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GROTHENDIECK GROUPS ARISING FROM CONTRAVARIANTLY FINITE SUBCATEGORIES

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To the memory of Maurice Auslander

Abstract

The subject of the paper is the study of the relative homological properties of a given additive category $\mathcal{C}$ in relation to a given contravariantly finite subcategory $\mathcal{X}$ in $\mathcal{C}$ under the assumption that any $\mathcal{X}$–epic has a kernel in $\mathcal{C}$. We introduce the notion of the Grothendieck group relative to the pair $(\mathcal{C},\mathcal{X})$ and also that of the Cartan map $c_{\mathcal{X}}$ relative to $(\mathcal{C},\mathcal{X})$ and we show that the cokernel of $c_{\mathcal{X}}$ is isomorphic to the corresponding Grothendieck group of the stable category $\mathcal{C}/\mathcal{J}_{\mathcal{X}}$ We also show that if the right $\mathcal{X}$–dimension of $\mathcal{C}$ is finite, then $c_{\mathcal{X}}$ is an isomorphism. In case $\mathcal{C}$ is a finite dimensional k–additive Krull–Schmidt category, we introduce the notion of the $\mathcal{X}$–dimension vector of an object of $\mathcal{C}$. We give criteria for when an indecomposable object is determined, up to isomorphism, by its $\mathcal{X}$–dimension vector.

1 Introduction

The notion of contravariantly finite subcategory was introduced in [2] by Auslander–Smalø for studying the problem of which subcategories of an artin algebra have almost split sequences. Since then, contravariantly finite subcategories turned to be very fruitful stimulating a long list of papers with substantial results on different aspects of the representation theory of finite dimensional algebras as also on relative homological algebra, cf. [3].

In [5], we exhibited how a contravariantly finite subcategory $\mathcal{X}$ of an additive category $\mathcal{C}$ induces on the stable category $\mathcal{C}/\mathcal{J}_{\mathcal{X}}$ a left triangulated structure assuming that any $\mathcal{X}$-epic has a kernel in $\mathcal{C}$. In particular this applies to the following case: Suppose that $\mathrm{mod}\,\Lambda$ is the category of finitely generated right Λ-modules over an artin algebra Λ and $\mathcal{P}^f_\Lambda$ is the full subcategory of the projective Λ- modules, then the stable category $\mathrm{mod}\,\Lambda/\mathcal{P}^f_\Lambda$ has a left triangulated structure.

In [1] Auslander-Reiten showed that the cokernel of the Cartan map is isomorphic to the Grothendieck group of the stable category $\mathrm{mod}\,\Lambda/\mathcal{P}^f_\Lambda$. The first task of the present paper is to extend this result. Namely, given an additive category $\mathcal{C}$ and a contravariantly finite subcategory $\mathcal{X}$ such that any $\mathcal{X}$-epic has a kernel in $\mathcal{C}$, we define a Cartan map relative to the pair $(\mathcal{C}, \mathcal{X})$ and we show that its cokernel is isomorphic to the Grothendieck group of the left triangulated category $\mathcal{C}/\mathcal{J}_{\mathcal{X}}$. We also show that if the relative to $\mathcal{X}$ right homological dimension of $\mathcal{C}$ is finite, then the Cartan map is an isomorphism. This result is analogous to the classical "reduction by resolution" result of Grothendieck for admissible subcategories of an abelian category, cf. [4], Theorem 4.2.

The next task of the paper is to give the relative to $\mathcal{X}$ homological interpretation of the Euler characteristic [9], Section 2.4., in case where $\mathcal{C}$ is a k-additive Krull-Schmidt category and $\mathcal{X}$ has finitely many indecomposable objects.

Finally, by introducing the notion of the $\mathcal{X}$-dimension vector for any object of $\mathcal{C}$ we give a criterion for deciding when an indecomposable object is determined, up to isomorphism, by its $\mathcal{X}$-dimension vector.

The contents of the paper section by section are as follows:

In Section 2, we introduce the notion of the right $\mathcal{X}$-resolution for any object of $\mathcal{C}$ and show that any such resolution is obtained in fact in a "unique way", see Lemma 2.5 ii. We prove the relative to $\mathcal{X}$ homological versions of "Schanuel's Lemma", (2.8) and "Horseshoe Lemma", (2.11). We define the notion of the right $\mathcal{X}$-dimension of $\mathcal{C}$ and we introduce the $\mathcal{X}$-relative Ext-functors.

In Section 3, we introduce the notion of the Grothendieck group and of the Cartan map $c_{\mathcal{X}}$ relative to the pair$(\mathcal{C}, \mathcal{X})$. We also reveal the notion of the Grothendieck group of a left triangulated category, a notion already introduced by Grothendieck cf. [7] for triangulated categories. We show that in case where we deal with the left triangulated structure of the stable category $\mathcal{C}/\mathcal{J}_{\mathcal{X}}$, its Grothendieck group is isomorphic to the cokernel of the

Cartan map $c_{\mathcal{X}}$, Theorem 3.19. We also show that if the $\mathcal{X}$-dimension of $\mathcal{C}$ is finite then $c_{\mathcal{X}}$ has to be an isomorphism, Theorem 3.20.

In Section 4, we deal with the case where $\mathcal{C}$ is a finite dimensional k-additive Krull-Schmidt category and $\mathcal{X}$ has finitely many non-isomorphic indecomposable objects. We define for any object in $\mathcal{C}$ its $\mathcal{X}$-dimension vector and also the Cartan matrix relative to the pair $(\mathcal{C}, \mathcal{X})$. In case this Cartan matrix is $\mathbb{Z}$-invertible, we introduce the $\mathcal{X}$-Euler characteristic of $(\mathcal{C}, \mathcal{X})$ and show its connection to the relative Ext-functors, Proposition 4.4. Moreover, we show that if the $\mathcal{X}$-dimension of $\mathcal{C}$ is finite, then this Cartan matrix has to be $\mathbb{Z}$-invertible, Proposition 4.7.

In Section 5, we deal with the question when an indecomposable object of $\mathcal{C}$ is determined by its $\mathcal{X}$-dimension vector. By using the main result of [6] we show that a particular full subcategory $\mathcal{D}_{\mathcal{X}}$ of $\mathcal{C}$ admits this property, Theorem 5.2. By applying this result in the case of $\operatorname{mod}\Lambda$, we obtain, Theorem 5.4, the relative to $\mathcal{X}$ analogon to the main result of [6].

2 Relative Homological Algebra and Contravariantly Finite Subcategories

In this paper we assume, unless otherwise stated, that all considered categories are skeletally small and additive. We also assume that their subcategories are full, additive, closed under isomorphisms and direct summands. Functors between additive categories are supposed to be additive. In the following, we shall denote by $\mathcal{AB}$ the category of abelian groups.

Given a category $\mathcal{C}$, we denote by $\mathcal{C}(A, B)$ the group of morphisms $\operatorname{Mor}_{\mathcal{C}}(A, B)$ from A to B. Given a morphism $\alpha : A \longrightarrow B$ and an object C, we denote by α^C (resp. by α_C) the induced group homomorphism $\mathcal{C}(C, \alpha) : \mathcal{C}(C, A) \longrightarrow \mathcal{C}(C, B)$ (resp. $\mathcal{C}(\alpha, C) : \mathcal{C}(B, C) \longrightarrow \mathcal{C}(A, C)$) in $\mathcal{AB}$.

Let $\mathcal{X}$ be a subcategory of $\mathcal{C}$ and

$$\mathbf{A} \;:\; \cdots \longrightarrow A_{i+1} \xrightarrow{\alpha_{i+1}} A_i \xrightarrow{\alpha_i} A_{i-1} \longrightarrow \cdots$$

be a complex of $\mathcal{C}$.

Definition 2.1. The complex $\mathbf{A}$ is said to be right *$\mathcal{X}$-exact* at the integer i, if the complex

$$\mathbf{A}^{\mathbf{X}} \;:\; \cdots \longrightarrow \mathcal{C}(X, A_{i+1}) \xrightarrow{\alpha_{i+1}^X} \mathcal{C}(X, A_i) \xrightarrow{\alpha_i^X} \mathcal{C}(X, A_{i-1}) \longrightarrow \cdots$$

is exact at the integer i, for any object X of $\mathcal{X}$. The complex $\mathbf{A}$ is said to be right *$\mathcal{X}$-exact*, if it is right $\mathcal{X}$-exact $\forall i \in \mathbb{Z}$.

Definition 2.2. A morphism $\alpha : A \longrightarrow B$ is said to be an *$\mathcal{X}$-epic*, if the complex $A \xrightarrow{\alpha} B \longrightarrow 0$ is right $\mathcal{X}$-exact. If A is an object of $\mathcal{X}$, then an $\mathcal{X}$-epic α is said to be a *right $\mathcal{X}$-approximation* of B.

The above definitions lead to the notion of contravariantly finite subcategory introduced in [2] by M. Auslander and S.O. Smalø, recalled below.

Definition 2.3. A subcategory $\mathcal{X}$ of $\mathcal{C}$ is said to be *contravariantly finite in* $\mathcal{C}$, if any object A of $\mathcal{C}$ has a right $\mathcal{X}$-approximation .

Definition 2.4. A right $\mathcal{X}$-exact complex of the form

$$\mathbf{X} \;:\; \cdots \longrightarrow X_{i+1} \xrightarrow{x_{i+1}} X_i \xrightarrow{x_i} X_{i-1} \longrightarrow \cdots \longrightarrow X_0 \xrightarrow{x_0} A \longrightarrow 0,$$

with X_i in $\mathcal{X}, \forall i \in \mathbb{N} \cup \{0\}$, is said to be a *right $\mathcal{X}$-resolution* of A.

In the remaining part of the paper we assume that

(i) *$\mathcal{X}$ is a subcategory contravariantly finite in $\mathcal{C}$* and

(ii) *any $\mathcal{X}$-epic has a kernel.*

As the next Lemma shows, the assumptions above imply that any object A of $\mathcal{C}$ has a right $\mathcal{X}$-resolution.

Lemma 2.5. *(i) Any object A of $\mathcal{C}$ has a right $\mathcal{X}$-resolution.*

(ii) If

$$\mathbf{X} \;:\; \cdots \longrightarrow X_{i+1} \xrightarrow{x_{i+1}} X_i \xrightarrow{x_i} X_{i-1} \longrightarrow \cdots \longrightarrow X_0 \xrightarrow{x_0} A \longrightarrow 0,$$

is a right $\mathcal{X}$-resolution of A, then $x_i = g_i \circ h_i, \forall i \in \mathbb{N}$, where $g_i : K_i \longrightarrow X_{i-1}$ is a kernel of x_{i-1} and $h_i : X_i \longrightarrow K_i$ is a right $\mathcal{X}$-approximation of K_i.

Proof. (i) Let $x_0 : X_0 \longrightarrow A$ be a right $\mathcal{X}$-approximation of A, $g_1 : K_1 \longrightarrow X_0$ be the kernel of x_0 and $h_1 : X_1 \longrightarrow K_1$ be a right $\mathcal{X}$-approximation of K_1. Since the sequences

$$0 \longrightarrow \mathcal{C}(Y, K_1) \xrightarrow{g_1^Y} \mathcal{C}(Y, X_0) \xrightarrow{x_0^Y} \mathcal{C}(Y, A) \longrightarrow 0 \text{ and } \mathcal{C}(Y, X_1) \xrightarrow{h_1^Y} \mathcal{C}(Y, K_1) \longrightarrow 0$$

are exact in $\mathcal{AB}$, the complex $X_1 \xrightarrow{g_1 \circ h_1} X_0 \xrightarrow{x_0} A \longrightarrow 0$ is right $\mathcal{X}$-exact.

