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STABLE EQUIVALENCES AND STABLE GROTHENDIECK GROUPS

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1. INTRODUCTION

It is well-known that one of the most basic discrete invariants of an abelian, resp. triangulated, category is its Grothendieck group. This useful characteristic is defined as the free abelian group with basis the isoclasses of objects modulo the relations given by the exact sequences, resp. triangles, so it is plainly invariant under exact equivalences. Our main purpose in this paper is to study the invariance of the Grothendieck group of a triangulated category under equivalences which are not necessarily exact. More generally we search for criteria such that a functor between triangulated or abelian categories induces an isomorphism on the level of Grothendieck groups. We investigate also the invariance under various types of equivalences of some related groups which reflect the complexity of the structure of the exact sequences or the triangles.

Most of the interesting triangulated categories which occur in practice arise from a model, in the sense that there is another category, the model, which is equipped with a distinguished class of morphisms, the weak equivalences, such that the triangulated category is obtained from the model by formally inverting the weak equivalences. For instance this is the case for the stable category of an exact Frobenius category and in particular for the derived category. Now even if two triangulated categories admit such models, a functor (usually an equivalence) which induces an isomorphism on the level of Grothendieck groups is not defined in general on the level of models. Hence it is useful to develop the theory in the setting of arbitrary triangulated categories. Actually in this paper we consider one-sided triangulated categories which include for instance many interesting classes of stable categories.

The article is organized as follows.

In Sec. 2 we recall the definition and the elementary properties of the Grothendieck group $\mathrm{K}_0(\mathcal{C})$ of a left triangulated category $\mathcal{C}$ and we give necessary and sufficient conditions such that an equivalence between two left triangulated categories induces an isomorphism on the level of Grothendieck groups. This is done by an analysis of the cone of a morphism and its behaviour under the given equivalence. Our main result in this section shows that the Grothendieck group of a Krull–Schmidt left triangulated category is invariant under a grade equivalence, i.e., an equivalence which commutes functorially with the loop functors.

In Sec. 3 we concentrate on stable categories induced by pairs $(\mathcal{C}, \mathcal{X})$ consisting of a Krull–Schmidt category $\mathcal{C}$ and a contravariantly finite subcategory $\mathcal{X}$ of $\mathcal{C}$ in the sense of Auslander-Smalø. Then under a mild condition on $\mathcal{C}$ the stable category $\mathcal{C}/\mathcal{X}$ admits the structure of a left triangulated category. Its Grothedieck group $\mathrm{K}_0(\mathcal{C}/\mathcal{X})$ is called the stable Grothendieck group. We construct some useful free presentations of $\mathrm{K}_0(\mathcal{C}/\mathcal{X})$ and we show that the Grothendieck group $\mathrm{K}_0(\mathrm{mod}(\mathcal{C}/\mathcal{X}))$ of the abelian category $\mathrm{mod}(\mathcal{C}/\mathcal{X})$ of finitely presented functors $(\mathcal{C}/\mathcal{X})^{op} \to \mathcal{A}b$ can be indentified with the kernel $\mathrm{E}(\mathcal{X})$ of the natural map $\mathrm{K}_0(\mathcal{C}, \oplus) \twoheadrightarrow \mathrm{K}_0(\mathcal{C}, \mathcal{X})$, where the latter is the Grothendieck group of $\mathcal{C}$ modulo the relations given by the $\mathcal{X}$-exact sequences. It follows that the group $\mathrm{E}(\mathcal{X})$ is invariant under stable equivalences. If $\mathcal{C}$ is abelian with enough projectives and $\mathcal{P}$ is the full subcategory of projectives of $\mathcal{C}$, then the stable Grothendieck group $\mathrm{K}_0(\mathcal{C}/\mathcal{P})$ is a classical object of study, appearing as the cokernel of the Cartan map $\mathrm{K}_0(\mathcal{P}, \oplus) \to \mathrm{K}_0(\mathcal{C})$. In this situation we can improve the main result of Sec. 2, by removing the functoriallity of the commutation of a given stable equivalence with the loop functors.

In Sec. 4 we interpret the Grothendieck group of a stable category as a suitable Waldhausen group and we indicate some consequences of this interpretation on the invariance of the stable Grothendieck group. We do this by establishing a bijective correspondence between pairs $(\mathcal{C}, \mathcal{X})$ where $\mathcal{C}$ is an additive category and $\mathcal{X}$ is a contravariantly finite subcategory such that any $\mathcal{X}$-epic admits a kernel, and additive Waldhausen categories of a special form.

In Sec. 5 we study other types of equivalences and their effect on Grothendieck groups and we apply the results of the previous sections to the stable category of finitely presented modules modulo projectives over a ring (usually an Artin algebra), generalizing some results of the literature. For instance we show that the stable Grothendieck group is invariant under derived equivalences, in particular under tilting, under stable equivalences which commute (not necessarily in a functorial way) with the loop functors (e.g., under general stable equivalences if the involved algebras are self-injective which are indecomposable or of Lowey length ≥ 2), and with the Auslander-Reiten operators in the representation-finite case.

A general convention used in the paper is that the composition of morphisms in a given category is meant in the diagrammatic order, i.e., the composition of $f : A \to B$, $g : B \to C$ is denoted by $f \circ g$. There is one exception: we use anti-diagrammatic order when we apply elements to morphisms in concrete categories.

2. LEFT TRIANGULATED CATEGORIES AND THEIR GROTHENDIECK GROUPS

Throughout this section we fix a left triangulated category $\mathcal{C} = (\mathcal{C}, \Omega, \Delta)$ and we assume that $\mathcal{C}$ is skeletally small, i.e., the collection $\mathrm{Iso}(\mathcal{C})$ of isoclasses of objects of $\mathcal{C}$ is a set.

We recall that a *left triangulated category* is a triple $(\mathcal{C}, \Omega, \Delta)$, where $\mathcal{C}$ is an additive category, $\Omega : \mathcal{C} \to \mathcal{C}$ is an additive endofunctor, and Δ is a class of diagrams in $\mathcal{C}$ of the form $(T) : \Omega(C) \to A \to B \to C$, satisfying all the axioms of a triangulated category [23], except that Ω is not necessarily an equivalence. The functor Ω is called the *loop functor* and the class of diagrams Δ is called the *left triangulation* of $\mathcal{C}$, see [5, 14] for more details. A diagram (T) as above is called *split*, if it is isomorphic to the diagram $\Omega(C) \stackrel{0}{\to} A \stackrel{(1_A,0)}{\to} A \oplus C \stackrel{{}^t(0,1_C)}{\to} C$. Since the latter is always in the left triangulation Δ, it follows that the class Δ_0 consisting of all split triangles, is a full subcategory of Δ.

If $\mathcal{U}$ is a class of objects in $\mathcal{C}$, then we denote by $\mathbb{Z}(\mathcal{U})$, the free abelian group with basis the set $\mathrm{Iso}(\mathcal{U})$ and by $(\ ) : \mathcal{U} \to \mathbb{Z}(\mathcal{U})$ the canonical map. Let $\mathcal{E}$ be a class of left triangles in Δ, closed under isomorphisms and such that $\Delta_0 \subseteq \mathcal{E} \subseteq \Delta$. We denote by $\langle \mathcal{E} \rangle$, the subgroup of $\mathbb{Z}(\mathcal{C})$ generated by all elements of the form $(A) - (B) + (C)$, whenever there exists a triangle $\Omega(C) \to A \to B \to C$ in $\mathcal{E}$.

