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Purity and Almost Split Morphisms in Abstract Homotopy Categories: A Unified Approach via Brown Representability

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Abstract. Our aim in this paper is to develop a theory of purity and to prove in a unified conceptual way the existence of almost split morphisms, almost split sequences and almost split triangles in abstract homotopy categories, a rather omnipresent class of categories of interest in representation theory. Our main tool for doing this is the classical Brown representability theorem.

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1. Introduction

Abstract homotopy categories were introduced by E. Brown in the mid-sixties as the proper framework for the study of the homotopy theory of CW-complexes. In this setting, he proved in [18] his celebrated representability theorem, a variant of which has recently found important applications in the stable module category of a modular group algebra and more generally in compactly generated triangulated categories, mainly through the work of Rickard [42] and Neeman [40]. Our main purpose in this paper is to develop a theory of purity and a theory of existence of almost split morphisms in an abstract homotopy category, using the Brown representability theorem as a main catalyzing tool.

This aim is justified by the fact that abstract homotopy categories are omnipresent in representation theory. They include module categories, locally finitely presented categories with products, compactly generated triangulated categories, categories of projective modules, and stable categories modulo projectives over left coherent and right perfect rings, categories of injective modules and stable categories modulo injectives over right Morita rings, (stable) categories of Cohen–Macaulay modules over Gorenstein rings, and many others. Abstract homotopy categories permit a unified treatment of many important phenomena occurring in various forms in the above examples, for instance pure homological algebra and its companion existence theory of almost split morphisms, two of the most fundamental concepts of modern representation theory. Our results which are stated in the setting of an abstract homotopy category, generalize and prove, in a unified conceptual way, most of the basic results concerning purity and almost split morphisms which are known to be true in the setting of the above-mentioned, more concrete and rather unrelated, examples. Moreover, we obtain many new results. We note that our approach was inspired by the recent work of H. Krause [37] in the triangulated case. The organization of the article is as follows.

In Section 2 and following [13], we recall some basic facts concerning homotopy colimits, abstract homotopy categories and Brown representability and we present a large list of examples of algebraic and topological categories which are abstract homotopy categories in a very natural way.

In Section 3, we introduce the class of *homological compact objects* in an abstract homotopy category $\mathcal{C}$. Roughly speaking, these are the objects which behave well with respect to arbitrary coproducts and weak cokernels. Using Brown representability, we associate to any homological compact object X and any maximal left ideal $\mathfrak{m}$ of $\operatorname{End}_{\mathcal{C}}(X)$, the $\mathfrak{m}$-*dual object* $\mathbb{D}_{\mathfrak{m}}(X)$ of X, which has local endomorphism ring and, in a sense, has the dual properties of X. This construction turns out to be fundamental, since it is the main tool for the proof of the main result of this section, which asserts that the $\mathfrak{m}$-dual object $\mathbb{D}_{\mathfrak{m}}(X)$ of any homological compact object X is the source of a left almost split morphism in $\mathcal{C}$. Moreover, if $\mathcal{C}$ has weak kernels, then any homological compact object with local endomorphism ring is the target of a right almost split morphism in $\mathcal{C}$.

In Section 4, we develop a theory of purity in an abstract homotopy category $\mathcal{C}$, which is based on a class of diagrams in $\mathcal{C}$, called *pure-sequences*, defined using a fixed subcategory $\mathcal{X} \subseteq \mathcal{C}$, called a *Whitehead subcategory*, which is part of the structure of $\mathcal{C}$. Using these sequences, we define pure-projective and pure-injective objects in $\mathcal{C}$. The main tools for the study of purity are the *representation categories* $\mathsf{L}(\mathcal{C})$, $\mathsf{D}(\mathcal{C})$ of $\mathcal{C}$. These are Grothendieck categories and, moreover, $\mathsf{L}(\mathcal{C})$ is a functor category and $\mathsf{D}(\mathcal{C})$ is locally coherent, and there are connecting *representation functors* $\mathbf{S}\colon \mathcal{C} \to \mathsf{L}(\mathcal{C})$ and $\mathbf{T}\colon \mathcal{C} \to \mathsf{D}(\mathcal{C})$, which reflect many important pure-theoretic properties of $\mathcal{C}$. Under reasonable conditions, we prove that $\mathcal{C}$ has enough pure-projective objects and pure-injective envelopes and, moreover, the functor $\mathbf{S}$ induces an equivalence between the pure-projective objects of $\mathcal{C}$ and the projective objects of $\mathsf{L}(\mathcal{C})$ and the functor $\mathbf{T}$ induces an equivalence between the pure-injective objects of $\mathcal{C}$ and the injective objects of $\mathsf{D}(\mathcal{C})$. In particular, it follows that any indecomposable pure-injective object in $\mathcal{C}$ has a local endomorphism ring.

The representation functors $\mathbf{S}$, $\mathbf{T}$ are not fully faithful in general. The failure of their fully faithfulness is measured by an ideal of morphisms in $\mathcal{C}$, called the ideal of *phantom maps*, studied in Section 5. The structure of the ideal of phantom maps is strongly connected with the pure homological behavior of $\mathcal{C}$. Indeed, in many cases $\mathcal{C}$ is phantomless iff $\mathcal{C}$ is pure-semisimple. We characterize the pure-semisimplicity, the *finite type* property and the naturally defined *Ziegler spectrum* of $\mathcal{C}$, in terms of properties of the representation categories $\mathsf{L}(\mathcal{C})$, $\mathsf{D}(\mathcal{C})$, and we present a variant of Auslander's correspondence [5] in this setting, which gives a bijective correspondence between Morita equivalence classes of representation-finite rings and equivalence classes of special abstract homotopy categories of finite type.

In Section 6, we study another useful version of Brown representability in an abstract homotopy category and its connection with flat approximations, the latter connection being inspired by a recent result of H. Krause [36] in the triangulated case. In most cases of interest, the representation functor $\mathbf{S}\colon \mathcal{C} \to \mathsf{L}(\mathcal{C})$ has its image in the full subcategory $\mathrm{Flat}(\mathcal{X})$ of flat contravariant functors from the Whitehead subcategory $\mathcal{X}$ mentioned above, to the category of Abelian groups. We give necessary (and in many cases sufficient) conditions such that $\mathbf{S}\colon \mathcal{C} \to \mathrm{Flat}(\mathcal{X})$ is a representation equivalence in the sense of Auslander [3], in terms of projective dimension of flat functors and flat approximations. This has important consequences if $\mathcal{C}$ is triangulated, see [12, 21, 39, 41]. Moreover, we show that the image $\mathrm{Im}\,\mathbf{S}$ of $\mathbf{S}$ is functorially finite in the representation category $\mathsf{L}(\mathcal{C})$, and the image $\mathrm{Im}\,\mathbf{T}$ of $\mathbf{T}$ is functorially finite in the representation category $\mathsf{D}(\mathcal{C})$.

In Section 7, we apply the theory of the previous sections to the most important examples of abstract homotopy categories, namely those of module categories, locally finitely presented categories with products and compactly generated triangulated categories. In all these examples, we recover, in a unified way, the main results about purity and the existence of almost split morphisms, sequences and triangles on these categories, proved by many authors following the fundamental work of M. Auslander and I. Reiten [7]. This section also contains new results. For instance, we prove that a pure-injective object in a compactly generated triangulated category $\mathcal{C}$, occurs as a source of a left almost split morphism in $\mathcal{C}$ iff it is a dual object of a compact object. If the compact objects of $\mathcal{C}$ form a Krull–Schmidt category, then we prove that a pure-injective object is the source of an Auslander–Reiten triangle in $\mathcal{C}$ iff it is the dual object of an indecomposable compact object.

A convention used in the paper is that we compose morphisms $f\colon A \to B$, $g\colon B \to C$ in a given category in the diagrammatic order, i.e. the composition of f, g is denoted by $f \circ g$. There two exceptions: we use the usual anti-diagrammatic order, when we compose functors and when we apply elements to morphisms in concrete categories. Our additive categories admit finite direct sums.

2. Abstract Homotopy Categories and Brown Representability

2.1. WEAK AND HOMOTOPY COLIMITS

Let $\mathcal{C}$ be an additive category and let $f\colon A \to B$ be a morphism in $\mathcal{C}$. We recall that a *weak cokernel* of f is a morphism $h\colon B \to C$ such that $f \circ h = 0$ and if $t\colon B \to D$ is a morphism with $f \circ t = 0$, then there exists a morphism $\alpha\colon C \to D$

such that $t = h \circ \alpha$. Equivalently, the sequence of functors $\mathcal{C}(C, -) \to \mathcal{C}(B, -) \to \mathcal{C}(A, -)$ is exact. The dual notion is *weak kernel*. We say that a diagram $A_0 \xrightarrow{f_0} A_1 \xrightarrow{f_1} A_2 \to \cdots$ in $\mathcal{C}$ is a *weak cokernel sequence* if f_{i+1} is a weak cokernel of f_i, for all $i \geqslant 0$.

Let $\mathcal{I}\colon I \to \mathcal{C}$ be a functor from a small category I. We use the notation

$$\mathcal{I}(i) = A_i \quad \text{and} \quad \mathcal{I}(i \to j) := \alpha_{ij}\colon A_i \to A_j.$$

A *weak colimit* of the functor $\mathcal{I}$ is an object A in $\mathcal{C}$ together with morphisms $f_i\colon A_i \to A$, which are compatible with the system $\{A_i, \alpha_{ij}\}$ in the sense that for any arrow $i \to j$ in I, we have $f_i = \alpha_{ij} \circ f_j$, and if $g_i\colon A_i \to B$ is another compatible family, then there exists a (not necessarily unique) morphism $\omega\colon A \to B$ such that $f_i \circ \omega = g_i$, for all $i \in I$. Hence, a weak colimit of $\mathcal{I}$, henceforth denoted by $\text{w.}\varinjlim A_i$, is defined as a genuine colimit except for the uniqueness property. In particular a weak colimit is not uniquely determined.

2.2. ABSTRACT HOMOTOPY CATEGORIES AND BROWN REPRESENTABILITY

Fix an additive category $\mathcal{C}$ with coproducts and weak cokernels. Then it is easy to see that $\mathcal{C}$ has weak colimits. We are especially interested in weak colimits of towers of objects, where a tower in $\mathcal{C}$ is a diagram of the form $A_0 \xrightarrow{f_0} A_1 \xrightarrow{f_1} A_2 \to \cdots$. We call a weak colimit of a tower, a *homotopy colimit* and we denote it by $\underrightarrow{\text{holim}}\, A_i$. A homotopy colimit of the tower above can be computed as a weak cokernel of the canonical morphism $\bigoplus_{i\geqslant 0} A_i \xrightarrow{1-f} \bigoplus_{i\geqslant 0} A_i$, induced by the morphisms $(1_{A_i}, -f_i)\colon A_i \to A_i \oplus A_{i+1} \hookrightarrow \bigoplus_{i\geqslant 0} A_i$. Hence, we have a weak cokernel sequence

$$\bigoplus_{i\geqslant 0} A_i \xrightarrow{1-f} \bigoplus_{i\geqslant 0} A_i \longrightarrow \underrightarrow{\text{holim}}\, A_i.$$

DEFINITION 2.1. Let $\mathcal{X} \subseteq \mathcal{C}$ be a full subcategory of $\mathcal{C}$. A weak colimit $\text{w.}\varinjlim A_i$ in $\mathcal{C}$ is called $\mathcal{X}$-*minimal*, if for any object $X \in \mathcal{X}$ the canonical morphism

$$\varinjlim \mathcal{C}(X, A_i) \longrightarrow \mathcal{C}(X, \text{w.}\varinjlim A_i)$$

is an isomorphism. We say that $\mathcal{X}$ is a *minimal subcategory* if every tower $A_0 \to A_1 \to A_2 \to \cdots$ in $\mathcal{C}$ has an $\mathcal{X}$-minimal homotopy colimit.

To proceed further we need the following concept borrowed from topology.

DEFINITION 2.2. A full subcategory $\mathcal{X} \subseteq \mathcal{C}$ is called a *Whitehead subcategory*, if $\mathcal{X}$ is skeletally small and a morphism $f\colon A \to B$ in $\mathcal{C}$ is invertible if $\mathcal{C}(X, f)\colon \mathcal{C}(X, A) \to \mathcal{C}(X, B)$ is invertible, for all $X \in \mathcal{X}$.

EXAMPLE 2.3. (α) Let $\mathcal{C}$ be the homotopy category of pointed connected CW-complexes and let $\mathcal{X} \subseteq \mathcal{C}$ be the full subcategory consisting of finite complexes. Then by a well-known result of J. H. C. Whitehead, $\mathcal{X}$ is a Whitehead subcategory of $\mathcal{C}$. This is of course the origin of the terminology [26].

(β) Let $\mathcal{C}$ be a locally small additive category with filtered colimits and let $\mathcal{X} \subseteq \mathcal{C}$ be a skeletally small full subcategory such that any object of $\mathcal{C}$ can be expressed as a filtered colimit of objects from $\mathcal{X}$. Then $\mathcal{X}$ is a Whitehead subcategory of $\mathcal{C}$ [38].

If $\mathcal{X}$ is a Whitehead subcategory of $\mathcal{C}$, then it is not difficult to see that any two $\mathcal{X}$-minimal weak colimits are isomorphic (noncanonically). Moreover if the colimit exists in $\mathcal{C}$, then it is isomorphic to the $\mathcal{X}$-minimal weak colimit.

DEFINITION 2.4 ([13]). An *abstract homotopy category* is an additive category which admits coproducts, weak cokernels and a minimal Whitehead subcategory.

Note that in general a minimal Whitehead subcategory $\mathcal{X} \subseteq \mathcal{C}$ is not uniquely determined. In many aspects the theory depends on the choice of $\mathcal{X}$.

DEFINITION 2.5. An additive functor $F\colon \mathcal{C}^{\mathrm{op}} \to \mathcal{A}b$ is called *half-exact* iff F sends a weak cokernel sequence $A \to B \to C$ in $\mathcal{C}$ to an exact sequence $F(C) \to F(B) \to F(A)$ in $\mathcal{A}b$. An additive functor $F\colon \mathcal{C} \to \mathcal{A}b$ is called half-exact, if F sends a weak cokernel sequence as above to an exact sequence $F(A) \to F(B) \to F(C)$ in $\mathcal{A}b$.

For any object $A \in \mathcal{C}$, the representable functor $\mathcal{C}(-, A)\colon \mathcal{C}^{\mathrm{op}} \to \mathcal{A}b$ is half-exact and sends coproducts to products. The following fundamental Brown representability theorem for abstract homotopy categories, asserts that the converse is true. This result is of central importance in what follows.

THEOREM 2.6 (Brown) ([18, 26, 13]). *Let $\mathcal{C}$ be an abstract homotopy category and let $F\colon \mathcal{C}^{\mathrm{op}} \to \mathcal{A}b$ be an additive functor. Then the following are equivalent.*

(α) *F is representable;*
(β) *F is half-exact and sends coproducts to products.*

Brown Representability has many consequences. First we note the following.

COROLLARY 2.7 ([13]). *An abstract homotopy category has arbitrary products.*

We recall that an object X in an additive category $\mathcal{C}$ is called *compact*, if the functor $\mathcal{C}(X, -)\colon \mathcal{C} \to \mathcal{A}b$ preserves all small coproducts. The full subcategory of $\mathcal{C}$ consisting of all compact objects is denoted by $\mathcal{C}^{\mathrm{b}}$. A full subcategory $\mathcal{X}$ of $\mathcal{C}$ is called a *compact subcategory* if $\mathcal{X} \subseteq \mathcal{C}^{\mathrm{b}}$. A useful consequence of Brown representability is the following adjoint functor theorem.

THEOREM 2.8 ([13]). *Let $\mathcal{C}$ be an abstract homotopy category and let $F\colon \mathcal{C} \to \mathcal{D}$ be an additive functor to an additive category $\mathcal{D}$. Then the following are equivalent.*

(α) *F admits a right adjoint $G\colon \mathcal{D} \to \mathcal{C}$;*
(β) *F preserves coproducts and weak cokernels.*

In case G exists and $\mathcal{C}$ admits a compact minimal Whitehead subcategory $\mathcal{X}$, then: G preserves coproducts iff F preserves compact objects.

Recall that an object X in an additive category $\mathcal{C}$ is called *finitely presented*, if the representable functor $\mathcal{C}(X, -)\colon \mathcal{C} \to \mathcal{A}b$ commutes with filtered colimits. The full subcategory of finitely presented objects of $\mathcal{C}$ is denoted by $\mathrm{f.p}(\mathcal{C})$. $\mathcal{C}$ is called *locally finitely presented* [17], if $\mathcal{C}$ has filtered colimits, $\mathrm{f.p}(\mathcal{C})$ is skeletally small and any object of $\mathcal{C}$ is a filtered colimit of finitely presented objects. If $\mathcal{C}$ is skeletally small, we denote by $\mathrm{Mod}(\mathcal{C})$ the category of additive functors $\mathcal{C}^{\mathrm{op}} \to \mathcal{A}b$. There is a well-defined tensor product functor $- \otimes_{\mathcal{C}} -\colon \mathrm{Mod}(\mathcal{C}) \times \mathrm{Mod}(\mathcal{C}^{\mathrm{op}}) \to \mathcal{A}b$, which satisfies all the usual properties. A functor F in $\mathrm{Mod}(\mathcal{C})$ is called *flat*, if $F \otimes_{\mathcal{C}} -\colon \mathrm{Mod}(\mathcal{C}^{\mathrm{op}}) \to \mathcal{A}b$ is exact. The full subcategory of $\mathrm{Mod}(\mathcal{C})$ consisting of all flat functors is denoted by $\mathrm{Flat}(\mathcal{C})$. By [17], $\mathrm{Flat}(\mathcal{C})$ is locally finitely presented and any locally finitely presented additive category arises in this way.

2.3. EXAMPLES

Although the primitive example of an abstract homotopy category in the non-additive setting is the homotopy category of CW-complexes, there is a host of interesting examples in the additive case.

(i) If Λ is a ring, then the category $\mathrm{Mod}(\Lambda)$ of right Λ-modules is an abstract homotopy category with minimal Whitehead subcategory, the full subcategory $\mathrm{mod}(\Lambda)$ of finitely presented modules, or the full subcategory $\mathcal{P}_\Lambda$ of finitely generated projective modules. More generally if $\mathcal{C}$ is skeletally small, then the functor category $\mathrm{Mod}(\mathcal{C})$ is an abstract homotopy category.
(ii) Let $\mathcal{F}$ be a locally finitely presented additive category with products. Then by [13], $\mathcal{F}$ is an abstract homotopy category with minimal Whitehead subcategory $\mathrm{f.p}(\mathcal{F})$. In particular for any skeletally small additive category $\mathcal{C}$ with weak cokernels, $\mathrm{Flat}(\mathcal{C})$ is an abstract homotopy category.
(iii) Let $\mathcal{C}$ be a compactly generated triangulated category [40]. Then $\mathcal{C}$ is an abstract homotopy category with minimal Whitehead subcategory, the full subcategory $\mathcal{C}^{\mathrm{b}}$ of compact objects [13]. More generally, let $\mathcal{C}$ be a right triangulated category with coproducts and suspension functor which is a right semi-equivalence in the sense of [2]. If $\mathcal{C}^{\mathrm{b}}$ is skeletally small and for all compact objects X of $\mathcal{C}$, $\mathcal{C}(X, A) = 0$ implies $A = 0$, then $\mathcal{C}$ is an abstract homotopy category. We refer to Section 7 for nontriangulated examples.

(iv) Let Λ be a left coherent and right perfect ring. Then the stable category $\underline{\mathrm{Mod}}(\Lambda)$ of right Λ-modules modulo projectives, is an abstract homotopy category with minimal Whitehead subcategory, the stable category $\underline{\mathrm{mod}}(\Lambda)$ induced by the finitely presented modules. Similarly the category $\mathbf{P}_\Lambda$ of projective modules is an abstract homotopy category with minimal Whitehead subcategory, the full subcategory of finitely generated projective modules [13].

(v) Let Λ be a right Morita ring, i.e. Λ is right Artinian and $\mathrm{Mod}(\Lambda)$ admits a finitely generated injective cogenerator. Then the stable category $\overline{\mathrm{Mod}}(\Lambda)$ of right Λ-modules modulo injectives, is an abstract homotopy category with minimal Whitehead subcategory, the stable category $\overline{\mathrm{mod}}(\Lambda)$ induced by the finitely generated modules. Similarly the category $\mathbf{I}_\Lambda$ of injective right Λ-modules is an abstract homotopy category with minimal Whitehead subcategory, the full subcategory of finitely generated injective modules [13].

One can also add to this list the exactly definable categories and their definable subcategories in the sense of [32]. By (iv) or (v), it follows that the stable category modulo projectives of a quasi-Frobenius ring, is an abstract homotopy category. Note that important examples of quasi-Frobenius rings are the group algebras of finite groups. The example (iii) of a compactly generated triangulated category, shows that we are supplied with many others abstract homotopy categories:

(vi) The stable homotopy category $\mathrm{Ho}(\mathcal{S})$ of spectra [39], is an abstract homotopy category with minimal Whitehead subcategory, the full subcategory $\mathrm{Ho}(\mathcal{S})^{\mathrm{b}}$ of finite spectra.

(vii) The unbounded derived category $\mathbf{D}(\Lambda)$ of all right Λ-modules over a ring Λ, is an abstract homotopy category with minimal Whitehead subcategory, the full subcategory $\mathbf{D}(\Lambda)^{\mathrm{b}}$ of perfect complexes.

(viii) Let H be a commutative Hopf algebra over a field. Then the homotopy category of complexes of injective comodules is an abstract homotopy category with as minimal Whitehead subcategory the full subcategory induced by the injective resolutions of the simple comodules [27]. If H is finite-dimensional, then the stable category of comodules modulo injectives, is an abstract homotopy category with minimal Whitehead subcategory, the stable category induced by the finite dimensional comodules [27].

(ix) We recall [11] that a ring Λ is called *right Gorenstein*, if any projective right module has finite injective dimension and any injective right module has finite projective dimension. Important examples of right (and left) Gorenstein rings include all Noetherian rings of finite selfinjective dimension. The category $\mathrm{CM}(\Lambda)$ of *Cohen–Macaulay modules* over Λ, consists of all modules A such that $A = \mathrm{Ker}(f_0)$, where $\cdots \to P^{-1} \to P^0 \xrightarrow{f_0} P^1 \to \cdots$ is an exact sequence of projective modules which remains exact if we apply to it the functor $(-, P)$, for any projective module P. The dual definition of the category CoCM(Λ) of *CoCohen–Macaulay modules* using injectives, is left to the reader.

If Λ is a left coherent and right perfect right Gorenstein ring, then the projectively stable category $\underline{\mathrm{CM}}(\Lambda)$ is an abstract homotopy category. If Λ is a right Morita right Gorenstein ring, then the injectively stable category $\overline{\mathrm{CoCM}}(\Lambda)$ is an abstract homotopy category. We refer to [11, 13] for details.