Let x_1 denote the composition $g_1 \circ h_1$. We consider the kernel $g_2 : K_2 \longrightarrow X_1$ of x_1 and a right $\mathcal{X}$-approximation h_2 of K_2 and continue inductively repeating the above construction.

(ii). Let $g_1 : K_1 \longrightarrow X_0$ be the kernel of x_0 and $h_1 : X_1 \longrightarrow K_1$ the (unique) morphism with $g_1 \circ h_1 = x_1$. We shall show that h_1 is a right $\mathcal{X}$-approximation of K_1.

For any object Y of $\mathcal{X}$, the commutative diagram

$$\begin{tikzcd} X_1 \arrow[r, "x_1"] \arrow[d, "h_1"'] & X_0 \arrow[r, "x_0"] \arrow[d, "id_{X_0}"'] & A \arrow[d, "id_A"'] \\ K_1 \arrow[r, "g_1"] & X_0 \arrow[r, "x_0"] & A \end{tikzcd}$$

yields, the commutative diagram

$$\begin{tikzcd} & \mathcal{C}(Y,X_1) \arrow[r, "x_1^Y"] \arrow[d, "h_1^Y"'] & \mathcal{C}(Y,X_0) \arrow[r, "x_0^Y"] \arrow[d, equal] & \mathcal{C}(X,A) \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & \mathcal{C}(Y,K_1) \arrow[r, "g_1^Y"] & \mathcal{C}(Y,X_0) \arrow[r, "x_0^Y"] & \mathcal{C}(Y,A) \arrow[r] & 0 \end{tikzcd}$$

in $\mathcal{AB}$ with exact rows.

By the Snake Lemma, h_1^Y is an epimorphism. Hence h_1 is a right $\mathcal{X}$-approximation of K_1. We repeat the above procedure using the kernel $g_2 : K_2 \to X_1$ of x_1 and continue inductively. $\square$

Definition 2.6. An object A of $\mathcal{C}$ is said to be of *finite right $\mathcal{X}$-dimension*, if there is a right $\mathcal{X}$-resolution

$$\mathbf{X} : 0 \to X_n \xrightarrow{x_n} X_{n-1} \to \cdots \to X_i \xrightarrow{x_i} X_{i-1} \to \cdots \to X_0 \xrightarrow{x_0} A \to 0.$$

of length n.

In this case we write $\mathrm{rt.\,dim}_{\mathcal{X}} A \leq n$. The least such integer n is called the *right $\mathcal{X}$-dimension* of A and is denoted by $\mathrm{rt.\,dim}_{\mathcal{X}} A = n$. Otherwise, A is said to have *infinite right $\mathcal{X}$-dimension* and we write $\mathrm{rt.\,dim}_{\mathcal{X}} A = \infty$.

Definition 2.7. The category $\mathcal{C}$ is said to be *right $\mathcal{X}$-regular*, if $\mathrm{rt.\,dim}_{\mathcal{X}} A < \infty$ for any object A of $\mathcal{C}$.

If there is some fixed $n \in \mathbb{N} \cup \{0\}$ with $\mathrm{rt.\,dim}_{\mathcal{X}} A \leq n$ for all objects A of $\mathcal{C}$, then $\mathcal{C}$ is said to have *finite right $\mathcal{X}$-dimension*. The least integer $n \geq 0$ with this property is said to be the *right $\mathcal{X}$-dimension* of the category $\mathcal{C}$. In this case we write $\mathrm{rt.\,dim}_{\mathcal{X}} \mathcal{C} = n$.

The next Lemma is the relative to $\mathcal{X}$ version of "Schanue'ls Lemma":

Lemma 2.8. *Let A and B two isomorphic objects of $\mathcal{C}$ of finite right $\mathcal{X}$-dimension.*

If

$$0 \to X_n \xrightarrow{x_n} X_{n-1} \to \cdots \to X_i \xrightarrow{x_i} X_{i-1} \to \cdots \to X_1 \xrightarrow{x_1} X_0 \xrightarrow{x_0} A \to 0$$

and

$$0 \longrightarrow Y_n \xrightarrow{y_n} Y_{n-1} \longrightarrow \cdots \longrightarrow Y_i \xrightarrow{y_i} Y_{i-1} \longrightarrow \cdots \longrightarrow Y_1 \xrightarrow{y_1} Y_0 \xrightarrow{y_0} B \longrightarrow 0$$

are two right $\mathcal{X}$-resolutions, then

$$X_0 \oplus Y_1 \oplus X_2 \oplus \cdots \cong Y_0 \oplus X_1 \oplus Y_2 \oplus \cdots .$$

Proof. By Lemma 2.5, $x_i = g_i \circ h_i, i \in \mathbb{N}$, (resp. $y_i = \ell_i \circ m_i, i \in \mathbb{N}$) where $g_i : K_i \longrightarrow X_{i-1}$ (resp. $\ell_i : L_i \longrightarrow Y_{i-1}$) is the kernel of x_{i-1} (resp. of y_{i-1}) and $h_i : X_i \longrightarrow K_i$ (resp. $m_i : Y_i \longrightarrow L_i$) is a right $\mathcal{X}$-approximation of K_i (resp. of L_i). Let σ_0 denote the isomorphism from A to B.

Since x_0 is a right $\mathcal{X}$-approximation of A and ℓ_1 is the kernel of y_0, there is the commutative diagram:

$$\begin{array}{ccccc} K_1 & \xrightarrow{g_1} & X_0 & \xrightarrow{x_0} & A \\ {\scriptstyle s_1}\downarrow & & {\scriptstyle t_0}\downarrow & & {\scriptstyle \sigma_0}\downarrow \\ L_1 & \xrightarrow{\ell_1} & Y_0 & \xrightarrow{y_0} & B \end{array}$$

One checks easily that $(t_0, \ell_1) : X_0 \oplus L_1 \longrightarrow Y_0$ is an $\mathcal{X}$-epic with kernel the morphism $\begin{pmatrix} g_1 \\ -s_1 \end{pmatrix} : K_1 \longrightarrow X_0 \oplus L_1$. Since Y_0 is an object of $\mathcal{X}$, there is a morphism $r : Y_0 \longrightarrow X_0 \oplus L_1$, satisfying $(t_0, \ell_1) \circ r = id_{Y_0}$.

Hence, $Y_0 \oplus K_1 \overset{\sigma_1}{\cong} X_0 \oplus L_1$.

The morphism $\begin{pmatrix} id_{Y_0} & 0 \\ 0 & h_1 \end{pmatrix} : Y_0 \oplus X_1 \longrightarrow Y_0 \oplus K_1$ (resp. $\begin{pmatrix} id_{X_0} & 0 \\ 0 & m_1 \end{pmatrix} :$ $X_0 \oplus Y_1 \longrightarrow X_0 \oplus L_1$) is a right $\mathcal{X}$-approximation of $Y_0 \oplus K_1$ (resp.of $X_0 \oplus L_1$) with kernel $\begin{pmatrix} 0 \\ g_2 \end{pmatrix} : K_2 \longrightarrow Y_0 \oplus K_1$ (resp. $\begin{pmatrix} 0 \\ \ell_2 \end{pmatrix} : L_2 \longrightarrow X_0 \oplus Y_1$).

We consider the right $\mathcal{X}$-resolutions:

$$0 \longrightarrow X_n \longrightarrow \cdots \longrightarrow X_2 \xrightarrow{\begin{pmatrix} 0 \\ x_2 \end{pmatrix}} Y_0 \oplus X_1 \xrightarrow{\begin{pmatrix} id_{Y_0} & 0 \\ 0 & h_1 \end{pmatrix}} Y_0 \oplus K_1 \longrightarrow 0$$

and

$$0 \longrightarrow Y_n \longrightarrow \cdots \longrightarrow Y_2 \xrightarrow{\begin{pmatrix} 0 \\ y_2 \end{pmatrix}} X_0 \oplus Y_1 \xrightarrow{\begin{pmatrix} id_{X_0} & 0 \\ 0 & m_1 \end{pmatrix}} X_0 \oplus L_1 \longrightarrow 0$$

and we conclude, as before, that $X_0 \oplus Y_1 \oplus K_2 \overset{\sigma_2}{\cong} Y_0 \oplus X_1 \oplus L_2$.

Hence, the Lemma follows by induction. □

Lemma 2.9. *Let*

$$(2.1)\qquad \begin{array}{ccccc} K & \xrightarrow{k} & D & \xrightarrow{d} & A \\ {\scriptstyle\eta}\downarrow & & {\scriptstyle\delta}\downarrow & & \downarrow{\scriptstyle\alpha} \\ L & \xrightarrow{\ell} & F & \xrightarrow{f} & B \end{array}$$

be a commutative diagram where d and f are $\mathcal{X}$-epics with kernels k and ℓ respectively. If α and η are $\mathcal{X}$-epics, then also δ is an $\mathcal{X}$-epic.

Proof. For any object X of $\mathcal{X}$, the diagram (2.1) yields a commutative diagram in $\mathcal{AB}$ with exact rows:

$$\begin{array}{ccccccccc} 0 & \to & \mathcal{C}(X,K) & \xrightarrow{k^X} & \mathcal{C}(X,D) & \xrightarrow{d^X} & \mathcal{C}(X,A) & \to & 0 \\ & & {\scriptstyle\eta^X}\downarrow & & {\scriptstyle\delta^X}\downarrow & & {\scriptstyle\alpha^X}\downarrow & & \\ 0 & \to & \mathcal{C}(X,L) & \xrightarrow{\ell^X} & \mathcal{C}(X,F) & \xrightarrow{f^X} & \mathcal{C}(X,B) & \to & 0 \end{array}$$

Since η^X and α^X are epimorphisms, δ^X is also an epimorphism. Hence δ is an $\mathcal{X}$-epic. □

The previous Lemma allows us to complete the diagram (2.1) with the kernels μ, ϵ and γ of the morphisms η, δ and α respectively. In this way we obtain the fully commutative diagram:

$$(2.2)\qquad \begin{array}{ccccc} M & \xrightarrow{m} & E & \xrightarrow{e} & C \\ {\scriptstyle\mu}\downarrow & & {\scriptstyle\epsilon}\downarrow & & {\scriptstyle\gamma}\downarrow \\ K & \xrightarrow{k} & D & \xrightarrow{d} & A \\ {\scriptstyle\eta}\downarrow & & {\scriptstyle\delta}\downarrow & & \downarrow{\scriptstyle\alpha} \\ L & \xrightarrow{\ell} & F & \xrightarrow{f} & B \end{array}$$

where the morphisms m and e are obtained by the kernel property. We leave for the reader the proof of the following:

Lemma 2.10. *In the diagram (2.2) the morphism m is the kernel of e and the morphism e is an $\mathcal{X}$-epic.*

Now, we are able to show the relative to $\mathcal{X}$ version of the "Horseshoe Lemma".