Definition 2.1. *The Grothendieck group of $\mathcal{C}$ with respect to the class of left triangles $\mathcal{E}$ is defined by:* $\mathrm{K}_0(\mathcal{C}, \mathcal{E}) = \mathbb{Z}(\mathcal{C})/\langle \mathcal{E} \rangle$.

Choosing $\mathcal{E}$ to be the class Δ of all left triangles, we obtain the Grothendieck group $\mathrm{K}_0(\mathcal{C}, \Delta) := \mathrm{K}_0(\mathcal{C})$. Choosing $\mathcal{E}$ to be the class Δ_0 of split triangles, we obtain the *split* Grothendieck group $K_0(\mathcal{C}, \Delta_0) := K_0(\mathcal{C}, \oplus)$, i.e., $K_0(\mathcal{C}, \oplus)$ is the Grothendieck group of the monoidal category $(\mathcal{C}, \oplus)$. Observe that the inclusions $\Delta_0 \subseteq \mathcal{E} \subseteq \Delta$ induce canonical epimorphisms $K_0(\mathcal{C}, \oplus) \twoheadrightarrow K_0(\mathcal{C}, \mathcal{E}) \twoheadrightarrow K_0(\mathcal{C})$. Their composition induces a short exact sequence

$$0 \to D(\mathcal{C}) \xrightarrow{i_{\mathcal{C}}} K_0(\mathcal{C}, \oplus) \xrightarrow{p_{\mathcal{C}}} K_0(\mathcal{C}) \to 0 \tag{2.1}$$

in $\mathcal{A}b$. We denote by $|\,| : \mathbb{Z}(\mathcal{C}) \to K_0(\mathcal{C}, \oplus)$ and by $[\,] : \mathbb{Z}(\mathcal{C}) \to K_0(\mathcal{C})$ the canonical morphisms. Plainly $D(\mathcal{C})$ is the subgroup of $K_0(\mathcal{C}, \oplus)$ generated by all elements of the form $|A| - |B| + |C|$, where $\Omega(C) \to A \to B \to C$ is a left triangle in Δ. Observe that $\forall A, B \in \mathcal{C}$: $[A] + [B] = [A \oplus B]$, since the split triangle $\Omega(B) \to A \to A \oplus B \to B$ is in Δ. Parts (i), (ii), (iii) of the next observation follow trivially from [6] and part (iv) is proved as in the case of abelian categories.

Lemma 2.2. *(i)* $\forall C \in \mathcal{C}$, $\forall n \in \mathbb{N}$: $[\Omega^n(C)] = (-1)^n [C]$.

(ii) *Any element of* $K_0(\mathcal{C})$ *is of the form* $[C]$, *for* $C \in \mathcal{C}$.

(iii) $K_0(\mathcal{C}) = 0$, *if* $\forall C \in \mathcal{C}$, *there exists* $n_C \in \mathbb{N}$: $\Omega^{n_C}(C) = 0$.

(iv) $[C_1] = [C_2]$ *in* $K_0(\mathcal{C})$ *iff there are triangles* $\Omega(A'') \to A' \to A_1 \to A''$ *and* $\Omega\,(A'') \to A' \to A_2 \to A''$ *in* $\mathcal{C}$ *such that:* $C_1 \oplus A_1 \cong C_2 \oplus A_2$.

2.1. Cones and Finitely Presented Functors

It is easy to see that if $f : B \to C$ is a morphism in $\mathcal{C}$, and if $\Omega(C) \xrightarrow{h} A \xrightarrow{g} B \xrightarrow{f} C$, $\Omega(C) \xrightarrow{h'} A' \xrightarrow{g'} B \xrightarrow{f} C$ are left triangles in Δ, then there exists an isomorphism $\alpha : A \to A'$ such that $\alpha \circ g' = g$ and $h \circ \alpha = h'$. In particular $|A| = |A'|$ in $K_0(\mathcal{C}, \oplus)$ and $[A] = [A']$ in $K_0(\mathcal{C})$. However α is not uniquely determined in general. We call A the **cone** of f and we denote it by $Cone(f)$, bearing in mind that $Cone(f)$ is uniquely determined up to a non-unique isomorphism.

We recall that an additive functor $F : \mathcal{C}^{op} \to \mathcal{A}b$ is called *finitely presented*, if there exists an exact sequence of functors $\mathcal{C}(-, A) \xrightarrow{\mathcal{C}(-,f)} \mathcal{C}(-, B) \to F \to 0$. Such a finitely presented functor is denoted always by F_f. We denote by $\mathrm{mod}(\mathcal{C})$ the category of finitely presented functors $\mathcal{C}^{op} \to \mathcal{A}b$ and by $\mathbb{Y} : \mathcal{C} \hookrightarrow \mathrm{mod}(\mathcal{C})$ the Yoneda embedding, defined by $\mathbb{Y}(A) = \mathcal{C}(-, A)$ and $\mathbb{Y}(f) = \mathcal{C}(-, f)$. It is well known that if idempotents split in $\mathcal{C}$, then $\mathbb{Y}$ induces an equivalence between $\mathcal{C}$ and the category $\mathsf{Proj}(\mathrm{mod}(\mathcal{C}))$ of finitely generated projective functors.

Recall that a *weak kernel* of a morphism $f : B \to C$ in $\mathcal{C}$ is a morphism $g : A \to B$ such that any morphism $h : X \to B$ with $h \circ f = 0$, factors through g, in a not nessecarily unique way. Equivalently the sequence of functors $\mathbb{Y}(A) \xrightarrow{\mathbb{Y}(g)} \mathbb{Y}(B) \xrightarrow{\mathbb{Y}(f)} \mathbb{Y}(C)$ is exact. A *weak-kernel sequence* in $\mathcal{C}$, is a diagram $\cdots \to A_{n-1} \xrightarrow{f_{n-1}} A_n \xrightarrow{f_n} A_{n+1} \to \cdots$, in which f_{n-1} is a weak kernel of f_n, $\forall n$. By the axioms of a left triangulated category, it follows that any morphism in $\mathcal{C}$ admits a weak kernel. Then it is well-known and easy to see that $\mathrm{mod}(\mathcal{C})$ is abelian. If $f : B \to C$ is any morphism in $\mathcal{C}$ then by embedding f in a left triangle $\Omega(C) \to A \to B \xrightarrow{f} C$ and developing the latter to the left, we obtain a weak kernel-sequence $\cdots \to \Omega(A) \to \Omega(B) \xrightarrow{-\Omega(f)} \Omega(C) \to A \to B \xrightarrow{f} C$. Applying to this sequence the Yoneda embedding $\mathbb{Y}$, we obtain a projective resolution of F_f:

$$\cdots \to \mathbb{Y}\Omega(B) \to \mathbb{Y}\Omega(C) \to \mathbb{Y}(A) \to \mathbb{Y}(B) \xrightarrow{\mathbb{Y}(f)} \mathbb{Y}(C) \to F_f \to 0 \quad (2.2)$$

Definition 2.3. *Let $\mathcal{P}$ be a full additive subcategory of an additive category $\mathcal{F}$. We say that $\mathcal{F}$* **admits cancelation with respect to** *$\mathcal{P}$, if $\forall F \in \mathcal{F}, \forall P \in \mathcal{P}$: $F \oplus P \cong F \Rightarrow P = 0$, equivalently if the canonical map $\mathrm{K}_0(\mathcal{P}, \oplus) \to \mathrm{K}_0(\mathcal{F}, \oplus)$ defined by $|P| \mapsto |P|$ is a monomorphism.*

Proposition 2.4. *If $\Omega(B) \xrightarrow{\Omega(f)} \Omega(C) \xrightarrow{\alpha} D \xrightarrow{\beta} B \xrightarrow{f} C$ is a weak-kernel sequence in $\mathcal{C}$, then: $|\mathbb{Y}(D)| = |\mathbb{Y}(Cone(f))|$ in $\mathrm{K}_0(\mathrm{mod}(\mathcal{C}), \oplus)$.*

If the category $\mathrm{mod}(\mathcal{C})$ admits cancelation with respect to $\mathsf{Proj}(\mathrm{mod}(\mathcal{C}))$, then: $|D| = |Cone(f)|$ in $\mathrm{K}_0(\mathcal{C}, \oplus)$ and $[D] = [Cone(f)]$ in $\mathrm{K}_0(\mathcal{C})$.

