}$-gl.dim $\mathscr{D}\leq 1$.

(b) F is $\mathfrak{R}$*-exact.*

(c) $F^{t+2}=0$.

(d) $\forall X\in\mathscr{D}$: $FX\in\mathscr{P}(\mathfrak{R})$.

If condition (i) *is true, then $\mathscr{D}\ltimes_t F$ is isomorphic to the free cleft extension $\mathscr{D}(F)$.*

Proof. Exactly as in Proposition 6.1. □

Corollary for Rings 8.3. *Let $\Lambda=\Gamma\ltimes_t M$ be the t-truncated cleft extension of Γ by M, $t\geq 0$. Then the following are equivalent*:

(i) *$\Lambda=\Gamma\ltimes_t M$ is right hereditary.*

(ii) (a) *Γ is right hereditary, and ${}_{\Gamma}M$ is flat.*

(b) $\otimes_{\Gamma}^{t+2}M=0$.

(c) $\forall X\in\mathrm{Mod}(\Gamma): X\otimes_{\Gamma}M$ *is a projective right Γ-module.*

Corollary 8.4. *The following are equivalent for the cleft extension $\mathscr{D}\ltimes_t F$*:

(i) $\mathfrak{F}$-gl.dim $\mathscr{D}\ltimes_t F\leq 2$.

(ii) (a) $F^{t+3}=\mathscr{L}_1^{\mathfrak{R}}F^{t+2}=\mathscr{L}_2^{\mathfrak{R}}F^i=0,\ \forall i=1,2,\ldots,t+1$.

(b) $\forall P\in\mathscr{P}(\mathfrak{R})$: $\mathscr{L}_1^{\mathfrak{R}}FF(P)=0$.

(c) $\forall X\in\mathscr{D}$: $F^{t+2}X,\ \mathscr{L}_1^{\mathfrak{R}}FX\in\mathscr{P}(\mathfrak{R})$.

(d) $\forall X\in\mathscr{D}$ *the naturally induced morphism φ^X is an epimorphism*:

$$\varphi^X:\mathscr{D}[\mathscr{L}_1^{\mathfrak{R}}F(X)\oplus F^{t+2}X,-]\to\mathscr{E}xt^2_{\mathfrak{R}}[FX,-]\oplus\mathscr{E}xt^3_{\mathfrak{R}}[X,-].$$

Proof. The proof is the same as in Corollary 6.4, using Theorem 6.3. □

Corollary for Rings 8.5. *Let $\Lambda=\Gamma\ltimes_t M$ be the t-truncated cleft extension of Γ by M, $t\geq 0$. Then the following are equivalent*:

(i) r.gl.dim $\Lambda\leq 2$.

(ii) (a) $\otimes_{\Gamma}^{t+3}M=\mathscr{T}or_1^{\Gamma}(M,M)=0$ *and* w.d $\otimes_{\Gamma}^{t+2}M=0$, w.d $\otimes_{\Gamma}^{i}M\leq 1$, $\forall i=1,2,\ldots,t+1$.

(b) $\forall X \in \mathrm{Mod}(\Gamma) : X \otimes_\Gamma^{t+2} M$, $\mathscr{T}or_1^\Gamma(X,M)$ *are projective right* Γ*-modules.*
(c) $\forall X \in \mathrm{Mod}(\Gamma)$, *the naturally induced morphism*:
$\mathrm{Hom}_\Gamma[\mathscr{T}or_1^\Gamma(X,M) \oplus X \otimes_\Gamma (\otimes_\Gamma^{t+2} M), -] \to \mathscr{E}xt_\Gamma^2[X \otimes_\Gamma M, -] \oplus \mathscr{E}xt_\Gamma^3[X, -]$
is an epimorphism.