Lemma 2.11. *Given a sequence $A \xrightarrow{\iota_{-1}} B \xrightarrow{\pi_{-1}} C$, where π_{-1} is an $\mathcal{X}$-epic* *and ι_{-1} is the kernel of π_{-1}, there is a commutative diagram*

$$\begin{tikzcd}
\vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] \\
X_1 \arrow[r, "\iota_1"] \arrow[d, "x_1"'] & X_1 \oplus Y_1 \arrow[r, "\pi_1"] \arrow[d, "z_1"'] & Y_1 \arrow[d, "y_1"'] \\
X_0 \arrow[r, "\iota_0"] \arrow[d, "x_0"'] & X_0 \oplus Y_0 \arrow[r, "\pi_0"] \arrow[d, "z_0"'] & Y_0 \arrow[d, "y_0"'] \\
A \arrow[r, "\iota_{-1}"] \arrow[d] & B \arrow[r, "\pi_{-1}"] \arrow[d] & C \arrow[d] \\
0 & 0 & 0
\end{tikzcd} \tag{2.3}$$

with ι_j (resp. π_j) $j = 0, 1, 2, \dots$ the canonical injection (resp. canonical projection) and where the columns are right $\mathcal{X}$-resolutions of A, B and C.

Proof. Let $x_0 : X_0 \longrightarrow A$ (resp. $y_0 : Y_0 \longrightarrow C$) be a right $\mathcal{X}$-approximation of A (resp. C) with kernel $g_1 : K_1 \longrightarrow X_0$ (resp. $\ell_1 : L_1 \longrightarrow Y_0$). Since π_{-1} is an $\mathcal{X}$-epic, there is a morphism $s_0 : Y_0 \longrightarrow B$ with $\pi_{-1} \circ s_0 = y_0$. Let $z_0 : X_0 \oplus Y_0 \longrightarrow B$ denote the morphism $(\iota_{-1} \circ x_0, s_0)$. The morphism z_0 is a right $\mathcal{X}$-approximation, since it is an $\mathcal{X}$-epimorphism by Lemma 2.9 and since $X_0 \oplus Y_0$ is an object of $\mathcal{X}$. Let $h_1 : H_1 \longrightarrow X_0 \oplus Y_0$ be the kernel of z_0.

The fully commutative diagram below is obtained in the same way as the diagram (2.2):

$$\begin{tikzcd}
K_1 \arrow[r, "m_1"] \arrow[d, "g_1"'] & H_1 \arrow[r, "n_1"] \arrow[d, "h_1"'] & L_1 \arrow[d, "\ell_1"'] \\
X_0 \arrow[r, "\iota_0"] \arrow[d, "x_0"'] & X_0 \oplus Y_0 \arrow[r, "\pi_0"] \arrow[d, "z_0"'] & Y_0 \arrow[d, "y_0"'] \\
A \arrow[r, "\iota_{-1}"] & B \arrow[r, "\pi_{-1}"] & C
\end{tikzcd}$$

Hence, by Lemma 2.10, the morphism n_1 is an $\mathcal{X}$-epic with kernel m_1.

We consider the sequence $K_1 \xrightarrow{m_1} H_1 \xrightarrow{n_1} L_1$ and continue inductively, repeating the above procedure. This leads to the construction of the promised right $\mathcal{X}$-resolutions, according to Lemma 2.5. □

We close this section with some observations concerning relative Ext-functors. The introduction of these functors will be justified in Section 4.

Let $f : A \to B$ be a morphism of $\mathcal{C}$ and

$$\mathbf{X} \; : \; \cdots \to X_{i+1} \xrightarrow{x_{i+1}} X_i \xrightarrow{x_i} X_{i-1} \to \cdots \to X_0 \xrightarrow{x_0} A \to 0,$$
$$\mathbf{Y} \; : \; \cdots \to Y_{i+1} \xrightarrow{y_{i+1}} Y_i \xrightarrow{y_i} Y_{i-1} \to \cdots \to Y_0 \xrightarrow{y_0} B \to 0$$

be right $\mathcal{X}$-resolutions of A and B respectively.

For any object A of $\mathcal{C}$,we denote by $\mathbf{X_d}$ the deleted right $\mathcal{X}$-resolution:

$$\cdots \to X_{i+1} \xrightarrow{x_{i+1}} X_i \xrightarrow{x_i} X_{i-1} \to \cdots \to X_0 \to 0.$$

One shows as in the classical "Comparison Theorem", that f induces a chain transformation from $\mathbf{X}$ to $\mathbf{Y}$ and that any two such chain transformations are chain homotopic.

Using deleted right $\mathcal{X}$-resolutions, one defines for any object D of $\mathcal{C}$ the contravariant functors:

$$\mathcal{E}\mathrm{xt}^n_{\mathcal{X}}(-, D) : \mathcal{C}^{\mathrm{op}} \to \mathcal{AB}, \forall n \geq 0$$

as the n-th right derived functors of $\mathcal{C}(-, D)$. We call these functors the *$\mathcal{X}$-relative Ext-functors.*

Obviously, if $\mathrm{rt.dim}_{\mathcal{X}} A \leq n$, then $\mathcal{E}\mathrm{xt}^j_{\mathcal{X}}(A, D) = 0, \forall j \geq n+1$ and any object D of $\mathcal{C}$.

By Lemma 2.11, we know that given a sequence $A \xrightarrow{\iota_{-1}} B \xrightarrow{\pi_{-1}} C$, with π_{-1} an $\mathcal{X}$-epic and ι_{-1} the kernel of π_{-1}, there are deleted right $\mathcal{X}$-resolutions $\mathbf{X_d}, \mathbf{Z_d}$ and $\mathbf{Y_d}$ of A, B and C respectively, such that there is an exact sequence of complexes of abelian groups:

$$0 \to \mathcal{C}(\mathbf{Y_d}, D) \xrightarrow{\pi \mathrm{D}} \mathcal{C}(\mathbf{Z_d}, D) \xrightarrow{\iota \mathrm{D}} \mathcal{C}(\mathbf{X_d}, D) \to 0$$

for any object D of $\mathcal{C}$. Hence, there is a long-exact sequence of $\mathcal{X}$-relative Ext-functors in $\mathcal{AB}$:

(2.4)

$$0 \to \mathcal{E}\mathrm{xt}^0_{\mathcal{X}}(C, D) \to \mathcal{E}\mathrm{xt}^0_{\mathcal{X}}(B, D) \to \mathcal{E}\mathrm{xt}^0_{\mathcal{X}}(A, D) \xrightarrow{\partial} \mathcal{E}\mathrm{xt}^1_{\mathcal{X}}(C, D) \to \cdots$$
$$\cdots \to \mathcal{E}\mathrm{xt}^i_{\mathcal{X}}(C, D) \to \mathcal{E}\mathrm{xt}^i_{\mathcal{X}}(B, D) \to \mathcal{E}\mathrm{xt}^i_{\mathcal{X}}(A, D) \xrightarrow{\partial} \mathcal{E}\mathrm{xt}^{i+1}_{\mathcal{X}}(C, D) \to \cdots$$

3 Stable Grothendieck Groups

Let $\mathcal{C}$ be a category and $\mathcal{X}$ a subcategory contravariantly finite in $\mathcal{C}$ such that any $\mathcal{X}$-epic has a kernel. In the present section, we shall introduce and study group structures related to the pair $(\mathcal{C}, \mathcal{X})$ and to the stable category

$\mathcal{C}/\mathcal{J}_{\mathcal{X}}$, cf. [5]. These group structures are similar to the classical Grothendieck groups studied in K–theory.

Let $\text{Iso}(\mathcal{C})$ be a set of representatives of the isomorphism classes of the objects of $\mathcal{C}$ and let $\mathcal{F}(\mathcal{C})$ be the free abelian group with base $\text{Iso}(\mathcal{C})$. For any object A of $\mathcal{C}$ the corresponding element of $\text{Iso}(\mathcal{C})$ will be denoted by (A). We also consider the subset $\text{Iso}(\mathcal{X})$ of $\text{Iso}(\mathcal{C})$ having as elements the representatives of the isomorphism classes of the objects of $\mathcal{X}$. Let $\mathcal{F}(\mathcal{X})$ be the free abelian group with base $\text{Iso}(\mathcal{X})$. Obviously, $\mathcal{F}(\mathcal{X}) \subseteq \mathcal{F}(\mathcal{C})$.

Let $\mathcal{F}_0(\mathcal{C},\mathcal{X})$ be the subgroup of $\mathcal{F}(\mathcal{C})$ generated by the set of formal sums

$$\Gamma(\mathcal{C},\mathcal{X}) = \{(A) - (B) + (C) : \text{ where there is an } \mathcal{X}\text{-epic } B \xrightarrow{g} C \text{ with kernel } A \xrightarrow{f} B\}$$

We denote by $\mathcal{K}_0(\mathcal{C},\mathcal{X})$ the quotient group $\mathcal{F}(\mathcal{C})/\mathcal{F}_0(\mathcal{C},\mathcal{X})$ and by $[A]$ the elements $(A) + \mathcal{F}_0(\mathcal{C},\mathcal{X})$ of $\mathcal{K}_0(\mathcal{C},\mathcal{X})$.

Definition 3.1. The group $\mathcal{K}_0(\mathcal{C},\mathcal{X})$ is said to be the *Grothendieck group* of the pair $(\mathcal{C},\mathcal{X})$.

Remark 3.2. Let A be an object of $\mathcal{C}$ with $\text{rt.}\dim_{\mathcal{X}} A \leq n$ and let

$$0 \longrightarrow X_n \xrightarrow{x_n} X_{n-1} \longrightarrow \cdots \longrightarrow X_i \xrightarrow{x_i} X_{i-1} \longrightarrow \cdots \longrightarrow X_1 \xrightarrow{x_1} X_0 \xrightarrow{x_0} A \longrightarrow 0$$

be a right $\mathcal{X}$-resolution of A, then $[A] = \sum_{i=0}^{n}(-1)^i[X_i]$. So, if $\mathcal{C}$ is right $\mathcal{X}$-regular, then the set $\{[X] : (X) \in \text{Iso}(\mathcal{X})\}$ forms a set of generators of $\mathcal{K}_0(\mathcal{C},\mathcal{X})$.

Remark 3.3. Given two objects A and B of $\mathcal{C}$, the canonical projection $A \oplus B \longrightarrow B$ is an $\mathcal{X}$-epic with kernel A. Hence, $[A \oplus B] = [A] + [B]$.

Let $\mathcal{F}_0(\mathcal{X},\mathcal{X})$ be the subgroup of $\mathcal{F}(\mathcal{C})$ generated by the set of formal sums

$$\Gamma(\mathcal{X},\mathcal{X}) = \{(A) - (B) + (C) : \text{ where there is an } \mathcal{X}\text{-epic } B \xrightarrow{g} C \text{ with kernel } A \xrightarrow{f} B \text{ and with } B, C \text{ objects of } \mathcal{X}\}$$

Lemma 3.4. *Let $g : B \longrightarrow C$ be an $\mathcal{X}$-epic with kernel $f : A \longrightarrow B$. If B and C are objects of $\mathcal{X}$, then $B \cong A \oplus C$ and A is an object of $\mathcal{X}$.*

Proof. Since g is an $\mathcal{X}$-epic, there is a morphism $h : C \longrightarrow B$ with $g \circ h = id_C$. Since f is a kernel of g, there is a morphism $k : B \longrightarrow A$ satisfying $f \circ k = id_B - h \circ g$. Now $A \underset{f}{\overset{k}{\leftrightarrows}} B \underset{h}{\overset{g}{\rightleftarrows}} C$ is a byproduct diagram and $B \cong A \oplus C$. Because $\mathcal{X}$ is closed under direct summands A is an object of $\mathcal{X}$. $\square$

Corollary 3.5. *The group $\mathcal{F}_0(\mathcal{X},\mathcal{X})$ is a subgroup of $\mathcal{F}(\mathcal{X})$ and of $\mathcal{F}_0(\mathcal{C},\mathcal{X})$.*

Proof. By 3.4, any element $(A)-(B)+(C)$ of $\Gamma(\mathcal{X},\mathcal{X})$ has the property that (A) is also in $\mathrm{Iso}(\mathcal{X})$. Hence, $\Gamma(\mathcal{X},\mathcal{X}) \subseteq \mathcal{F}(\mathcal{X})$.