Proof. Let $\Omega(C) \xrightarrow{h} A \xrightarrow{g} B \xrightarrow{f} C$ be a left triangle in $\mathcal{C}$. Then we have the following exact sequences in $\mathrm{mod}(\mathcal{C})$:

$$\mathbb{Y}\Omega(B) \xrightarrow{\mathbb{Y}\Omega(f)} \mathbb{Y}\Omega(C) \xrightarrow{\mathbb{Y}(h)} \mathbb{Y}(A) \xrightarrow{\mathbb{Y}(g)} \mathbb{Y}(B) \xrightarrow{\mathbb{Y}(f)} \mathbb{Y}(C) \to F_f \to 0,$$

$$\mathbb{Y}\Omega(B) \xrightarrow{\mathbb{Y}\Omega(f)} \mathbb{Y}\Omega(C) \xrightarrow{\mathbb{Y}(\alpha)} \mathbb{Y}(D) \xrightarrow{\mathbb{Y}(\beta)} \mathbb{Y}(B) \xrightarrow{\mathbb{Y}(f)} \mathbb{Y}(C) \to F_f \to 0.$$

The above exact sequences induce the following exact commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & Coker(\mathbb{Y}\Omega(f)) & \longrightarrow & \mathbb{Y}(A) & \longrightarrow & Ker(\mathbb{Y}(f)) & \longrightarrow & 0 \\
 & & \kappa \downarrow & & \lambda \downarrow & & \parallel \downarrow & & \\
0 & \longrightarrow & Coker(\mathbb{Y}\Omega(f)) & \longrightarrow & \mathbb{Y}(D) & \longrightarrow & Ker(\mathbb{Y}(f)) & \longrightarrow & 0
\end{array}$$

By Schanuel's lemma, it follows that $Coker(\mathbb{Y}\Omega(f)) \oplus \mathbb{Y}(A) \cong Coker(\mathbb{Y}\Omega(f)) \oplus \mathbb{Y}(D)$. Hence in the Grothendieck group $K_0(\text{mod}(\mathcal{C}), \oplus)$ we have $|Coker(\mathbb{Y}\Omega(f))| + |\mathbb{Y}(A)| = |Coker(\mathbb{Y}\ \Omega(f))| + |\mathbb{Y}(D)| \Rightarrow |\mathbb{Y}(A)| = |\mathbb{Y}(D)|$. If $\text{mod}(\mathcal{C})$ admits cancelation with respect to $\mathsf{Proj}(\text{mod}(\mathcal{C}))$, then $|\mathbb{Y}(A)| = |\mathbb{Y}(D)|$ in $K_0(\mathsf{Proj}(\text{mod}(\mathcal{C})), \oplus)$. The Yoneda functor $\mathbb{Y} : \mathcal{C} \hookrightarrow \text{mod}(\mathcal{C})$ induces a morphism $\mathbb{Y}_* : K_0(\mathcal{C}, \oplus) \to K_0(\mathsf{Proj}(\text{mod}(\mathcal{C})), \oplus)$, defined by $\mathbb{Y}_*(|A|) = |\mathbb{Y}(A)|$. Then $\mathbb{Y}_*(|A|) = \mathbb{Y}_*(|D|)$. Now the cofinality principle ensures that the inclusion of an additive category into its idempotent (or Karoubian) completion induces a monomorphism on the level of the split Grothendieck groups [21]. Since $\mathsf{Proj}(\text{mod}(\mathcal{C}))$ is the idempotent completion of $\mathcal{C} \approx \mathbb{Y}(\mathcal{C})$, the morphism $\mathbb{Y}_* : K_0(\mathcal{C}, \oplus) \to K_0(\mathsf{Proj}\ (\text{mod}(\mathcal{C})), \oplus)$ is a monomorphism. Then we have $|Cone(f)| = |A| = |D|$ in $K_0(\mathcal{C}, \oplus)$ and $[Cone(f)] = p_{\mathcal{C}}(|Cone(f)|) = p_{\mathcal{C}}(|D|) = [D]$ in $K_0(\mathcal{C})$. □

We recall that an additive category $\mathcal{F}$ is called a *Krull-Schmidt category*, if any object of $\mathcal{F}$ is a finite coproduct of indecomposable objects with local endomorphism ring. It is well-known that for a skeletally small Krull-Schmidt category $\mathcal{F}$, the group $K_0(\mathcal{F}, \oplus)$ is free with basis the set of isoclasses of indecomposable objects.

Lemma 2.5. *If $\mathcal{C}$ is a Krull-Schmidt category, then* $\text{mod}(\mathcal{C})$ *admits cancelation with respect to* $\mathsf{Proj}(\text{mod}(\mathcal{C}))$.

Proof. Let F be a finitely presented functor in $\text{mod}(\mathcal{C})$ and assume that $F \oplus \mathbb{Y}(C) \cong F$, where $C \in \mathcal{C}$. Since $\mathcal{C}$ is a Krull-Schmidt category, it is well-known that any finitely generated functor admits a projective cover. If $\rho : \mathbb{Y}(A) \twoheadrightarrow F$ is a projective cover, then so is $\rho \oplus 1_{\mathbb{Y}(C)} : \mathbb{Y}(A) \oplus \mathbb{Y}(C) \to F \oplus \mathbb{Y}(C)$. Since $F \oplus \mathbb{Y}(C) \cong F$, by the uniquenness of the projective covers we infer that there exists an isomorphism $\mathbb{Y}(A) \oplus \mathbb{Y}(C) \cong \mathbb{Y}(A) \Rightarrow \mathbb{Y}(A \oplus C) \cong \mathbb{Y}(A) \Rightarrow A \oplus C \cong A$. Since $\mathcal{C}$ is Krull-Schmidt this implies trivially that $C = 0$, i.e., $\mathbb{Y}(C) = 0$. □

Remark 2.6. Lemma 2.5 holds in the more general case in which the endomorphism ring $\text{End}_{\mathcal{C}}(C)$ of any object $C \in \mathcal{C}$ is F-semiperfect, i.e., $\text{End}_{\mathcal{C}}(C)/\mathbf{r}$ is Von Neumann regular and idempotents can be lifted modulo the Jacobson radical $\mathbf{r}$.