With the notation preceding the Proposition 6.6 we have the following:

Corollary 8.6. *The following are equivalent for the cleft extension* $\mathscr{D} \ltimes_t F$:
(i) $\mathscr{D} \ltimes_t F$ *is* $\mathfrak{F}$*-Frobenius.*
(ii) *The following are true* $\forall P \in \mathscr{P}(\mathfrak{R})$, $\forall I \in \mathscr{I}(\mathfrak{R})$, $\forall i = 0, 1, \ldots, t$:
(a) $F^{t+1}P$, $Ker(\sigma_{F^iP}) \in \mathscr{I}(\mathfrak{R})$.
(b) $G^{t+1}I$, $Coker(\rho_{G^iI}) \in \mathscr{P}(\mathfrak{R})$.
(c) σ_{F^iP} *is epic and* $GKer(\sigma_P \circ G\sigma_{FP} \circ \cdots \circ G^t\sigma_{F^tP}) = 0$.
(d) ρ_{G^iI} *is monic and* $FCoker(F^t\rho_{G^tI} \circ \cdots \circ F\rho_{GI} \circ \rho_I) = 0$.
Hence if the category $\mathscr{D}$ *is* $\mathfrak{R}$*-Frobenius, then the category* $\mathscr{D} \ltimes_t \mathrm{Id}_{\mathscr{D}}$ *of* $(t+1)$*-nilpotent endomorphisms of* $\mathscr{D}$ *is* $\mathfrak{F}$*-Frobenius.*

Proof. The proof is as in Corollary 6.7, using Lemma 5.3 and the t-truncated Σ-associated complex of the objects $\mathbf{T}(P)$, $\mathbf{H}(I)$. □

We recall now the vanishing condition

$$(\dagger) \qquad \mathscr{L}_i^{\mathfrak{R}} F^j(FP) = 0, \quad \forall i,j \geq 1, \ \forall P \in \mathscr{P}(\mathfrak{R})$$

of the last section. We define the *t-truncated nilpotency class* $c_t(F)$ of F to be ∞ if F is non-nilpotent, otherwise $c_t(F) = min\{m \geq 0$: $F^{\phi(m+1)} = 0\}$.

Theorem 8.7. *Let* $\mathscr{D} \ltimes_t F$ *be the t-truncated extension of* $\mathscr{D}$ *by* F, $t \geq 0$.
(1) *The following are equivalent*:
(i) *The natural morphisms* $\xi^i : \mathscr{L}_i^{\mathfrak{F}}\mathbf{C} \to \mathscr{L}_i^{\Sigma}\mathbf{C}$, $\forall i \geq 0$ *are isomorphisms.*
(ii) *The functor* F *is* $\mathfrak{R}$*-exact.*
If one of the above statements is true, then $\forall (X, f) \in \mathscr{D} \ltimes_t F$, $\forall n \geq 1$:

$$\mathscr{L}_{2n-1}^{\mathfrak{F}}\mathbf{C}(X, f) \cong Ker(F^{\phi(2n-2)}f)/Im(F^{\phi(2n-1)}f_t),$$

$$\mathscr{L}_{2n}^{\mathfrak{F}}\mathbf{C}(X, f) \cong Ker(F^{\phi(2n-1)}f_t)/Im(F^{\phi(2n)}f),$$

$$\mathscr{L}_n^{\mathfrak{F}}\mathbf{CZ} \cong \mathscr{L}_n^{\Sigma}\mathbf{CZ} \cong F^{\phi(n)}, \quad \forall n \geq 0.$$

(2) *Suppose that condition* $(\dagger)$ *is true for* F. *The following are equivalent*:
(i) $\mathfrak{F}$-gl.dim $\mathscr{D} \ltimes_t F < \infty$.
(ii) $\mathfrak{R}$-gl.dim $\mathscr{D} < \infty$ *and* F *is nilpotent.*
In any case we have the bounds:

$$\mathfrak{R}\text{-gl.dim}\, \mathscr{D} \leq \mathfrak{F}\text{-gl.dim}\, \mathscr{D} \ltimes_t F \leq c_t(F) + 2 \cdot \mathfrak{R}\text{-gl.dim}\, \mathscr{D}.$$