For any two objects $(A),(B)$ of $\mathrm{Iso}(\mathcal{C})$ it holds $(A)-(A\oplus B)+(B) \in \mathcal{F}_0(\mathcal{C},\mathcal{X})$. So, again by 3.4, $\Gamma(\mathcal{X},\mathcal{X}) \subseteq \mathcal{F}_0(\mathcal{C},\mathcal{X})$. ■

Definition 3.6. The group $\mathcal{K}_0(\mathcal{X},\oplus) := \mathcal{F}(\mathcal{X})/\mathcal{F}_0(\mathcal{X},\mathcal{X})$ is said to be the *Grothendieck group* of $\mathcal{X}$.

We shall denote the elements $(X)+\mathcal{F}_0(\mathcal{X},\mathcal{X})$ of $K_0(\mathcal{X},\oplus)$ by $\|X\|$.

Remark 3.7. For any $(A),(B),(C) \in \mathrm{Iso}(\mathcal{X})$, the formal sum $(A)-(B)+(C)$ of $\mathcal{F}(\mathcal{X})$ belongs to $\Gamma(\mathcal{X},\mathcal{X})$ if and only if $B \simeq A\oplus C$, because of Lemma 3.4. Hence, $\|A\oplus B\| = \|A\| + \|B\|$.

Let $\iota_0 : \mathcal{F}(\mathcal{X}) \longrightarrow \mathcal{F}(\mathcal{C})$, $\iota_1 : \mathcal{F}(\mathcal{X},\mathcal{X}) \longrightarrow \mathcal{F}(\mathcal{X})$ and $\iota_2 : \mathcal{F}(\mathcal{X},\mathcal{X}) \longrightarrow \mathcal{F}_0(\mathcal{C},\mathcal{X})$ be the canonical inclusions.

The short exact sequences

$$0 \longrightarrow \mathcal{F}_0(\mathcal{C},\mathcal{X}) \xrightarrow{s} \mathcal{F}(\mathcal{C}) \xrightarrow{p} \mathcal{K}_0(\mathcal{C},\mathcal{X}) \longrightarrow 0 \tag{3.1}$$

and

$$0 \longrightarrow \mathcal{F}_0(\mathcal{X},\mathcal{X}) \xrightarrow{\iota_1} \mathcal{F}(\mathcal{X}) \xrightarrow{w} \mathcal{K}_0(\mathcal{X},\oplus) \longrightarrow 0 \tag{3.2}$$

yield in $\mathcal{AB}$ the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{F}_0(\mathcal{X},\mathcal{X}) & \xrightarrow{\iota_1} & \mathcal{F}(\mathcal{X}) & \xrightarrow{w} & \mathcal{K}_0(\mathcal{X},\oplus) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \iota_2} & & \downarrow{\scriptstyle \iota_0} & & \downarrow{\scriptstyle c_\mathcal{X}} & & \\ 0 & \longrightarrow & \mathcal{F}_0(\mathcal{C},\mathcal{X}) & \xrightarrow{s} & \mathcal{F}(\mathcal{C}) & \xrightarrow{p} & \mathcal{K}_0(\mathcal{C},\mathcal{X}) & \longrightarrow & 0 \end{array} \tag{3.3}$$

where the homomorphism $c_\mathcal{X}$ is canonically induced by ι_0.

Definition 3.8. The group homomorphism $c_\mathcal{X} : \mathcal{K}_0(\mathcal{X},\oplus) \longrightarrow K_0(\mathcal{C},\mathcal{X})$ is said to be the *Cartan map* of the pair $(\mathcal{C},\mathcal{X})$.

In the following we are interested in $\mathrm{Coker}\, c_\mathcal{X}$. As we shall see $\mathrm{Coker}\, c_\mathcal{X}$ appears as "Grothendieck group" of the left triangulated structure of the stable category $\mathcal{C}/\mathcal{J}_\mathcal{X}$. We start by introducing "Grothendieck groups" for any left triangulated category. Firstly, we recall some facts about left triangulated categories developed in [5].

Let Ω be an endofunctor of a category $\mathcal{D}$ and let $\mathrm{LTR}(\mathcal{D},\Omega)$ be the category with objects the diagrams of the form $\Omega C \xrightarrow{f} A \xrightarrow{g} B \xrightarrow{h} C$ and with set of morphisms from $\Omega C \xrightarrow{f} A \xrightarrow{g} B \xrightarrow{h} C$ to $\Omega C'' \xrightarrow{f'} A' \xrightarrow{g'} B' \xrightarrow{h'} C'$, the triples (α, β, γ) of morphisms of $\mathcal{D}$ from (A, B, C) to (A', B', C'), which make the diagram below commutative:

$$\begin{tikzcd}
\Omega C \arrow[r, "f"] \arrow[d, "\Omega\gamma"'] & A \arrow[r, "g"] \arrow[d, "\alpha"'] & B \arrow[r, "h"] \arrow[d, "\beta"'] & C \arrow[d, "\gamma"'] \\
\Omega C'' \arrow[r, "f'"] & A' \arrow[r, "g'"] & B' \arrow[r, "h'"] & C''
\end{tikzcd}$$

The objects of $\mathrm{LTR}(\mathcal{D}, \Omega)$ are said to be the *left triangles* of $(\mathcal{D}, \Omega)$.

Whenever it is convenient the objects $\Omega C \xrightarrow{f} A \xrightarrow{g} B \xrightarrow{h} C$ of $\mathrm{LTR}(\mathcal{D}, \Omega)$ will be denoted by (A, B, C, f, g, h).

Definition 3.9. A full subcategory Δ of $\mathrm{LTR}(\mathcal{D}, \Omega)$ is said to be a *left triangulation* of $(\mathcal{D}, \Omega)$ if it is closed under isomorphisms and satisfies the following four axioms:

LT1i For any object A of $\mathcal{D}$, the left triangle $0 \xrightarrow{0} A \xrightarrow{1_A} A \xrightarrow{0} 0$ belongs to Δ.

LT1ii For any morphism $x : B \to C$ of $\mathcal{D}$, there is a left triangle in Δ of the form (A, B, C, f, g, x).

LT2 For any left triangle (A, B, C, f, g, h) in Δ, the left triangle $(\Omega C, A, B, -\Omega h, f, g)$ is also in Δ.

LT3 For any two left triangles (A, B, C, f, g, h), (A', B', C', f', g', h') in Δ and any two morphisms $\beta : B \to B'$ and $\gamma : C \to C'$ of $\mathcal{D}$ with $\gamma \circ h = h' \circ \beta$, there is a morphism $\alpha : A \to A'$ of $\mathcal{D}$ such that the triple (α, β, γ) is a morphism from the first triangle to the second.

LT4 For any two left triangles (A, B, C, f, g, h) and (E, C, D, i, ℓ, k) in Δ, there is a third left triangle $(F, B, D, j, m, k \circ h)$ in Δ and two morphisms $\alpha : A \to F$ and $\beta : F \to E$ of $\mathcal{D}$, such that the diagram below is fully commutative and where the second column from the left, i.e. $\Omega E \xrightarrow{f \circ \Omega \ell} A \xrightarrow{\alpha} F \xrightarrow{\beta} E$, is a left triangle in Δ.

$$\begin{tikzcd}
 & \Omega E \arrow[d, "f\circ\Omega\ell"'] & & & \\
\Omega C \arrow[r, "f"] \arrow[d, "\Omega k"'] & A \arrow[r, "g"] \arrow[d, "\alpha"'] & B \arrow[r, "h"] \arrow[d, equal] & C \arrow[d, "k"'] \\
\Omega D \arrow[r, "\jmath"] \arrow[d, equal] & F \arrow[r, "m"] \arrow[d, "\beta"'] & B \arrow[r, "k\circ h"] \arrow[d, "h"'] & D \arrow[d, equal] \\
\Omega D \arrow[r, "\imath"] & E \arrow[r, "\ell"] & C \arrow[r, "k"] & D
\end{tikzcd}$$

Definition 3.10. The triple $(\mathcal{D}, \Omega, \Delta)$ is said to be a *left triangulated category*.

Definition 3.11. A covariant functor $\mathcal{G} : \mathcal{D} \longrightarrow \mathcal{D}'$ is said to be a *triangle equivalence* from the left triangulated category $(\mathcal{D}, \Omega, \Delta)$ to the left triangulated category $(\mathcal{D}', \Omega', \Delta')$, if it is an equivalence and there is a natural isomorphism $\phi : \mathcal{G} \circ \Omega \longrightarrow \Omega' \circ \mathcal{G}$, such that if (A, B, C, f, g, h) is a left triangle of Δ, then the left triangle $(\mathcal{G}A, \mathcal{G}B, \mathcal{G}C, \mathcal{G}f \circ \phi_C^{-1}, \mathcal{G}g, \mathcal{G}h)$ is in Δ'.

We also make the following definition for later use:

Definition 3.12. Given a left triangulated category $(\mathcal{D}, \Omega, \Delta)$, the functor Ω is said to be *locally nilpotent* if for any object A of $\mathcal{D}$, there is an $n(A) \in \mathbb{N}$, such that $\Omega^{n(A)} A = 0$.

Now, we are ready to construct a "Grothendieck group" for any left triangulated category. A notion which was introduced already in by A. Grothendieck for triangulated categories, see [7] and [8].

Let $(\mathcal{D}, \Omega, \Delta)$ be a left triangulated category and let
$\mathrm{Iso}(\mathcal{D}) = \{(A) : A \text{ object of } \mathcal{D}\}$ be a set of representatives of the isomorphism classes of the objects of $\mathcal{D}$.

Let $\mathcal{F}(\mathcal{D})$ be the free abelian group with base $\mathrm{Iso}(\mathcal{D})$ and $\mathcal{F}_0(\mathcal{D}, \Omega, \Delta)$ be the subgroup of $\mathcal{F}(\mathcal{D})$ generated by the set of formal sums

$$\Gamma(\mathcal{D}, \Omega, \Delta) = \{(A) - (B) + (C) : \text{ where there is a left triangle } (A, B, C, f, g, h) \text{ in } \Delta\}$$

We denote the quotient group $\mathcal{F}(\mathcal{D})/\mathcal{F}_0(\mathcal{D}, \Omega, \Delta)$ by $\mathcal{K}_0(\mathcal{D}, \Omega, \Delta)$ and its elements $(A) + \mathcal{F}_0(\mathcal{D}, \Omega, \Delta)$ by $[A]$.

Definition 3.13. The group $\mathcal{K}_0(\mathcal{D}, \Omega, \Delta)$ is said to be the *Grothendieck group* of the left triangulated category $(\mathcal{D}, \Omega, \Delta)$.