Note that any category of the above list admits weak kernels, except possibly in the example (ii) of a locally finitely presented category with products. In what follows when we view a category of the above list as an abstract homotopy category, we will assume tacitly that the choice of a minimal Whitehead subcategory is as described above. We call this choice, the *natural choice*. If $\mathcal{C}$ is an abstract homotopy category with $\mathcal{X} \subseteq \mathcal{C}$ as a minimal Whitehead subcategory, then without any loss of generality, we shall assume always that $\mathcal{X}$ is full additive and closed under direct summands. If $\mathcal{C}$ admits cokernels, we may (and will) replace weak cokernels by cokernels and homotopy or filtered weak colimits by genuine colimits.

3. Almost Split Morphisms

Throughout this section we fix an abstract homotopy category $\mathcal{C}$ and let $\mathcal{X}$ be a fixed minimal Whitehead subcategory of $\mathcal{C}$. In the sequel we follow ideas of M. Auslander [4], W. Crawley-Boevey [17] and H. Krause [37].

3.1. DUAL OBJECTS

We fix throughout a compact object $X \in \mathcal{C}$. We denote by $\Lambda_X := \mathrm{End}_{\mathcal{C}}(X)$ the endomorphism ring of X. For any additive functor $F\colon \mathcal{C}^{\mathrm{op}} \to \mathcal{A}b$, define a left Λ_X-module structure on the Abelian group $F(X)$ as follows. If $\rho \in \Lambda_X$ and $x \in F(X)$, then $\rho \star x := F(\rho)(x)$. In particular, $\mathcal{C}(X, A)$ is a left Λ_X-module for any object $A \in \mathcal{C}$. Let $\mathfrak{m}$ be a maximal left ideal of Λ_X, let $S_{\mathfrak{m}}(X) := \Lambda_X/\mathfrak{m}$ be the corresponding simple left Λ_X-module and let $I_{\mathfrak{m}}$ be its injective envelope. Consider the additive functor

$$H_X := [\mathcal{C}(X, -), I_{\mathfrak{m}}]\colon \mathcal{C}^{\mathrm{op}} \to \mathcal{A}b, \qquad H_X(A) := \mathrm{Hom}_{\Lambda_X}\big(\mathcal{C}(X, A), I_{\mathfrak{m}}\big).$$

Since X is compact, the functor H_X sends coproducts to products. We want this functor to be representable. For this purpose we introduce the following concept.

DEFINITION 3.1. A compact object X is called *homological* if the functor $\mathcal{C}(X, -)\colon \mathcal{C} \to \mathcal{A}b$ is half-exact. The full subcategory of $\mathcal{C}$ consisting of all homological compact objects is denoted by $\mathcal{H}(\mathcal{C}^{\mathrm{b}})$.

It follows that if $X \in \mathcal{H}(\mathcal{C}^{\mathrm{b}})$, then the functor H_X is half-exact and converts coproducts to products. Hence, by Brown representability, there exists an object $\mathbb{D}_{\mathfrak{m}}(X) \in \mathcal{C}$, unique up to isomorphism, and a natural isomorphism:

$$\omega\colon\ [\mathcal{C}(X,-), I_{\mathfrak{m}}] \xrightarrow{\cong} \mathcal{C}(-, \mathbb{D}_{\mathfrak{m}}(X)).$$

From now on we assume that $X \in \mathcal{H}(\mathcal{C}^{\mathrm{b}})$.

LEMMA 3.2. *The object $\mathbb{D}_{\mathfrak{m}}(X)$ has a local endomorphism ring.*

Proof. Using the isomorphism ω, we have:

$$\mathrm{End}_{\mathcal{C}}(\mathbb{D}_{\mathfrak{m}}(X)) = [\mathcal{C}(X, \mathbb{D}_{\mathfrak{m}}(X)), I_{\mathfrak{m}}] \cong [[\mathcal{C}(X, X), I_{\mathfrak{m}}], I_{\mathfrak{m}}] = \mathrm{End}_{\Lambda_X}(I_{\mathfrak{m}}).$$

Since it is the injective envelope of the simple left Λ_X-module $S_{\mathfrak{m}}(X)$, $I_{\mathfrak{m}}$, hence $\mathbb{D}_{\mathfrak{m}}(X)$, has a local endomorphism ring. □

DEFINITION 3.3. The object $\mathbb{D}_{\mathfrak{m}}(X)$ is called the $\mathfrak{m}$-*dual object* of X with respect to the maximal left ideal $\mathfrak{m}$ of Λ_X.

Clearly, the class of $\mathfrak{m}$-dual objects of $\mathcal{C}$ corresponds bijectively with the disjoint union $\bigsqcup\{\mathrm{Max}(\Lambda_X) \mid X \in \mathrm{Iso}(\mathcal{H}(\mathcal{C}^{\mathrm{b}}))\}$, where $\mathrm{Max}(\Lambda_X)$ is the set of maximal left ideals of Λ_X. Here $\mathrm{Iso}(\mathcal{Y})$ denotes the collection of isoclasses of objects of $\mathcal{Y} \subseteq \mathcal{C}$.

Generalizing this construction, let $\rho\colon \Gamma \to \Lambda_X$ be a ring morphism and let I be an injective left Γ-module. As above, for any additive functor $F\colon \mathcal{C}^{\mathrm{op}} \to \mathcal{A}b$, the Abelian group $F(X)$ admits, via the ring morphism ρ, a left Γ-module structure and Brown representability implies that there exists an object $\mathbb{D}_I(X) \in \mathcal{C}$, unique up to isomorphism, equipped with a natural isomorphism:

$$\omega\colon\ \mathrm{Hom}_{\Gamma}[\mathcal{C}(X,-), I] \xrightarrow{\cong} \mathcal{C}(-, \mathbb{D}_I(X)).$$

We call $\mathbb{D}_I(X)$ the I-*dual object* of X with respect to the injective Γ-module I. The notation $\mathbb{D}_I(X) \in \mathcal{C}$ means that a homological compact object X in $\mathcal{C}$ is fixed and I is an injective left Γ-module, where $\rho\colon \Gamma \to \Lambda_X$ is a fixed ring morphism. The isomorphism ω is a powerful tool for producing new interesting objects.

EXAMPLE 3.4. (i) Choose $\Gamma = \mathbb{Z}$ and $I = \mathbb{Q}/\mathbb{Z}$; in this case we use the notation $\mathbb{D}_{\mathbb{Q}/\mathbb{Z}}(X)$. For example if $\mathcal{C} = \mathrm{Mod}(\Lambda)$, then for $X = \Lambda$ and any right Λ-module A, the above isomorphism reduces to: $(A_\Lambda, \mathbb{Q}/\mathbb{Z}) \cong {}_\Lambda(A, \mathbb{D}_{\mathbb{Q}/\mathbb{Z}}(\Lambda))$.

(ii) Choose $\Gamma = \mathbb{Z}$, let p be a prime integer and let I be the injective envelope of the simple $\mathbb{Z}$-module $\mathbb{Z}/(p)$, i.e. I is the Prüfer group $\mathbb{Z}_{p^\infty}$. Then the above isomorphism produces the p-*dual object* $\mathbb{D}_p(X)$, with endomorphism ring, the commutative ring $\mathrm{End}_{\mathbb{Z}}(\mathbb{Z}_{p^\infty})$ of p-adic integers. Choosing $I = \mathbb{Q}$, we have the *rational dual object* $\mathbb{D}_{\mathbb{Q}}(X)$ of X, with endomorphism ring $\mathbb{Q}$.

(iii) If $\mathrm{Ho}(\mathcal{S})$ is the stable homotopy category of spectra and X is a finite spectrum, then $\mathbb{D}_{\mathbb{Q}/\mathbb{Z}}(X)$ is the Brown–Comenetz dual of the Spanier–Whitehead dual of X [19]. The p-dual object $\mathbb{D}_p(X)$ is a p-local spectrum [39], for any prime p. If S^n is the nth-suspension of the sphere spectrum S^0, then let $\pi_n(A) := \mathrm{Ho}(\mathcal{S})(S^n, A)$ be the stable homotopy groups of A. Then

$$(\pi_n(A), \mathbb{Z}_{p^\infty}) \cong \mathrm{Ho}(\mathcal{S})(A, \mathbb{D}_p(S^n)), \qquad (\pi_n(A), \mathbb{Q}) \cong \mathrm{Ho}(\mathcal{S})(A, \mathbb{D}_{\mathbb{Q}}(S^n))$$

and

$$\pi_n(\mathbb{D}_{\mathbb{Q}/\mathbb{Z}}(S^n)) = \mathbb{Q}/\mathbb{Z}, \quad \forall n \geqslant 0.$$

We recall that an additive functor $F\colon \mathcal{C}^{\mathrm{op}} \to \mathcal{A}b$ is called *finitely presented* if there exists an exact sequence of functors $\mathcal{C}(-, A) \to \mathcal{C}(-, B) \to F \to 0$. Let $\mathrm{mod}(\mathcal{C})$ be the category of finitely presented contravariant additive functors. It is well-known that $\mathrm{mod}(\mathcal{C})$ is Abelian iff $\mathcal{C}$ admits weak kernels, see [6].

LEMMA 3.5. *If $\mathbb{D}_I(X)$ is an I-dual object of X, then for every object $F \in$* $\mathrm{mod}(\mathcal{C})$*:*

$$\mathrm{Hom}_\Gamma(F(X), I) \xrightarrow{\cong} [F, \mathcal{C}(-, \mathbb{D}_I(X))].$$

If $\mathcal{C}$ has weak kernels, then $\mathcal{C}(-, \mathbb{D}_I(X))$ is an injective object in $\mathrm{mod}(\mathcal{C})$.

Proof. Choose a presentation $\mathcal{C}(-, A) \to \mathcal{C}(-, B) \to F \to 0$ of F. Then $\mathcal{C}(X, A) \to \mathcal{C}(X, B) \to F(X) \to 0$ is exact in $\mathrm{Mod}(\Gamma^{\mathrm{op}})$. Hence, the sequence $0 \to [F(X), I] \to [\mathcal{C}(X, B), I] \to [\mathcal{C}(X, A), I]$ is exact. Using the isomorphism ω, it follows that $[F(X), I]$ is isomorphic to the kernel of $\mathcal{C}(B, \mathbb{D}_I(X)) \to \mathcal{C}(A, \mathbb{D}_I(X))$ which is isomorphic to $[F, \mathcal{C}(-, \mathbb{D}_I(X))]$. If $\mathcal{C}$ has weak kernels, then $\mathrm{mod}(\mathcal{C})$ is Abelian and the above isomorphisms show that $\mathcal{C}(-, \mathbb{D}_I(X))$ is an injective object in $\mathrm{mod}(\mathcal{C})$. □

We denote by $\mathbf{S}\colon \mathcal{C} \to \mathrm{Mod}(\mathcal{X})$, the restricted Yoneda functor defined by $\mathbf{S}(A) = \mathcal{C}(-, A)|_{\mathcal{X}}$. In particular $\mathbf{S}|_{\mathcal{X}}$ is the Yoneda embedding $\mathcal{X} \hookrightarrow \mathrm{Mod}(\mathcal{X})$.

LEMMA 3.6. *For any $F \in$* $\mathrm{Mod}(\mathcal{X})$ *there exists an isomorphism:*

$$\mathrm{Hom}_\Gamma(F(X), I) \xrightarrow{\cong} [F, \mathbf{S}(\mathbb{D}_I(X))].$$

In particular, $\mathbf{S}(\mathbb{D}_I(X))$ *is an injective object in* $\mathrm{Mod}(\mathcal{X})$.

Proof. First let F be finitely presented, and choose a presentation $\mathcal{X}(-, Y_1) \to \mathcal{X}(-, Y_0) \to F \to 0$. By Yoneda's lemma the exact sequence

$$0 \to [F, \mathbf{S}(\mathbb{D}_I(X))] \to [\mathcal{X}(-, Y_0), \mathbf{S}(\mathbb{D}_I(X))] \to [\mathcal{X}(-, Y_1), \mathbf{S}(\mathbb{D}_I(X))]$$

is isomorphic to the sequence

$$0 \to [F, \mathbf{S}(\mathbb{D}_I(X))] \to \mathcal{C}(Y_0, \mathbb{D}_I(X)) \to \mathcal{C}(Y_1, \mathbb{D}_I(X)).$$

As in the proof of Lemma 3.5, the kernel of the last morphism is isomorphic via ω to $\mathrm{Hom}_\Gamma(F(X), I)$. Now let $F \in \mathrm{Mod}(\mathcal{X})$ be arbitrary. It is well known that F is a filtered colimit $\varinjlim F_i$ of finitely presented functors. Then

$$\begin{aligned}\mathrm{Hom}_\Gamma(F(X), I) &= \mathrm{Hom}_\Gamma(\varinjlim F_i(X), I) = \varprojlim \mathrm{Hom}_\Gamma(F_i(X), I)\\ &\cong \varprojlim [F_i, \mathbf{S}(\mathbb{D}_I(X))] \cong [\varinjlim F_i, \mathbf{S}(\mathbb{D}_I(X))]\\ &= [F, \mathbf{S}(\mathbb{D}_I(X))].\end{aligned}$$

The injectivity of $\mathbf{S}(\mathbb{D}_I(X))$ in $\mathrm{Mod}(\mathcal{X})$ follows trivially from these isomorphisms. □

COROLLARY 3.7. *For every object $A \in \mathcal{C}$, the functor* $\mathbf{S}$ *induces a canonical isomorphism*

$$\mathbf{S}_{A,\mathbb{D}_I(X)}\colon \mathcal{C}(A, \mathbb{D}_I(X)) \xrightarrow{\cong} [\mathbf{S}(A), \mathbf{S}(\mathbb{D}_I(X))], \qquad f \mapsto \mathbf{S}(f).$$

Proof. Choosing $F = \mathbf{S}(A)$ in Lemma 3.6, we infer that we have an isomorphism $\mathrm{Hom}_\Gamma(\mathcal{C}(X, A), I) \cong [\mathbf{S}(A), \mathbf{S}(\mathbb{D}_I(X))]$. Clearly, the composite isomorphism

$$\mathcal{C}(A, \mathbb{D}_I(X)) \xrightarrow{\omega_X^{-1}} \mathrm{Hom}_\Gamma(\mathcal{C}(X, A), I) \to [\mathbf{S}(A), \mathbf{S}(\mathbb{D}_I(X))]$$

coincides with $\mathbf{S}_{A,\mathbb{D}_I(X)}$. □

If $\mathcal{Y}$ is a class of objects of $\mathcal{C}$, then $\mathrm{Add}(\mathcal{Y})$, resp. $\mathrm{add}(\mathcal{Y})$, resp. $\mathrm{Prod}(\mathcal{Y})$, denotes the full subcategory of $\mathcal{C}$ consisting of all direct summands of arbitrary coproducts, resp. finite coproducts, resp. products, of objects of $\mathcal{Y}$.

COROLLARY 3.8. *If $\mathcal{X} \subseteq \mathcal{H}(\mathcal{C}^{\mathrm{b}})$, then $\mathbb{D}_{\mathbb{Q}/\mathbb{Z}}(\mathcal{X}) = \{\mathbb{D}_{\mathbb{Q}/\mathbb{Z}}(X); X \in \mathcal{X}\}$ is a 'coWhitehead' subcategory of $\mathcal{C}$, i.e. $\mathbb{D}_{\mathbb{Q}/\mathbb{Z}}(\mathcal{X})$ is skeletally small and $f\colon A \to B$ is invertible in $\mathcal{C}$ iff $\mathcal{C}(f, \mathbb{D}_{\mathbb{Q}/\mathbb{Z}}(X))$ is invertible, for all $X \in \mathcal{X}$. Moreover $E := \prod_{X\in \mathrm{Iso}(\mathcal{X})} \mathbb{D}_{\mathbb{Q}/\mathbb{Z}}(X)$ is a cogenerator of $\mathcal{C}$ and $\mathbf{S}(E)$ is an injective cogenerator of* $\mathrm{Mod}(\mathcal{X})$.

For instance, if $\mathcal{C} = \mathrm{Mod}(\Lambda)$ and $\mathcal{X} = \mathcal{P}_\Lambda$, then we recover the well-known fact that $\mathbb{D}_{\mathbb{Q}/\mathbb{Z}}(\Lambda) = \mathrm{Hom}_{\mathbb{Z}}(\Lambda, \mathbb{Q}/\mathbb{Z})$ is an injective cogenerator for $\mathrm{Mod}(\Lambda)$.

Assume from now on that the homological compact object X is in $\mathcal{X}$ and consider a maximal left ideal $\mathfrak{m}$ of Λ_X. It is not difficult to see that defining

$$\mathcal{X}(-, X)_{\mathfrak{m}}(Y) := \{\phi \in \mathcal{X}(Y, X) \mid \forall X \xrightarrow{\theta} Y : \theta \circ \phi \in \mathfrak{m}\},$$

we obtain a right ideal in $\mathcal{X}$, i.e. an additive subfunctor $\mathcal{X}(-, X)_{\mathfrak{m}} \hookrightarrow \mathcal{X}(-, X)$. The corresponding quotient functor is denoted by $S_{X,\mathfrak{m}} := \mathcal{X}(-, X)/\mathcal{X}(-, X)_{\mathfrak{m}}$. By [4] the functor $S_{X,\mathfrak{m}}$ is simple and any simple functor in $\mathrm{Mod}(\mathcal{X})$ arises in this way.

Consider the evaluation

$$\omega_X\colon [\mathcal{C}(X,X), I_\mathfrak{m}] = \mathrm{Hom}_{\Lambda_X}(\Lambda_X, I_\mathfrak{m}) \xrightarrow{\cong} \mathcal{C}(X, \mathbb{D}_\mathfrak{m}(X))$$

of the isomorphism ω at X. Let $\pi\colon \Lambda_X \twoheadrightarrow \Lambda_X/\mathfrak{m}$ be the canonical projection and let $\mu\colon \Lambda_X/\mathfrak{m} \rightarrowtail I_\mathfrak{m}$ be the canonical inclusion. We denote the image of the composition $\pi \circ \mu$ under ω_X by h_X. Hence, $h_X\colon X \to \mathbb{D}_\mathfrak{m}(X)$ is the morphism in $\mathcal{C}$ defined by $h_X := \omega_X(\pi \circ \mu)$. Observe that by construction $h_X \neq 0$.

THEOREM 3.9. (i) *The functor* $\mathbf{S}(\mathbb{D}_\mathfrak{m}(X)) \in \mathrm{Mod}(\mathcal{X})$ *is the injective envelope of the simple functor* $S_{X,\mathfrak{m}}$.

(ii) *If* $\mathcal{X} \subseteq \mathcal{H}(\mathcal{C}^{\mathrm{b}})$ *and* $E \in \mathrm{Mod}(\mathcal{X})$ *is the injective envelope of a simple functor, then there exists* $X \in \mathcal{X}$ *and a maximal left ideal* $\mathfrak{m}$ *of* Λ_X *such that* $E = \mathbf{S}(\mathbb{D}_\mathfrak{m}(X))$.

Proof. (i) By Corollary 3.7, the objects $\mathbb{D}_\mathfrak{m}(X)$, $\mathbf{S}(\mathbb{D}_\mathfrak{m}(X))$ have isomorphic endomorphism rings. Hence, by Lemma 3.2, the ring $\mathrm{End}(\mathbf{S}(\mathbb{D}_\mathfrak{m}(X)))$ is local.

Consider the morphism $\mathbf{S}(h_X)\colon \mathbf{S}(X) \to \mathbf{S}(\mathbb{D}_\mathfrak{m}(X))$ in $\mathrm{Mod}(\mathcal{X})$ and let $\kappa\colon F \to \mathbf{S}(X)$ be its kernel and G its image. We claim that G is the simple functor $S_{X,\mathfrak{m}}$. Indeed, since $\mathbf{S}(X) = \mathcal{X}(-,X)$, it suffices to show that $F = \mathcal{X}(-,X)_\mathfrak{m}$. Let $\alpha\colon A \to X$ be any morphism with $A \in \mathcal{X}$. Then ω induces a commutative diagram:

$$\begin{array}{ccc} H_X(X) & \xrightarrow{\omega_X} & \mathcal{C}(X, \mathbb{D}_\mathfrak{m}(X)) \\ {\scriptstyle H_\mathcal{X}(\alpha)}\downarrow & & \downarrow{\scriptstyle \alpha^*} \\ H_X(A) & \xrightarrow{\omega_A} & \mathcal{C}(A, \mathbb{D}_\mathfrak{m}(X)), \end{array} \qquad (\dagger)$$

where $\alpha^* = \mathcal{C}(\alpha, \mathbb{D}_\mathfrak{m}(X))$. Then $\alpha^*\omega_X(\pi\circ\mu) = \alpha^*(h_X) = \alpha \circ h_X = \mathcal{C}(A, h_X)(\alpha)$. On the other hand, $H_X(\alpha)(\pi \circ \mu)\colon \mathcal{C}(X,A) \to I_\mathfrak{m}$ is the map defined by $H_X(\alpha)(\pi \circ \mu)(\theta) = \pi \circ \mu(\theta \circ \alpha)$. Since ω_A is invertible, by the commutativity of the above diagram, we have $\alpha \in F(A)$ iff $\alpha \circ h_X = 0$ iff for all $\theta\colon X \to A$, we have $\pi \circ \mu(\theta \circ \alpha) = 0$. Obviously this last condition holds iff $\theta \circ \alpha \in \mathfrak{m}$. Hence, $F = \mathcal{X}(-,X)_\mathfrak{m}$ and $G = \mathrm{Im}\,\mathbf{S}(h_X)$ is the simple functor $S_{X,\mathfrak{m}}$. Since $\mathrm{End}_\mathcal{C}(\mathbf{S}(\mathbb{D}_\mathfrak{m}(X)))$ is local, the inclusion $S_{X,\mathfrak{m}} \hookrightarrow \mathbf{S}(\mathbb{D}_\mathfrak{m}(X))$ is an injective envelope.

(ii) Assume that E is the injective envelope of a simple functor S in $\mathrm{Mod}(\mathcal{X})$. Then S is of the form $S_{X,\mathfrak{m}}$ for an object $X \in \mathcal{X}$ and a maximal left ideal $\mathfrak{m}$ of $\Lambda_X = \mathrm{End}_\mathcal{C}(X)$. Since $\mathcal{X} \subseteq \mathcal{H}(\mathcal{C}^{\mathrm{b}})$, we can perform the construction of the $\mathfrak{m}$-dual object $\mathbb{D}_\mathfrak{m}(X)$ of X with respect to $\mathfrak{m}$ and then obviously $E \cong \mathbf{S}(\mathbb{D}_\mathfrak{m}(X))$. □

3.2. ALMOST SPLIT MORPHISMS

We recall that a morphism $f\colon B \to C$ in a given additive category $\mathcal{C}$ is called *right almost split*, if f is not a split epimorphism and any morphism $g\colon A \to C$

which is not a split epimorphism factors through f. The morphism f is called *right minimal*, if any morphism $\alpha\colon B \to B$ such that $\alpha \circ f = f$ is an automorphism. A right minimal right almost split morphism is called a *minimal right almost split* morphism. The dual notions are left almost split morphism, left minimal morphism and minimal left almost split morphism. We refer to [4] for details and more information concerning these morphisms.