(3) *Suppose that the vanishing condition* (†) *is true for F. Then*

$$\mathscr{L}_n^{\mathfrak{F}}\mathbf{CZ} \cong \bigoplus_{p+q=n} \mathscr{L}_q^{\mathfrak{R}} F^{\phi(p)}, \quad \forall n \geq 0,$$

$$\mathscr{E}xt_{\mathfrak{F}}^n[\mathbf{Z}(X), \mathbf{Z}(Y)] \cong \bigoplus_{p+q=n} \mathscr{R}_{\mathfrak{R}}^p \mathscr{D}[F^{\phi(q)}X, Y], \quad \forall n \geq 0,$$

$$\mathfrak{F}\text{-gl.dim}\, \mathscr{D} \ltimes_t F = sup \left\{ n \geq 0\colon \bigoplus_{p+q=n+1} \mathscr{R}_{\mathfrak{R}}^p \mathscr{D}[F^{\phi(q)}-,-] = 0 \right\}.$$

Proof. (1) Since the statement is independent of $\tilde{\eta}$, this is true by Theorem 4.7. The last assertion follows from the description of ${}^t\Sigma^\bullet_{\mathbf{Z}(X)}$. Part (2) follows directly from Theorem 7.5, and the proof of part (3) is similar to the proof of Theorems 7.10 and 7.13. The necessary modifications are left to the reader. □

Corollary for Rings 8.8. *Let $\Lambda = \Gamma \ltimes_t M$ be a t-truncated cleft extension of Γ by M, $t \geq 0$. Assume that $\mathscr{T}or_i^\Gamma[M, \otimes_\Gamma^j M] = 0$, $\forall i, j \geq 1$.*

(1)

$$\mathscr{T}or_n^\Lambda[\mathbf{Z}-, \Gamma] \cong \bigoplus_{p+q=n} \mathscr{T}or_p^\Gamma[-, \otimes_\Gamma^{\phi(q)} M], \quad \forall n \geq 0.$$

In particular: $\mathscr{T}or_n^\Lambda[\Gamma, \Gamma] \cong \otimes_\Gamma^{\phi(n)} M$. *If in addition* $\mathscr{T}or_q^\Gamma[-, \otimes_\Gamma^p M] = 0$, $\forall p, q \geq 1$: $p + q = n$, *then* $\mathscr{T}or_i^\Lambda[-, \Gamma] \cong \mathscr{T}or_i^{(\Lambda,\Gamma)}[-, \Gamma]$, $\quad \forall i \geq n$.

(2)

$$\mathscr{E}xt_\Lambda^n[\mathbf{Z}-, \mathbf{Z}-] \cong \bigoplus_{p+q=n} \mathscr{R}^p \operatorname{Hom}_\Gamma[- \otimes_\Gamma^{\phi(q)} M, -],$$

$$\mathscr{T}or_n^\Lambda[\mathbf{Z}-, \mathbf{Z}-] \cong \bigoplus_{p+q=n} \mathscr{L}_p[- \otimes_\Gamma^{\phi(q)} M \otimes_\Gamma -],$$

$$\text{r.gl.dim}\, \Lambda = sup\{n \geq 0\colon \mathscr{R}^p \operatorname{Hom}_\Gamma[- \otimes_\Gamma^{\phi(q)} M, -] = 0,\ \forall p + q = n + 1\},$$

$$\text{w.gl.dim}\, \Lambda = sup\{n \geq 0\colon \mathscr{L}_p[- \otimes_\Gamma^{\phi(q)} M -] = 0,\ \forall p + q = n + 1\}.$$

(3) *Suppose that ${}_\Gamma M$ is flat. Then we have the following*:

$$\mathscr{E}xt_\Lambda^n[\mathbf{Z}-, \mathbf{Z}-] \cong \bigoplus_{p+q=n} \mathscr{E}xt_\Gamma^p[- \otimes_\Gamma^{\phi(q)} M, -],$$

$$\mathscr{T}or_n^\Lambda[\mathbf{Z}-, \mathbf{Z}-] \cong \bigoplus_{p+q=n} \mathscr{T}or_p^\Gamma[- \otimes_\Gamma^{\phi(q)} M, -],$$

$$\text{r.gl.dim}\, \Lambda = sup\{n \geq 0\colon \mathscr{E}xt_\Gamma^p[- \otimes_\Gamma^{\phi(q)} M, -] = 0,\ \forall p + q = n + 1\},$$

$$\text{w.gl.dim}\, \Lambda = sup\{n \geq 0\colon \mathscr{T}or_p^\Gamma[- \otimes_\Gamma^{\phi(q)} M, -] = 0,\ \forall p + q = n + 1\}.$$