Remark 3.14. Any triangle equivalence $\mathcal{G}: \mathcal{D} \longrightarrow \mathcal{D}'$ between two left triangulated categories $(\mathcal{D}, \Omega, \Delta)$ and $(\mathcal{D}', \Omega', \Delta')$, induces a group isomorphism between $\mathcal{F}_0(\mathcal{D}, \Omega, \Delta)$ and $\mathcal{F}_0(\mathcal{D}', \Omega', \Delta')$. Hence, $\mathcal{G}$ induces an isomorphism between the corresponding Grothendieck groups.

Lemma 3.15. *(i) Let (A, B, C, f, g, h) be a left triangle in Δ, then g (resp. f) is a weak kernel of h (resp. g). Moreover, if $g = 0$ (resp. $f = 0$), then h (resp. g) is a monomorphism.*

(ii) For any two objects A and B of $\mathcal{D}$, the left triangle $(A, A \oplus B, B, 0, \iota_A, \pi_B)$ belongs to Δ, where ι_A is the canonical injection and π_B is the canonical projection .

Proof. (i) By LT1i, the left triangle $(C, C, 0, 0, id_C, 0)$ belongs to Δ. Hence by LT2, the left triangle $(0, C, C, 0, 0, id_C)$ belongs also to Δ. We consider the commutative diagram:

$$\begin{array}{ccccccc}
\Omega C & \xrightarrow{f} & A & \xrightarrow{g} & B & \xrightarrow{h} & C \\
{\scriptstyle id_{\Omega C}}\downarrow & & {\scriptstyle \ell}\downarrow & & {\scriptstyle h}\downarrow & & {\scriptstyle id_C}\downarrow \\
\Omega C & \xrightarrow{0} & 0 & \xrightarrow{0} & C & \xrightarrow{id_C} & C
\end{array}$$

where ℓ exists by LT3. Hence, $h \circ g = 0$.

Let $k : X \longrightarrow B$ be a morphism with $h \circ k = 0$. We consider the triangle $(X, X, 0, 0, id_X, 0)$ and the commutative diagram:

$$\begin{array}{ccccccc}
0 & \xrightarrow{0} & X & \xrightarrow{id_X} & X & \xrightarrow{0} & 0 \\
{\scriptstyle 0}\downarrow & & {\scriptstyle m}\downarrow & & {\scriptstyle k}\downarrow & & {\scriptstyle 0}\downarrow \\
\Omega C & \xrightarrow{f} & A & \xrightarrow{g} & B & \xrightarrow{h} & C
\end{array}$$

where m exists by LT3. Hence, $g \circ m = k$. Obviously, if $g = 0$ then $k = 0$. So, in this case h is a monomorphism.

The remaining part of (i) follows applying the same arguments as above on the left triangle $(\Omega C, A, B, -\Omega h, f, g)$, which belongs to Δ because of LT2.

(ii) By LT1ii, there is a left triangle $\Omega B \xrightarrow{f} K \xrightarrow{g} A \oplus B \xrightarrow{\pi_B} B$ in Δ. Since $\Omega \pi_B \circ f = 0$, the morphism f equals zero and the first part of our Lemma implies that g is a monomorphism. Hence, (K, g) is a kernel of π_B and so there is an isomorphism $\sigma : K \longrightarrow A$ with $\iota_A \circ \sigma = g$. Since, $(\sigma, id_{A \oplus B}, id_A)$ is an isomorphism from $(K, A \oplus B, B, 0, g, \pi_B)$ to $(A, A \oplus B, B, 0, \iota_A, \pi_B)$, this last left triangle belongs also to Δ. □

Corollary 3.16. *(i) If $[A]$ is an element of $\mathcal{K}_0(\mathcal{D},\Omega,\Delta)$, then*

$$[A]=(-1)^n[\Omega^n A],\ \forall n\in\mathbb{N}\cup\{0\}.$$

(ii) If $[A]$ and $[B]$ are elements of $\mathcal{K}_0(\mathcal{D},\Omega,\Delta)$, then $[A]+[B]=[A\oplus B]$.

(iii) If Ω is locally nilpotent, then $\mathcal{K}_0(\mathcal{D},\Omega,\Delta)$ is the trivial group.

Proof. (i) Given an object A of $\mathcal{D}$, the left triangle $\Omega A \xrightarrow{-id_{\Omega A}} \Omega A \xrightarrow{0} 0 \xrightarrow{0} A$ belongs to Δ by LT1i and LT2 . Hence, $[A]=-[\Omega A]$ and so $[A]=(-1)^n[\Omega^n A]$.

(ii) By Lemma 3.15(ii) the left triangle $(A, A\oplus B, B, 0, \iota_A, \pi_B)$ belongs to Δ for any two objects A and B of $\mathcal{D}$. Hence, $[A]+[B]=[A\oplus B]$.

(iii) Obvious. □

Now, we are ready for a closer study of the Grothendieck groups of stable categories.

Let $\mathcal{C}$ be an additive category and a subcategory $\mathcal{X}$ contravariantly finite in $\mathcal{C}$, such that any $\mathcal{X}$–epic has a kernel. We recall, that the stable category $\mathcal{C}_{\mathcal{X}}=\mathcal{C}/\mathcal{J}_{\mathcal{X}}$ has the same objects as $\mathcal{C}$ and that for any two objects A and B of $\mathcal{C}_{\mathcal{X}}$ the set of morphisms $\mathcal{C}_{\mathcal{X}}(A,B)$ equals the factorgroup $\mathcal{C}(A,B)/\mathcal{X}(A,B)$, where $\mathcal{X}(A,B)$ is the subgroup of $\mathcal{C}(A,B)$ having as elements the morphisms of $\mathcal{C}$ which factor over some object of $\mathcal{X}$. The composition in $\mathcal{C}_{\mathcal{X}}$ is induced by the corresponding composition in $\mathcal{C}$.

Let $\mathcal{F}$ be the projection functor $\mathcal{C}\longrightarrow\mathcal{C}_{\mathcal{X}}$. The image $\mathcal{F}A$ of any object A of $\mathcal{C}$ will be denoted by $\underline{A}$ and the image $\mathcal{F}f$ of any morphism f of $\mathcal{C}$ will be denoted by $\underline{f}$.

In [5], we constructed a functor $\Omega_{\mathcal{X}}:\mathcal{C}_{\mathcal{X}}\longrightarrow\mathcal{C}_{\mathcal{X}}$ and a left triangulation $\Delta_{\mathcal{X}}$ of $\mathcal{C}_{\mathcal{X}}$, such that the triple $(\mathcal{C}_{\mathcal{X}},\Omega_{\mathcal{X}},\Delta_{\mathcal{X}})$ forms a left triangulated category. We recall the definition of $\Omega_{\mathcal{X}}$ on the objects of $\mathcal{C}_{\mathcal{X}}$ for later use. We refer for further details to [5]. For any object A of $\mathcal{C}$ we consider a right $\mathcal{X}$-approximation $g:X\longrightarrow A$ and its kernel $f:K\longrightarrow X$. We define $\Omega_{\mathcal{X}}\underline{A}:=\underline{K}$.

Definition 3.17. The factor group

$$\mathcal{K}_0(\mathcal{C}_{\mathcal{X}},\Omega_{\mathcal{X}},\Delta_{\mathcal{X}}):=\mathcal{F}(\mathcal{C}_{\mathcal{X}},\Omega_{\mathcal{X}},\Delta_{\mathcal{X}})/\mathcal{F}_0(\mathcal{C}_{\mathcal{X}},\Omega_{\mathcal{X}},\Delta_{\mathcal{X}})$$

is said to be the *Grothendieck group* of the stable category $\mathcal{C}_{\mathcal{X}}$.

Let $\pi_0:\mathcal{F}(\mathcal{C})\longrightarrow\mathcal{F}(\mathcal{C}_{\mathcal{X}})$ be the epimorphism defined by $(A)\mapsto(\underline{A})$. Its kernel equals the canonical inclusion $\iota_0:\mathcal{F}(\mathcal{X})\longrightarrow\mathcal{F}(\mathcal{C})$.

Lemma 3.18. *The restriction of π_0 to $\mathcal{F}_0(\mathcal{C},\mathcal{X})$ defines a group epimorphism $\pi_1:\mathcal{F}_0(\mathcal{C},\mathcal{X})\longrightarrow\mathcal{F}_0(\mathcal{C}_{\mathcal{X}},\Omega_{\mathcal{X}},\Delta_{\mathcal{X}})$*

Proof. By [5], Prop. 2.10, any $\mathcal{X}$-epic $g : B \to C$ with kernel $f : A \to B$ induces a left triangle $(\underline{A}, \underline{B}, \underline{C}, \underline{f}, \underline{g}, \underline{h})$ in $\Delta_{\mathcal{X}}$. Conversely, the same Proposition implies that given a left triangle in $\Delta_{\mathcal{X}}$, there is an $\mathcal{X}$-epimorphism $\ell : E \to F$ with kernel $m : D \to E$, such that the induced left triangle $(\underline{D}, \underline{E}, \underline{F}, \underline{k}, \underline{\ell}, \underline{m})$ is isomorphic to the given one. Hence, the restriction of π_0 on $\Gamma(\mathcal{C}, \mathcal{X})$ defines a surjective map $\Gamma(\mathcal{C}, \mathcal{X}) \to \Gamma(\mathcal{C}_{\mathcal{X}}, \Omega_{\mathcal{X}}, \Delta_{\mathcal{X}})$, $(A) - (B) + (C) \mapsto (\underline{A}) - (\underline{B}) + (\underline{C})$ and so π_0 restricted on $\mathcal{F}_0(\mathcal{C}, \mathcal{X})$ defines an epimorphism $\pi_1 : \mathcal{F}_0(\mathcal{C}, \mathcal{X}) \to \mathcal{F}_0(\mathcal{C}_{\mathcal{X}}, \Omega_{\mathcal{X}}, \Delta_{\mathcal{X}})$. □

The epimorphisms π_0, π_1 and the short exact sequences

$$0 \to \mathcal{F}_0(\mathcal{C}_{\mathcal{X}}, \Omega_{\mathcal{X}}, \Delta_{\mathcal{X}}) \xrightarrow{t} \mathcal{F}(\mathcal{C}_{\mathcal{X}}, \Omega_{\mathcal{X}}, \Delta_{\mathcal{X}}) \xrightarrow{q} \mathcal{K}_0(\mathcal{C}_{\mathcal{X}}, \Omega_{\mathcal{X}}, \Delta_{\mathcal{X}}) \to 0. \tag{3.4}$$

and (3.1) yield in $\mathcal{AB}$ the following commutative diagram:

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
 & & \mathcal{F}_0(\mathcal{C}, \mathcal{X}) \arrow[r, "\pi_1"] \arrow[d, "s"] & \mathcal{F}_0(\mathcal{C}_{\mathcal{X}}, \Omega_{\mathcal{X}}, \Delta_{\mathcal{X}}) \arrow[r] \arrow[d, "t"] & 0 \\
0 \arrow[r] & \mathcal{F}(\mathcal{X}) \arrow[r, "\iota_0"] \arrow[d, "r"] & \mathcal{F}(\mathcal{C}) \arrow[r, "\pi_0"] \arrow[d, "p"] & \mathcal{F}(\mathcal{C}_{\mathcal{X}}) \arrow[r] \arrow[d, "q"] & 0 \\
0 \arrow[r] & \operatorname{Ker}\pi \arrow[r, "\iota"] & \mathcal{K}_0(\mathcal{C}, \mathcal{X}) \arrow[r, "\pi"] \arrow[d] & \mathcal{K}_0(\mathcal{C}_{\mathcal{X}}, \Omega_{\mathcal{X}}, \Delta_{\mathcal{X}}) \arrow[r] \arrow[d] & 0 \\
 & & 0 & 0 &
\end{tikzcd} \tag{3.5}$$

where π and r are canonically induced by π_0 and p respectively. Since π_0 is an epimorphism, π is also an epimorphism.