Our main result in this section is the following theorem.

THEOREM 3.10. *Let* $0 \neq X \in \mathcal{H}(\mathcal{C}^{\mathrm{b}})$ *and let* $\Lambda_X = \mathrm{End}_{\mathcal{C}}(X)$ *be its endomorphism ring. Then there exists an object* $\mathbb{D}_{\mathfrak{m}}(X)$ *in* $\mathcal{C}$ *with local endomorphism ring and a nonzero morphism* $h_X\colon X \to \mathbb{D}_{\mathfrak{m}}(X)$ *satisfying the following properties:*

(i) *Assume that the endomorphism ring* Λ_X *is local.*

(α) *If* $\alpha\colon A \to X$ *is a nonsplit epimorphism, then* $\alpha \circ h_X = 0$*. Any weak kernel* $f_X\colon A_X \to X$ *of* h_X *in* $\mathcal{C}$ *is a right almost split morphism.*

(β) *If* $\mathcal{C}$ *has weak kernels, then the image* S_X *of the morphism* $\mathcal{C}(-, h_X)\colon \mathcal{C}(-, X) \to \mathcal{C}(-, \mathbb{D}_{\mathfrak{m}}(X))$ *in the Abelian category* mod($\mathcal{C}$) *is a simple functor and the projection* $\mathcal{C}(-, X) \to S_X$ *is a projective cover.*

(ii) (α) *Assume that* $X \in \mathcal{X}$ *or* Λ_X *is local and* $\mathcal{C}$ *admits weak kernels. If* $\beta\colon \mathbb{D}_{\mathfrak{m}}(X) \to B$ *is a nonsplit monomorphism, then* $h_X \circ \beta = 0$*. Any weak cokernel* $g^X\colon \mathbb{D}_{\mathfrak{m}}(X) \to B^X$ *of* h_X *in* $\mathcal{C}$ *is a left almost split morphism.*

(β) *If* $\mathcal{C}$ *has weak kernels, then the inclusion* $S_X \hookrightarrow \mathcal{C}(-, \mathbb{D}_{\mathfrak{m}}(X))$ *is an injective envelope in* mod($\mathcal{C}$).

Proof. Let $\mathfrak{m}$ be a maximal left ideal of Λ_X and consider the nonzero morphism $h_X\colon X \to \mathbb{D}_{\mathfrak{m}}(X)$ constructed in Subsection 3.1. If Λ_X is local, it follows that $\mathfrak{m}$ is the unique maximal left ideal of Λ_X, i.e. $\mathfrak{m} = \mathcal{J}ac(\Lambda_X)$.

(i)(α) Let $\alpha\colon A \to X$ be a morphism in $\mathcal{C}$ which is not a split epimorphism. Consider the commutative diagram (†) in the proof of Theorem 3.9 above, where $\alpha^* = \mathcal{C}(\alpha, \mathbb{D}_{\mathfrak{m}}(X))$. Then we have $\alpha^*\omega_X(\pi \circ \mu) = \alpha^*(h_X) = \alpha \circ h_X$. On the other hand, $H_X(\alpha)(\pi \circ \mu)\colon \mathcal{C}(X, A) \to I_{\mathfrak{m}}$ is the map defined by $H_X(\alpha)(\pi \circ \mu)(\beta) = \pi \circ \mu(\beta \circ \alpha)$. The morphism $\beta \circ \alpha$ is an element of the local ring Λ_X. Since α is nonsplit epic, it follows that $\beta \circ \alpha$ is a noninvertible element of Λ_X. Hence, $\beta \circ \alpha \in \mathcal{J}ac(\Lambda_X)$, or equivalently $\pi(\beta \circ \alpha) = 0$. Hence, $H_X(\alpha)(\pi \circ \mu)(\beta) = 0$. Since β was arbitrary, it follows that $H_X(\alpha)(\pi \circ \mu) = 0$. By the commutativity of the diagram (†), it follows that $\alpha \circ h_X = 0$.

If $f_X\colon A_X \to X$ is a weak kernel of h_X in $\mathcal{C}$, then f_X is nonsplit epic, since $h_X \neq 0$. Let $\alpha\colon A \to X$ be a nonsplit epic. Since $\alpha \circ h_X = 0$, it follows that α factors through f_X. Hence, f_X is a right almost split morphism.

(i) (β) Let $\kappa\colon G \hookrightarrow \mathcal{C}(-, X)$ be the kernel of $\mathcal{C}(-, h_X)\colon \mathcal{C}(-, X) \to \mathcal{C}(-, \mathbb{D}_{\mathfrak{m}}(X))$ in mod($\mathcal{C}$). We claim that G is a proper maximal subfunctor of $\mathcal{C}(-, X)$. Indeed, let $i\colon F \hookrightarrow \mathcal{C}(-, X)$ be a proper subfunctor and let $\mathcal{C}(-, D) \to \mathcal{C}(-, C) \xrightarrow{\epsilon} F \to 0$ be a presentation of F. Then, by the Yoneda lemma, the morphism $\epsilon \circ i\colon \mathcal{C}(-, C) \to \mathcal{C}(-, X)$ is of the form $\mathcal{C}(-, \alpha)$, where $\alpha\colon C \to X$. Since F is a proper subfunctor of $\mathcal{C}(-, X)$, it follows easily that α is nonsplit epic. Hence, by (i)(α), we have $\alpha \circ h_X = 0$. Then $\mathcal{C}(-, \alpha) \circ \mathcal{C}(-, h_X) = 0$ and this implies that $i \circ \mathcal{C}(-, h_X) = 0$. Then i factors through κ and this means that G contains F as a subfunctor. Hence, G is a maximal proper subfunctor of $\mathcal{C}(-, X)$. Consequently, $S_X := \operatorname{Im} \mathcal{C}(-, h_X)$ is a simple functor in $\operatorname{mod}(\mathcal{C})$ and then obviously the projection $\mathcal{C}(-, X) \to S_X$ is a projective cover.

(ii)(α) If $X \in \mathcal{X}$, then by Theorem 3.9, $\mathbf{S}(\mathbb{D}_{\mathrm{m}}(X))$ is the injective envelope of the simple image $S_{X,\mathrm{m}}$ of the morphism $\mathbf{S}(h_X)$. Let $\beta\colon \mathbb{D}_{\mathrm{m}}(X) \to B$ be a morphism in $\mathcal{C}$ which is not a split monomorphism. Then $\mathbf{S}(\beta)$ is not a monomorphism. Indeed otherwise $\mathbf{S}(\beta)$ will be a split monomorphism, since $\mathbf{S}(\mathbb{D}_{\mathrm{m}}(X))$ is injective. Using Corollary 3.7, it is easy to see that this implies that β is a split monomorphism and this is not true. Hence, $\mathbf{S}(\beta)$ is not a monomorphism. Since $S_{X,\mathrm{m}} \hookrightarrow \mathbf{S}(\mathbb{D}_{\mathrm{m}}(X))$ is an injective envelope, this implies that the composition $S_{X,\mathrm{m}} \hookrightarrow \mathbf{S}(\mathbb{D}_{\mathrm{m}}(X)) \to \mathbf{S}(B)$ is zero. Then $\mathbf{S}(h_X) \circ \mathbf{S}(\beta) = 0$. Since $X \in \mathcal{X}$, it follows that $h_X \circ \beta = 0$. If Λ_X is local and $\mathcal{C}$ admits weak kernels, then the assertion follows directly from the fact that $\mathcal{C}(-, h_X)$ has simple image and $\mathcal{C}(-, \mathbb{D}_{\mathrm{m}}(X))$ is injective in $\operatorname{mod}(\mathcal{C})$.

Let $g^X\colon \mathbb{D}_{\mathrm{m}}(X) \to B^X$ be a weak cokernel of h_X in $\mathcal{C}$. Then g^X is not a split monomorphism, since $h_X \neq 0$. If $\beta\colon \mathbb{D}_{\mathrm{m}}(X) \to B$ is not a split monomorphism, then by the above arguments we have that $h_X \circ \beta = 0$, hence β factors through g^X. This shows that g^X is a left almost split morphism.

(ii)(β) Since $\mathbb{D}_{\mathrm{m}}(X)$ has local endomorphism ring, it follows by Yoneda that the functor $\mathcal{C}(-, \mathbb{D}_{\mathrm{m}}(X))$ is an indecomposable injective object in $\operatorname{mod}(\mathcal{C})$. Hence, the inclusion $\mu\colon S_X \hookrightarrow \mathcal{C}(-, \mathbb{D}_{\mathrm{m}}(X))$ is an injective envelope. □

COROLLARY 3.11. *Let $\mathcal{C}$ be an abstract homotopy category with weak kernels. Then for any object $X \in \mathcal{H}(\mathcal{C}^{\mathrm{b}})$ with local endomorphism ring, there exists a right almost split morphism $f_X\colon A_X \to X$ and a left almost split morphism $g^X\colon \mathbb{D}_{\mathrm{m}}(X) \to B^X$.*

COROLLARY 3.12. *If $\mathcal{C}$ admits kernels and cokernels, then for any homological compact object X with local endomorphism ring, there exists a minimal right almost split morphism $A_X \to X$ and a minimal left almost split morphism $\mathbb{D}_{\mathrm{m}}(X) \to B^X$.*

Proof. Choose f_X to be the kernel and g^X to be the cokernel of h_X. □

The next result relates the existence of almost split morphisms in $\mathcal{C}$ with properties of projective or injective functors in $\operatorname{Mod}(\mathcal{X})$.

THEOREM 3.13. *Assume that $\mathcal{C}$ has weak kernels and $\mathcal{X} \subseteq \mathcal{H}(\mathcal{C}^{\mathrm{b}})$.*

(i) *If E is an object of $\mathcal{C}$, then for the following statements:*

(α) $\mathbf{S}(E)$ *is the injective envelope of a simple functor in* $\operatorname{Mod}(\mathcal{X})$;

(β) $E \cong \mathbb{D}_{\mathfrak{m}}(X)$, *where* $X \in \mathcal{X}$ *and* $\mathfrak{m}$ *is a maximal left ideal of* Λ_X;
(γ) *the functor* $\mathbf{S}(E)$ *is injective in* $\mathrm{Mod}(\mathcal{X})$ *and there exists a left almost split morphism* $E \to B$ *in* $\mathcal{C}$,

we have (α) $\Leftrightarrow$ (β) $\Rightarrow$ (γ). *If, in addition, any injective object of* $\mathrm{Mod}(\mathcal{X})$ *is in the image of* $\mathbf{S}$ *and if for injective objects* $\mathbf{S}(I), \mathbf{S}(J)$, *the canonical morphism* $\mathcal{C}(I, J) \to (\mathbf{S}(I), \mathbf{S}(J))$ *is surjective, then we have also* (γ) $\Rightarrow$ (α).

(ii) *If* P *is an object in* $\mathcal{C}$, *then the following are equivalent:*

(α) $\mathbf{S}(P)$ *is the projective cover of a simple functor in* $\mathrm{Mod}(\mathcal{X})$;
(β) $P \in \mathcal{X}$ *and* P *has a local endomorphism ring*;
(γ) *The functor* $\mathbf{S}(P)$ *is projective in* $\mathrm{Mod}(\mathcal{X})$ *and there exists a right almost split morphism* $A \to P$ *in* $\mathcal{C}$.

Proof. (i) By Theorem 3.9 we have (α) $\Leftrightarrow$ (β) and by Theorem 3.10(ii)(α) we have (β) $\Rightarrow$ (γ). Assume now that the additonal hypotheses are true and let $g\colon E \to B$ be a left almost split morphism in $\mathcal{C}$, where $\mathbf{S}(E)$ is an injective functor. Consider the morphism $\mathbf{S}(g)\colon \mathbf{S}(E) \to \mathbf{S}(B)$ and let $\mu\colon S \to \mathbf{S}(E)$ be its kernel. It suffices to show that S is a simple functor. Let $\alpha\colon S \to G$ be any morphism in $\mathrm{Mod}(\mathcal{X})$ and let $\nu\colon G \to E(G)$ be an injective envelope. Since $E(G)$ is injective, there exists a morphism $\sigma\colon \mathbf{S}(E) \to E(G)$ such that $\alpha \circ \nu = \mu \circ \sigma$. By hypothesis, there exists an object $I \in \mathcal{C}$ such that $\mathbf{S}(I) = E(G)$ and a morphism $\sigma'\colon E \to I$ such that $\mathbf{S}(\sigma') = \sigma$. If σ is a split monomorphism, then obviously α is a monomorphism. If σ is not a split monomorphism, then obviously the same is true for σ'. Since g is left almost split, there exists $\tau\colon B \to I$ such that $\sigma' = g \circ \tau$. Then $\sigma = \mathbf{S}(g) \circ \mathbf{S}(\tau)$ and this implies that $\mu \circ \sigma = \alpha \circ \nu = 0$. Then $\alpha = 0$, since ν is a monomorphism. Since any morphism $\alpha\colon S \to G$ is zero or a monomorphism, S is a simple functor.

(ii) We use tacitly the fact that since $\mathcal{X}$ is compact, the functor $\mathbf{S}$ induces an equivalence between $\mathrm{Add}(\mathcal{X})$ and the category of projective objects in $\mathrm{Mod}(\mathcal{X})$ [12]. (α) $\Rightarrow$ (β) Since simple functors are finitely generated, it follows that $\mathbf{S}(P)$ is finitely generated, i.e. $P \in \mathcal{X}$. Then it is easy to see that $\mathrm{End}_{\mathcal{C}}(P)$ is local. (β) $\Rightarrow$ (γ) This follows by Corollary 3.11. (γ) $\Rightarrow$ (α) Since P is the target of a right almost split morphism, by [10], $\mathrm{End}_{\mathcal{C}}(P)$ is local. Since $\mathbf{S}(P)$ is projective, it is easy to see that $\mathbf{S}(P)$ is finitely generated, i.e. $P \in \mathcal{X}$. Then the projection $\mathcal{X}(-, P) \to S_{P,\mathfrak{m}}$ is a projective cover, where $\mathfrak{m}$ is the unique maximal ideal of $\mathrm{End}_{\mathcal{C}}(P)$. □

Our last result in this section indicates a useful factorization property.

COROLLARY 3.14. *Let* $X \in \mathcal{H}(\mathcal{C}^{\mathrm{b}})$, *let* $\rho\colon \Gamma \to \mathrm{End}_{\mathcal{C}}(X)$ *be a ring morphism and let* I *be an injective cogenerator of* $\mathrm{Mod}(\Gamma^{\mathrm{op}})$. *If* $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} D$ *is a weak cokernel sequence in* $\mathcal{C}$. *Then any morphism* $X \to B$ *factors through* f *iff any morphism* $C \to \mathbb{D}_I(X)$ *factors through* h.

Proof. Since $X \in \mathcal{H}(\mathcal{C}^{\mathrm{b}})$, the sequence

$$\mathcal{C}(X, A) \overset{f_*}{\to} \mathcal{C}(X, B) \overset{g_*}{\to} \mathcal{C}(X, C) \overset{h_*}{\to} \mathcal{C}(X, D)$$

is exact in $\mathrm{Mod}(\Gamma^{\mathrm{op}})$, where $f_* = \mathcal{C}(X, f)$ and so on. Then any morphism $X \to B$ factors through f iff $g_* = 0$. Since I is injective, using the isomorphism ω it follows that the sequence $\mathcal{C}(D, \mathbb{D}_I(X)) \xrightarrow{h^*} \mathcal{C}(C, \mathbb{D}_I(X)) \xrightarrow{g^*} \mathcal{C}(B, \mathbb{D}_I(X)) \xrightarrow{f^*} \mathcal{C}(A, \mathbb{D}_I(X))$ is exact in $\mathrm{Mod}(\Gamma^{\mathrm{op}})$, where $h^* := \mathcal{C}(h, \mathbb{D}_I(X))$, and so on. Then any morphism $C \to \mathbb{D}_I(X)$ factors through h iff $g^* = 0$. But g^* is isomorphic to the morphism (g_*, I). Since I is a cogenerator, we have $g_* = 0$ iff $g^* = 0$. □

4. Purity in Abstract Homotopy Categories

Throughout this section we fix an abstract homotopy category $\mathcal{C}$ with a fixed minimal Whitehead subcategory $\mathcal{X} \subseteq \mathcal{C}$. We assume always that $\mathcal{X}$ is compact, i.e. $\mathcal{X} \subseteq \mathcal{C}^{\mathrm{b}}$, and $\mathcal{X}$ has weak cokernels and the inclusion $\mathcal{X} \hookrightarrow \mathcal{C}$ preserves them.

4.1. PURITY

A *sequence* in $\mathcal{C}$ is a diagram (E): $A \overset{g}{\to} B \overset{f}{\to} C$ such that $g \circ f = 0$. A sequence (E) is called *pure* if for any $X \in \mathcal{X}$, the induced sequence $0 \to \mathcal{C}(X, A) \to \mathcal{C}(X, B) \to \mathcal{C}(X, C) \to 0$ is exact in $\mathcal{A}b$. The class of pure sequences is denoted by $\mathcal{E}$. The terminology 'pure' will be justified in Section 7.

We call an object $P \in \mathcal{C}$ *pure-projective*, if for any pure-sequence $A \overset{g}{\to} B \overset{f}{\to} C$, the induced sequence

$$0 \to \mathcal{C}(P, A) \to \mathcal{C}(P, B) \to \mathcal{C}(P, C) \to 0$$

is exact in $\mathcal{A}b$. The full subcategory of $\mathcal{C}$ consisting of all pure-projectives is denoted by $\mathcal{P}(\mathcal{E})$. We say that $\mathcal{C}$ has *enough pure-projectives*, if for every $C \in \mathcal{C}$ there exists a pure-sequence $A \to P \to C$, with $P \in \mathcal{P}(\mathcal{E})$. Such a sequence is called a *pure-projective presentation* of C. A *pure-projective cover* of C, is a right minimal morphism f included in a pure-projective presentation $A \to P \overset{f}{\to} C$ in $\mathcal{C}$. We say that $\mathcal{C}$ has pure-projective covers, if any object of $\mathcal{C}$ admits a pure-projective cover. We leave to the reader to formulate the dual notions concerning pure-injectivity. The full subcategory of pure-injective objects is denoted by $\mathcal{I}(\mathcal{E})$. Finally we call an abstract homotopy category $\mathcal{C}$ *pure-semisimple*, if any pure-sequence $A \to B \to C$ in $\mathcal{C}$ splits, i.e. if it is isomorphic to the sequence $A \xrightarrow{(1_A, 0)} A \oplus C \xrightarrow{{}^{t}(0, 1_C)} C$.

4.2. REPRESENTATION CATEGORIES

Since by definition $\mathcal{X}$ is skeletally small, we can consider the category $\mathrm{Mod}(\mathcal{X})$, resp. $\mathrm{Mod}(\mathcal{X}^{\mathrm{op}})$, of contravariant, resp. covariant, additive functors from $\mathcal{X}$ to $\mathcal{A}b$.

We consider also the category $\mathrm{mod}(\mathcal{X}^{\mathrm{op}})$ of finitely presented covariant functors and we set $\mathcal{B}(\mathcal{X}) := \mathrm{mod}(\mathcal{X}^{\mathrm{op}})^{\mathrm{op}}$.

Since $\mathcal{X}$ has weak cokernels and split idempotents, it follows that $\mathcal{B}(\mathcal{X})$ is a skeletally small Abelian category with enough injectives and any injective is of the form $\mathbb{Y}(X)$, where $\mathbb{Y}\colon \mathcal{X} \hookrightarrow \mathcal{B}(\mathcal{X})$ is the (covariant) Yoneda embedding. Consider the category $\mathrm{Flat}(\mathcal{B}(\mathcal{X}))$ of flat functors $\mathcal{B}(\mathcal{X})^{\mathrm{op}} \to \mathcal{A}b$ and set:

$$\mathsf{L}(\mathcal{C}) := \mathrm{Mod}(\mathcal{X}) \quad \text{and} \quad \mathsf{D}(\mathcal{C}) := \mathrm{Flat}(\mathcal{B}(\mathcal{X})).$$

In other words $\mathsf{D}(\mathcal{C})$ is the *conjugate* category $\widehat{\mathrm{Mod}(\mathcal{X}^{\mathrm{op}}})$ of $\mathrm{Mod}(\mathcal{X}^{\mathrm{op}})$ in the sense of J. E. Roos [43]. We identify $\mathcal{B}(\mathcal{X})$ with the full subcategory of finitely presented objects of $\mathsf{D}(\mathcal{C})$, via the embedding $M \mapsto (-, M)$, and usually we view the objects of $\mathcal{B}(\mathcal{X})$ as finitely presented functors $\mathcal{X} \to \mathcal{A}b$.

We recall that a locally finitely presented Grothendieck category $\mathcal{G}$ is called *locally coherent*, if the full subcategory $\mathrm{f.p}(\mathcal{G})$ of finitely presented objects of $\mathcal{G}$ is Abelian. It follows that $\mathsf{D}(\mathcal{C})$ is locally coherent and can be identified with the category of left exact functors $\mathcal{B}(\mathcal{X})^{\mathrm{op}} \to \mathcal{A}b$.