We leave to the reader to state the corresponding results about $\mathfrak{F}$-f.gl.dim $\mathscr{D} \ltimes_t F$ and f.gl.dim $\Gamma \ltimes_t M$, applying for example Theorem 5.11, or using the techniques developed so far. The isomorphisms of part (2) in Corollary 8.8 take a more familiar form under some mild conditions on Γ, X, Y:

Proposition 8.9. *Suppose that Γ is a k-algebra over a commutative ring k, which acts centrally on M. Let Γ be k-projective, $\mathscr{T}or_i^{\Gamma}[M, \otimes_{\Gamma}^{j} M] = 0$, $\forall i, j \geq 1$, and let $\Lambda = \Gamma \ltimes_t M$ be the t-truncated cleft extension. Let $X, Y \in \mathrm{Mod}(\Gamma)$, $W \in \mathrm{Mod}(\Gamma^{op})$ and let $\Gamma^e = \Gamma \otimes_k \Gamma^{op}$ be the enveloping ring of Γ over k.*

(1) *If $\mathscr{E}xt_k^n[X, Y] = 0$, $\forall n \geq 1$, then*

$$\mathscr{E}xt_{\Lambda}^n[\mathbf{Z}(X), \mathbf{Z}(Y)] \cong \bigoplus_{p+q=n} \mathscr{E}xt_{\Gamma^e}^p[\otimes_{\Gamma}^{\phi(q)} M, \mathrm{Hom}_k(X, Y)], \quad \forall n \geq 0.$$

(2) *If $\mathscr{T}or_n^k[X, W] = 0$, $\forall n \geq 1$, then*

$$\mathscr{T}or_n^{\Lambda}[\mathbf{Z}(X), \mathbf{Z}(W)] \cong \bigoplus_{p+q=n} \mathscr{T}or_p^{\Gamma^e}[\otimes_{\Gamma}^{\phi(q)} M, X \otimes_k W], \quad \forall n \geq 0.$$

Proof. It is easy to see that we have an natural isomorphism, $\forall q \geq 1$:

$$\mathrm{Hom}_{\Gamma^e}[\otimes_{\Gamma}^{q} M, \mathrm{Hom}_k(X, Y)] \cong \mathrm{Hom}_{\Gamma}[X \otimes_{\Gamma} (\otimes_{\Gamma}^{q} M), Y].$$

From this isomorphism, the hypotheses and standard arguments [8], we have

$$\mathscr{R}^p \, \mathrm{Hom}_{\Gamma}[X \otimes_{\Gamma}^{q} M, Y] \cong \mathscr{E}xt_{\Gamma^e}^p[\otimes_{\Gamma}^{q} M, \mathrm{Hom}_k(X, Y)].$$

Hence by Theorem 8.8, part (1) follows and (2) is similar. $\square$

The *t-truncated tensor nilpotent class* of the bimodule M is defined by

$$c_t(M) = \min\{n \geq 0: \otimes_{\Gamma}^{\phi(n+1)} M = 0\}.$$

Corollary 8.10. *Let Γ be a finite-dimensional k-algebra over an algebraically closed field k, which acts centrally on M. Let $\mathscr{T}or_i^{\Gamma}[M, \otimes_{\Gamma}^{j} M] = 0$, $\forall i, j \geq 1$, and consider the t-truncated cleft extension $\Lambda = \Gamma \ltimes_t M$ of Γ by M, $t \geq 0$. Then*

$$\mathrm{gl.dim}\, \Lambda = \max\{c_t(M),\ \mathrm{p.d}_{\Gamma^e}(\otimes_{\Gamma}^{\phi(i)} M) + i;\ i = 0, 1, \ldots, c_t(M)\}.$$

Proof. Follows from the previous proposition and the fact that in our case all simple Γ^e-modules are of the form $\mathrm{Hom}_k(S, T)$ for simple Γ-modules S, T. $\square$

The following is a version of the Strong No Loops Conjecture [20,27].