With the assumptions made at the beginning of our paper we have:

Theorem 3.19. *The cokernel of the Cartan map $c_{\mathcal{X}} : \mathcal{K}_0(\mathcal{X}, \oplus) \to \mathcal{K}_0(\mathcal{C}, \mathcal{X})$ is isomorphic to the Grothendieck group $\mathcal{K}_0(\mathcal{C}_{\mathcal{X}}, \Omega_{\mathcal{X}}, \Delta_{\mathcal{X}})$.*

Proof. Because of the diagram (3.3) we obtain $\operatorname{Im} c_{\mathcal{X}} = \operatorname{Im}(p \circ \iota_0)$. By the diagram (3.5), $\operatorname{Im}(p \circ \iota_0) = \operatorname{Im}(\iota \circ r)$. Since π_1 is an epimorphism, the application of the "Snake Lemma" on the two bottom rows of diagram (3.5) implies that r is an epimorphism. Hence, $\operatorname{Im} c_{\mathcal{X}} = \operatorname{Ker}\pi$. □

The next Theorem gives a sufficient condition for the Cartan map $c_{\mathcal{X}}$ to be an isomorphism. With the assumptions made at the beginning of our paper we have:

Theorem 3.20. *Let $\mathcal{C}$ be a category and $\mathcal{X}$ be a subcategory contravariantly finite in $\mathcal{C}$. If $\mathcal{C}$ is right $\mathcal{X}$-regular, then the Cartan map $c_{\mathcal{X}} : \mathcal{K}_0(\mathcal{X}, \oplus) \longrightarrow K_0(\mathcal{C}, \mathcal{X})$ is an isomorphism.*

Proof. Since $\mathcal{C}$ is right $\mathcal{X}$-regular, the functor $\Omega_{\mathcal{X}}$ is locally nilpotent. By Corollary 3.16(iii), the Grothendieck group $K_0(\mathcal{C}_{\mathcal{X}}, \Omega_{\mathcal{X}}, \Delta_{\mathcal{X}})$ is the trivial group and so $c_{\mathcal{X}}$ is an epimorphism.

Let

$$0 \longrightarrow X_n \xrightarrow{x_n} X_{n-1} \longrightarrow \cdots \longrightarrow X_i \xrightarrow{x_i} X_{i-1} \longrightarrow \cdots \longrightarrow X_0 \xrightarrow{x_0} A \longrightarrow 0.$$

be a right $\mathcal{X}$-resolution of an object A of $\mathcal{C}$. We claim that the assignment $(A) \mapsto \sum_{i=0}^{n}(-1)^i \|X_i\|$ determines a well-defined homomorphism $\delta_{\mathcal{X}} : \mathcal{F}(\mathcal{C}) \longrightarrow \mathcal{K}_0(\mathcal{X}, \oplus)$. Indeed, given another right $\mathcal{X}$-resolution

$$0 \longrightarrow Y_n \xrightarrow{y_n} Y_{n-1} \longrightarrow \cdots \longrightarrow Y_i \xrightarrow{y_i} Y_{i-1} \longrightarrow \cdots \longrightarrow Y_1 \xrightarrow{y_1} Y_0 \xrightarrow{y_0} A \longrightarrow 0$$

of A, we obtain by Lemma 2.8 and Remark 3.7 that

$$\sum_{i \text{ even}} \|X_i\| + \sum_{i \text{ odd}} \|Y_i\| = \sum_{i \text{ even}} \|Y_i\| + \sum_{i \text{ odd}} \|X_i\|, \ 0 \leq i \leq n.$$

So, $\sum_{i=0}^{n}(-1)^i \|X_i\| = \sum_{i=0}^{n}(-1)^i \|Y_i\|$ and $\delta_{\mathcal{X}}$ is well-defined.

Now, Lemma 2.11 implies that $\delta_{\mathcal{X}}((A) - (B) + (C)) = 0$ for any element of $\Gamma(\mathcal{C}, \mathcal{X})$. Hence, there is a homomorphism $d_{\mathcal{X}} : \mathcal{K}_0(\mathcal{C}, \mathcal{X}) \longrightarrow \mathcal{K}_0(\mathcal{X}, \oplus)$ induced by $\delta_{\mathcal{X}}$. For any object X of $\mathcal{X}$ we have $d_{\mathcal{X}}([X]) = \|X\|$, since $0 \longrightarrow X \xrightarrow{id_X} X \longrightarrow 0$ is trivially a right $\mathcal{X}$-resolution of X.

We claim that the composition $d_{\mathcal{X}} \circ c_{\mathcal{X}}$ is the identity homomorphism of $\mathcal{K}_0(\mathcal{X}, \oplus)$. The set $\Gamma(\|\mathcal{X}\|) = \{\|X\| : (X) \in \mathrm{Iso}(\mathcal{X})\}$ forms a set of generators of $\mathcal{K}_0(\mathcal{X}, \oplus)$. For any element $\|X\|$ of $\Gamma(\|\mathcal{X}\|)$ we have $d_{\mathcal{X}} \circ c_{\mathcal{X}}(\|X\|) = d_{\mathcal{X}}([X]) = \|X\|$. So $d_{\mathcal{X}} \circ c_{\mathcal{X}} = id_{\mathcal{K}_0(\mathcal{X}, \oplus)}$. This implies that $c_{\mathcal{X}}$ is a monomorphism and completes the proof. □

4 Cartan Matrices and Euler Characteristic

Let k be a field, $\mathcal{C}$ be a finite dimensional k-additive Krull-Schmidt category, cf. [9] Section 2.2. and $\{X_1, \ldots, X_m\}$ be a finite family of non-isomorphic indecomposable objects of $\mathcal{C}$. Let X denote the direct sum $X_1 \oplus \cdots \oplus X_m$. The full subcategory $\mathcal{X} = \mathrm{add}\, X$ is contravariantly finite in $\mathcal{C}$, cf. [1] Prop.1.9.

In the present section, we shall deal with subcategories of the above form with the property that any $\mathcal{X}$-epic has a kernel and we shall define a quadratic matrix behaving analogously to the classical Cartan matrix. If this matrix is $\mathbb{Z}$–invertible, we shall introduce a bilinear form $\mathcal{K}_0(\mathcal{C},\mathcal{X}) \times \mathcal{K}_0(\mathcal{C},\mathcal{X}) \longrightarrow \mathbb{Z}$ and shall give an interpretation of the corresponding quadratic form using relative homological terms.

Definition 4.1. For any object A of $\mathcal{C}$, the m–tuple

$$\underline{\dim}_{\mathcal{X}} A := (\dim_k \mathcal{C}(X_1, A), \dim_k \mathcal{C}(X_2, A), \ldots, \dim_k \mathcal{C}(X_m, A))$$

is said to be the *$\mathcal{X}$-dimension vector* of A.

Lemma 4.2. *The assignment* $[A] \mapsto \underline{\dim}_{\mathcal{X}} A$ *determines a group homomorphism* $\underline{\dim}_{\mathcal{X}} : \mathcal{K}_0(\mathcal{C},\mathcal{X}) \longrightarrow \mathbb{Z}^m$.

Proof. We consider the group homomorphism $\phi : \mathcal{F}(\mathcal{C}) \longrightarrow \mathbb{Z}^m$ defined by $(A) \mapsto \underline{\dim}_{\mathcal{X}} A, \forall (A) \in \operatorname{Iso}(\mathcal{C})$. For any generator $(A) - (B) + (C)$ of $\Gamma(\mathcal{C},\mathcal{X})$ we have $\phi((A) - (B) + (C)) = 0$, since there is an $\mathcal{X}$–epic $B \xrightarrow{g} C$ with kernel $A \xrightarrow{f} B$, yielding $\forall i = 1, \ldots, m$, the short exact sequence $0 \longrightarrow \mathcal{C}(X_i, A) \xrightarrow{f^{X_i}} \mathcal{C}(X_i, B) \xrightarrow{g^{X_i}} \mathcal{C}(X_i, C) \longrightarrow 0$ in $\mathcal{AB}$.

Hence, there is a well defined group homomorphism $\underline{\dim}_{\mathcal{X}} : \mathcal{K}_0(\mathcal{C},\mathcal{X}) \longrightarrow \mathbb{Z}^m$ induced by ϕ. □

For an object A of $\mathcal{C}$ we have $[A] = 0$ if and only if $\underline{\dim}_{\mathcal{X}} A = 0$. Indeed, if $\underline{\dim}_{\mathcal{X}} A = 0$, then the zero morphism $0 \longrightarrow A$ is a right $\mathcal{X}$–approximation of A, implying that $(A) \in \mathcal{F}_0(\mathcal{C},\mathcal{X})$.

Definition 4.3. The $m \times m$ matrix $C_{\mathcal{X}} = (c_{ij})$ with $c_{ij} = \dim_k \mathcal{C}(X_i, X_j)$ is said to be the *Cartan matrix* of the pair $(\mathcal{C},\mathcal{X})$.

Suppose that the Cartan matrix $C_{\mathcal{X}}$ is $\mathbb{Z}$–invertible, then $C_{\mathcal{X}}^{-1}$ defines a bilinear form:

$$\langle -, - \rangle_{\mathcal{X}} : \mathcal{K}_0(\mathcal{C},\mathcal{X}) \times \mathcal{K}_0(\mathcal{C},\mathcal{X}) \longrightarrow \mathbb{Z}^m \text{ by } \langle [A], [B] \rangle_{\mathcal{X}} = \underline{\dim}_{\mathcal{X}} A\, C_{\mathcal{X}}^{-t} (\underline{\dim}_{\mathcal{X}} B)^t$$

The corresponding quadratic form $q_{\mathcal{X}}([A]) = \langle [A], [A] \rangle_{\mathcal{X}}$ is said to be the *$\mathcal{X}$-Euler characteristic* of the pair $(\mathcal{C},\mathcal{X})$.

Our next Proposition is the relative homological version of the corresponding one in [9], Section 2.4.

Proposition 4.4. *Let A and B be two objects of $\mathcal{C}$. If* $\operatorname{rt.dim}_{\mathcal{X}} A < \infty$, *then*

$$\langle [A], [B] \rangle_{\mathcal{X}} = \sum_{j \geq 0} (-1)^j \dim_k \mathcal{E}\mathrm{xt}^j_{\mathcal{X}}(A, B) \tag{4.1}$$

Proof. We proceed by induction on $n = \mathrm{rt.\,dim}_{\mathcal{X}} A$.

For $d = 0$, there is a $\mathcal{X}$-resolution $0 \to Q \to A$ with Q an object of $\mathcal{X}$. Hence, $[A] = [Q]$ and $\underline{\dim}_{\mathcal{X}} A = \underline{\dim}_{\mathcal{X}} Q$. Moreover, $\mathcal{E}\mathrm{xt}^j_{\mathcal{X}}(A, B) = 0$, $\forall j \geq 1$ and $\mathcal{E}\mathrm{xt}^0_{\mathcal{X}}(A, B) = \mathcal{C}(Q, B)$. Hence, the right hand side of (4.1) equals $\dim_k \mathcal{C}(Q, B)$. Suppose that $Q = 0$, then obviously both sides of (4.1) equal zero. Suppose that $Q \neq 0$, then without loss of generality we may assume that $Q = X_i$ for some $i = 1 \dots m$. Since $C_{\mathcal{X}}^{-1}(\underline{\dim}_{\mathcal{X}} X_i)^t = \epsilon(i)$, where $\epsilon(i)$ denotes the column vector with 1 at the i-th position and zero anywhere else, the left hand side of (4.1) equals $\underline{\dim}_{\mathcal{X}} B \epsilon(i) = \dim_k \mathcal{C}(X_i, B)$. Hence, again both sides coincide.