As in Section 3, let $\mathbf{S}\colon \mathcal{C} \to \mathsf{L}(\mathcal{C})$ and let $\mathbf{S}(A) = \mathcal{C}(-, A)|_{\mathcal{X}}$ be the restricted Yoneda functor. Let $\mathbf{i}\colon \mathrm{Flat}(\mathcal{B}(\mathcal{X})) \hookrightarrow \mathrm{Mod}(\mathcal{B}(\mathcal{X}))$ be the inclusion functor. By a well-known result of Gabriel [23], $\mathbf{i}$ admits an exact left adjoint $\Theta\colon \mathrm{Mod}(\mathcal{B}(\mathcal{X})) \to \mathrm{Flat}(\mathcal{B}(\mathcal{X}))$. On the other hand the Yoneda embedding $\mathbb{Y}\colon \mathcal{X} \hookrightarrow \mathcal{B}(\mathcal{X})$ induces a fully faithful right exact functor $\mathbb{Y}^!\colon \mathrm{Mod}(\mathcal{X}) \hookrightarrow \mathrm{Mod}(\mathcal{B}(\mathcal{X}))$ which is a left adjoint (given by the Kan construction) of the induced restriction functor $\mathbb{Y}_*\colon \mathrm{Mod}(\mathcal{B}(\mathcal{X})) \to \mathrm{Mod}(\mathcal{X})$, which is defined by $\mathbb{Y}_*(F) = F\mathbb{Y}$. It is well known that $\mathbb{Y}^!$ is defined as follows: $\mathbb{Y}^!(F)(M) = F \otimes_{\mathcal{X}} \mathcal{B}(\mathcal{X})[M, \mathbb{Y}(-)]$. In particular, $\mathbb{Y}^!$ preserves flatness, so it induces a fully faithful functor $\mathbb{Y}^!\colon \mathrm{Flat}(\mathcal{X}) \to \mathsf{D}(\mathcal{C})$. Then we have adjoint pairs $(\mathbb{Y}^!, \mathbb{Y}_*)\colon \mathsf{L}(\mathcal{C}) \to \mathrm{Mod}(\mathcal{B}(\mathcal{X}))$ and $(\Theta, \mathbf{i})\colon \mathrm{Mod}(\mathcal{B}(\mathcal{X})) \to \mathsf{D}(\mathcal{C})$. We denote by $\mathbf{T}\colon \mathcal{C} \to \mathsf{D}(\mathcal{C})$ the composite functor

$$\mathbf{T} := \Theta\mathbb{Y}^!\mathbf{S}\colon \mathcal{C} \xrightarrow{\mathbf{S}} \mathsf{L}(\mathcal{C}) \xrightarrow{\mathbb{Y}^!} \mathrm{Mod}(\mathcal{B}(\mathcal{X})) \xrightarrow{\Theta} \mathsf{D}(\mathcal{C}).$$

We recall that an object M in a locally coherent category $\mathcal{G}$ is called *FP-injective*, if $\mathcal{E}xt^1_{\mathcal{G}}(F, M) = 0$, for any finitely presented object $F \in \mathcal{G}$. The full subcategory of FP-injective objects of $\mathcal{G}$ is denoted by $\mathrm{FP\,Inj}\,\mathcal{G}$. If $\mathcal{A}$ is a skeletally small Abelian category, then $\mathcal{E}x(\mathcal{A}^{\mathrm{op}}, \mathcal{A}b)$ denotes the category of exact functors $\mathcal{A}^{\mathrm{op}} \to \mathcal{A}b$.

The following result is basic in what follows.

LEMMA 4.1. *The image of* $\mathbf{S}$ *lies in the full subcategory* $\mathrm{Flat}\,\mathsf{L}(\mathcal{C})$ *of flat functors of* $\mathsf{L}(\mathcal{C})$ *and the functor* $\mathbb{Y}^!$ *induces an equivalence*

$$\mathrm{Flat}\,\mathsf{L}(\mathcal{C}) \xrightarrow{\approx} \mathcal{E}x(\mathcal{B}(\mathcal{X})^{\mathrm{op}}, \mathcal{A}b).$$

The functor $\mathbf{T}$ *is isomorphic to* $\mathbb{Y}^!\mathbf{S}$ *and there is an identification*

$$\mathcal{E}x(\mathcal{B}(\mathcal{X})^{\mathrm{op}}, \mathcal{A}b) = \mathrm{FP\,Inj}\,\mathsf{D}(\mathcal{C}).$$

In particular $\mathbf{T}$ *induces a functor* $\mathbf{T}\colon \mathcal{C} \to \mathrm{FP\,Inj}\,\mathsf{D}(\mathcal{C}) \hookrightarrow \mathsf{D}(\mathcal{C})$.

Proof. Since $\mathcal{X}$ has weak cokernels, it is easy to see [14] that a functor $F \in \mathrm{Mod}(\mathcal{X})$ is flat iff for any weak cokernel sequence $X \to Y \to Z$ in $\mathcal{X}$, the sequence $F(Z) \to F(Y) \to F(X)$ is exact, i.e. F is half-exact. Then for all $A \in \mathcal{C}$, the functor $\mathbf{S}(A) = \mathcal{C}(-, A)|_{\mathcal{X}}$ is flat since it is half-exact and $\mathcal{X}$ is closed under weak cokernels in $\mathcal{C}$. For a proof that the functor $\mathbb{Y}^!$: $\mathrm{Flat}\,\mathsf{L}(\mathcal{C}) \to \mathcal{E}x(\mathcal{B}(\mathcal{X})^{\mathrm{op}}, \mathcal{A}b)$ is an equivalence we refer to [14]. Since $\mathrm{Im}\,\mathbf{S}$ consists of flat functors and $\mathbb{Y}^!$ preserves flatness, from the definition of $\mathbf{T}$ it follows directly that $\mathbf{T}$ is isomorphic to $\mathbb{Y}^!\mathbf{S}$. The last assertion is well known. We include a proof for completeness sake. If E: $\mathcal{B}(\mathcal{X})^{\mathrm{op}} \to \mathcal{A}b$ is exact, then there exists a flat functor F: $\mathcal{X}^{\mathrm{op}} \to \mathcal{A}b$ such that $E = \mathbb{Y}^!(F)$. Write $F = \varinjlim \mathbf{S}(X_i)$ as a filtered colimit of representable functors over $\mathcal{X}$. Since $\mathbb{Y}^!$ commutes with filtered colimits, $E = \varinjlim \mathbb{Y}^!\mathbf{S}(X_i)$. By the definition of $\mathbb{Y}^!$ it follows that $\mathbb{Y}^!\mathbf{S}(X_i) = (-, \mathbb{Y}(X_i))$, which obviously is an FP-injective object in $\mathsf{D}(\mathcal{C})$. Then E is FP-injective, since by [46] a filtered colimit of FP-injective objects in a locally coherent category is again FP-injective. Conversely if E is FP-injective, let $0 \to M_1 \to M_2 \to M_3 \to 0$ be an exact sequence in $\mathcal{B}(\mathcal{X})$. By the construction of cokernels in $\mathsf{D}(\mathcal{C})$, see [23], it follows that $0 \to (-, M_1) \to (-, M_2) \to (-, M_3) \to 0$ is exact in $\mathsf{D}(\mathcal{C})$. Since E is FP-injective and the $(-, M_i)$ are finitely presented, the sequence $0 \to ((-, M_3), E) \to ((-, M_2), E) \to ((-, M_1), E) \to 0$ is exact, so its isomorphic copy $0 \to E(M_3) \to E(M_2) \to E(M_1) \to 0$ is exact, i.e. E is exact. □

The following is a direct consequence of the above lemma.

COROLLARY 4.2. *If* $0 \to F_1 \to F_2 \to F_3 \to 0$ *is an exact sequence of flat functors in* $\mathsf{L}(\mathcal{C})$*, then the sequence* $0 \to \mathbb{Y}^!(F_1) \to \mathbb{Y}^!(F_2) \to \mathbb{Y}^!(F_3) \to 0$ *is exact in* $\mathsf{D}(\mathcal{C})$.

From now on we identify the functors $\mathbf{T}, \mathbb{Y}^!\mathbf{S}$: $\mathcal{C} \to \mathsf{D}(\mathcal{C})$.

Remark 4.3. For all $A \in \mathcal{C}$, for all $X \in \mathcal{X}$, and all $M \in \mathcal{B}(\mathcal{X})$:

$$\mathbf{T}(A)(\mathbb{Y}(X)) = \mathbf{S}(A) \otimes_{\mathcal{X}} [\mathbb{Y}(X), \mathbb{Y}(-)] = \mathbf{S}(A) \otimes_{\mathcal{X}} (X, -) \cong \mathcal{C}(X, A),$$
$$\mathbf{T}(X)(M) = \mathbf{S}(X) \otimes_{\mathcal{X}} [M, \mathbb{Y}(-)] = [M, \mathbb{Y}(X)] = M(X).$$

Remark 4.4. It is easy to see that $\mathbf{S}$ preserves products. Using that $\mathcal{X}$ consists of compact objects it follows that $\mathbf{S}$ preserves coproducts. Since $\mathbb{Y}^!$ is a left adjoint, it follows that $\mathbf{T}$ preserves coproducts. Since a product of FP-injective objects is again FP-injective, it follows directly by Lemma 4.1 that $\mathbf{T}$ preserves products.

We call $\mathsf{L}(\mathcal{C}), \mathsf{D}(\mathcal{C})$, the *representation categories* of $\mathcal{C}$, with respect to the fixed Whitehead subcategory $\mathcal{X}$. The categories $\mathsf{L}(\mathcal{C}), \mathsf{D}(\mathcal{C})$ are Grothendieck abstract homotopy categories and together with the connecting *representation functors* $\mathbf{S}$: $\mathcal{C} \to \mathsf{L}(\mathcal{C})$, $\mathbf{T}$: $\mathcal{C} \to \mathsf{D}(\mathcal{C})$, they are very useful for the study of purity in $\mathcal{C}$.

Our aim in this section is, using the representation categories, to prove that an abstract homotopy category $\mathcal{C}$ satisfying some additional conditions, has enough pure-projectives and pure-injective envelopes. The usefullness of the representation categories in the study of purity, emerges from the following result.

LEMMA 4.5. *Let* (E): $A \to B \to C$ *be a sequence in* $\mathcal{C}$. *Then* (E) *is pure iff the sequence* $0 \to \mathbf{S}(A) \to \mathbf{S}(B) \to \mathbf{S}(C) \to 0$ *is short exact in* $\mathsf{L}(\mathcal{C})$ *iff the sequence* $0 \to \mathbf{T}(A) \to \mathbf{T}(B) \to \mathbf{T}(C) \to 0$ *is short exact in* $\mathsf{D}(\mathcal{C})$.

Proof. The first assertion is clear. If (E) is pure, then $0 \to \mathbf{S}(A) \to \mathbf{S}(B) \to \mathbf{S}(C) \to 0$ is an exact sequence of flat functors in $\mathsf{L}(\mathcal{C})$. Since $\mathbf{T} = \mathbb{Y}^{!}\mathbf{S}$, by Corollary 4.2, the sequence $0 \to \mathbf{T}(A) \to \mathbf{T}(B) \to \mathbf{T}(C) \to 0$ is exact in $\mathsf{D}(\mathcal{C})$. Conversely, if the last sequence is exact, then since its terms are FP-injective objects and for all $X \in \mathcal{X}$, the object $\mathbf{T}(X) = (-, \mathbb{Y}(X))$ is finitely presented, the sequence

$$0 \to (\mathbf{T}(X), \mathbf{T}(A)) \to (\mathbf{T}(X), \mathbf{T}(B)) \to (\mathbf{T}(X), \mathbf{T}(C)) \to 0$$

is exact. Obviously this sequence is isomorphic to the sequence

$$0 \to (\mathbf{S}(X), \mathbf{S}(A)) \to (\mathbf{S}(X), \mathbf{S}(B)) \to (\mathbf{S}(X), \mathbf{S}(C)) \to 0,$$

or, equivalently, to the sequence

$$0 \to \mathcal{C}(X, A) \to \mathcal{C}(X, B) \to \mathcal{C}(X, C) \to 0.$$

Hence, the sequence (E) is pure. □

To proceed further in an efficient way we need to impose an axiom. First it is convenient to introduce some terminology.

DEFINITION 4.6. A morphism g: $B \to C$ in $\mathcal{C}$ is called *pure-epic*, if $\mathbf{S}(f)$ is an epimorphism in $\mathsf{L}(\mathcal{C})$. A morphism f: $A \to B$ in $\mathcal{C}$ is called *pure-monic*, if $\mathbf{T}(f)$ is a monomorphism in $\mathsf{D}(\mathcal{C})$.

AXIOM 4.7. Any pure-epic $B \to C$, resp. pure-monic $A \to B$, can be included in a pure-sequence $A \to B \to C$ in $\mathcal{C}$.

From now on, we assume throughout that Axiom 4.7 holds in $\mathcal{C}$.

4.3. PURE-PROJECTIVES

The following result shows the existence of enough pure-projective objects in $\mathcal{C}$ and describes their structure.

PROPOSITION 4.8. $\mathcal{C}$ *has enough pure-projectives and* $\mathcal{P}(\mathcal{E}) = \mathrm{Add}(\mathcal{X})$. *Moreover, for all* $P \in \mathcal{P}(\mathcal{E})$ *and* $A \in \mathcal{C}$, *the canonical morphism* $\mathcal{C}(P, A) \to [\mathbf{S}(P), \mathbf{S}(A)]$ *is an isomorphism and the functor* $\mathbf{S}$: $\mathcal{C} \to \mathsf{L}(\mathcal{C})$ *induces an equivalence* $\mathbf{S}$: $\mathcal{P}(\mathcal{E}) \xrightarrow{\approx} \mathrm{Proj}\,\mathsf{L}(\mathcal{C})$.

Proof. Clearly $\mathrm{Add}(\mathcal{X}) \subseteq \mathcal{P}(\mathcal{E})$. Let C be an object in $\mathcal{C}$. Then the set of morphisms $I_C := \{X \to C \mid X \in \mathrm{Iso}(\mathcal{X})\}$ induces a morphism $g_C\colon \bigoplus_{a \in I_C} X_a \to C$. By construction the morphism $\mathcal{C}(X, g_C)$ is an epimorphism for all $X \in \mathcal{X}$; equivalently $\mathbf{S}(g_C)$ is an epimorphism and then g_C is pure-epic. By Axiom 4.7, there exists a pure-sequence $A \to \bigoplus_{a \in I_C} X_a \xrightarrow{g_C} C$ in $\mathcal{C}$. Since $\bigoplus_{a \in I_C} X_a \in \mathcal{P}(\mathcal{E})$, it follows that $\mathcal{C}$ has enough pure-projectives. If $C \in \mathcal{P}(\mathcal{E})$, then the above pure-sequence splits, hence $C \in \mathrm{Add}(\mathcal{X})$ as a direct summand of $\bigoplus_{a \in I_C} X_a$. We conclude that $\mathcal{P}(\mathcal{E}) = \mathrm{Add}(\mathcal{X})$. Now let $A \in \mathcal{C}$ and $X \in \mathcal{X}$. Then obviously the canonical morphism $\mathbf{S}_{X,A}\colon \mathcal{C}(X, A) \to [\mathbf{S}(X), \mathbf{S}(A)]$ is invertible. Since $\mathbf{S}$ preserves coproducts, this isomorphism can be extended to an isomorphism $\mathbf{S}_{T,A}\colon \mathcal{C}(T, A) \to [\mathbf{S}(T), \mathbf{S}(A)]$, for any T being a coproduct of objects of $\mathcal{X}$. From this it follows that $\mathbf{S}_{P,A}$ is an isomorphism for any $P \in \mathcal{P}(\mathcal{E}) = \mathrm{Add}(\mathcal{X})$. Finally, since $\mathcal{X}$ is compact, it is trivial to see that the functor $\mathbf{S}\colon \mathrm{Add}(\mathcal{X}) \to \mathrm{Proj}\,\mathsf{L}(\mathcal{C})$ is an equivalence [12] and the last assertion follows. □

We recall that a functor category $\mathrm{Mod}(\mathcal{Y})$ is called *perfect*, if any functor has a projective cover, equivalently if any flat functor is projective.

COROLLARY 4.9. *$\mathcal{C}$ is pure-semisimple iff any object of $\mathcal{C}$ has a pure-projective cover iff the category $\mathsf{L}(\mathcal{C})$ is perfect. In this case the functor $\mathbf{S}$ induces an equivalence $\mathbf{S}\colon \mathcal{C} \xrightarrow{\approx} \mathrm{Flat}\,\mathsf{L}(\mathcal{C}) = \mathrm{Proj}\,\mathsf{L}(\mathcal{C})$.*

Proof. If $\mathcal{C}$ is pure-semisimple then by Proposition 4.8, $\mathcal{C} = \mathrm{Add}(\mathcal{X})$. By [13], the functor category $\mathsf{L}(\mathcal{C})$ is perfect and $\mathbf{S}\colon \mathcal{C} \xrightarrow{\approx} \mathrm{Flat}\,\mathsf{L}(\mathcal{C}) = \mathrm{Proj}\,\mathsf{L}(\mathcal{C})$ is an equivalence. Trivially, $\mathcal{C}$ then has pure-projective covers. Conversely if $\mathcal{C}$ has projective covers then for all $A \in \mathcal{C}$, let $B \to P \xrightarrow{f} A$ be a sequence with f a pure-projective cover. Using Proposition 4.8, we see easily that $\mathbf{S}(f)\colon \mathbf{S}(P) \to \mathbf{S}(A)$ is a projective cover. It is well known that the only flat functors admitting projective covers are the projective ones. Since $\mathbf{S}(A)$ is flat, $\mathbf{S}(A)$ is projective or equivalently A is pure-projective. This implies that $\mathcal{C}$ is pure-semisimple. It remains to show that $\mathcal{C}$ is pure-semisimple, if $\mathsf{L}(\mathcal{C})$ is perfect. But for all $A \in \mathcal{C}$, $\mathbf{S}(A)$ is flat, hence projective. It follows that A is pure-projective and then $\mathcal{C}$ is pure-semisimple. □

4.4. PURE-INJECTIVES

We would like to know under what conditions the abstract homotopy category $\mathcal{C}$ admits pure-injective envelopes.

LEMMA 4.10. *For all $A \in \mathcal{C}$ and for all $I \in \mathcal{I}(\mathcal{E})$, the canonical morphism $\mathbf{S}_{A,I}\colon \mathcal{C}(A, I) \to \mathsf{L}(\mathcal{C})[\mathbf{S}(A), \mathbf{S}(I)]$ is an isomorphism. In particular the functors $\mathbf{S}\colon \mathcal{I}(\mathcal{E}) \to \mathsf{L}(\mathcal{C})$ and $\mathbf{T}\colon \mathcal{I}(\mathcal{E}) \to \mathsf{D}(\mathcal{C})$ are fully faithful.*

Proof. Let A, I be in $\mathcal{C}$ with I be a pure-injective object. By Proposition 4.8, we can choose pure-projective presentations $B \to P_0 \to A$ and $C \to P_1 \to B$.

Then we have an exact sequence $\mathbf{S}(P_1) \to \mathbf{S}(P_0) \to \mathbf{S}(A) \to 0$ in $\mathsf{L}(\mathcal{C})$. Since I is pure-injective, we have an exact sequence $0 \to \mathcal{C}(A, I) \to \mathcal{C}(P_0, I) \to \mathcal{C}(P_1, I)$ and an exact sequence

$$0 \to [\mathbf{S}(A), \mathbf{S}(I)] \to [\mathbf{S}(P_0), \mathbf{S}(I)] \to [\mathbf{S}(P_1), \mathbf{S}(I)].$$

Then we have the following exact commutative diagram:

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathcal{C}(A, I) & \longrightarrow & \mathcal{C}(P_0, I) & \longrightarrow & \mathcal{C}(P_1, I) \\ & & \downarrow{\scriptstyle \mathbf{S}_{A,I}} & & \downarrow{\scriptstyle \mathbf{S}_{P_0,I}} & & \downarrow{\scriptstyle \mathbf{S}_{P_1,I}} \\ 0 & \longrightarrow & [\mathbf{S}(A), \mathbf{S}(I)] & \longrightarrow & [\mathbf{S}(P_0), \mathbf{S}(I)] & \longrightarrow & [\mathbf{S}(P_1), \mathbf{S}(I)] \end{array}$$

Since $\mathbf{S}_{P,A}$ is invertible, for all $P \in \mathcal{P}(\mathcal{E})$, it follows directly that the canonical morphism $\mathbf{S}_{A,I}$ is invertible. Hence, $\mathbf{S}\colon \mathcal{I}(\mathcal{E}) \to \mathsf{L}(\mathcal{C})$ is fully faithful. Since $\mathbb{Y}^!$ is fully faithful, it follows that the functor $\mathbf{T} = \mathbb{Y}^!\mathbf{S}\colon \mathcal{I}(\mathcal{E}) \to \mathsf{D}(\mathcal{C})$ is fully faithful. □

Viewing $\mathrm{Flat}\,\mathsf{L}(\mathcal{C})$ as an abstract homotopy category with minimal Whitehead subcategory $\mathbb{Y}(\mathcal{X})$, it follows easily that a sequence $F_1 \to F_2 \to F_3$ in $\mathrm{Flat}\,\mathsf{L}(\mathcal{C})$ is pure iff the sequence $0 \to F_1 \to F_2 \to F_3 \to 0$ is exact in $\mathsf{L}(\mathcal{C})$.

PROPOSITION 4.11. *Any injective object E in $\mathsf{D}(\mathcal{C})$ is of the form $\mathbb{Y}^!(F)$, where F is a pure-injective object in* $\mathrm{Flat}\,\mathsf{L}(\mathcal{C})$.

Proof. Since E is FP-injective, by Lemma 4.1 there exists a flat functor $F \in \mathsf{L}(\mathcal{C})$ such that $\mathbb{Y}^!(F) = E$. Let $F_1 \rightarrowtail F_2 \twoheadrightarrow F_3$ be a pure sequence in $\mathrm{Flat}\,\mathsf{L}(\mathcal{C})$. By Corollary 4.2, $\mathbb{Y}^!(F_1) \rightarrowtail \mathbb{Y}^!(F_2) \twoheadrightarrow \mathbb{Y}^!(F_3)$ is a short exact sequence in $\mathsf{D}(\mathcal{C})$. Since $\mathbb{Y}^!(F)$ is injective,

$$(\mathbb{Y}^!(F_3), \mathbb{Y}^!(F)) \rightarrowtail (\mathbb{Y}^!(F_2), \mathbb{Y}^!(F)) \twoheadrightarrow (\mathbb{Y}^!(F_1), \mathbb{Y}^!(F))$$

is a short exact sequence in $\mathcal{A}b$. Since $\mathbb{Y}^!$ is fully faithful, the sequence $(F_3, F) \rightarrowtail (F_2, F) \twoheadrightarrow (F_1, F)$ is short exact. It follows that F is a pure-injective object in $\mathrm{Flat}\,\mathsf{L}(\mathcal{C})$. □

THEOREM 4.12. *Assume that any pure-injective in* $\mathrm{Flat}\,\mathsf{L}(\mathcal{C})$ *is of the form* $\mathbf{S}(I)$, *for some pure-injective $I \in \mathcal{C}$. Then $\mathcal{C}$ has pure-injective envelopes and the functor* $\mathbf{T}\colon \mathcal{C} \to \mathsf{D}(\mathcal{C})$ *induces an equivalence:* $\mathbf{T}\colon \mathcal{I}(\mathcal{E}) \xrightarrow{\approx} \mathrm{Inj}\,\mathsf{D}(\mathcal{C})$.

Proof. We show that $\mathbf{T}(I)$ is injective in $\mathsf{D}(\mathcal{C})$ for any pure-injective I in $\mathcal{C}$. Let $\mu\colon \mathbf{T}(I) \to E$ be the injective envelope of $\mathbf{T}(I)$ in $\mathsf{D}(\mathcal{C})$. By Proposition 4.11, there exists a pure-injective object F in $\mathrm{Flat}\,\mathsf{L}(\mathcal{C})$, such that $\mathbb{Y}^!(F) = E$. By hypothesis, there exists a pure-injective object $J \in \mathcal{C}$ such that $\mathbf{S}(J) = F$. Since $E = \mathbf{T}(J)$, by Lemma 4.10 the morphism μ is of the form $\mathbf{T}(\alpha)$, where $\alpha\colon I \to J$. Since $\mathbf{T}(\alpha)$ is monic, α is pure-monic. Since I is pure-injective in $\mathcal{C}$, α is split monic. Then obviously $\mathbf{T}(\alpha)$ is invertible, so $\mathbf{T}(I)$ is injective in $\mathsf{D}(\mathcal{C})$. The above argument also shows that if E is an injective object of $\mathsf{D}(\mathcal{C})$, then $E \cong \mathbf{T}(I)$, for a pure-injective object I in $\mathcal{C}$. So by Lemma 4.10, $\mathbf{T}\colon \mathcal{I}(\mathcal{E}) \to \mathrm{Inj}\,\mathsf{D}(\mathcal{C})$ is an equivalence.