Corollary 2.7. *If $\mathcal{C}$ is a Krull-Schmidt category and $\Omega(B) \xrightarrow{\Omega(f)} \Omega(C) \xrightarrow{\alpha} D \xrightarrow{\beta} B \xrightarrow{f} C$ is a weak kernel sequence in $\mathcal{C}$, then: $D \cong Cone(f)$.*

Proof. By Proposition 2.4 and Lemma 2.5, we have $|D| = |Cone(f)|$ in $K_0(\mathcal{C}, \oplus)$. Since $\mathcal{C}$ is Krull-Schmidt, this implies trivially that $Cone(f) \cong D$. □

In general the Yoneda embedding $\mathbb{Y} : \mathcal{C} \to \mathrm{mod}(\mathcal{C})$ does not induces a group homomorphism $\mathbb{Y}_* : \mathrm{K}_0(\mathcal{C}) \to \mathrm{K}_0(\mathrm{mod}(\mathcal{C}))$, except if any triangle in $\mathcal{C}$ splits. We correct this situation as follows. Let $\mathcal{G}$ be the subgroup of $\mathrm{K}_0(\mathrm{mod}(\mathcal{C}))$, generated by all elements of the form $[F_f] + [F_{\Omega(f)}]$, where f is a morphism in $\mathcal{C}$ and set $\widetilde{\mathrm{K}}_0(\mathrm{mod}(\mathcal{C})) = \mathrm{K}_0(\mathrm{mod}(\mathcal{C}))/\mathcal{G}$. We call $\widetilde{\mathrm{K}}_0(\mathrm{mod}(\mathcal{C}))$ the *reduced* Grothendieck group of $\mathrm{mod}(\mathcal{C})$ and we denote by $\| \, \| : \mathrm{K}_0(\mathrm{mod}\,(\mathcal{C})) \to \widetilde{\mathrm{K}}_0(\mathrm{mod}(\mathcal{C}))$ the natural map. It is easily checked that $||F_{\Omega^n f}|| = (-1)^n ||F_f||$. Let $\Omega(C) \to A \to B \to C$ be a triangle in $\mathcal{C}$. Then the resolution (2.2) shows that $[F_f] = [\mathbb{Y}(C)] - [\mathbb{Y}(B)] + [\mathbb{Y}(A)] - [F_{\Omega(f)}]$ in $\mathrm{K}_0(\mathrm{mod}(\mathcal{C}))$. Hence $[F_f] + [F_{\Omega(f)}] = [\mathbb{Y}(C)] - [\mathbb{Y}(B)] + [\mathbb{Y}(A)]$. It follows that $[\mathbb{Y}(C)] - [\mathbb{Y}(B)] + [\mathbb{Y}(A)] = 0$ in $\widetilde{\mathrm{K}}_0(\mathrm{mod}(\mathcal{C}))$, so the assignement $[C] \mapsto ||\mathbb{Y}(C)||$ defines a group homomorphism $||Y(\cdot)|| : \mathrm{K}_0(\mathcal{C}) \to \widetilde{\mathrm{K}}_0(\mathrm{mod}(\mathcal{C}))$.

2.2. Equivalences

If $\mathcal{C} = (\mathcal{C}, \Omega, \Delta)$ and $\mathcal{C}' = (\mathcal{C}', \Omega', \Delta')$ are left triangulated categories, then a **grade functor** $\mathbf{F} : \mathcal{C} \to \mathcal{C}'$ is a pair $(\mathbf{F}, \omega) : \mathcal{C} \to \mathcal{C}'$ where $\mathbf{F} : \mathcal{C} \to \mathcal{C}'$ is an additive functor and $\omega : \mathbf{F}\Omega \to \Omega'\mathbf{F}$ is a natural isomorphism. An **exact functor** $\mathbf{F} : \mathcal{C} \to \mathcal{C}'$ is a grade functor $(\mathbf{F}, \omega)$ such that for any left triangle $\Omega(C) \xrightarrow{h} A \xrightarrow{g} B \xrightarrow{f} C$ in Δ, the diagram $\Omega'\mathbf{F}(C) \xrightarrow{\omega_C^{-1}\mathbf{F}(h)} \mathbf{F}(A) \xrightarrow{\mathbf{F}(g)} \mathbf{F}(B) \xrightarrow{\mathbf{F}(f)} \mathbf{F}(C)$ is a left triangle in Δ'. A grade, resp. exact, functor $(\mathbf{F}, \omega) : \mathcal{C} \to \mathcal{C}'$ is called a **grade**, resp. **triangle**, **equivalence**, if $\mathbf{F}$ is an equivalence of categories.

It is trivial to see that if $\mathbf{F} : \mathcal{C} \to \mathcal{C}'$ is an exact functor between left triangulated categories, then $\mathbf{F}$ induces a morphism $\mathrm{K}_0(\mathbf{F}) : \mathrm{K}_0(\mathcal{C}) \to \mathrm{K}_0(\mathcal{C}')$ defined by $\mathrm{K}_0(\mathbf{F})([A]) = [\mathbf{F}(A)]$ and a morphism $\widetilde{\mathrm{K}}_0(\mathrm{mod}\ (\mathcal{C})) \to \widetilde{\mathrm{K}}_0(\mathrm{mod}(\mathcal{C}'))$. For instance the loop functor $\Omega : \mathcal{C} \to \mathcal{C}$ is exact and $\mathrm{K}_0(\Omega) = -\mathrm{Id}$. Plainly if $\mathbf{F}$ is a triangle equivalence, then $\mathrm{K}_0(\mathbf{F})$ is invertible. However our main purpose is to study when $\mathbf{F}$ induces an isomorphism $\mathrm{K}_0(\mathbf{F})$, in case $\mathbf{F}$ is not necessarily an exact equivalence.

If $\mathbf{F} : \mathcal{C} \to \mathcal{C}'$ is an additive functor, then we have an induced morphism $\widetilde{\mathbf{F}} : \mathrm{K}_0(\mathcal{C}, \oplus) \to \mathrm{K}_0(\mathcal{C}', \oplus)$, defined by $\widetilde{\mathbf{F}}(|C|) = |\mathbf{F}(C)|$, and an exact diagram:

$$(\dagger) \qquad \begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{D}(\mathcal{C}) & \xrightarrow{\ i\ } & \mathrm{K}_0(\mathcal{C}, \oplus) & \xrightarrow{\ p\ } & \mathrm{K}_0(\mathcal{C}) & \longrightarrow & 0 \\ & & & & \Big\downarrow{\scriptstyle \widetilde{\mathbf{F}}} & & & & \\ 0 & \longrightarrow & \mathrm{D}(\mathcal{C}') & \xrightarrow{\ i'\ } & \mathrm{K}_0(\mathcal{C}', \oplus) & \xrightarrow{\ p'\ } & \mathrm{K}_0(\mathcal{C}') & \longrightarrow & 0 \end{array}$$

Trivially $\widetilde{\mathbf{F}}$ is invertible, if $\mathbf{F}$ is an equivalence.

Theorem 2.8. *Let $\mathbf{F} : \mathcal{C} \to \mathcal{C}'$ be an equivalence between the left triangulated categories $\mathcal{C}, \mathcal{C}'$, with quasi-inverse $\mathbf{G}$. Then the following statements are equivalent.*

(i) The assignement $[A] \mapsto [\mathbf{F}(A)]$ induces an isomorphism $\mathbf{F}^* : \mathrm{K}_0(\mathcal{C}) \to \mathrm{K}_0(\mathcal{C}')$.
(ii) The assignement $|A| \mapsto |\mathbf{F}(A)|$ induces an isomorphism $\mathbf{F}_* : \mathrm{D}_0(\mathcal{C}) \to \mathrm{D}_0(\mathcal{C}')$.
(iii) For any morphism f in $\mathcal{C}$: $[\mathbf{F}(Cone(f))] = [Cone(\mathbf{F}(f))]$ in $\mathrm{K}_0(\mathcal{C}')$ and for any morphism g in $\mathcal{C}'$: $[\mathbf{G}(Cone(g))] = [Cone(\mathbf{G}(g))]$ in $\mathrm{K}_0(\mathcal{C})$.