Corollary 8.11. *Let S be a simple object of $\mathscr{D}$ without selfextensions, i.e. we have $\mathscr{E}xt_{\mathfrak{R}}^1[S, S] = 0$, and consider the t-truncated extension $\mathscr{D} \ltimes_t F$ of $\mathscr{D}$ by F. If the vanishing condition (†) is true for F, for the simple object $\mathbf{Z}(S)$ of $\mathscr{D} \ltimes_t F$, consider* *the following statements*:

(i) $\mathscr{E}xt^1_{\mathfrak{F}}[\mathbf{Z}(S),\mathbf{Z}(S)] \neq 0$.
(ii) $\mathscr{E}xt^1_{\Sigma}[\mathbf{Z}(S),\mathbf{Z}(S)] \neq 0$.
(iii) $\mathscr{E}xt^i_{\Sigma}[\mathbf{Z}(S),\mathbf{Z}(S)] \neq 0,\ \forall i \geq 0$.
(iv) $\mathscr{E}xt^i_{\mathfrak{F}}[\mathbf{Z}(S),\mathbf{Z}(S)] \neq 0,\ \forall i \geq 0$.
(v) $\mathfrak{F}$-p.d $\mathbf{Z}(S) = \infty$.

Then (i) $\Leftrightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) $\Rightarrow$ (v). *In particular if* $\mathscr{D}$ *is semisimple, then for any simple object* $\mathbf{Z}(S)$ *of* $\mathscr{D} \ltimes_t F$ *with* $\mathscr{E}xt^1_{\mathscr{D}\ltimes_t F}[\mathbf{Z}(S),\mathbf{Z}(S)] \neq 0$, *we have*

$$\mathscr{E}xt^i_{\mathscr{D}\ltimes_t F}[\mathbf{Z}(S),\mathbf{Z}(S)] = \mathscr{D}[F^{\phi(i)}S,S] \neq 0, \quad \forall i \geq 0$$

and $\mathrm{p.d}_{\mathscr{D}\ltimes_t F}\mathbf{Z}(S) = \infty$.

Proof. Since $\mathscr{E}xt^1_{\mathfrak{F}}[\mathbf{Z}(S),\mathbf{Z}(S)] \cong \mathscr{E}xt^1_{\Sigma}[\mathbf{Z}(S),\mathbf{Z}(S)] \oplus \mathscr{E}xt^1_{\mathfrak{R}}[S,S]$, the hypothesis implies that (i) $\Leftrightarrow$ (ii). From the t-truncated Σ-associated complex ${}^t\Sigma^{\bullet}_{\mathbf{Z}(S)}$, we see that $\forall i \geq 0$: $\mathscr{E}xt^i_{\Sigma}[\mathbf{Z}(S),\mathbf{Z}(S)] = \mathscr{D}[F^{\phi(i)}S,S]$ and this is a direct summand of $\mathscr{E}xt^i_{\mathfrak{F}}[\mathbf{Z}(S),\mathbf{Z}(S)]$ as follows from part (3) of the previous theorem. Hence (iii) $\Rightarrow$ (iv). Since S is simple and F is right exact, trivially (ii) $\Rightarrow$ (iii). The implication (iv) $\Rightarrow$ (v) is obvious. $\square$

Corollary for Rings 8.12. *Let* $\Lambda = \Gamma \ltimes_t M$ *be a* t*-truncated cleft extension of* Γ *by* M, $t \geq 0$, *and suppose that* $\mathscr{T}or^{\Gamma}_i[M, \otimes^j_{\Gamma} M] = 0$, $\forall i,j \geq 1$. *Let* S *be a simple* Γ*-module with* $\mathscr{E}xt^1_{\Gamma}[S,S] = 0$. *If for the simple* Λ*-module* $\mathbf{Z}(S)$ *we have* $\mathscr{E}xt^1_{\Lambda}[\mathbf{Z}(S),\mathbf{Z}(S)] \neq 0$, *then* $\mathscr{E}xt^i_{\Lambda}[\mathbf{Z}(S),\mathbf{Z}(S)] \neq 0$, $\forall i \geq 0$ *and* $\mathrm{p.d}\,\mathbf{Z}(S) = \infty$.