Assume that our claim is true for all objects A, such that $\mathrm{rt.\,dim}_{\mathcal{X}} A < d$. Let A be an object of $\mathcal{C}$ with $\mathrm{rt.\,dim}_{\mathcal{X}} A = d$. There is a sequence $A' \xrightarrow{f} Q \xrightarrow{g} A$, where Q is an object of $\mathcal{X}$ and g is a $\mathcal{X}$-epic with kernel f. Hence, $[A] = [Q] - [A']$ and $\underline{\dim}_{\mathcal{X}} A = \underline{\dim}_{\mathcal{X}} Q - \underline{\dim}_{\mathcal{X}} A'$.

We have:

$$\begin{aligned} \langle [A], [B] \rangle_{\mathcal{X}} &= \langle [Q], [B] \rangle_{\mathcal{X}} - \langle [A'], [B] \rangle_{\mathcal{X}} = \\ &\sum_{j \geq 0} (-1)^j \dim_k \mathcal{E}\mathrm{xt}^j_{\mathcal{X}}(Q, B) - \sum_{j \geq 0} (-1)^j \dim_k \mathcal{E}\mathrm{xt}^j_{\mathcal{X}}(A', B). \end{aligned} \tag{4.2}$$

The sequence $A' \xrightarrow{f} Q \xrightarrow{g} A$, yields the long exact sequence

$$\cdots \to \mathcal{E}\mathrm{xt}^i_{\mathcal{X}}(A, B) \to \mathcal{E}\mathrm{xt}^i_{\mathcal{X}}(Q, B) \to \mathcal{E}\mathrm{xt}^i_{\mathcal{X}}(A', B) \xrightarrow{\partial} \mathcal{E}\mathrm{xt}^{i+1}_{\mathcal{X}}(A, B) \to \cdots$$

discussed at the end of Section 2. Since $\dim_k$ acts additively on this long exact sequence, we obtain for the last term of (4.2):

$$\begin{aligned} &\sum_{j \geq 0} (-1)^j \dim_k \mathcal{E}\mathrm{xt}^j_{\mathcal{X}}(Q, B) - \sum_{j \geq 0} (-1)^j \dim_k \mathcal{E}\mathrm{xt}^j_{\mathcal{X}}(A', B) = \\ &\sum_{j \geq 0} (-1)^j \dim_k \mathcal{E}\mathrm{xt}^j_{\mathcal{X}}(A, B). \end{aligned}$$

□

Let Γ denote the k-algebra $\mathrm{End}_{\mathcal{C}}(X)$, $\mathrm{mod}\,\Gamma$ the category of finitely generated right Γ-modules and $\mathcal{P}^f_\Gamma$ the full subcategory of $\mathrm{mod}\,\Gamma$ with objects the projective finitely generated Γ-modules. Let $\mathcal{G}$ denote the functor $\mathcal{C}(X, -) : \mathcal{C} \to \mathrm{mod}\,\Gamma$. It is well-known that $\mathcal{G}$ restricted on $\mathcal{X}$ induces an equivalence $\sigma : \mathcal{X} \xrightarrow{\sim} \mathcal{P}^f_\Gamma$.

The $\mathcal{X}$-relative Ext-functors, which appear in formula (4.1) are related with the absolute Ext-functors of $\mathrm{mod}\,\Gamma$ as follows:

Proposition 4.5. *For any object B of $\mathcal{C}$ and $\forall i \geq 0$, the functors $\mathcal{E}\mathrm{xt}^i_{\mathcal{X}}(-, B)$ and $\mathrm{Ext}^i_\Gamma(\mathcal{G}-, \mathcal{G}B)$ are natural isomorphic .*

Proof. We give a brief proof since all arguments are standard. Let

$$\cdots \longrightarrow Y_{i+1} \xrightarrow{y_{i+1}} Y_i \xrightarrow{y_i} Y_{i-1} \longrightarrow \cdots \longrightarrow Y_0 \xrightarrow{y_0} B \longrightarrow 0,$$

be a right $\mathcal{X}$-resolution of B.

For any object X of $\mathcal{X}$ we obtain from the resolution above the diagram

$$\begin{tikzcd}
\cdots \arrow[r] & \mathcal{C}(X,Y_1) \arrow[r, "y_1^X"] \arrow[d] & \mathcal{C}(X,Y_0) \arrow[r, "y_0^X"] \arrow[d] & \mathcal{C}(X,B) \arrow[r] \arrow[d, "\phi_{X,B}"'] & 0 \\
\cdots \arrow[r] & \mathrm{Hom}_\Gamma(\mathcal{G}X,\mathcal{G}Y_1) \arrow[r, "\mathcal{G}y_1{}^{\mathcal{G}X}"] & \mathrm{Hom}_\Gamma(\mathcal{G}X,\mathcal{G}Y_0) \arrow[r, "\mathcal{G}y_0{}^{\mathcal{G}X}"] & \mathrm{Hom}_\Gamma(\mathcal{G}X,\mathcal{G}B) \arrow[r] & 0
\end{tikzcd}$$

This is a fully commutative diagram with exact rows. Moreover, the vertical homomorphisms $\mathcal{C}(X, Y_i) \longrightarrow \mathrm{Hom}_\Gamma(\mathcal{G}X, \mathcal{G}Y_i)$ are isomorphisms for all $i \geq 0$, hence the morphism $\phi_{X,B}$ is also an isomorphism. We leave for the reader to verify that $\phi_{X,B}$ is natural on both arguments.

Let A and B be objects of $\mathcal{C}$ and let

$$\cdots \longrightarrow Z_{i+1} \xrightarrow{z_{i+1}} Z_i \xrightarrow{z_i} Z_{i-1} \longrightarrow \cdots \longrightarrow Z_0 \xrightarrow{z_0} A \longrightarrow 0, \tag{4.3}$$

be a right $\mathcal{X}$-resolution of A. By using this resolution we obtain the diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{C}(Z_0,B) \arrow[r, "z_{1,B}"] \arrow[d, "\phi_{Z_0,B}"'] & \mathcal{C}(Z_1,B) \arrow[r, "z_{2,B}"] \arrow[d, "\phi_{Z_1,B}"'] & \mathcal{C}(Z_2,B) \arrow[r] \arrow[d, "\phi_{Z_2,B}"'] & \cdots \\
0 \arrow[r] & \mathrm{Hom}_\Gamma(\mathcal{G}Z_0,\mathcal{G}B) \arrow[r, "\mathcal{G}z_{1,\mathcal{G}B}"] & \mathrm{Hom}_\Gamma(\mathcal{G}Z_1,\mathcal{G}B) \arrow[r, "\mathcal{G}z_{2,\mathcal{G}B}"] & \mathrm{Hom}_\Gamma(\mathcal{G}Z_2,\mathcal{G}B) \arrow[r] & \cdots
\end{tikzcd}$$

where the rows are complexes and the vertical maps are isomorphisms.

This shows that the i-th right derived functor $\mathcal{E}\mathrm{xt}^i_{\mathcal{X}}(-, B)$ of $\mathcal{C}(-, B)$ is isomorphic to the i-th right derived functor $\mathrm{Ext}^i_\Gamma(\mathcal{G}-, \mathcal{G}B)$ of $\mathrm{Hom}_\Gamma(\mathcal{G}-, \mathcal{G}B)$. □

Remark 4.6. For a given object A of $\mathcal{C}$, we describe explicitly the induced isomorphism ψ from $\mathcal{E}\mathrm{xt}^0_{\mathcal{X}}(A, B) = \mathrm{Ker}\, z_{1,B} = \{g : Z_0 \longrightarrow B : g \circ z_1 = 0\}$ to $\mathrm{Ext}^0_\Gamma(\mathcal{G}A, \mathcal{G}B) = \mathrm{Ker}\, \mathcal{G}z_{1,\mathcal{G}B} = \mathrm{Hom}_\Gamma(\mathcal{G}A, \mathcal{G}B)$ for later use.

For any element $g \in \mathcal{E}\mathrm{xt}^0_{\mathcal{X}}(A, B)$ we have $\mathcal{G}g \circ Gz_1 = 0$. Since $\mathcal{G}z_{0,\mathcal{G}B}$ is the cokernel of $\mathcal{G}z_{1,\mathcal{G}B}$, there is a uniquely determined morphism $m : \mathcal{G}A \longrightarrow \mathcal{G}B$ satisfying $m \circ \mathcal{G}z_{0,\mathcal{G}B} = \mathcal{G}g$. The assignment $g \mapsto m$ defines the isomorphism $\psi : \mathcal{E}\mathrm{xt}^0_{\mathcal{X}}(A, B) \longrightarrow \mathrm{Hom}_\Gamma(\mathcal{G}A, \mathcal{G}B)$.

Our next Proposition relates the right $\mathcal{X}$-dimension of $\mathcal{C}$ with the global dimension of Γ.

Proposition 4.7. *If any morphism between the objects of $\mathcal{X}$ has a weak kernel in $\mathcal{C}$, then*

$$\mathrm{rt.\,dim}_{\mathcal{X}}\mathcal{C} \leq \mathrm{gl.\,dim}\,\Gamma \leq 2 + \mathrm{rt.\,dim}_{\mathcal{X}}\mathcal{C}$$

Proof. Since, any right $\mathcal{X}$-resolution of an object A of $\mathcal{C}$, gives rise to a projective resolution of $\mathcal{G}A$, the left-hand inequality is true.

For the right-hand inequality, it is enough to show that if $\mathcal{G}Y_1 \xrightarrow{f_1} \mathcal{G}Y_0 \xrightarrow{f_0} M$ is a projective presentation of a Γ-module M, then $\mathrm{Ker}\, f_1$ is isomorphic to $\mathcal{G}L$ for some object L of $\mathcal{C}$. Let $g_1 : Y_1 \to Y_0$ be a morphism of $\mathcal{C}$ with $\mathcal{G}g_1 = f_1$. We consider the complex $Y_3 \xrightarrow{g_3} Y_2 \xrightarrow{g_2} Y_1 \xrightarrow{g_1} Y_0$, where $g_i = h_i \circ \ell_{i+1}, i = 1, 2$ with $h_i : L_i \to Y_i$ a weak kernel of g_i and $\ell_{i+1} : Y_{i+1} \to L_i$ a right $\mathcal{X}$-approximation. The exactness of $\mathcal{G}Y_3 \xrightarrow{\mathcal{G}g_3} \mathcal{G}Y_2 \xrightarrow{\mathcal{G}g_2} \mathcal{G}Y_1 \xrightarrow{\mathcal{G}g_1} \mathcal{G}Y_0$ yields $\mathrm{Ker}\,\mathcal{G}g_1 \simeq \mathcal{G}Y_2/\,\mathrm{Im}\,\mathcal{G}g_3$. But $\mathrm{Im}\,\mathcal{G}g_3 \simeq \mathrm{Im}\,\mathcal{G}h_2 \simeq \mathrm{Ker}\,\mathcal{G}\ell_2$ and $\mathcal{G}Y_2/\,\mathrm{Ker}\,\mathcal{G}\ell_2 \simeq \mathrm{Im}\,\mathcal{G}\ell_2 \simeq \mathcal{G}L_1$. □

Corollary 4.8. *If any morphism between objects of $\mathcal{X}$ has a weak kernel in $\mathcal{C}$ and $\mathrm{rt.\,dim}_{\mathcal{X}}\mathcal{C} \leq n$, then the Cartan matrix $C_{\mathcal{X}}$ is $\mathbb{Z}$-invertible*

Proof. The Cartan matrix $C_{\mathcal{X}}$ coincides with the Cartan matrix of Γ, which is $\mathbb{Z}$-invertible because $\mathrm{gl.\,dim}\,\Gamma \leq n + 2$. □

Remark 4.9. Suppose that Λ is a finite dimensional k-algebra of finite representation type and let $\{X_1, X_2, \ldots, X_m\}$ be a complete set of non-isomorphic indecomposable Λ-modules. Then $\mathcal{X} = \mathrm{add}(X_1 \oplus X_2 \oplus \cdots \oplus X_m)$ equals to $\mathrm{mod}\,\Lambda$ and $\mathrm{rt.\,dim}_{\mathcal{X}}\mathrm{mod}\,\Lambda = 0$. In this case, the Lemma above specializes to the well-known fact that the Auslander-algebra of $\mathrm{mod}\,\Lambda$ has global dimension ≤ 2.