Proof. (iii) $\Rightarrow$ (i) Consider the group homomorphism $\phi : \mathbb{Z}(\mathcal{C}) \to \mathrm{K}_0(\mathcal{C}')$ defined by $\phi((C)) = [\mathbf{F}(C)]$, uniquely induced by the map $\mathrm{Iso}(\mathcal{C}) \to \mathrm{K}_0(\mathcal{C}')$ sending the class of C to $[\mathbf{F}(C)]$. If $(A) - (B) + (C) \in \mathbb{Z}(\mathcal{C})$, where $\Omega(C) \to A \to B \xrightarrow{f} C$ is a triangle in $\mathcal{C}$, then $\phi((A) - (B) + (C)) = [\mathbf{F}(A)] - [\mathbf{F}(B)] + [\mathbf{F}(C)] = [\mathbf{F}(Cone(f))] - [\mathbf{F}(B)] + [\mathbf{F}(C)] = [Cone(\mathbf{F}(f))] - [\mathbf{F}(B)] + [\mathbf{F}(C)] = 0$, since in $\mathcal{C}'$ we have the triangle $\Omega'\mathbf{F}(C) \to Cone(\mathbf{F}(f)) \to \mathbf{F}(B) \xrightarrow{\mathbf{F}(f)} \mathbf{F}(C)$. Hence there exists a unique homomorphism $\mathbf{F}^* : \mathrm{K}_0(\mathcal{C}) \to \mathrm{K}_0(\mathcal{C}')$ defined by $\mathbf{F}^*([C]) = [\mathbf{F}(C)]$. Similarly using the quasi-inverse $\mathbf{G}$, we see that there exists a unique homomorphism $\mathbf{G}^* : \mathrm{K}_0(\mathcal{C}') \to \mathrm{K}_0(\mathcal{C})$ defined by $\mathbf{G}^*([A]) = [\mathbf{G}(A)]$. Plainly $\mathbf{F}^*$ is the inverse of $\mathbf{G}^*$.

(i) $\Rightarrow$ (iii) Let $\Omega(C) \to A \to B \xrightarrow{f} C$ be a left triangle in Δ. Then we have a relation $[B] = [A] + [C] = [Cone(f)] + [C]$ in $\mathrm{K}_0(\mathcal{C})$. Since $\mathbf{F}^*$ is a group morphism, we have $[\mathbf{F}(B)] = [\mathbf{F}(A)] + [\mathbf{F}(C)] = [\mathbf{F}(Cone(f))] + [\mathbf{F}(C)]$ in $\mathrm{K}_0(\mathcal{C}')$. Let $\Omega'\mathbf{F}(C) \to D \to \mathbf{F}(B) \xrightarrow{\mathbf{F}(f)} \mathbf{F}(C)$ be a left triangle in Δ'. Since we have also $[\mathbf{F}(B)] = [D] + [\mathbf{F}(C)] = [Cone(\mathbf{F}(f))] + [\mathbf{F}(C)]$, it follows that $[Cone(\mathbf{F}(f))] = [\mathbf{F}(Cone(f))]$. The proof of the remaining assertion is similar.

(i) $\Rightarrow$ (ii) The isomorphism $\mathbf{F}^* : \mathrm{K}_0(\mathcal{C}) \to \mathrm{K}_0(\mathcal{C}')$ completes the right square of the diagram (†) above in a unique way. Hence there exists a unique isomorphism $\mathbf{F}_* : \mathrm{D}(\mathcal{C}) \to \mathrm{D}(\mathcal{C}')$ making the above diagram commutative. Obviously $\mathbf{F}_*$ is defined by $\mathbf{F}_*(|A|) = |\mathbf{F}(A)|$. The direction (ii) $\Rightarrow$ (i) is similar. □

The following result is a useful refinement of the above Theorem.

Proposition 2.9. *Let $\mathbf{F} : \mathcal{C} \to \mathcal{C}'$ be an equivalence between the left triangulated categories $(\mathcal{C}, \Omega, \Delta)$ and $(\mathcal{C}', \Omega', \Delta')$. If for any morphism f in $\mathcal{C}$ it holds: $|\mathbf{F}(Cone(f))| = |Cone(\mathbf{F}(f))|$ in $\mathrm{K}_0(\mathcal{C}', \oplus)$, then the assignement $[A] \mapsto [\mathbf{F}(A)]$ induces an isomorphism $\mathbf{F}^* : \mathrm{K}_0(\mathcal{C}) \to \mathrm{K}_0(\mathcal{C}')$.*

Proof. The hypothesis implies that $[Cone(\mathbf{F}(f))] = [\mathbf{F}(Cone(f))]$, for any morphism $f : B \to C$ in $\mathcal{C}$. As in the above Theorem, this implies that $\widetilde{\mathbf{F}}$

induces a morphism $\mathbf{F}_* : \mathrm{D}(\mathcal{C}) \to \mathrm{D}(\mathcal{C}')$ making the left square of the diagram (†), commutative. Obviously $\mathbf{F}_*$ is a monomorphism. We show that $\mathbf{F}_*$ is also an epimorphism. To prove this it is sufficient to prove that $\mathbf{F}_*$ is surjective on generators. So let $|A'| - |B'| + |C'|$ be a generator of $\mathrm{D}(\mathcal{C}')$, where $\Omega'(C') \to A' \to B' \xrightarrow{f'} C'$ is a left triangle in $\mathcal{C}'$. Let $B, C \in \mathcal{C}$ and let $f : B \to C$ be a morphism in $\mathcal{C}$ such that $\mathbf{F}(B) = B'$, $\mathbf{F}(C) = C'$, $\mathbf{F}(f) = f'$. Then in $\mathrm{K}_0(\mathcal{C}', \oplus)$ we have $|\mathbf{F}(Cone(f))| = |Cone(\mathbf{F}(f))| = |Cone(f')| = |A'|$. Consider the element $|Cone(f)| - |B| + |C| \in \mathrm{K}_0(\mathcal{C}, \oplus)$, which lies in $\mathrm{D}(\mathcal{C})$, since we have a triangle $\Omega(C) \to Cone(f) \to B \xrightarrow{f} C$ in $\mathcal{C}$. Then $\mathbf{F}_*(|Cone(f)| - |B| + |C|) = |\mathbf{F}(Cone(f))| - |\mathbf{F}(B)| + \mathbf{F}(C)| = |A'| - |B'| + |C'|$. We infer that $\mathbf{F}_*$ is surjective, hence an isomorphism. By the diagram (†), it follows that there exists a unique isomorphism $\mathbf{F}^* : \mathrm{K}_0(\mathcal{C}) \to \mathrm{K}_0(\mathcal{C}')$ such that the whole diagram commutes. It is easy to see that indeed $\mathbf{F}^*$ is given by $\mathbf{F}^*([A]) = [\mathbf{F}(A)]$. □

The following main result of this section shows that the Grothendieck group of a Krull-Schmidt left triangulated category is invariant under grade equivalences.