Now let A be in $\mathcal{C}$ and let $\mu\colon \mathbf{T}(A) \to \mathbf{T}(I)$ be an injective envelope in $\mathsf{D}(\mathcal{C})$. Since μ is of the form $\mathbf{T}(f)$, for a pure-monic $f\colon A \to I$, by Axiom 4.7, there exists a pure-sequence $A \xrightarrow{f} I \to C$ in $\mathcal{C}$, which is a pure-injective copresentation of A. If $\rho\colon I \to I$ is a morphism in $\mathcal{C}$ such that $f \circ \rho = f$, then $\mathbf{T}(f) \circ \mathbf{T}(\rho) = \mathbf{T}(f)$. Since $\mathbf{T}(f) = \mu$ is an injective envelope, $\mathbf{T}(\rho)$ is an automorphism. Since $\mathbb{Y}^!$ is fully faithful, $\mathbf{S}(\rho)$ is an automorphism. Then ρ is an automorphism, since $\mathbf{S}$ reflects isomorphisms. Hence, $\mathcal{C}$ has pure-injective envelopes. □

If X is a homological compact object, then choosing $\Gamma = \mathbb{Z}$ and $I = \mathbb{Q}/\mathbb{Z}$ in Subsection 3.1, it follows that the dual object $\mathbb{D}_{\mathbb{Q}/\mathbb{Z}}(X)$ of X exists in $\mathcal{C}$ and there exists a natural isomorphism $\omega\colon [\mathcal{C}(X,-), \mathbb{Q}/\mathbb{Z}] \xrightarrow{\cong} \mathcal{C}(-, \mathbb{D}_{\mathbb{Q}/\mathbb{Z}}(X))$.

THEOREM 4.13. *Assume that $\mathcal{X} \subseteq \mathcal{H}(\mathcal{C}^{\mathrm{b}})$ and that $\mathbb{Y}^!$ is exact on exact sequences in $\mathsf{L}(\mathcal{C})$ of the form*

$$\mathbf{S}(A) \xrightarrow{\mathbf{S}(f)} \mathbf{S}(B) \xrightarrow{\mathbf{S}(g)} \mathbf{S}(C).$$

Then we have the following:

(i) *$\mathcal{C}$ has pure-injectives envelopes and $\mathcal{I}(\mathcal{E}) = \mathrm{Prod}(\mathbb{D}_{\mathbb{Q}/\mathbb{Z}}(\mathcal{X}))$.*
(ii) *$\mathbf{T}$ induces an equivalence $\mathbf{T}\colon \mathcal{I}(\mathcal{E}) \xrightarrow{\approx} \mathrm{Inj}\,\mathsf{D}(\mathcal{C})$.*

Proof. The isomorphism ω above shows directly that any dual object of the form $\mathbb{D}_{\mathbb{Q}/\mathbb{Z}}(X)$ with $X \in \mathcal{X}$, is pure-injective. In particular, $\mathrm{Prod}(\mathbb{D}_{\mathbb{Q}/\mathbb{Z}}(\mathcal{X})) \subseteq \mathcal{I}(\mathcal{E})$.

Now let $A \in \mathcal{C}$ and consider the set of morphisms $J_A := \{A \to \mathbb{D}_{\mathbb{Q}/\mathbb{Z}}(X) \mid X \in \mathrm{Iso}(\mathcal{X})\}$. Since $\mathcal{C}$ as an abstract homotopy category, has products, the set of morphisms J_A induces a morphism $f_A\colon A \to \prod_{J_A} \mathbb{D}_{\mathbb{Q}/\mathbb{Z}}(X)$ in $\mathcal{C}$, with the property that $\mathcal{C}(f_A, \mathbb{D}_{\mathbb{Q}/\mathbb{Z}}(X))$ is an epimorphism, for all $X \in \mathcal{X}$. We set $E := \prod_{J_A} \mathbb{D}_{\mathbb{Q}/\mathbb{Z}}(X)$. Then the morphism

$$[\mathcal{C}(X, f_A), \mathbb{Q}/\mathbb{Z}]\colon [\mathcal{C}(X, E), \mathbb{Q}/\mathbb{Z}] \to [\mathcal{C}(X, A), \mathbb{Q}/\mathbb{Z}]$$

is isomorphic via ω to the morphism $\mathcal{C}(f_A, \mathbb{D}_{\mathbb{Q}/\mathbb{Z}}(X))\colon \mathcal{C}(E, \mathbb{D}_{\mathbb{Q}/\mathbb{Z}}(X)) \to \mathcal{C}(A, \mathbb{D}_{\mathbb{Q}/\mathbb{Z}}(X))$. Since $\mathcal{C}(f_A, \mathbb{D}_{\mathbb{Q}/\mathbb{Z}}(X))$ is an epimorphism and $\mathbb{Q}/\mathbb{Z}$ is an injective cogenerator in $\mathcal{A}b$, it follows that for all $X \in \mathcal{X}$ the morphism $\mathcal{C}(X, f_A)\colon \mathcal{C}(X, A) \to \mathcal{C}(X, E)$ is a monomorphism. Hence, $\mathbf{S}(f_A)\colon \mathbf{S}(A) \to \mathbf{S}(E)$ is a monomorphism. By hypothesis, $\mathbf{T}(f_A)$ is also a monomorphism. Then by Axiom 4.7, there exists a pure-sequence $A \to E \to C$ in $\mathcal{C}$, which is a pure-injective copresentation of A. Hence, $\mathcal{C}$ has enough pure-injectives. If A is pure-injective, then the morphism f_A splits, hence A is a direct summand of $E = \prod_{J_A} \mathbb{D}_{\mathbb{Q}/\mathbb{Z}}(X)$. It follows that $\mathcal{I}(\mathcal{E}) = \mathrm{Prod}(\mathbb{D}_{\mathbb{Q}/\mathbb{Z}}(\mathcal{X}))$.

Let E be an injective object in $\mathsf{D}(\mathcal{C})$ and consider the functor $F\colon \mathcal{C}^{\mathrm{op}} \to \mathcal{A}b$ defined by $F(A) = (\mathbf{T}(A), E)$. Since $\mathcal{X} \subseteq \mathcal{H}(\mathcal{C}^{\mathrm{b}})$, $\mathbf{S}$ sends a weak cokernel sequence $A \to B \to C$ in $\mathcal{C}$ to the exact sequence $\mathbf{S}(A) \to \mathbf{S}(B) \to \mathbf{S}(C)$ in $\mathsf{L}(\mathcal{C})$. By hypothesis on $\mathbb{Y}^!$, the sequence $\mathbf{T}(A) \to \mathbf{T}(B) \to \mathbf{T}(C)$ is exact in $\mathsf{D}(\mathcal{C})$ and since E is injective in $\mathsf{D}(\mathcal{C})$, the functor F is half-exact. Plainly F sends coproducts to products, hence by Brown representability, there exists an isomorphism:

$$\xi\colon\ F = (\mathbf{T}(-), E) \xrightarrow{\cong} \mathcal{C}(-, I_E).$$

Since $\mathbf{T}$ sends pure-sequences in $\mathcal{C}$ to short exact sequences in $\mathsf{D}(\mathcal{C})$ and since E is injective in $\mathsf{D}(\mathcal{C})$, it follows that I_E is pure-injective. Then $\xi_{|\mathcal{X}}\colon (\mathbf{T}(-), E)|_{\mathcal{X}} \cong \mathbf{S}(I_E)$ and $\mathbb{Y}^!(\xi_{|\mathcal{X}})\colon\ \mathbb{Y}^!(\mathbf{T}(-), E)|_{\mathcal{X}} \cong \mathbf{T}(I_E)$. It is not difficult to see that the functor $\mathbb{Y}^!(\mathbf{T}(-), E)|_{\mathcal{X}}$ is isomorphic to the functor E. Hence, $\mathbf{T}(I_E) \cong E$.

Next we show that $\mathbf{T}(I)$ is injective in $\mathsf{D}(\mathcal{C})$, for any pure-injective object I in $\mathcal{C}$. Let $\alpha\colon\ \mathbf{T}(I) \to E$ be the injective envelope of $\mathbf{T}(I)$ in $\mathsf{D}(\mathcal{C})$. By the above argument, there exists a pure-injective object I_E such that $\mathbf{T}(I_E) \cong E$. Since $\mathbf{T}_{A,I_E}$ is invertible, for all $A \in \mathcal{C}$, there exists a morphism $f\colon\ I \to I_E$ such that $\mathbf{T}(f) = \alpha$. Since $\mathbf{T}(f)$ is a monomorphism, there exists a pure-sequence $I \to I_E \to C$ in $\mathcal{C}$. Since I is pure-injective, this sequence splits and I is a direct summand of I_E. Then $\mathbf{T}(I)$ is a direct summand of $\mathbf{T}(I_E) = E$, hence $\mathbf{T}(I)$ is injective in $\mathsf{D}(\mathcal{C})$. Then by Lemma 4.10, $\mathbf{T}$ induces an equivalence $\mathbf{T}\colon\ \mathcal{I}(\mathcal{E}) \xrightarrow{\approx} \operatorname{Inj}\mathsf{D}(\mathcal{C})$. The existence of pure-injective envelopes in $\mathcal{C}$ is proved in the same way as in Theorem 4.12. □

COROLLARY 4.14. *Let E be an object in $\mathcal{C}$.*

(1) *If $\mathcal{C}$ satisfies the assumptions of Theorems* 4.12 *or* 4.13 *and E is indecomposable and pure-injective, then E has local endomorphism ring.*

(2) *If $\mathcal{C}$ satisfies the assumptions of Theorem* 4.13 *then for the statements:*

(i) $E \cong \mathbb{D}_{\mathfrak{m}}(X)$*, where $X \in \mathcal{X}$ and $\mathfrak{m}$ is a maximal left ideal of* $\operatorname{End}_{\mathcal{C}}(X)$*;*

(ii) *E is pure-injective and there exists a left almost split morphism $E \to B$ in $\mathcal{C}$;*

(iii) $\mathbf{T}(E)$ *is the injective envelope of a simple functor in* $\mathsf{D}(\mathcal{C})$.

we have (i) ⇒ (ii) ⇒ (iii).

Proof. (1) By hypothesis, $\mathbf{T}(E)$ is indecomposable injective in $\mathsf{D}(\mathcal{C})$ and $\operatorname{End}_{\mathcal{C}}(E) \cong \operatorname{End}_{\mathsf{D}(\mathcal{C})}(\mathbf{T}(E))$. Then $\operatorname{End}_{\mathcal{C}}(E)$ is local by [28]. (2) (i) ⇒ (ii) Follows from part (ii) of Theorem 3.10. (ii) ⇒ (iii) The proof is identical with the proof of $(\beta) \Rightarrow (\gamma)$ in part (i) of Theorem 3.13, using that $\mathbf{T}\colon\ \mathcal{I}(\mathcal{E}) \to \operatorname{Inj}\mathsf{D}(\mathcal{C})$ is an equivalence. □

5. Phantomless Abstract Homotopy Categories, Categories of Finite Type and Ziegler Spectra

Throughout this section we keep the setting, the assumptions and the notations introduced in Section 4. In particular we assume that Axiom 4.7 holds.

5.1. PHANTOM MAPS

An important class of morphisms in an abstract homotopy category $\mathcal{C}$ is the class of phantom morphisms, i.e. morphisms which are invisible in the representation categories $\mathsf{L}(\mathcal{C}), \mathsf{D}(\mathcal{C})$. In a sense their complexity measures how far is $\mathcal{C}$ from being locally finitely presented. More precisely:

DEFINITION 5.1. A morphism $f\colon B \to C$ is called *phantom*, if $\mathcal{C}(X, f) = 0$, for all $X \in \mathcal{X}$. $\mathcal{C}$ is called *phantomless*, if there are no nonzero phantom morphisms in $\mathcal{C}$. Equivalently the functor $\mathbf{S}\colon \mathcal{C} \to \mathsf{L}(\mathcal{C})$ or the functor $\mathbf{T}\colon \mathcal{C} \to \mathsf{D}(\mathcal{C})$ is faithful.

Let $\mathrm{Ph}(A, B)$ be the set of all phantom morphisms from A to B. Plainly $\mathrm{Ph}(\mathcal{C}) = \bigcup_{A,B\in\mathcal{C}} \mathrm{Ph}(A, B)$ concides with the kernel-ideal $\ker \mathbf{S} := \{f \mid \mathbf{S}(f) = 0\}$ of $\mathbf{S}$, hence it is an ideal in $\mathcal{C}$, i.e. $\mathrm{Ph}(-, -)$ is an additive subfunctor of $\mathcal{C}(-, -)$. Since $\mathcal{X}$ is a Whitehead subcategory of $\mathcal{C}$, it follows that $\mathbf{S}$ reflects isomorphisms. Hence, $\mathrm{Ph}(\mathcal{C})$ is contained in the Jacobson radical of $\mathcal{C}$. Observe that by Proposition 4.8 and Lemma 4.10, $\mathrm{Ph}(P, B) = 0 = \mathrm{Ph}(A, I)$, for all $P \in \mathcal{P}(\mathcal{E})$ and for all $I \in \mathcal{I}(\mathcal{E})$. To study phantom morphisms it is useful to impose on $\mathcal{C}$ the following axiom.

AXIOM 5.2. If $K \xrightarrow{g} P \xrightarrow{f} A$ is a pure-projective presentation in $\mathcal{C}$, then f is a weak cokernel of g.

Note that this axiom holds in all the examples listed in Section 2. The next lemma shows that the phantomless property of $\mathcal{C}$ forces $\mathbf{S}$ to be full.

LEMMA 5.3. *Assume that Axiom* 5.2 *holds in* $\mathcal{C}$. *If* $\mathcal{C}$ *is phantomless, then the functor* $\mathbf{S}\colon \mathcal{C} \to \mathsf{L}(\mathcal{C})$ *is full and any object of* $\mathcal{C}$ *is a filtered colimit of objects of* $\mathcal{X}$.

Proof. Let $K \xrightarrow{g} P \xrightarrow{f} A$ be a pure-projective presentation of A in $\mathcal{C}$ and let $h\colon A \to C$ be a weak cokernel of f. Then h is phantom, hence $h = 0$. It follows that f is an epimorphism. Since f is a weak cokernel of g, for any $B \in \mathcal{C}$, we have the following exact commutative diagram:

$$\begin{array}{ccccccc}
0 \longrightarrow & \mathcal{C}(A, B) & \longrightarrow & \mathcal{C}(P, B) & \longrightarrow & \mathcal{C}(K, B) \\
& \downarrow{\scriptstyle \mathbf{S}_{A,B}} & & \downarrow{\scriptstyle \mathbf{S}_{P,B}} & & \downarrow{\scriptstyle \mathbf{S}_{K,B}} \\
0 \longrightarrow & [\mathbf{S}(A), \mathbf{S}(B)] & \longrightarrow & [\mathbf{S}(P), \mathbf{S}(B)] & \longrightarrow & [\mathbf{S}(K), \mathbf{S}(B)]
\end{array}$$

Since $\mathcal{C}$ is phantomless, $\mathbf{S}_{A,B}, \mathbf{S}_{K,B}$ are monomorphisms and since P is pure-projective, $\mathbf{S}_{P,B}$ is invertible. A simple diagram chasing shows that $\mathbf{S}_{A,B}$ is invertible. Hence $\mathbf{S}$ is fully faithful. Now fix an object $A \in \mathcal{C}$ and consider the category $\mathcal{U}_A$ with objects, morphisms $\alpha\colon X \to A$ with $X \in \mathcal{X}$. A morphism in $\mathcal{U}_A$ from $\alpha\colon X_1 \to A$ to $\beta\colon X_2 \to A$ is a morphism $f\colon X_1 \to X_2$ such that $f \circ \beta = \alpha$. Since $\mathbf{S}$ is fully faithful, the inclusion $\mathcal{X} \hookrightarrow \mathcal{C}$ is dense in the sense of [45]. Then it is well known that A is a colimit of the functor $\mathcal{U}_A \to \mathcal{C}$ defined by sending $\alpha\colon X \to A$ to X. Using that $\mathcal{X}$ is closed under weak cokernels in $\mathcal{C}$, it is not difficult to see that the category $\mathcal{U}_A$ is filtered. So any object of $\mathcal{C}$ is a filtered colimit of objects of $\mathcal{X}$. □

PROPOSITION 5.4. *The following are equivalent.*

(i) $\mathcal{C}$ *is a locally finitely presented category and* $\text{f.p}(\mathcal{C}) = \mathcal{X}$;
(ii) *the functor* $\mathbf{S}$ *induces an equivalence* $\mathcal{C} \overset{\approx}{\to} \text{Flat}\,\mathsf{L}(\mathcal{C})$;
(iii) *the functor* $\mathbf{T}$ *induces an equivalence* $\mathcal{C} \overset{\approx}{\to} \text{FP Inj}\,\mathsf{D}(\mathcal{C})$;
(iv) $\mathcal{C}$ *is phantomless, Axiom 5.2 holds and any functor* $\mathcal{I}\colon I \to \mathcal{X}$ *from a skeletally small filtered category* I*, has an* $\mathcal{X}$*-minimal weak colimit in* $\mathcal{C}$.

Proof. The implication (i) $\Rightarrow$ (ii) is obvious and by Lemma 4.1 it directly follows that (ii) $\Leftrightarrow$ (iii). (ii) $\Rightarrow$ (iv) Clearly $\mathcal{C}$ is phantomless and Axiom 5.2 holds in $\mathcal{C}$. Moreover, any functor $\mathcal{I}\colon I \to \mathcal{C}$ admits a filtered colimit, which obviously is an $\mathcal{X}$-minimal weak colimit, since $\mathcal{X}$ consists of finitely presented objects. (iv) $\Rightarrow$ (i) Since $\mathcal{C}$ is phantomless by Lemma 5.3, $\mathbf{S}\colon \mathcal{C} \to \mathsf{L}(\mathcal{C})$ is fully faithful. Let F be a flat functor over $\mathcal{X}$. Then F is a filtered colimit $\varinjlim \mathbf{S}(X_i)$, where $X_i = \mathcal{I}(i)$ and $\mathcal{I}\colon I \to \mathcal{X}$ is a functor from a skeletally small filtered category I. By hypothesis, there exists in $\mathcal{C}$ an $\mathcal{X}$-minimal weak colimit $\text{w.}\varinjlim X_i$. Then plainly $\mathbf{S}(\text{w.}\varinjlim X_i) \cong \varinjlim \mathbf{S}(X_i) = F$ and $\mathbf{S}\colon \mathcal{C} \to \text{Flat}\,\mathsf{L}(\mathcal{C})$ is surjective on objects. Hence $\mathbf{S}$ is an equivalence. □

Remark 5.5. If any morphism in $\mathcal{C}$ is a weak kernel, then any epimorphism in $\mathcal{C}$ splits. Indeed, if $f\colon B \to C$ is an epimorphism and a weak kernel, then f is a weak kernel of its weak cokernel which is zero. Then 1_C factors through f, i.e. f splits.

LEMMA 5.6. *If any morphism in* $\mathcal{C}$ *is a weak kernel, then:* $\mathcal{C}^{\mathrm{b}} = \text{f.p}(\mathcal{C})$.

Proof. Let X be a compact object and let $\{A_i; i \in I\}$ be a filtered system in $\mathcal{C}$, with filtered colimit $\varinjlim A_i$, so we have an exact sequence

$$\bigoplus_{i\to j} A_i \overset{g}{\to} \bigoplus_{i\in I} A_i \overset{f}{\to} \varinjlim A_i \to 0.$$

Since g is a weak kernel of f and since by Remark 5.5, f is a split epimorphism, it follows that the sequence

$$\mathcal{C}\Big(X, \bigoplus_{i\to j} A_i\Big) \to \mathcal{C}\Big(X, \bigoplus_{i\in I} A_i\Big) \to \mathcal{C}(X, \varinjlim A_i) \to 0$$

is exact. Since X is compact, it follows directly that the canonical morphism $\varinjlim \mathcal{C}(X, A_i) \to \mathcal{C}(X, \varinjlim A_i)$ is invertible, hence X is finitely presented. The converse is obvious. □

THEOREM 5.7. *If any morphism in* $\mathcal{C}$ *is a weak kernel, then the following statements are equivalent.*

(i) *$\mathcal{C}$ is phantomless;*
(ii) *$\mathcal{C}$ is pure-semisimple;*
(iii) *$\mathcal{C}$ is a locally finitely presented category and $\mathcal{X} = \mathcal{C}^{\mathrm{b}}$.*

Proof. If $\mathcal{C}$ is phantomless, then for all $A \in \mathcal{C}$ choose a pure-epic $f\colon P \to A$ with P pure-projective. Since any weak cokernel of f is phantom, it follows that f is an epimorphism, which splits by Remark 5.5. Hence, A is pure-projective and then $\mathcal{C}$ is pure-semisimple. If $\mathcal{C}$ is pure-semisimple, then by Corollary 4.8, $\mathcal{C} = \mathrm{Add}(\mathcal{X})$ is locally finitely presented. It is easy to see that $\mathrm{Add}(\mathcal{X})^{\mathrm{b}} = \mathrm{add}(\mathcal{X})$. Since by our general assumptions $\mathcal{X}$ is closed under direct summands, we infer that $\mathcal{C}^{\mathrm{b}} = \mathcal{X}$. If $\mathcal{C}$ is locally finitely presented and $\mathcal{C}^{\mathrm{b}} = \mathcal{X}$, then by Lemma 5.6, $\mathcal{X} = \mathrm{f.p}(\mathcal{C})$. Then $\mathbf{S}\colon \mathcal{C} \to \mathrm{Flat}\,\mathsf{L}(\mathcal{C})$ is an equivalence, in particular $\mathcal{C}$ is phantomless. □

5.2. ABSTRACT HOMOTOPY CATEGORIES OF FINITE TYPE AND ZIEGLER SPECTRA

We recall [23] that a Grothendieck category $\mathcal{G}$ is called locally Artinian, resp. locally Noetherian, resp. locally finite, if $\mathcal{G}$ has a set of Artinian, resp. Noetherian, resp. finite length, generators. Recall also that an additive category $\mathcal{A}$ is called a *Krull–Schmidt category* if any object of $\mathcal{A}$ is a finite coproduct of indecomposable objects and any indecomposable object has a local endomorphism ring. Finally, recall that an Abelian category is called a *length category* if any of its objects has finite length.