5 Indecomposable Objects Determined by their Dimension Vectors

In the present section we keep the same assumptions and notation as in Section 4. Our aim is to study when given an indecomposable object A of $\mathcal{C}$, the Γ-module $\mathcal{G}A$ remains indecomposable. This enables us to develop criteria for when indecomposable objects of $\mathcal{C}$ are determined, up to isomorphism, by their $\mathcal{X}$-dimension vectors linking this question with results developed in representation theory of finite dimensional-algebras.

Let $\mathrm{Gen}\,\mathcal{X}$ be the full subcategory of $\mathcal{C}$ with objects those A of $\mathcal{C}$ for which there is an epimorphism $Z' \to A$ with Z' an object of $\mathcal{X}$. Obviously, any right $\mathcal{X}$-approximation of an object A of $\mathrm{Gen}\,\mathcal{X}$ is an epimorphism.

Let $\mathcal{D}_{\mathcal{X}}$ be the full subcategory of $\mathcal{C}$ formed by the objects A of $\operatorname{Gen}\mathcal{X}$ for which there is a right $\mathcal{X}$-approximation $Z'' \xrightarrow{f} A$ with $\operatorname{Ker} f$ also in $\operatorname{Gen}\mathcal{X}$.

We shall show that under certain assumptions, the indecomposable objects of $\mathcal{C}$ which are in $\mathcal{D}_{\mathcal{X}}$ are determined, up to isomorphism by their $\mathcal{X}$–dimension vectors.

Lemma 5.1. *If any morphism between the objects of $\mathcal{X}$ has a cokernel in $\mathcal{C}$, then the functor $\mathcal{G}$ restricted on $\mathcal{D}_{\mathcal{X}}$ is a fully faithful functor.*

Proof. Let A be an object of $\mathcal{D}_{\mathcal{X}}$. Because of the definition of $\mathcal{D}_{\mathcal{X}}$, there is a right $\mathcal{X}$–presentation $Z_1 \xrightarrow{z_1} Z_0 \xrightarrow{z_0} A \longrightarrow 0$ of A, where z_0 is an epimorphism and $\operatorname{Ker} z_0$ is an object of $\operatorname{Gen}\mathcal{X}$. By Lemma 2.5, $z_1 = f_1 \circ h_0$, where $f_1 : \operatorname{Ker} z_0 \longrightarrow Z_0$ is the kernel of z_0 and $h_0 : Z_1 \longrightarrow \operatorname{Ker} z_0$ is a right $\mathcal{X}$–approximation. The hypothesis of our Lemma implies that z_0 is the cokernel of f_1.

Given an object B of $\mathcal{C}$ we consider the relative cohomology group $\mathcal{E}\mathrm{xt}^0_{\mathcal{X}}(A,B)$. The assignment $\ell \mapsto \ell \circ z_0$ determines a homomorphism $\omega : \mathcal{C}(A,B) \longrightarrow \mathcal{E}\mathrm{xt}^0_{\mathcal{X}}(A,B)$ which is monomorphism because z_0 is an epimorphism. For any $g \in \mathcal{E}\mathrm{xt}^0_{\mathcal{X}}(A,B)$ we have $g \circ z_1 = g \circ f_1 \circ h_0 = 0$. This implies $g \circ f_1 = 0$, since h_0 is an epimorphism. Hence, there is a morphism $\ell : A \longrightarrow B$ with $\ell \circ z_0 = g$ and so ω is an isomorphism.

Let $\psi : \mathcal{E}\mathrm{xt}^0_{\mathcal{X}}(A,B) \longrightarrow \operatorname{Hom}_\Gamma(\mathcal{G}A, \mathcal{G}B)$ be the isomorphism described in Remark 4.6. We evaluate the composition $\psi \circ \omega : \mathcal{C}(A,B) \longrightarrow \operatorname{Hom}_\Gamma(\mathcal{G}A,\mathcal{G}B)$ on an element $\ell \in \mathcal{C}(A,B)$. We have $\psi \circ \omega(\ell) = \psi(\ell \circ z_0) = m$, where $m : \mathcal{G}A \longrightarrow \mathcal{G}B$ is uniquely determined by the property $m \circ \mathcal{G}z_0 = \mathcal{G}\ell \circ \mathcal{G}z_0$. Since $\mathcal{G}z_0$ is an epimorphism, $m = \mathcal{G}\ell$. The proof is completed. $\square$

We recall that given a finite -dimensional k–algebra Γ and a finite–dimensional indecomposable Γ–module M, its dimension vector $\underline{\dim} M$ is defined to be the m-tuple

$$(\dim_k \operatorname{Hom}_\Gamma(P_1, M), \dim_k \operatorname{Hom}_\Gamma(P_2, M), \ldots, \dim_k \operatorname{Hom}_\Gamma(P_m, M)),$$

where $\{P_1, P_2, \ldots, P_m\}$ is a complete set of non-isomorphic indecomposable projective Γ–modules of finite k–dimension.

The fact that $\mathcal{G} : \mathcal{D}_{\mathcal{X}} \longrightarrow \operatorname{mod}\Gamma$ is fully faithful enable us to prove the next Theorem, which is based on the main result of [6].

Theorem 5.2. *Suppose that any morphism between the objects of $\mathcal{X}$ has a cokernel in $\mathcal{C}$. Let A and B be objects of $\mathcal{D}_{\mathcal{X}}$ which are indecomposable as objects of $\mathcal{C}$ with $\underline{\dim}_{\mathcal{X}} A = \underline{\dim}_{\mathcal{X}} B$. If $\mathcal{G}A$ belongs to a postprojective component of the Auslander–Reiten quiver of Γ, then $A \simeq B$.*

Proof. Since the restriction of $\mathcal{G}$ on $\mathcal{D}_{\mathcal{X}}$ is a fully faithful functor, the Γ-modules $\mathcal{G}A$ and $\mathcal{G}B$ are indecomposable. Moreover, for the dimension vectors we have

$$\underline{\dim}\mathcal{G}A = \underline{\dim}_{\mathcal{X}}A = \underline{\dim}_{\mathcal{X}}B = \underline{\dim}\mathcal{G}B$$

By our assumption $\mathcal{G}A$ is in a postprojective component of Γ. Hence, by the main result of [6], the Γ-module $\mathcal{G}A$ is isomorphic to $\mathcal{G}B$. Using again the fact that $\mathcal{G}$ is a fully faithful functor restricted on $\mathcal{D}_{\mathcal{X}}$, we obtain that A is isomorphic to B. □

Suppose that $\mathcal{C}$ is the category $\operatorname{mod}\Lambda$ of finitely generated Λ-modules over a finite-dimensional k-algebra Λ. Let $\{X_1, X_2, \ldots, X_m\}$ be a set of non-isomorphic indecomposable Λ-modules, X be their direct sum and $\mathcal{X}$ be the full subcategory $\operatorname{add} X$. Let Γ be the k-algebra $\operatorname{End}_\Lambda(X)$ and let $\mathcal{G}$ be the functor $\operatorname{Hom}_\Lambda(X, -) : \operatorname{mod}\Lambda \longrightarrow \operatorname{mod}\Gamma$. In this case the following facts can be easily proven:

Remark 5.3. Suppose that A is an object of $\mathcal{D}_{\mathcal{X}}$.

(i) If $f : Y \longrightarrow A$ is a minimal right $\mathcal{X}$-approximation of A, then $\mathcal{G}f : \mathcal{G}Y \longrightarrow \mathcal{G}A$ is a projective cover of $\mathcal{G}A$ and conversely any projective cover of $\mathcal{G}A$ determines a minimal right $\mathcal{X}$-approximation of A.

(ii) If $f : Y \longrightarrow A$ is a minimal right $\mathcal{X}$-approximation of A, then $\operatorname{Ker} f$ is an object of $\operatorname{Gen}\mathcal{X}$.

A right $\mathcal{X}$-presentation $Z_1 \xrightarrow{z_1} Z_0 \xrightarrow{z_0} A$ is said to be a *minimal right $\mathcal{X}$-presentation* if z_0 and $k_1 : Z_1 \longrightarrow \operatorname{Ker} z_0$ are minimal right $\mathcal{X}$-approximations, where $i_0 : \operatorname{Ker} z_0 \longrightarrow Z_0$ is the canonical monomorphism and $z_1 = i_0 \circ k_1$. Our next Theorem is based also on the main result of [6]. We leave for the reader the proof, which can be easily derived using the arguments of the Remark above.

Theorem 5.4. *Let A and B be two indecomposable Λ-modules which belong to $\mathcal{D}_{\mathcal{X}}$. Let $Z_1 \longrightarrow Z_0 \longrightarrow A$ be a minimal right $\mathcal{X}$-presentation of A. If $\mathcal{G}A$ belongs to a postprojective component of the Auslander–Reiten quiver of* $\operatorname{End}_\Lambda(X)$, *then the following are equivalent:*

(i) $A \simeq B$.

(ii) $\underline{\dim}_{\mathcal{X}}A = \underline{\dim}_{\mathcal{X}}B$.

(iii) B has a minimal $\mathcal{X}$-presentation $Z_1 \longrightarrow Z_0 \longrightarrow B$.

Remark 5.5. A more general statement, than that of Lemma 5.1, is true for the category $\mathcal{C}$ and the subcategory $\mathcal{D}_{\mathcal{X}}$:

The functor $\mathcal{G} : \mathcal{C} \longrightarrow \operatorname{mod} \operatorname{End}_{\mathcal{C}}(X)$ has a left adjoint functor $\mathcal{F}$ if and only if any morphism between the objects of $\mathcal{X}$ has a cokernel in $\mathcal{C}$.

Using this statement we are able to detect the subcategory $\mathcal{D}_{\mathcal{X}}$ in the following sense:

An object A of $\mathcal{C}$ is in $\mathcal{D}_{\mathcal{X}}$ if and only if the counit $\tau_A : \mathcal{FG}A \longrightarrow A$ of the adjoint pair $(\mathcal{F}, \mathcal{G})$ is an isomorphism.

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