Theorem 2.10. *Let $\mathbf{F} : \mathcal{C} \to \mathcal{C}'$ be a grade equivalence between the left triangulated categories $\mathcal{C}, \mathcal{C}'$. If $\mathcal{C}'$ is Krull-Schmidt, then the assignement $[A] \mapsto [\mathbf{F}(A)]$ induces an isomorphism $\mathbf{F}^* : \mathrm{K}_0(\mathcal{C}) \to \mathrm{K}_0(\mathcal{C}')$ and an isomorphism $\mathbf{F}_* : \mathrm{D}(\mathcal{C}) \to \mathrm{D}(\mathcal{C}')$. Further $\mathbf{F}$ induces an isomorphism $\widetilde{\mathrm{K}}_0(\mathrm{mod}(\mathcal{C})) \xrightarrow{\cong} \widetilde{\mathrm{K}}_0(\mathrm{mod}(\mathcal{C}'))$.*

Proof. Let $f : B \to C$ be a morphism in $\mathcal{C}$ and let $\Omega(C) \to A \to B \xrightarrow{f} C$ be a left triangle in Δ. Consider the sequence $\mathbf{F}\Omega(B) \xrightarrow{\mathbf{F}\Omega(f)} \mathbf{F}\Omega(C) \to \mathbf{F}(A) \to \mathbf{F}(B) \xrightarrow{\mathbf{F}(f)} \mathbf{F}(C)$ in $\mathcal{C}'$. Since $\mathbf{F}$ commutes functorially with the loop functors Ω, Ω', the above sequence is isomorphic to $\Omega'\mathbf{F}(B) \xrightarrow{\Omega'\mathbf{F}(f)} \Omega'\mathbf{F}(C) \to \mathbf{F}(A) \to \mathbf{F}(B) \xrightarrow{\mathbf{F}(f)} \mathbf{F}(C)$. The last diagram is a weak kernel sequence in $\mathcal{C}'$, since $\mathbf{F}$ as an equivalence preserves weak kernels. Then by Corollary 2.7, $|Cone(\mathbf{F}(f))| = |\mathbf{F}(Cone(f))| = |\mathbf{F}(A)|$ in $\mathrm{K}_0(\mathcal{C}', \oplus)$ and by Proposition 2.9 we infer that the assignement $[C] \mapsto [\mathbf{F}(C)]$ induces an isomorphism $\mathbf{F}^* : \mathrm{K}_0(\mathcal{C}) \to \mathrm{K}_0(\mathcal{C}')$. Finally for any morphism f in $\mathcal{C}$, we have: $[F_{\mathbf{F}(f)}] + [F_{\Omega'\mathbf{F}(f)}] = [F_{\mathbf{F}(f)}] + [F_{\mathbf{F}\Omega(f)}]$ in $\mathrm{K}_0(\mathrm{mod}(\mathcal{C}'))$. This implies that the isomorphism $\mathrm{K}_0(\mathrm{mod}(\mathcal{C})) \xrightarrow{\cong} \mathrm{K}_0(\mathrm{mod}(\mathcal{C}'))$ induced by $\mathbf{F}$, induces an isomorphism $\widetilde{\mathrm{K}}_0(\mathrm{mod}(\mathcal{C})) \xrightarrow{\cong} \widetilde{\mathrm{K}}_0(\mathrm{mod}(\mathcal{C}'))$ in the reduced Grothendieck groups. □

We shall see in the next section that in some specific situations it is possible to remove the functoriality of the isomorphism $\mathbf{F}\Omega(C) \cong \Omega'\mathbf{F}(C)$, $\forall C \in \mathcal{C}$.

3. GROTHENDIECK GROUPS OF STABLE CATEGORIES AND STABLE EQUIVALENCES

Throughout this section we fix a skeletally small additive category $\mathcal{C}$ with split idempotents. Let $\mathcal{X} \subseteq \mathcal{C}$ be a full additive subcategory of $\mathcal{C}$ closed under direct summands and isomorphisms. We denote by $\mathcal{C}/\mathcal{X}$ the induced stable category. We recall that the objects of $\mathcal{C}/\mathcal{X}$ are the objects of $\mathcal{C}$. The morphisms of $\mathcal{C}/\mathcal{X}$ are equivalence classes of morphisms, two morphisms being equivalent if their difference factorizes through an object of $\mathcal{X}$. We denote by $\underline{C}$ the object $C \in \mathcal{C}$ viewed as an object in $\mathcal{C}/\mathcal{X}$ and by $\underline{f}$, the equivalence class of a morphism f in $\mathcal{C}$. Setting $\pi(C) = \underline{C}$ and $\pi(f) = \underline{f}$, we obtain the additive *projection* functor $\pi : \mathcal{C} \to \mathcal{C}/\mathcal{X}$.

3.1. Stable Categories as Left Triangulated Categories

We call a morphism $f : B \to C$ in $\mathcal{C}$, an $\mathcal{X}$-*epic*, if the morphism $\mathcal{C}(\mathcal{X}, f) : \mathcal{C}(\mathcal{X}, B) \to \mathcal{C}(\mathcal{X}, C)$ is surjective in $\mathcal{A}b$. We recall [2] that $\mathcal{X}$ is called *contravariantly finite* in $\mathcal{C}$, if for any object $C \in \mathcal{C}$, there exists an $\mathcal{X}$-epic $\chi_C : X_C \to C$ with $X_C \in \mathcal{X}$; such a morphism χ_C is called a *right $\mathcal{X}$-approximation* of C. The pair $(\mathcal{C}, \mathcal{X})$ is called a *left homotopy pair* [7], if $\mathcal{X}$ is contravariantly finite in $\mathcal{C}$ and any $\mathcal{X}$-epic in $\mathcal{C}$ has a kernel. We entrust to the reader to formulate the dual notions of $\mathcal{X}$-monic, covariantly finite subcategory and right homotopy pair.

Throughout this section we fix a left homotopy pair $(\mathcal{C}, \mathcal{X})$. By [5], the stable category $\mathcal{C}/\mathcal{X}$ is equipped in a natural way with a left triangulated structure $(\mathcal{C}/\mathcal{X}, \Omega_{\mathcal{X}}, \Delta_{\mathcal{X}})$. We recall briefly the definition of the loop functor $\Omega_{\mathcal{X}}$ and the left triangulation $\Delta_{\mathcal{X}}$, and we refer to [5] for details. Let $C, D \in \mathcal{C}$ and let $0 \to K_C \xrightarrow{g_C} X_C \xrightarrow{\chi_C} C$, $0 \to K_D \xrightarrow{g_D} X_D \xrightarrow{\chi_D} D$ be left exact sequences in $\mathcal{C}$, where χ_C, resp. χ_D, is a right $\mathcal{X}$-approximation of C, resp. D. Then any morphism $\alpha : C \to D$ induces a commutative diagram

$$\begin{array}{ccccccc}
0 & \longrightarrow & K_C & \xrightarrow{g_C} & X_C & \xrightarrow{\chi_C} & C \\
 & & \downarrow{\scriptstyle \kappa_\alpha} & & \downarrow{\scriptstyle \chi_\alpha} & & \downarrow{\scriptstyle \alpha} \\
0 & \longrightarrow & K_D & \xrightarrow{g_D} & X_D & \xrightarrow{\chi_D} & D
\end{array}$$