Proposition 5.4 and Gabriel's theory [23], consult also [44], admit the following corollary:

COROLLARY 5.8. *$\mathcal{C}$ is pure-semisimple iff $\mathsf{D}(\mathcal{C})$ is locally Noetherian iff $\mathcal{B}(\mathcal{X})$ is a Noetherian Abelian category iff the functor $\mathbf{T}$ induces an equivalence $\mathbf{T}\colon \mathcal{C} \xrightarrow{\approx} \mathrm{Inj}\,\mathsf{D}(\mathcal{C})$ iff the functor $\mathbf{S}$ induces an equivalence $\mathbf{S}\colon \mathcal{C} \xrightarrow{\approx} \mathrm{Proj}\,\mathsf{L}(\mathcal{C})$. If this is the case, then $\mathcal{X}$ is a Krull–Schmidt category and any object of $\mathcal{C}$ is in a unique way a coproduct of objects of $\mathcal{X}$ having a local endomorphism ring.*

DEFINITION 5.9. An abstract homotopy category $\mathcal{C}$ is called *of finite type* iff its representation category $\mathsf{D}(\mathcal{C})$ is locally finite.

PROPOSITION 5.10. *The following are equivalent.*

(i) *$\mathcal{C}$ is of finite type;*
(ii) *$\mathcal{B}(\mathcal{X})$ is an Abelian length category;*
(iii) *the functor category $\mathrm{Mod}(\mathcal{X}^{\mathrm{op}})$ is locally finite;*
(iv) *$\mathcal{C}$ is pure semisimple and $\mathcal{B}(\mathcal{X})$ is Artinian.*

If $\mathcal{X}$ is Abelian with a finite number of nonisomorphic simple objects, then $\mathcal{C}$ is of finite type iff $\mathcal{X}$ is a length category with a finite number of nonisomorphic indecomposable objects.

Proof. By Gabriel's theory [23], $\mathsf{D}(\mathcal{C})$ is locally finite iff the category of finitely presented objects of $\mathsf{D}(\mathcal{C})$, which is equivalent to $\mathcal{B}(\mathcal{X})$, is Artinian and Noetherian, hence (i) $\Leftrightarrow$ (ii). The equivalence (i) $\Leftrightarrow$ (iv) follows from Corollary 5.8. Since $\mathcal{B}(\mathcal{X}) = \mathrm{mod}(\mathcal{X}^{\mathrm{op}})^{\mathrm{op}}$, the equivalence (ii) $\Leftrightarrow$ (iii) follows. The last assertion follows from the above equivalences and well-known results of M. Auslander [5]. □

Recall from [14] that given a category $\mathcal{K}$, there exists an Abelian category $\mathcal{A}(\mathcal{K})$, the *free Abelian category* of $\mathcal{K}$, and a functor $F\colon \mathcal{K} \to \mathcal{A}(\mathcal{K})$ such that any functor $\mathcal{K} \to \mathcal{U}$ to an Abelian category $\mathcal{U}$ has an, essentially unique, exact factorization through F. By [14], an Abelian category $\mathcal{A}$ is free iff $\mathcal{A}$ is an *Auslander category*, i.e. if $\mathcal{A}$ has enough projectives and injectives and has global dimension $\leqslant 2$ and dominant dimension $\geqslant 2$ in the sense that any projective P admits an exact coresolution $0 \to P \to I_0 \to I_1$, where I_0, I_1 are projective-injective objects. In this case $\mathcal{A}$ is the free Abelian category of the full subcategory of projective-injective objects.

Recall that a ring Λ is called *representation-finite* if Λ is right Artinian and the set of isoclasses of indecomposable finitely presented right Λ-modules is finite. Let $\mathbf{F}$ be the collection of Morita equivalence classes of representation-finite rings and let $\mathbf{G}$ be the collection of equivalence classes of abstract homotopy categories of finite type with minimal Whitehead subcategory $\mathcal{X}$, such that the Abelian category $\mathcal{B}(\mathcal{X})$ is free with a finite number of nonisomorphic simple objects. The following consequence of Proposition 5.10, is a variant of Auslander's correspondence [5].

COROLLARY 5.11. *The map* $\chi\colon \mathbf{G} \to \mathbf{F}$ *defined by* $\chi(\mathcal{C}) = \Lambda$ *is a bijection, where* $\Lambda = \mathrm{End}_{\mathcal{C}}(Y)$, $Y = \mathrm{add}(\mathcal{Y})$ *and* $\mathcal{Y}$ *is the full subcategory of projective-injective objects of* $\mathcal{B}(\mathcal{X})$. *The inverse map is given by* $\psi(\Lambda) = \mathrm{Mod}(\Lambda)$.

Assume now that the hypotheses of Theorem 4.12 or Theorem 4.13 hold. Since the functor $\mathbf{T}$ induces an equivalence between the pure-injectives of $\mathcal{C}$ and the injectives of $\mathsf{D}(\mathcal{C})$, it follows that the isoclasses of indecomposable pure-injective objects of $\mathcal{C}$ form a set, since it is well-known [23] that this is true for the isoclasses of indecomposable injective objects in the locally coherent category $\mathsf{D}(\mathcal{C})$.

DEFINITION 5.12. The *Ziegler spectrum* $\mathrm{Zsp}(\mathcal{C})$ of $\mathcal{C}$ is defined to be the set of isoclasses of indecomposable pure-injective objects of $\mathcal{C}$.

Following [35] we equip the Ziegler spectrum with a topology in the following way. Let Φ be a class of morphisms in $\mathcal{X}$. An object $I \in \mathrm{Zsp}(\mathcal{C})$ is called Φ-*injective*, if $\mathcal{C}(\phi, I)\colon \mathcal{C}(Y, I) \to \mathcal{C}(X, I)$ is an epimorphism for any $\phi\colon X \to Y$ in Φ. We denote by $\mathbf{U}_{\Phi}$ the subset of $\mathrm{Zsp}(\mathcal{C})$ consisting of all Φ-injectives. The next result reduces the study of $\mathrm{Zsp}(\mathcal{C})$ to the study of spectra of more familiar abstract homotopy categories, for which there exists already a rich theory. Its proof is identical with the proof of the corresponding statement for module categories [31, 35].

COROLLARY 5.13. *The subsets* $\{\mathbf{U}_\Phi \mid \Phi$ *is a class of morphisms in* $\mathcal{X}\}$ *form the closed sets of a topology in* Zsp($\mathcal{C}$)*, the* Ziegler topology *of* $\mathcal{C}$*, and there is a bijection between the open sets of* Zsp($\mathcal{C}$) *and the Serre subcategories of* $\mathcal{B}(\mathcal{X})$*, given by*

$$\mathcal{O} \longmapsto \big\{M \in \mathcal{B}(\mathcal{X}) \mid \{E \in \mathrm{Zsp}(\mathcal{C}) : \mathsf{D}(\mathcal{C})(M, \mathbf{T}(E)) \neq 0\} \subseteq \mathcal{O}\big\}.$$

The space Zsp($\mathcal{C}$) *is homeomorphic with the Ziegler spectrum of the abstract homotopy category* Flat $\mathsf{L}(\mathcal{C})$ *and with the spectrum* [31] *of the category* $\mathsf{D}(\mathcal{C})$.

6. Representation Equivalences and Flat Approximations

Throughout this section we fix an abstract homotopy category $\mathcal{C}$ equipped with a minimal compact Whitehead subcategory $\mathcal{X} \subseteq \mathcal{C}$ closed under weak cokernels. We assume throughout that $\mathcal{C}$ satisfies Axioms 4.7 and 5.2 of Sections 4, 5.

Although $\mathcal{C}$ is not necessarily equivalent to Flat $\mathsf{L}(\mathcal{C})$, there is another kind of equivalence between these categories which is useful in many cases. Recall [3] that a functor $F\colon \mathcal{C} \to \mathcal{G}$ is called a *representation equivalence*, if F is full, surjective on objects and reflects isomorphisms. Since $\mathcal{X}$ is closed under weak cokernels in $\mathcal{C}$, by Lemma 4.1, we have $\mathrm{Im}\,\mathbf{S} \subseteq$ Flat $\mathsf{L}(\mathcal{C})$. We would like to know under what conditions the representation functor $\mathbf{S}\colon \mathcal{C} \to$ Flat $\mathsf{L}(\mathcal{C})$ is a representation equivalence, so that $\mathcal{C}$ will be as close as possible to locally finitely presented, if it is not phantomless. Another motivation for the study of this question comes from algebraic topology. Since $\mathcal{X}$ is closed under weak cokernels in $\mathcal{C}$, by [14], a functor $F\colon \mathcal{X}^{\mathrm{op}} \to \mathcal{A}b$ is flat iff F is half-exact. Then $\mathbf{S}\colon \mathcal{C} \to$ Flat $\mathsf{L}(\mathcal{C})$ is a representation equivalence iff any half-exact functor $F\colon \mathcal{X}^{\mathrm{op}} \to \mathcal{A}b$ is of the form $\mathbf{S}(C)$ and any morphism $F \to G$ of half-exact functors is of the form $\mathbf{S}(\alpha)$. Hence the question above deals with the problem of 'extending' half-exact functors $\mathcal{X}^{\mathrm{op}} \to \mathcal{A}b$ to half-exact (representable) functors $\mathcal{C}^{\mathrm{op}} \to \mathcal{A}b$, thus it is related to another version of the classical Adams–Brown representability [1, 18] a generalization of which in the triangulated case is treated in detail in [12, 21, 22, 36, 39, 41].

6.1. FLAT APPROXIMATIONS

We assume that $\mathcal{C}$ has weak kernels and satisfies the assumptions of Theorem 4.12 or Theorem 4.13, so that $\mathcal{C}$ admits pure-injective envelopes and $\mathbf{T}\colon \mathcal{I}(\mathcal{E}) \to$ Inj $\mathsf{D}(\mathcal{C})$ is an equivalence. Recall that a full subcategory $\mathcal{Y}$ of an additive category $\mathcal{A}$ is called *contravariantly finite* [8], if for any object $A \in \mathcal{A}$, there exists a morphism $f_A\colon Y_A \to A$ with $Y_A \in \mathcal{Y}$, such that any morphism $Y \to A$ with Y in $\mathcal{Y}$ factors through f_A. The dual notion is *covariantly finite*. If $\mathcal{Y}$ is both contravariantly and covariantly finite, then it is called *functorially finite*.

THEOREM 6.1. $\mathrm{Im}\,\mathbf{S}$ *is functorially finite in* $\mathsf{L}(\mathcal{C})$*, i.e. any functor* $M \in \mathsf{L}(\mathcal{C})$ *has a right* $\mathrm{Im}\,\mathbf{S}$*-approximation* $p_M\colon \mathbf{S}(A) \to M$ *and a left* $\mathrm{Im}\,\mathbf{S}$*-approximation* $i_M\colon M \to \mathbf{S}(B)$*. Moreover,* $\mathrm{Im}\,\mathbf{T}$ *is functorially finite in* $\mathsf{D}(\mathcal{C})$.

Proof. Let M be in $\mathsf{L}(\mathcal{C})$ and consider the functor $\mathbb{Y}^!(M) \in \mathsf{D}(\mathcal{C})$. Choose a minimal injective presentation

$$0 \to \mathbb{Y}^!(M) \xrightarrow{\mu'} \mathbf{T}(I_0) \xrightarrow{f'} \mathbf{T}(I_1)$$

of $\mathbb{Y}^!(M)$ in $\mathsf{D}(\mathcal{C})$. By our assumptions, f' is of the form $\mathbf{T}(f)$, where $f\colon I_0 \to I_1$ is a morphism in $\mathcal{C}$. Since $\mathbb{Y}_*$ is exact and $\mathbb{Y}_*\mathbb{Y}^! = \mathrm{Id}_{\mathsf{L}(\mathcal{C})}$, we have an exact sequence

$$0 \to M \xrightarrow{\mu} \mathbf{S}(I_0) \xrightarrow{\mathbf{S}(f)} \mathbf{S}(I_1),$$

where $\mu := \mathbb{Y}_*(\mu')$. Let $g\colon A \to I_0$ be a weak kernel of f in $\mathcal{C}$. Since $\mathbf{S}(g) \circ \mathbf{S}(f) = 0$, there exists a unique morphism $p_M\colon \mathbf{S}(A) \to M$ such that $p_M \circ \mu = \mathbf{S}(g)$. If $\alpha\colon \mathbf{S}(B) \to M$ is a morphism in $\mathsf{L}(\mathcal{C})$, then by Lemma 4.10, the morphism $\mathbb{Y}^!(\alpha) \circ \mathbb{Y}^!(\mu)\colon \mathbf{T}(B) \to \mathbf{T}(I_0)$ is of the form $\mathbf{T}(h)$, where $h\colon B \to I_0$. Then obviously $\mathbf{T}(h) \circ \mathbf{T}(f) = 0$. Hence, by Lemma 4.10, we have $h \circ f = 0$. Since g is a weak kernel of f in $\mathcal{C}$, there exists a morphism $\gamma\colon B \to A$ such that $\gamma \circ g = h$. Then $\mathbf{S}(\gamma) \circ \mathbf{S}(g) = \mathbf{S}(h)$. Since $\mathbb{Y}^!(\alpha) \circ \mathbb{Y}^!(\mu) = \mathbf{T}(h) = \mathbb{Y}^!\mathbf{S}(h)$ and $\mathbb{Y}^!$ is faithful, we have $\alpha \circ \mu = \mathbf{S}(h) = \mathbf{S}(\gamma) \circ \mathbf{S}(g)$. Then $\mathbf{S}(\gamma) \circ p_M \circ \mu = \mathbf{S}(\gamma) \circ \mathbf{S}(g) = \alpha \circ \mu$ and this implies that $\mathbf{S}(\gamma) \circ p_M = \alpha$, since μ is a monomorphism. This shows that p_M is a right $\mathrm{Im}\,\mathbf{S}$-approximation. Hence, $\mathrm{Im}\,\mathbf{S}$ is contravariantly finite in $\mathsf{L}(\mathcal{C})$. Now let $\mathbf{S}(P_1) \xrightarrow{\mathbf{S}(f)} \mathbf{S}(P_0) \xrightarrow{\epsilon} M \to 0$ be the start of a projective resolution of M and let $P_1 \xrightarrow{f} P_0 \xrightarrow{g} A$ be a weak cokernel sequence in $\mathcal{C}$. Then there exists a unique morphism $i_M\colon M \to \mathbf{S}(A)$ such that $\epsilon \circ i_M = \mathbf{S}(g)$. If $h\colon M \to \mathbf{S}(B)$ is a morphism in $\mathsf{L}(\mathcal{C})$, then $\mathbf{S}(f) \circ \epsilon \circ h = 0$. Since P_0 is pure-projective, the morphism $\epsilon \circ h$ is of the form $\mathbf{S}(\tau)$, where $\tau\colon P_0 \to B$. Hence, $\mathbf{S}(f) \circ \mathbf{S}(\tau) = 0$ and then $f \circ \tau = 0$. Hence, there exists $\sigma\colon A \to B$ such that $g \circ \sigma = \tau$. Then $\mathbf{S}(g) \circ \mathbf{S}(\sigma) = \mathbf{S}(\tau) \Rightarrow \epsilon \circ i_M \circ \mathbf{S}(\sigma) = \epsilon \circ h \Rightarrow i_M \circ \mathbf{S}(\sigma) = h$. Hence, i_M is a left $\mathrm{Im}\,\mathbf{S}$-approximation.

Using the adjoint pair $(\mathbb{Y}^!, \mathbb{Y}_*)$ and the contravariant finiteness of $\mathrm{Im}\,\mathbf{S}$ it is easy to see that $\mathrm{Im}\,\mathbf{T}$ is contravariantly finite in $\mathsf{D}(\mathcal{C})$. Since $\mathbf{T}$ preserves products and $\mathrm{Im}\,\mathbf{T}$ consists of FP-injective objects, it follows that $\mathrm{Im}\,\mathbf{T}$ is closed under products and pure-subobjects, for the purity in $\mathsf{D}(\mathcal{C})$ induced by its finitely presented objects. Then it follows by [35], that $\mathrm{Im}\,\mathbf{T}$ is covariantly finite in $\mathsf{D}(\mathcal{C})$. □

COROLLARY 6.2. *If* $\mathbf{S}\colon \mathcal{C} \to \mathrm{Flat}\,\mathsf{L}(\mathcal{C})$*, resp.* $\mathbf{T}\colon \mathcal{C} \to \mathrm{FP\,Inj}\,\mathsf{D}(\mathcal{C})$*, is surjective on objects, then* $\mathrm{Flat}\,\mathsf{L}(\mathcal{C})$*, resp.* $\mathrm{FP\,Inj}\,\mathsf{D}(\mathcal{C})$*, is functorially finite in* $\mathsf{L}(\mathcal{C})$*, resp.* $\mathsf{D}(\mathcal{C})$.

COROLLARY 6.3. *Let* $\mathcal{P}$ *be a skeletally small additive category with weak cokernels. Then the category* $\mathrm{Flat}(\mathcal{P})$ *is contravariantly finite in* $\mathrm{Mod}(\mathcal{P})$ *iff* $\mathrm{Flat}(\mathcal{P})$ *admits weak kernels. In particular if* Λ *is a left coherent ring, then the category* $\mathrm{Flat}(\Lambda)$ *of flat right* Λ*-modules is contravariantly finite in* $\mathrm{Mod}(\Lambda)$ *iff* $\mathrm{Flat}(\Lambda)$ *admits weak kernels.*

Proof. If Flat($\mathcal{P}$) is contravariantly finite, then Flat($\mathcal{P}$) has weak kernels by [13]. Since $\mathcal{P}$ has weak cokernels, the category Flat($\mathcal{P}$) is an abstract homotopy category and as such satisfies the assumptions of Theorem 4.12. If in addition Flat ($\mathcal{P}$) has weak kernels, then since in our case **S** is an equivalence, the contravariant finiteness of Flat ($\mathcal{P}$) follows from Corollary 6.2. The last assertion follows from the well-known observation that the category $\mathcal{P}_\Lambda$ of finitely generated projective right Λ-modules has weak cokernels iff Λ is left coherent. □

6.2. REPRESENTATION EQUIVALENCES

The following result shows a connection between representation equivalences and flat approximations, first observed by H. Krause [36] in case $\mathcal{C}$ is a compactly generated triangulated category.

Let $p_M\colon \mathbf{S}(A) \to M$ be the right Im **S**-approximation and $i_M\colon M \to \mathbf{S}(B)$ be the left Im **S**-approximation of $M \in \mathsf{L}(\mathcal{C})$, constructed in Theorem 6.1.

PROPOSITION 6.4. *If for all functors $M \in \mathsf{L}(\mathcal{C})$, p_M is right minimal or i_M is left minimal, then the functor* $\mathbf{S}\colon \mathcal{C} \to \operatorname{Flat}\mathsf{L}(\mathcal{C})$ *is a representation equivalence.*

Proof. Let

$$\mathbf{S}(P_1) \xrightarrow{\mathbf{S}(f)} \mathbf{S}(P_0) \xrightarrow{\epsilon} M \to 0$$

be the start of a projective resolution of M, such that $i_M\colon M \to \mathbf{S}(A)$ is a minimal left flat approximation of M, where $g\colon P_0 \to A$ is a weak cokernel of f and $\mathbf{S}(g) = \epsilon \circ i_M$. If M is flat, then obviously i_M is invertible. Hence, **S** is surjective on flat functors. If $\alpha\colon \mathbf{S}(A) \to \mathbf{S}(A')$ is a morphism, then by construction and hypothesis, we can choose projective presentations

$$\mathbf{S}(P_1) \xrightarrow{\mathbf{S}(f)} \mathbf{S}(P_0) \xrightarrow{\mathbf{S}(g)} \mathbf{S}(A) \to 0 \quad \text{and} \quad \mathbf{S}(P_1') \xrightarrow{\mathbf{S}(f')} \mathbf{S}(P_0') \xrightarrow{\mathbf{S}(g')} \mathbf{S}(A') \to 0$$

such that g, resp. g', is a weak cokernel of f, resp. f'. Then α induces morphisms $\gamma\colon P_1 \to P_1'$ and $\beta\colon P_0 \to P_0'$ such that $\gamma \circ f' = f \circ \beta$ and $\mathbf{S}(\beta) \circ \mathbf{S}(g') = \mathbf{S}(g) \circ \alpha$. By the first relation, there exists δ such that $g \circ \delta = \beta \circ g'$. Obviously then $\mathbf{S}(\delta) = \alpha$, hence **S** is full. Since **S** reflects isomorphisms, it is a representation equivalence. The proof for p_M is similar. □

Remark 6.5. H. Krause [36] proved that if $\mathcal{C}$ is a compactly generated triangulated category and $\mathcal{X} = \mathcal{C}^{\mathrm{b}}$, then the converse holds for $p_M\colon \mathbf{S}(A) \to M$. The corresponding result for the morphism i_M is not true in general (even in the setting of [36]). However, it is not difficult to see that it is true if any torsionless functor, i.e. a subfunctor of a projective functor, has a projective cover.

We define the *pure-projective dimension* p.p.d A of an object $A \in C$ inductively as follows. If A is pure-projective, then p.p.d $A = 0$. If p.p.d $A > 0$, then p.p.d $A \leqslant n$ if there exists a pure sequence $K \to P \to A$ in $\mathcal{C}$ such that p.p.d $K < n$ and P is pure-projective. Then define p.p.d $A = n$ if p.p.d $A \leqslant n$ and p.p.d $A \nleqslant n - 1$. If p.p.d $A \neq n$, for all $n \geqslant 0$, then define p.p.d $A = \infty$. Finally define the *pure-global dimension* p.gl.dim $\mathcal{C}$ of $\mathcal{C}$ by p.gl.dim $\mathcal{C} = \sup\{\text{p.p.d } A \mid A \in \mathcal{C}\}$.

Remark 6.6. Since $\mathbf{S}$ reflects isomorphisms it is easy to see (compare [12]) that p.d $\mathbf{S}(A) =$ p.p.d A for all $A \in \mathcal{C}$. Hence,

$$\text{p.gl.dim}\, \mathcal{C} \leqslant \sup\{\text{p.d } F \mid F \in \text{Flat}\, \mathsf{L}(\mathcal{C})\}.$$

We recall [44] that the *weight* $w(\mathcal{X})$ of $\mathcal{X}$ is the cardinality of the disjoint union of the sets $\mathcal{C}(X_1, X_2)$, where X_1, X_2 run over the isoclasses of objects of $\mathcal{X}$. By a result of Simson [44], if $w(\mathcal{X}) \leqslant \aleph_t$ for some $t \geqslant 0$, then $\sup\{\text{p.d } F;\ F \in \text{Flat}\, \mathsf{L}(\mathcal{C})\} \leqslant t + 1$. Hence, in this case p.gl.dim $\mathcal{C} \leqslant t + 1$. It follows that if $\mathcal{X}$ has a countable skeleton, i.e. if $w(\mathcal{X}) \leqslant \aleph_0$, then p.gl.dim $\mathcal{C} \leqslant 1$.