Let Λ be a finite-dimensional k-algebra over the algebraically closed field k. Let $\{S_1, S_2, \ldots, S_n\}$ be a complete set of isomorphism classes of simple right Λ-modules and let P_i be the projective cover of S_i. The *Cartan matrix* $C(\Lambda)$ of Λ is the $n \times n$ matrix with entries $C_{ij}(\Lambda) = \dim_k \mathrm{Hom}_{\Lambda}[P_i, P_j]$. It is conjectured by Eilenberg that if Λ has finite global dimension, then $\mathrm{Det}\, C(\Lambda) = 1$.

Proposition 8.13. *Let* Γ *be a semisimple* k*-algebra over the algebraically closed field* k *and let* $\Lambda = \Gamma \ltimes_t M$ *be a* t*-truncated cleft extension of* Γ *by* M, $t \geq 0$. *If* $\mathrm{gl.dim}\,\Lambda < \infty$, *then* $\mathrm{Det}\, C(\Lambda) = 1$.

Proof. First we show that $\mathrm{C}_{ii}(\Lambda) = 1$, $\forall i = 1, \ldots, n$. From the adjoint pair $(\mathbf{T},\mathbf{U})$ we have that $\mathrm{Hom}_{\Lambda}[\mathbf{T}(S_i),\mathbf{T}(S_i)] = \mathrm{Hom}_{\Gamma}[S_i, S_i \oplus FS_i \oplus F^2S_i \oplus \cdots \oplus F^{t+1}S_i]$. If $\mathrm{Hom}_{\Lambda}[S_i, F^kS_i] \neq 0$ for some $k = 1,2,\ldots,t+1$, then we have a monomorphism $S_i \hookrightarrow F^kS_i$ which splits since Γ is semisimple. Hence S_i is a direct summand of F^kS_i. Consider the trivial extension $\mathscr{E} := \mathscr{D} \ltimes_0 F^k$, where $\mathscr{D} = \mathrm{mod}(\Gamma)$. Since Λ has finite global dimension, we have that F, hence F^k is nilpotent. This implies that also that $\mathscr{E}$ has finite global dimension. Now $\mathscr{E}xt^1_{\mathscr{E}}[\mathbf{Z}(S_i),\mathbf{Z}(S_i)] = \mathscr{D}[F^kS_i, S_i] \neq 0$. Since $\mathbf{Z}(S_i)$ is a simple object in $\mathscr{E}$, from Corollary 8.11, we have that $\mathrm{p.d}\mathbf{Z}(S_i) = \infty$ and this is impossible. Hence $\mathrm{Hom}_{\Lambda}[S_i, F^kS_i] = 0$, $\forall k = 1, \ldots, t+1$. This shows that $\mathrm{Hom}_{\Lambda}[\mathbf{T}(S_i),\mathbf{T}(S_i)] = \mathrm{Hom}_{\Gamma}[S_i,S_i]$, hence $C_{ii}(\Lambda) = 1$, $\forall i = 1, \ldots, n$.

Next, we show that if $i \neq j$ and $C_{ij}(\Lambda) \neq 0$, then $C_{ji}(\Lambda) = 0$. Indeed otherwise we have as above nonzero split monics $S_i \hookrightarrow F^k S_j$ and $S_j \hookrightarrow F^m S_i$, where $1 \leq k, m \leq t+1$. Then we have a nonzero split monic $S_j \hookrightarrow F^m S_i \hookrightarrow F^{k+m} S_j$. But then $\mathrm{Hom}_\Gamma[F^{k+m}S_j, S_j] \neq 0$. By the above argument this is impossible. Since $C_{ii}(\Lambda) = 1$, $\forall i = 1, \ldots, n$ and for $i \neq j$, $C_{ij}(\Lambda) \neq 0$ implies $C_{ji}(\Lambda) = 0$, we can arrange the simples in such a way that $C(\Lambda)$ is an upper triangular matrix with all diagonal entries equal to 1. Then obviously we have $\mathrm{Det}\, C(\Lambda) = 1$. □

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