Setting $\Omega_{\mathcal{X}}(\underline{C}) = \underline{K_C}$, and $\Omega_{\mathcal{X}}(\underline{\alpha}) = \underline{\kappa_\alpha}$, we obtain the well–defined additive loop functor $\Omega_{\mathcal{X}} : \mathcal{C}/\mathcal{X} \to \mathcal{C}/\mathcal{X}$. The triangulation $\Delta_{\mathcal{X}}$ of the pair $(\mathcal{C}/\mathcal{X}, \Omega_{\mathcal{X}})$

is defined as follows. Let $0 \to A \xrightarrow{g} B \xrightarrow{f} C$ be a left exact sequence in $\mathcal{C}$, where f is an $\mathcal{X}$-epic. Then there exists an exact commutative diagram

$$\begin{array}{ccccccc}
0 & \longrightarrow & K_C & \xrightarrow{g_C} & X_C & \xrightarrow{\chi_C} & C \\
 & & h\downarrow & & \gamma\downarrow & & \|\downarrow \\
0 & \longrightarrow & A & \xrightarrow{g} & B & \xrightarrow{f} & C
\end{array}$$

which induces a diagram $\Omega_{\mathcal{X}}(\underline{C}) \xrightarrow{\underline{h}} \underline{A} \xrightarrow{\underline{g}} \underline{B} \xrightarrow{\underline{f}} \underline{C}$ in $\mathcal{C}/\mathcal{X}$. Then $\Delta_{\mathcal{X}}$ is defined to be the collection of all such diagrams together with their isomorphic copies.

Any morphism $f : B \to C$ in $\mathcal{C}$ can be made to be an $\mathcal{X}$-epic as follows. Let $\chi_C : X_C \to C$ be a right $\mathcal{X}$-approximation and set $Cyl(f) = B \oplus X_C$. Then ${}^t(f, \chi_C) : Cyl(f) \to C$ is an $\mathcal{X}$-epic, hence it induces a left exact sequence $0 \to Cone(f) \to Cyl(f) \xrightarrow{{}^t(f,\chi_C)} C$ in $\mathcal{C}$, where now $Cone(f)$ denotes the pull-back of f and χ_C:

$$\begin{array}{ccccccc}
0 & \longrightarrow & K_C & \xrightarrow{h} & Cone(f) & \xrightarrow{g} & B \\
 & & \|\downarrow & & \lambda\downarrow & & f\downarrow \\
0 & \longrightarrow & K_C & \xrightarrow{\kappa_C} & X_C & \xrightarrow{\chi_C} & C
\end{array}$$

Plainly $\underline{Cyl(f)} = \underline{B}$ in $\mathcal{C}/\mathcal{X}$ and we have a triangle $\Omega_{\mathcal{X}}(\underline{C}) \to \underline{Cone(f)} \to \underline{B} \xrightarrow{\underline{f}} \underline{C}$ in $\Delta_{\mathcal{X}}$. Hence for any morphism f in $\mathcal{C}$ we have: $\underline{Cone(f)} \cong Cone(\underline{f})$.

Remark 3.1. (1) If $f : B \to C$ is a morphism in $\mathcal{C}$ and $\chi_{Cone(f)} : X_{Cone(f)} \to Cone(f)$ is a right $\mathcal{X}$-approximation of $Cone(f)$, then the morphisms $\chi_{Cone(f)} \circ g : X_{Cone(f)} \to B$ and $\chi_{Cone(f)} \circ g \circ f : X_{Cone(f)} \to C$ are right $\mathcal{X}$-approximations of B, C respectively. Indeed this follows easily from the above pull-back diagram.

(2) If the morphism f is an $\mathcal{X}$-epic with kernel A, then the pull-back property forces λ to be also an $\mathcal{X}$-epic. Since X_C is in $\mathcal{X}$, it follows that λ splits and we have an isomorphism $Cone(f) \cong A \oplus X_C$. Hence: $\underline{Cone(f)} \cong Cone(\underline{f}) \cong \underline{A}$.

(3) $\forall B, C \in \mathcal{C}$: $\underline{B} \cong \underline{C} \Leftrightarrow B \oplus X_C \cong C \oplus X_B$.

3.2. Presentations of Grothendieck Groups

We denote by $\mathrm{K}_0(\mathcal{C}/\mathcal{X})$ the Grothendieck group of the left triangulated category $\mathcal{C}/\mathcal{X}$, and we call it the **stable Grothendieck group** of $\mathcal{C}$. There is another group which is closely related to the stable Grothendieck group. We define $\mathrm{K}_0(\mathcal{C}, \mathcal{X})$ to be the quotient of $\mathbb{Z}(\mathcal{C})$ modulo the subgroup generated by all elements of the form $(A) - (B) + (C)$, whenever there exists a left exact sequence $0 \to A \to B \xrightarrow{f} C$ in $\mathcal{C}$ with f an $\mathcal{X}$-epic. From now on such a sequence is called an $\mathcal{X}$-*exact sequence*. Then it is proved in [6] that there exists an exact sequence

$$0 \to \mathrm{Ker}(\mathbf{c}_{\mathcal{X}}) \to \mathrm{K}_0(\mathcal{X}, \oplus) \xrightarrow{\mathbf{c}_{\mathcal{X}}} \mathrm{K}_0(\mathcal{C}, \mathcal{X}) \to \mathrm{K}_0(\mathcal{C}/\mathcal{X}) \to 0 \tag{3.1}$$

where $\mathbf{c}_{\mathcal{X}}$ is the **Cartan morphism** defined by $\mathbf{c}_{\mathcal{X}}(|X|) = [X]$. If any object $C \in \mathcal{C}$ admits a finite $\mathcal{X}$-resolution $0 \to X_n \to \cdots \to X_1 \to X_0 \to C \to 0$ in the sense that $X_t \in \mathcal{X}, \forall t \geq 0$ and the sequence remains exact after the application of $\mathcal{C}(X, -)$, $\forall X \in \mathcal{X}$, then by [6], $\mathbf{c}_{\mathcal{X}}$ is invertible and $\mathrm{K}_0(\mathcal{C}/\mathcal{X}) = 0$.

Consider the naturally induced short exact sequences

$$0 \to \mathrm{E}(\mathcal{X}) \to \mathrm{K}_0(\mathcal{C}, \oplus) \xrightarrow{\varpi_{\mathcal{X}}} \mathrm{K}_0(\mathcal{C}, \mathcal{X}) \to 0 \tag{3.2}$$

$$0 \to \mathrm{D}(\mathcal{X}) \to \mathrm{K}_0(\mathcal{C}/\mathcal{X}, \oplus) \xrightarrow{\pi_{\mathcal{X}}} \mathrm{K}_0(\mathcal{C}/\mathcal{X}) \to 0 \tag{3.3}$$