LEMMA 6.7. *If* p.p.d $A \leqslant 1$*, then for all* $B \in \mathcal{C}$*, the canonical morphism* $\mathbf{S}_{A,B}$: $\mathcal{C}(A, B) \to [\mathbf{S}(A), \mathbf{S}(B)]$ *is an epimorphism. Hence, the functor* $\mathbf{S}$ *is full if* p.gl.dim $\mathcal{C} \leqslant 1$.

Proof. Let $P_1 \xrightarrow{g} P_0 \xrightarrow{f} A$ be a pure sequence in $\mathcal{C}$ with P_1, P_0 pure-projective and let $B \in \mathcal{C}$. Since f is a weak cokernel of g, the sequence $\mathcal{C}(A, B) \to \mathcal{C}(P_0, B) \to \mathcal{C}(P_1, B)$ is exact. Since the sequence $0 \to [\mathbf{S}(A), \mathbf{S}(B)] \to [\mathbf{S}(P_0), \mathbf{S}(B)] \to [\mathbf{S}(P_1), \mathbf{S}(B)]$ is exact and for $i = 0, 1$, the morphisms $\mathbf{S}_{P_i,B}$: $\mathcal{C}(P_i, B) \to [\mathbf{S}(P_i), \mathbf{S}(B)]$, are invertible, it follows that $\mathbf{S}_{A,B}$: $\mathcal{C}(A, B) \to [\mathbf{S}(A), \mathbf{S}(B)]$ is an epimorphism. □

PROPOSITION 6.8. *If any flat functor* F *in* $\mathsf{L}(\mathcal{C})$ *has projective dimension* p.d $F \leqslant 1$*, then the functor* $\mathbf{S}$: $\mathcal{C} \to \text{Flat}\, \mathsf{L}(\mathcal{C})$ *is a representation equivalence.*

Proof. Since $\mathbf{S}$ reflects isomorphisms, and by Remark 6.6 and Lemma 6.7, $\mathbf{S}$ is full, it remains to show that $\mathbf{S}$ is surjective on flat functors. Let F be a flat functor and let $0 \to R_1 \to R_0 \to F \to 0$ be a projective resolution of F in $\mathsf{L}(\mathcal{C})$. Then the morphism $R_1 \to R_0$ is of the form $\mathbf{S}(g)$: $\mathbf{S}(P_1) \to \mathbf{S}(P_0)$, for a morphism g: $P_1 \to P_0$ in $\mathcal{C}$. Since $\mathbb{Y}^!$ is exact on exact sequences of flat functors, it follows that $\mathbf{T}(g)$ is a monomorphism, i.e. g is pure-monic. By Axiom 4.7, there exists a pure sequence $P_1 \xrightarrow{g} P_0 \to A$ in $\mathcal{C}$. Then obviously $\mathbf{S}(A) \cong F$. □

COROLLARY 6.9. $\mathbf{S}$: $\mathcal{C} \to \text{Flat}\, \mathsf{L}(\mathcal{C})$ *is a representation equivalence, if* $w(\mathcal{X}) \leqslant \aleph_0$.

As a consequence we have the following result on flat approximations.

COROLLARY 6.10. *If any flat functor* F *in* $\mathsf{L}(\mathcal{C})$ *has* p.d $F \leqslant 2$ (*e.g., if the weight of* $\mathcal{X}$ *is of cardinality* $w(\mathcal{X}) \leqslant \aleph_1$)*, then* Flat $\mathsf{L}(\mathcal{C})$ *is contravariantly finite in* $\mathsf{L}(\mathcal{C})$.

Proof. By Corollary 6.2, it suffices to show that $\mathbf{S}$ is surjective on flat functors. Let

$$0 \to \mathbf{S}(P_2) \xrightarrow{\mathbf{S}(f_2)} \mathbf{S}(P_1) \xrightarrow{\mathbf{S}(f_1)} \mathbf{S}(P_0) \xrightarrow{\mathbf{S}(f_0)} F \to 0$$

be a projective resolution of the flat functor F in $\mathsf{L}(\mathcal{C})$. Then the morphism f_2 is pure-monic, hence by Axiom 4.7, there exists a pure sequence $P_2 \xrightarrow{f_2} P_1 \xrightarrow{g} A$ in $\mathcal{C}$. Then $\mathbf{S}(A) \cong \operatorname{Im}(\mathbf{S}(f_1))$. Since $\text{p.p.d}\, A \leqslant 1$, by Lemma 6.7 the inclusion morphism $\mathbf{S}(A) \hookrightarrow \mathbf{S}(P_0)$ comes from a morphism $h\colon A \to P_0$ and plainly h is pure-monic, since the cokernel of $\mathbf{S}(h)$ is flat and $\mathbb{Y}^!$ is exact on short exact sequences of flat functors. By Axiom 4.7 there exists a pure-sequence $A \xrightarrow{h} P_0 \to B$ in $\mathcal{C}$. Then obviously $\mathbf{S}(B) \cong F$. □

The converse of Proposition 6.8 is not true in general. For instance if $\mathcal{C}$ is a locally finitely presented category with products, then $\mathbf{S}\colon \mathcal{C} \to \text{Flat}(\text{f.p}(\mathcal{C}))$ is an equivalence but a flat functor over $\text{f.p}(\mathcal{C})$ can be of arbitrary large projective dimension. However the converse is true if in $\mathcal{C}$ any epimorphism splits and the pure-projective presentations are *closed under weak push-outs along phantom morphisms*, in the following sense. If $K \xrightarrow{g} P \xrightarrow{f} A$ is a pure-projective presentation of A and $h\colon K \xrightarrow{f} B$ is a phantom morphism in $\mathcal{C}$, then there exists a commutative diagram:

$$\begin{array}{ccccc} K & \xrightarrow{g} & P & \xrightarrow{f} & A \\ {\scriptstyle h}\downarrow & & {\scriptstyle \gamma}\downarrow & & \downarrow{\scriptstyle \|} \\ B & \xrightarrow{\alpha} & D & \xrightarrow{\beta} & A \end{array}$$

such that the lower sequence is pure and the left square is a *weak push-out* diagram, i.e. the morphism ${}^{\mathrm{t}}(\gamma, \alpha)\colon P \oplus B \to D$ is a weak cokernel of $(g, -h)\colon K \to P \oplus B$. Note that these conditions hold trivially if $\mathcal{C}$ is a triangulated category.

THEOREM 6.11. *If any epimorphism in $\mathcal{C}$ splits and the pure-projective presentations are closed under weak push-outs along phantom morphisms, then the functor* $\mathbf{S}\colon \mathcal{C} \to \text{Flat}\,\mathsf{L}(\mathcal{C})$ *is a representation equivalence iff* $\text{p.d}\, F \leqslant 1$, *for all* $F \in \text{Flat}\,\mathsf{L}(\mathcal{C})$.

Proof. ($\Rightarrow$) Since any flat functor F is of the form $\mathbf{S}(A)$, by Remark 6.6 it suffices to show that $\text{p.p.d}\, A \leqslant 1$ for all $A \in \mathcal{C}$. Let

$$K \xrightarrow{g} P \xrightarrow{f} A \quad \text{and} \quad L \xrightarrow{g'} Q \xrightarrow{f'} K$$

be pure-projective presentations and let $h\colon K \to B$ be a weak cokernel of f'. Consider the weak push-out of h, g, as in the above diagram. Since f' is pure-epic, h is a phantom morphism or equivalently $\mathbf{S}(h) = 0$. Applying the functor $\mathbf{S}$ to the diagram above, we have the following exact commutative diagram in $\mathsf{L}(\mathcal{C})$:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbf{S}(K) & \xrightarrow{\mathbf{S}(g)} & \mathbf{S}(P) & \xrightarrow{\mathbf{S}(f)} & \mathbf{S}(A) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle \mathbf{S}(\gamma)} & & \| & & \\ 0 & \longrightarrow & \mathbf{S}(B) & \xrightarrow{\mathbf{S}(\alpha)} & \mathbf{S}(D) & \xrightarrow{\mathbf{S}(\beta)} & \mathbf{S}(A) & \longrightarrow & 0 \end{array}$$

Then there exists a morphism $\delta'\colon \mathbf{S}(A) \to \mathbf{S}(D)$ such that $\mathbf{S}(f) \circ \delta' = \mathbf{S}(\gamma)$. Since $\mathbf{S}$ is full, there exists $\delta\colon A \to D$ such that $\mathbf{S}(\delta) = \delta'$. It follows easily that $\mathbf{S}(\delta) \circ \mathbf{S}(\beta) = 1_{\mathbf{S}(A)}$. Since $\mathbf{S}$ reflects isomorphisms, we have that $\delta \circ \beta$ is invertible. Let $\omega\colon \mathbf{S}(D) \to F$ be the cokernel of $\mathbf{S}(\gamma)$. By the above diagram, there exists a unique isomorphism $\zeta\colon \mathbf{S}(B) \to F$ such that $\zeta = \mathbf{S}(\alpha) \circ \omega$. Consider the morphism $\omega \circ \zeta^{-1}\colon \mathbf{S}(D) \to \mathbf{S}(B)$. By the fullness of $\mathbf{S}$, there exists a morphism $\phi\colon D \to B$ such that $\mathbf{S}(\phi) = \omega \circ \zeta^{-1}$. Then $\mathbf{S}(\alpha) \circ \mathbf{S}(\phi) = \mathbf{S}(\alpha) \circ \omega \circ \zeta^{-1} = 1_{\mathbf{S}(B)}$. Since $\mathbf{S}$ reflects isomorphisms, we have that $\alpha \circ \phi$ is invertible. Now observe that $\mathbf{S}(\gamma) \circ \mathbf{S}(\phi) = \mathbf{S}(\gamma) \circ \omega \circ \zeta^{-1} = 0$. Hence, $\gamma \circ \phi$ is phantom. Since P is pure-projective it follows that $\gamma \circ \phi = 0$. Then $h = h \circ \alpha \circ \phi = g \circ \gamma \circ \phi = 0$. This implies that f' is an epimorphism since its weak cokernel is zero. By hypothesis f' splits. Hence, K is pure-projective as a direct summand of Q. This implies that p.p.d $A \leqslant 1$. □

Theorem 6.11, together with Corollary 6.9, generalizes the well-known result of Brown [18] and Adams [1], see also [39], on the homotopy category of pointed CW-complexes to the nonstable case and answers a question of Heller [26]. Since the assumptions of Theorem 6.11 hold in triangulated categories, we recover the generalizations of Adams–Brown representability theorem to triangulated categories studied in [12, 21, 41].

7. Applications

7.1. LOCALLY FINITELY PRESENTED CATEGORIES

Let $\mathcal{F}$ be a locally finitely presented additive category with products. We view $\mathcal{F}$ as an abstract homotopy category with minimal Whitehead subcategory f.p($\mathcal{F}$). Clearly f.p($\mathcal{F}$) is compact and closed under (weak) cokernels in $\mathcal{F}$. Moreover, it is not difficult to see that $\mathcal{F}$ satisfies Axiom 4.7 and the condition imposed in Theorem 4.12. Then $\mathcal{F}$ is phantomless and the theory of Section 4 applied to $\mathcal{F}$ recovers the theory of purity developed by Crawley-Boevey [17], and by Gruson and Jensen [24], Simson [44] and others (see [28]), if in addition $\mathcal{F}$ is Abelian. In particular, $\mathcal{F}$ has enough pure-projective objects and admits pure-injective envelopes. Concerning almost split morphisms, we have the following direct consequence of Theorem 3.10.

THEOREM 7.1. *Let X be a homological finitely presented object in $\mathcal{F}$. Then the* $\mathfrak{m}$*-dual object* $\mathbb{D}_{\mathfrak{m}}(X)$ *is pure-injective and there exists a left almost split morphism* $g_X\colon \mathbb{D}_{\mathfrak{m}}(X) \to F^X$ *in* $\mathcal{F}$*. If* $\mathcal{F}$ *has weak kernels and* X *has a local endomorphism ring, then there exists a right almost split morphism* $f_X\colon F_X \to X$ *in* $\mathcal{F}$*.*

Theorem 7.1 and the results of Sections 3 and 4 can be applied to the locally finitely presented category $\mathcal{F}$ of flat quasi-coherent sheaves for a nonsingular irreducible curve/surface X over a field, where f.p$(\mathcal{F})$ is the category of vector bundles over X [17]. Similar remarks are hold for categories of rational G-modules with good filtrations over an algebraic group G [17]. Details are left to the interested reader.

7.2. RIGHT TRIANGULATED CATEGORIES WITH A RIGHT SEMI-EQUIVALENCE SUSPENSION FUNCTOR

Let $\mathcal{C}$ be a right triangulated [2] (or suspended [30]) category with suspension functor $\Sigma\colon \mathcal{C} \to \mathcal{C}$. Recall that Σ is called a *right semi-equivalence* [2], if Σ is fully faithful and $\operatorname{Im}\Sigma$ is closed under extensions in the following sense: if $\Sigma(A) \to X \to \Sigma(C) \to \Sigma^2(A)$ is a triangle in $\mathcal{C}$, then $X = \Sigma(B)$ for some B in $\mathcal{C}$. The following are nontrivial examples of right triangulated categories with a right semi-equivalence suspension functor, which are not necessarily triangulated.

EXAMPLE 7.2. (1) Let $(\mathcal{C}^{\leqslant 0}, \mathcal{C}^{\geqslant 0})$ be a t-structure in a triangulated category $\mathcal{C}$. Then $\mathcal{C}^{\leqslant 0}$ is right triangulated with a right semi-equivalence suspension functor.

(2) Let $\mathcal{C}$ be an Abelian category with enough injectives and let $\mathcal{I}$ be the full subcategory of injective objects of $\mathcal{C}$. By [2], a full subcategory $\mathcal{X}/\mathcal{I}$ of $\mathcal{C}/\mathcal{I}$ is right triangulated with right semi-equivalence suspension functor iff $\mathcal{X}$ is closed under extensions, cokernels of monomorphisms and $\mathcal{I} \subseteq \mathcal{X} \subseteq \mathcal{I}^{\perp}$, where $\mathcal{I}^{\perp} := \{A \in \mathcal{C} \mid \mathcal{E}xt^n(X, A) = 0, \forall X \in \mathcal{I}, \forall n \geqslant 1\}$. For instance, we can take $\mathcal{X} = \mathcal{I}^{\perp}$.

Throughout we fix a right triangulated category $\mathcal{C}$ with a right semi-equivalence suspension functor Σ. We assume that $\mathcal{C}$ has coproducts and Σ preserves coproducts. By [2], the functor $\mathcal{C}(A, -)\colon \mathcal{C} \to \mathcal{A}b$ is half-exact, for all $A \in \mathcal{C}$. Hence, $\mathcal{H}(\mathcal{C}^{\mathrm{b}}) = \mathcal{C}^{\mathrm{b}}$ and then by Lemma 3.11 of [13], we have the following consequence.

LEMMA 7.3. *Any compact subcategory of* $\mathcal{C}$ *is minimal.*

The above lemma and the adjoint functor theorem 2.8 admit the following.

COROLLARY 7.4. *If* $\mathcal{C}$ *contains a compact Whitehead subcategory, then* $\mathcal{C}$ *is an abstract homotopy category and the suspension functor* Σ *admits a right adjoint* Ω*.*

From now on we assume that $\mathcal{C}$ contains a compact Whitehead subcategory $\mathcal{X}$ and we view $\mathcal{C}$ as an abstract homotopy category. The following important class of triangles was introduced by Happel [25], generalizing the fundamental notion of an almost split sequence in an Abelian category, due to Auslander and Reiten [7].

DEFINITION 7.5. A triangle $A \xrightarrow{g} B \xrightarrow{f} C \to \Sigma(A)$ in $\mathcal{C}$ is called an *Auslander–Reiten triangle*, if g is left almost split and f is right almost split.

THEOREM 7.6. *If X is a compact object in $\mathcal{C}$ with local endomorphism ring, then there exists an Auslander–Reiten triangle:*

$$\mathbb{D}_{\mathfrak{m}}(X) \xrightarrow{g^X} A \xrightarrow{f_X} \Sigma(X) \xrightarrow{-\Sigma(h_X)} \Sigma\mathbb{D}_{\mathfrak{m}}(X)$$

and an Auslander–Reiten triangle $E \to B \to X \xrightarrow{h_X} \mathbb{D}_{\mathfrak{m}}(X)$, if $\mathbb{D}_{\mathfrak{m}}(X) \in \operatorname{Im}\Sigma$.

Proof. Consider the morphism $h_X \colon X \to \mathbb{D}_{\mathfrak{m}}(X)$ constructed in Theorem 3.10 and let

$$X \xrightarrow{h_X} \mathbb{D}_{\mathfrak{m}}(X) \xrightarrow{g^X} A \xrightarrow{f_X} \Sigma(X)$$

be a triangle in $\mathcal{C}$. By Theorem 3.10, g^X is left almost split. Since Σ is fully faithful, $\operatorname{End}_{\mathcal{C}}(\Sigma(X)) \cong \operatorname{End}_{\mathcal{C}}(X)$ is local. Then by [2], f_X is right almost split. Hence, the triangle

$$(*)\colon\ \mathbb{D}_{\mathfrak{m}}(X) \xrightarrow{g^X} A \xrightarrow{f_X} \Sigma(X) \xrightarrow{-\Sigma(h_X)} \Sigma\mathbb{D}_{\mathfrak{m}}(X)$$

is an Auslander–Reiten triangle. If $\mathbb{D}_{\mathfrak{m}}(X) \in \operatorname{Im}\Sigma$, let $E \in \mathcal{C}$ be such that $\Sigma(E) = \mathbb{D}_{\mathfrak{m}}(X)$. Since $\operatorname{Im}\Sigma$ is closed under extensions, $A = \Sigma(B)$, for some $B \in \mathcal{C}$. Moreover,

$$\operatorname{End}_{\mathcal{C}}(E) \cong \operatorname{End}_{\mathcal{C}}(\Sigma(E)) \cong \operatorname{End}_{\mathcal{C}}(\mathbb{D}_{\mathfrak{m}}(X))$$

is local. Then by [2], there exists a triangle

$$(\dagger)\colon\ E \xrightarrow{\alpha} B \xrightarrow{\beta} X \xrightarrow{h_X} \Sigma(E)$$

in $\mathcal{C}$ and the morphism β is a weak kernel of h_X. Hence, by Theorem 3.10, β is a right almost split morphism. Then by [2], it follows that $(\dagger)$ is an Auslander–Reiten triangle in $\mathcal{C}$. □

Since any compact object in $\mathcal{C}$ is homological, Corollary 3.14 implies the following generalization of a result of H. Krause [37].

COROLLARY 7.7. *Let X be a compact object in $\mathcal{C}$, let $\rho\colon \Gamma \to \Lambda_X := \operatorname{End}_{\mathcal{C}}(X)$ be a ring morphism and let I be an injective left Γ-module. If $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} \Sigma(A)$ is a triangle in $\mathcal{C}$, then there exist isomorphisms:*

$$\operatorname{Hom}_{\Lambda_X}[\operatorname{Coker}\mathcal{C}(X, \Sigma(f)), I] \cong \operatorname{Coker}\mathcal{C}(f, \Omega\mathbb{D}_I(X)) \cong \operatorname{Ker}\mathcal{C}(g, \mathbb{D}_I(X)).$$

If I is a cogenerator, then any morphism $X \to C$ factors through g iff any morphism $A \to \Omega\mathbb{D}_I(X)$ factors through f. In particular $X \in \mathcal{X}$ iff $\mathbb{D}_I(X)$ is pure-injective.

It is not difficult to see that if Σ preserves pure monics, then a sequence $A \to B \to C$ in $\mathcal{C}$ is pure iff there exists a triangle $A \to B \to C \xrightarrow{h} \Sigma(A)$ in $\mathcal{C}$ such that $0 \to \mathcal{C}(X, A) \to \mathcal{C}(X, B) \to \mathcal{C}(X, C) \to 0$ is exact, for all $X \in \mathcal{X}$.

7.3. TRIANGULATED CATEGORIES

Let $\mathcal{C}$ be a triangulated category which admits all small coproducts. We assume that $\mathcal{C}$ is *compactly generated* [40]. This means that $\mathcal{C}$ admits a set of compact objects $\mathcal{S}$ such that if $\mathcal{C}(\Sigma^n(S), A) = 0$, for all $S \in \mathcal{S}$ and $n \in \mathbb{Z}$, then $A = 0$. By [13], this is equivalent to say that the full subcategory $\mathcal{C}^{\mathrm{b}}$ of compact objects is a (minimal) Whitehead subcategory of $\mathcal{C}$, i.e. $\mathcal{C}$ is an abstract homotopy category. The purity in $\mathcal{C}$ based on the Whitehead subcategory $\mathcal{C}^{\mathrm{b}}$ is denoted by $\mathcal{E}$ and coincides with (and recovers) the theory of purity and phantom maps developed in [12, 20, 34, 39].

It is not difficult to see [14] that $\mathcal{C}$ enjoys the very useful property that the representation categories of $\mathcal{C}$ coincide: $\mathsf{D}(\mathcal{C}) = \mathrm{Mod}(\mathcal{C}^{\mathrm{b}}) = \mathsf{L}(\mathcal{C})$. Hence, Theorem 7.6 admits the following consequence, in which the existence of Auslander–Reiten triangles and pure-injective envelopes is due to H. Krause.

THEOREM 7.8 ([37, 12, 34]). *Let X be a compact object in $\mathcal{C}$ with local endomorphism ring. Then there exists an Auslander–Reiten triangle*

$$\Sigma^{-1}\mathbb{D}_{\mathfrak{m}}(X) \to A \to X \xrightarrow{h_X} \mathbb{D}_{\mathfrak{m}}(X).$$

Moreover, $\mathcal{C}$ has enough pure-projectives, pure-injective envelopes and there exist equivalences

$$\mathcal{P}(\mathcal{E}) \approx \mathrm{Add}(\mathcal{C}^{\mathrm{b}}) \quad \textit{and} \quad \mathcal{I}(\mathcal{E}) \approx \mathrm{Prod}(\mathbb{D}_{\mathbb{Q}/\mathbb{Z}}(\mathcal{C}^{\mathrm{b}})).$$

$\mathcal{C}$ is pure-semisimple iff $\mathcal{C}$ has pure-projective covers iff $\mathcal{C}$ is phantomless iff $\mathcal{C}$ is locally finitely presented iff $\mathrm{Mod}(\mathcal{C}^{\mathrm{b}})$ *is locally Noetherian iff* $\mathrm{Mod}(\mathcal{C}^{\mathrm{b}})$ *is perfect.*

Let $\mathbf{S}\colon \mathcal{C} \to \mathsf{L}(\mathcal{C}) = \mathrm{Mod}(\mathcal{C}^{\mathrm{b}})$ be the restricted Yoneda functor, defined by $\mathbf{S}(C) = \mathcal{C}(-, C)|_{\mathcal{C}^{\mathrm{b}}}$. The following result characterizes the pure-injective objects of $\mathcal{C}$ occurring as a source of a left almost split morphism in $\mathcal{C}$ and, under the assumption that $\mathcal{C}^{\mathrm{b}}$ is a Krull–Schmidt category, gives necessary and sufficient conditions for the existence of Auslander–Reiten triangles in $\mathcal{C}$ starting at a pure-injective object.

THEOREM 7.9. *For an object E in $\mathcal{C}$, the following are equivalent:*

(i) *E is pure-injective and there exists a left almost split morphism $E \to B$ in $\mathcal{C}$;*
(ii) *$\mathbf{S}(E)$ is the injective envelope of a simple functor;*
(iii) *$E \cong \mathbb{D}_{\mathfrak{m}}(X)$, where X is compact and $\mathfrak{m}$ is a maximal left ideal of* $\mathrm{End}_{\mathcal{C}}(X)$.

If $\mathcal{C}^{\mathrm{b}}$ is a Krull–Schmidt category, then the above are also equivalent to:

(iv) *E is pure-injective and there exists an Auslander–Reiten triangle $E \to B \to P \to \Sigma(E)$ in $\mathcal{C}$.*
(v) *$E \cong \mathbb{D}_{\mathfrak{m}}(X)$, for an indecomposable compact object X and a maximal left ideal $\mathfrak{m}$ of $\mathrm{End}_{\mathcal{C}}(X)$.*

Proof. That the first three statements are equivalent is a direct consequence of Theorems 3.10, 3.13 and 4.13, since the hypotheses imposed in these results hold trivially in $\mathcal{C}$. Assume that $\mathcal{C}^{\mathrm{b}}$ is a Krull–Schmidt category. (i) $\Rightarrow$ (iv) By part (iii), $\mathbf{S}(E)$ is the injective envelope of a simple functor, which by Section 3, is of the form $S_{X,\mathfrak{m}}$, where $X \in \mathcal{C}^{\mathrm{b}}$. Since $\mathcal{C}^{\mathrm{b}}$ is Krull–Schmidt, we can choose X to be indecomposable. By Theorem 3.9, the injective envelope of $S_{X,\mathfrak{m}}$ is isomorphic to $\mathbf{S}(\mathbb{D}_{\mathfrak{m}}(X))$. Hence, $\mathbf{S}(\mathbb{D}_{\mathfrak{m}}(X)) \cong \mathbf{S}(E)$ and this implies that $E \cong \mathbb{D}_{\mathfrak{m}}(X)$. If $\Sigma^{-1}(\mathbb{D}_{\mathfrak{m}}(X)) \to A \to X \to E$ is the Auslander–Reiten triangle of Theorem 7.8 ending at X, then $E \to \Sigma(A) \to \Sigma(X) \to \Sigma(E)$ is the desired Auslander–Reiten triangle starting at E. (iv) $\Rightarrow$ (v) Let $E \xrightarrow{g} B \xrightarrow{f} P \xrightarrow{h} \Sigma(E)$ be an Auslander–Reiten triangle in $\mathcal{C}$. If P is not compact, then any morphism $\alpha\colon X \to P$ with $X \in \mathcal{C}^{\mathrm{b}}$ is not a split epimorphism, hence α factors through f. Then the triangle is pure and since E is pure-injective, g is split monic and this is not true. Hence $P \in \mathcal{C}^{\mathrm{b}}$ and then by using (iii), we have an isomorphism $E \cong \mathbb{D}_{\mathfrak{m}}(X)$, where $X = \Sigma(P)$, since the end terms of an Auslander–Reiten triangle are uniquely determined up to isomorphism [25]. The implication (v) $\Rightarrow$ (iii) is trivial. □

We refer to [15] for necessary and sufficient conditions for the existence of Auslander–Reiten triangles in $\mathcal{C}^{\mathrm{b}}$.

7.4. SPECTRA AND COMPLEXES

The results of Subsection 7.3 are directly applicable to the stable homotopy category $\mathrm{Ho}(\mathcal{S})$ of spectra [39], where $\mathrm{Ho}(\mathcal{S})^{\mathrm{b}}$ is the full subcategory of finite spectra and to the unbounded derived category $\mathbf{D}(\Lambda)$ of a ring Λ, where $\mathbf{D}(\Lambda)^{\mathrm{b}}$ is, up to equivalence, the homotopy category $\mathcal{H}^{\mathrm{b}}(\mathcal{P}_{\Lambda})$ of perfect complexes, i.e. bounded complexes with components finitely generated projective modules. For instance, we have the following.

COROLLARY 7.10. (1) *For any finite spectrum X with local endomorphism ring, there exists an Auslander–Reiten triangle $\Sigma^{-1}\mathbb{D}_{\mathfrak{m}}(X) \to A \to X \to \mathbb{D}_{\mathfrak{m}}(X)$ in* $\mathrm{Ho}(\mathcal{S})$. *Moreover, for any finite spectrum X, the $\mathfrak{m}$-dual spectrum $\mathbb{D}_{\mathfrak{m}}(X)$ is pure-injective and occurs as the source of an almost split morphism in* $\mathrm{Ho}(\mathcal{S})$.

(2) *For any perfect complex $X^{\bullet}$ in $\mathbf{D}(\Lambda)$ with local endomorphism ring, there exists an Auslander–Reiten triangle $\mathbb{D}_{\mathfrak{m}}(X^{\bullet})[-1] \to A^{\bullet} \to X^{\bullet} \to \mathbb{D}_{\mathfrak{m}}(X^{\bullet})$ in* $\mathbf{D}(\Lambda)$. *Moreover, for any perfect complex $X^{\bullet}$, the $\mathfrak{m}$-dual complex $\mathbb{D}_{\mathfrak{m}}(X^{\bullet})$ is pure-injective and occurs as the source of an almost split morphism in* $\mathbf{D}(\Lambda)$.

(3) *If $\mathcal{H}^{\mathrm{b}}(\mathcal{P}_{\Lambda})$ is a Krull–Schmidt category, then for a pure-injective complex $E^{\bullet} \in \mathbf{D}(\Lambda)$, there exists an Auslander–Reiten triangle $E^{\bullet} \to A^{\bullet} \to B^{\bullet} \to E^{\bullet}[1]$*

in $\mathbf{D}(\Lambda)$ *iff there exists an indecomposable perfect complex* $X^\bullet$*, such that* $E^\bullet = \mathbb{D}_{\mathfrak{m}}(X^\bullet)$*, in which case* $B^\bullet = X^\bullet[1]$.

It is easy to see $\mathcal{H}^{\mathrm{b}}(\mathcal{P}_\Lambda)$ is Krull–Schmidt, if Λ is an Artin R-algebra. If I is the injective envelope of $R/\mathcal{J}ac(R)$ and $\mathrm{D} = \mathrm{Hom}_R(-, I)$ is the usual duality of Artin algebras, then for any perfect complex $X^\bullet$ and any complex $A^\bullet \in \mathbf{D}(\Lambda)$, the isomorphism ω of Subsection 3.1, reduces to the rather well-known isomorphism: $\mathrm{D}(X^\bullet, A^\bullet) \cong (A^\bullet, \mathbb{D}_I(X^\bullet))$, see [25]. It follows that $\mathbb{D}_I(X^\bullet) \cong X^\bullet \otimes^L_\Lambda \mathrm{D}(\Lambda)$, where $-\otimes^L_\Lambda \mathrm{D}(\Lambda)\colon \mathbf{D}(\Lambda) \to \mathbf{D}(\Lambda)$ is the total left derived functor of the Nakayama functor $- \otimes_\Lambda \mathrm{D}(\Lambda)\colon \mathrm{Mod}(\Lambda) \to \mathrm{Mod}(\Lambda)$.

Since the endomorphism ring $\mathbb{Z}$ of the sphere spectrum $S^0 \in \mathrm{Ho}(\mathcal{S})^{\mathrm{b}}$ is not semiperfect, it follows that $\mathrm{Ho}(\mathcal{S})^{\mathrm{b}}$ is not a Krull–Schmidt category. However the full subcategory $\mathrm{Ho}(\mathcal{S})^{\mathrm{b}}_p$ of p-local finite spectra [39], where p is a prime, is a Krull–Schmidt subcategory of the category $\mathrm{Ho}(\mathcal{S})_p$ of p-local spectra, so one can characterize the p-local pure-injective spectra occuring as a source of an Auslander–Reiten triangle in the p-local category, along the lines of Theorem 7.9.

7.5. MODULE CATEGORIES

If Γ is an associative ring, then we view the category $\mathrm{Mod}(\Gamma)$ of right Γ-modules as an abstract homotopy category with minimal Whitehead subcategory, the full subcategory $\mathrm{mod}(\Gamma)$ of finitely presented modules. Obviously $\mathrm{Mod}(\Gamma)$ satisfies the conditions imposed on Theorem 4.12. The representation categories

$$\mathsf{L}(\Gamma) := \mathsf{L}(\mathrm{Mod}(\Gamma)) = \mathrm{Mod}(\mathrm{mod}(\Gamma))$$

and

$$\mathsf{D}(\Gamma) := \mathsf{D}(\mathrm{Mod}(\Gamma)) = \mathrm{Mod}(\mathrm{mod}(\Gamma^{\mathrm{op}})^{\mathrm{op}})$$

have been used for a long time as an indispensable tool for the study of purity of modules. The description of the functor $\mathbf{S}$ is clear and it is easy to see that $\mathbf{T}$ is isomorphic to the functor $M \mapsto M \otimes_\Gamma -\colon \mathrm{mod}(\Gamma^{\mathrm{op}}) \to \mathcal{A}b$. The theory of purity developed in Section 4, applied to $\mathrm{Mod}(\Gamma)$ coincides and recovers the well-known and extensively studied purity in the sense of Cohn. In particular the Ziegler spectrum as defined in Section 5, coincides with the Ziegler spectrum defined by Ziegler [47] and studied extensively by Krause [35] and others. Theorem 4.12 and Corollary 5.8 recover the well-known result that $\mathrm{Mod}(\Gamma)$ has pure-injective envelopes and that Γ is right pure-semisimple iff $\mathrm{Mod}(\Gamma)$ has pure-projective covers iff $\mathsf{L}(\Gamma)$ is perfect iff $\mathsf{D}(\Gamma)$ is locally Noetherian [44, 28]. Proposition 5.10 recovers the well-known result that Γ is representation-finite iff $\mathsf{D}(\Gamma)$ is locally finite [5].

We recall the fundamental concept of an almost split sequence.

DEFINITION 7.11 ([7]). An exact sequence $0 \to A \xrightarrow{g} B \xrightarrow{f} C \to 0$ is called an *almost split sequence*, if f is right almost split and g is left almost split.

We recall that the *transpose* $\mathrm{Tr}(X)$ of a finitely presented right Γ-module X is defined by the exact sequence

$$\mathrm{Hom}_\Gamma(P_0, \Gamma) \to \mathrm{Hom}_\Gamma(P_1, \Gamma) \to \mathrm{Tr}(X) \to 0$$

in $\mathrm{mod}(\Gamma^{\mathrm{op}})$, where $P_1 \to P_0 \to X \to 0$ is a projective presentation of X in $\mathrm{mod}(\Gamma)$. The following consequence of Theorem 3.10 is the original basic existence result of almost split morphisms and almost split sequences due to M. Auslander and I. Reiten [4, 7]. We include a proof as a sample application of Brown representability and the machinery of abstract homotopy categories. For part (β) it is essential to work not in $\mathrm{Mod}(\Gamma)$ but in a more suitable abstract homotopy category.

THEOREM 7.12 ([4]). (α) *Let P be a projective Γ-module with local endomorphism ring. Then there exists a nonzero map h_P: $P \to \mathbb{D}_{\mathfrak{m}}(P)$ in* $\mathrm{Mod}(\Gamma)$*, such that $\mathbb{D}_{\mathfrak{m}}(P)$ is an indecomposable injective module,* $\mathrm{Im}(h_P) = S_{\mathfrak{m},P}$ *is a simple module, the inclusion* $\mathrm{Rad}(P) = \mathrm{Ker}(h_P) \hookrightarrow P$ *is minimal right almost split and the projection* $\mathbb{D}_{\mathfrak{m}}(P) \twoheadrightarrow \mathrm{Coker}(h_P) = \mathbb{D}_{\mathfrak{m}}(P)/\mathrm{Soc}(\mathbb{D}_{\mathfrak{m}}(P))$ *is minimal left almost split.*

(β) *Let X be a nonprojective finitely presented module with local endomorphism ring. Then there exists a functor* $\mathbb{D}_{\mathfrak{m}}(-, \underline{X})$: $\underline{\mathrm{mod}}(\Gamma)^{\mathrm{op}} \to \mathcal{A}b$*, an isomorphism of functors*

$$\mathbb{D}_{\mathfrak{m}}(-, \underline{X})(?) \cong \mathcal{E}xt^1_\Gamma[?, \mathrm{Hom}_{\Lambda_X}(\mathrm{Tr}(X), I_{\mathfrak{m}})]$$

and an almost split sequence:

$$0 \to \mathrm{Hom}_{\Lambda_X}[\mathrm{Tr}(X), I_{\mathfrak{m}}] \to A \to X \to 0,$$

where $\Lambda_X = \mathrm{End}(\underline{X})$ *and* $\mathfrak{m}$ *is a maximal ideal of* Λ_X*. Further,* $\mathrm{Hom}_{\Lambda_X}[\mathrm{Tr}(X), I]$ *is pure-injective in* $\mathrm{Mod}(\Gamma)$*. If Γ is an Artin algebra, then* $\mathrm{Hom}_{\Lambda_X}[\mathrm{Tr}(X), I_{\mathfrak{m}}] \cong \mathrm{D}\,\mathrm{Tr}(X)$ *and there exists an isomorphism of functors*

$$\mathbb{D}_{\mathfrak{m}}(-, \underline{X})(?) \cong \mathrm{D}\,\underline{\mathrm{Hom}}_\Gamma[X, ?] \cong \mathcal{E}xt^1_\Gamma[?, \mathrm{D}\,\mathrm{Tr}(X)]\colon\ \underline{\mathrm{mod}}(\Gamma)^{\mathrm{op}} \longrightarrow \mathcal{A}b,$$

where $\mathrm{D}\,\mathrm{Tr}$ *is the usual 'dual of the transpose'* [7].

Proof. (α) Since P has local endomorphism ring, it follows that P is finitely generated, in particular P is compact. Viewing $\mathrm{Mod}(\Gamma)$ as an Abelian abstract homotopy category, with minimal Whitehead subcategory the full subcategory $\mathcal{P}_\Gamma$ of finitely generated projective modules, all the assertions are consequences of Theorem 3.10.

(β) Let $\underline{\mathrm{mod}}(\Gamma)$ be the stable category of finitely presented modules modulo projectives. We view the category $\mathrm{Mod}(\underline{\mathrm{mod}}(\Gamma))$ of contravariant additive functors $\underline{\mathrm{mod}}(\Gamma)^{\mathrm{op}} \to \mathcal{A}b$ as an abstract homotopy category in which the functor $(-, \underline{X})$ is a homological compact projective object. Since $\Lambda_X := \mathrm{End}(\underline{X})$ is a factor ring of

$\mathrm{End}(X)$, it is a local ring with unique maximal ideal $\mathfrak{m}$. By Brown representability, there exists a functor $\mathbb{D}_{\mathfrak{m}}(-,\underline{X})$ in $\mathrm{Mod}(\underline{\mathrm{mod}}(\Gamma))$ and a natural isomorphism

$$\omega\colon\ \mathrm{Hom}_{\Lambda_X}[((-,\underline{X}),?),I_{\mathfrak{m}}]\xrightarrow{\cong}[?,\mathbb{D}_{\mathfrak{m}}(-,\underline{X})].$$

Then for any finitely presented Γ-module C, $\omega_{(-,\underline{C})}$ induces isomorphisms:

$$\begin{aligned}&\mathrm{Hom}_{\Lambda_X}[((-,\underline{X}),(-,\underline{C})),I_{\mathfrak{m}}]\\&\quad\cong \mathrm{Hom}_{\Lambda_X}[(\underline{X},\underline{C}),I_{\mathfrak{m}}]\\&\quad\cong[(-,\underline{C}),\mathbb{D}_{\mathfrak{m}}(-,\underline{X})]\cong\mathbb{D}_{\mathfrak{m}}(-,\underline{X})(\underline{C}).\end{aligned}$$

But it is well known [7] that we have isomorphisms (2):

$$\begin{aligned}&\mathrm{Hom}_{\Lambda_X}[(\underline{X},\underline{C}),I_{\mathfrak{m}}]\\&\quad\cong\mathrm{Hom}_{\Lambda_X}[\mathcal{T}or_1^{\Gamma}(\mathrm{Tr}(X),C),I_{\mathfrak{m}}]\\&\quad\cong\mathcal{E}xt^1_{\Gamma}[C,\mathrm{Hom}_{\Lambda_X}(\mathrm{Tr}(X),I_{\mathfrak{m}})].\end{aligned}$$

It follows that we have an isomorphism of functors

$$\mathbb{D}_{\mathfrak{m}}(-,\underline{X})(?)\cong\mathcal{E}xt^1_{\Gamma}[?,\mathrm{Hom}_{\Lambda_X}(\mathrm{Tr}(X),I_{\mathfrak{m}})].\tag{3}$$

By Yoneda's lemma the morphism $h_{(-,\underline{X})}\colon (-,\underline{X})\to\mathbb{D}_{\mathfrak{m}}(-,\underline{X})$ constructed in Theorem 3.10 corresponds to an element Δ_1 of $\mathbb{D}_{\mathfrak{m}}(-,\underline{X})(\underline{X})$. Under the isomorphism (3), Δ_1 corresponds to an element Δ_2 of $\mathcal{E}xt^1_{\Gamma}[X,\mathrm{Hom}_{\Lambda_X}(\mathrm{Tr}(X),I_{\mathfrak{m}})]$. Using the properties of $h_{(-,\underline{X})}$ in Theorem 3.10, it is trivial to check that Δ_2 represents an almost split sequence

$$0\to\mathrm{Hom}_{\Lambda_X}[\mathrm{Tr}(X),I_{\mathfrak{m}}]\to A\to X\to 0$$

in $\mathrm{Mod}(\Gamma)$.

Now let

$$(E)\colon\ 0\to A\xrightarrow{f}B\xrightarrow{g}C\to 0$$

be a pure-exact sequence in $\mathrm{Mod}(\Gamma)$. Then

$$(X,g)\colon\ (X,B)\to(X,C)$$

is an epimorphism. This implies that

$$(\underline{X},\underline{g})\colon\ (\underline{X},\underline{B})\to(\underline{X},\underline{C})$$

is an epimorphism, hence

$$[(\underline{X},\underline{g}),I_{\mathfrak{m}}]\colon\ [(\underline{X},\underline{C}),I_{\mathfrak{m}}]\to[(\underline{X},\underline{B}),I_{\mathfrak{m}}]$$

is a monomorphism. The pure-exact sequence (E) induces a long exact sequence

$$\begin{aligned}0\ \to\ &(C,\mathrm{Hom}_{\Lambda_X}[\mathrm{Tr}(X),I_{\mathfrak{m}}])\to(B,\mathrm{Hom}_{\Lambda_X}[\mathrm{Tr}(X),I_{\mathfrak{m}}])\\\xrightarrow{f^*}\ &(A,\mathrm{Hom}_{\Lambda_X}[\mathrm{Tr}(X),I_{\mathfrak{m}}])\to\mathcal{E}xt^1_{\Gamma}(C,\mathrm{Hom}_{\Lambda_X}[\mathrm{Tr}(X),I_{\mathfrak{m}}])\\\xrightarrow{g^*}\ &\mathcal{E}xt^1_{\Gamma}(B,\mathrm{Hom}_{\Lambda_X}[\mathrm{Tr}(X),I_{\mathfrak{m}}])\to\cdots.\end{aligned}$$

Using isomorphism (2), we have that g^* is isomorphic to the morphism $[(\underline{X}, g), I_{\mathfrak{m}}]$ which is a monomorphism; so f^* is an epimorphism. This shows that any morphism $A \to \mathrm{Hom}_{\Lambda_X}[\mathrm{Tr}(X), I_{\mathfrak{m}}]$ factors through f, i.e. $\mathrm{Hom}_{\Lambda_X}[\mathrm{Tr}(X), I_{\mathfrak{m}}]$ is pure-injective.

If Γ is an Artin algebra over a commutative Artin ring R, then we can choose $I_{\mathfrak{m}}$ to be the injective envelope I of $R/\mathcal{J}ac(R)$. Then the functor $\mathrm{D} = \mathrm{Hom}_R(-, I)$ is the usual duality of Artin algebras. It follows directly from the construction that there exists an isomorphism $\mathbb{D}_I(-, \underline{X})(?) \cong \mathrm{D}\,\underline{\mathrm{Hom}}_\Gamma[X, ?] \cong \mathcal{E}xt^1_\Gamma[?, \mathrm{D}\,\mathrm{Tr}(X)]$. □

Using Corollary 3.14 and the isomorphisms of the proof of Theorem 7.12, we have the following well-known basic result of M. Auslander.

COROLLARY 7.13 ([4]). *Let* $X \in \mathrm{mod}(\Gamma)$*, let* $\rho\colon \Lambda \to \mathrm{End}_\Gamma(X)$ *be a ring morphism and let* I *be an injective* Λ^{op}*-module. If* $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ *is a short exact sequence in* $\mathrm{Mod}(\Gamma)$*, then there exists an isomorphism*

$$\mathrm{Hom}_\Lambda[\mathrm{Coker}(X, g), I] \cong \mathrm{Coker}[f, \mathrm{Hom}_\Lambda(\mathrm{Tr}(X), I)]$$

which is functorial with respect to short exact sequences. In particular, if the Λ^{op}*-module* I *is an injective cogenerator, then every morphism* $X \to C$ *factors through* g *iff every morphism* $A \to \mathrm{Hom}_\Lambda(\mathrm{Tr}(X), I)$ *factors through* f.

Working stably as in Theorem 7.12, one can characterize along the lines of Theorem 7.9 the pure-injective modules occurring as a source of a left split morphism, see [16]. If $\mathrm{mod}(\Gamma)$ is a Krull–Schmidt category, one can characterize the pure-injective noninjective modules occurring as a source of an almost split sequence, see [33].

We leave it to the reader to apply the theory of Sections 3, 4, 5 to the other examples of abstract homotopy categories listed in Section 2, for instance to stable module categories (examples (iv), (v) of Section 2). See also the recent paper by P. Jørgensen [29] for a discussion of phantom maps in the setting of stable categories. Finally, note that one can obtain the existence of almost split morphisms and (relative) almost split sequences in a dualizing variety [9], and the existence of Serre–Grothendieck duality for derived categories of quasi-coherent sheaves over a smooth projective algebraic variety, using Brown representability and the machinery of abstract homotopy categories. Details are left to the reader.

Added in Proof. Prof. D. Simson kindly informed the author that the first use of Brown's abstract homotopy categories in an algebraic context seems to be in D. Simson and A. Tyc, Brown's theorem for cohomology theories on categories of chain complexes, *Commentationes Math.* **18** (1975), 285–296. In that paper various categories of complexes of projective modules over a ring are proved to be abstract homotopy categories, hence they satisfy the Brown representability theorem.

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