The following exact commutative diagram of short exact sequences:

$$(\dagger) \qquad \begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{Ker}(\mathbf{c}_{\mathcal{X}}) & \longrightarrow & \mathrm{K}_0(\mathcal{X}, \oplus) & \longrightarrow & \mathrm{Im}(\mathbf{c}_{\mathcal{X}})) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathrm{E}(\mathcal{X}) & \longrightarrow & \mathrm{K}_0(\mathcal{C}, \oplus) & \longrightarrow & \mathrm{K}_0(\mathcal{C}, \mathcal{X}) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathrm{D}(\mathcal{X}) & \longrightarrow & \mathrm{K}_0(\mathcal{C}/\mathcal{X}, \oplus) & \longrightarrow & \mathrm{K}_0(\mathcal{C}/\mathcal{X}) & \longrightarrow & 0
\end{array}$$

includes all the above information and it induces an exact sequence in $\mathcal{A}b$:

$$0 \to \mathrm{Ker}(\mathbf{c}_{\mathcal{X}}) \to \mathrm{E}(\mathcal{X}) \xrightarrow{\omega_{\mathcal{X}}} \mathrm{K}_0(\mathcal{C}/\mathcal{X}, \oplus) \to \mathrm{K}_0(\mathcal{C}/\mathcal{X}) \to 0 \tag{3.4}$$

where $\omega_{\mathcal{X}}$ is the composition $\mathrm{E}(\mathcal{X}) \to \mathrm{D}(\mathcal{X}) \to \mathrm{K}_0(\mathcal{C}/\mathcal{X}, \oplus)$. Clearly $\mathrm{E}(\mathcal{X})$ is the subgroup of $\mathrm{K}_0(\mathcal{C}, \oplus)$ generated by all elements of the form $|A| - |B| + |C| \in \mathrm{K}_0(\mathcal{C}, \oplus)$ for which there exists an $\mathcal{X}$-exact sequence $0 \to A \xrightarrow{g} B \xrightarrow{f} C$ in $\mathcal{C}$. Similarly $\mathrm{D}(\mathcal{X})$ is the subgroup of $\mathrm{K}_0(\mathcal{C}/\mathcal{X}, \oplus)$ generated by all elements of the form $|\underline{A}| - |\underline{B}| + |\underline{C}| \in \mathrm{K}_0(\mathcal{C}/\mathcal{X}, \oplus)$ for which there exists a left triangle $\Omega_{\mathcal{X}}(\underline{C}) \to \underline{A} \to \underline{B} \to \underline{C}$ in $\Delta_{\mathcal{X}}$.

Remark 3.2. If $\mathcal{C}$ is Krull-Schmidt, then the categories $\mathcal{X}, \mathcal{C}/\mathcal{X}$ also enjoy the Krull-Schmidt property. It follows in this case that the "split" groups $\mathrm{K}_0(\mathcal{C}, \oplus)$, $\mathrm{K}_0(\mathcal{X}, \oplus)$, $\mathrm{K}_0(\mathcal{C}/\mathcal{X}, \oplus)$ are free abelian on the set of isoclasses of their indecomposable objects and moreover $\mathrm{K}_0(\mathcal{C}, \oplus) \cong \mathrm{K}_0\ (\mathcal{X}, \oplus) \oplus \mathrm{K}_0\ (\mathcal{C}/\mathcal{X}, \oplus)$. It follows that $\mathrm{E}(\mathcal{X}), \mathrm{D}(\mathcal{X})$, $\mathrm{Ker}(\mathbf{c}_{\mathcal{X}})$ are free abelian, the exact sequence (3.4) is a free resolution of the stable group $\mathrm{K}_0(\mathcal{C}/\mathcal{X})$ and: $\mathrm{E}(\mathcal{X}) \cong \mathrm{D}(\mathcal{X}) \oplus \mathrm{Ker}(\mathbf{c}_{\mathcal{X}})$.

From now on we denote by $\mathsf{A}(\mathcal{X})$ the free group with basis the subset

$$\{|A| - |B| + |C| \in \mathrm{K}_0(\mathcal{C}, \oplus) \mid \text{whenever } 0 \to A \to B \to C \text{ is } \mathcal{X}\text{- exact in } \mathcal{C}\}.$$

Consider the canonical epimorphism $\mathsf{A}(\mathcal{X}) \twoheadrightarrow \mathrm{E}(\mathcal{X})$. Composing this morphism with the composition $\mathrm{E}(\mathcal{X}) \twoheadrightarrow \mathrm{D}(\mathcal{X}) \hookrightarrow \mathrm{K}_0(\mathcal{C}/\mathcal{X}, \oplus)$, we obtain a morphism $\alpha_{\mathcal{X}} : \mathsf{A}(\mathcal{X}) \to \mathrm{K}_0(\mathcal{C}/\mathcal{X})$ and an exact sequence

$$\mathsf{A}(\mathcal{X}) \xrightarrow{\alpha_{\mathcal{X}}} \mathrm{K}_0(\mathcal{C}/\mathcal{X}, \oplus) \to \mathrm{K}_0(\mathcal{C}/\mathcal{X}) \to 0 \tag{3.5}$$

If $\mathcal{C}$ is a Krull-Schmidt category, then (3.5) is a free presentation of the stable Grothendieck group, which is some cases it is more useful to consider.

3.3. Stable Equivalences

We fix two left homotopy pairs $(\mathcal{C}, \mathcal{X})$ and $(\mathcal{D}, \mathcal{Y})$ and we regard the stable categories $\mathcal{C}/\mathcal{X}$, $\mathcal{D}/\mathcal{Y}$ as left triangulated categories. We fix also a stable equivalence $\mathbf{F} : \mathcal{C}/\mathcal{X} \xrightarrow{\approx} \mathcal{D}/\mathcal{Y}$, which is not necessarily exact. The next result shows that the group $\mathsf{A}(\mathcal{X})$ is invariant under stable equivalence.

Proposition 3.3. *The functor* $\mathbf{F}$ *induces an isomorphism* $\mathbf{F}_2 : \mathsf{A}(\mathcal{X}) \xrightarrow{\cong} \mathsf{A}(\mathcal{Y})$.

Proof. Let $|A| - |B| + |C|$ be a generator of $\mathsf{A}(\mathcal{X})$, which comes from an $\mathcal{X}$-exact sequence $0 \to A \xrightarrow{g} B \xrightarrow{f} C$ in $\mathcal{C}$. Let $A', B' \in \mathcal{D}$ such that $\mathbf{F}(\underline{B}) = \underline{B}'$, $\mathbf{F}(\underline{C}) = \underline{C}'$ and let $f' : B' \to C'$ be a morphism in $\mathcal{D}$ such that $\mathbf{F}(\underline{f}) = \underline{f}'$. Then f' induces a $\mathcal{Y}$-exact sequence (1) $0 \to Cone(f') \to Cyl(f') \xrightarrow{\iota(f', \psi_{C'})} C'$ in $\mathcal{D}$, so the element $\alpha(f') := |Cone(f')| - |Cyl(f')| + |C'|$ is in $\mathsf{A}(\mathcal{Y})$. We set $\mathbf{F}_2(|A| - |B| + |C|) = \alpha(f')$ and we show that this element is independent of the choices we made. Let $A'', B'' \in \mathcal{D}$ be such that $\mathbf{F}(\underline{B}) = \underline{B''}$, $\mathbf{F}(\underline{C}) = \underline{C''}$ and let $f'' : B'' \to C''$ be a morphism in $\mathcal{D}$ such that $\mathbf{F}(\underline{f}) = \underline{f''}$. Consider the $\mathcal{Y}$-exact sequence (2) $0 \to Cone(f'') \to Cyl(f'') \overset{\iota(f'',\psi_{C''})}{\longrightarrow} C''$ in $\mathcal{D}$, which defines the element $\alpha(f'') \in \mathsf{A}(\mathcal{Y})$. Then in $\mathcal{D}/\mathcal{Y}$ we have: $\underline{Cyl(f')} = \underline{Cyl(f'')}$, $\underline{C'} = \underline{C''}$. Since $(1), (2)$ induce isomorphic triangles in $\Delta_{\mathcal{Y}}$, we have $\underline{Cone(f')} \cong \underline{Cone(f'')}$. By Remark 3.1, we have isomorphisms:

$$C' \oplus Y_{C''} \cong C'' \oplus Y_{C'}, \qquad B' \oplus Y_{B''} \cong B'' \oplus Y_{B'},$$

$$Cone(f') \oplus Y_{Cone(f'')} \cong Cone(f'') \oplus Y_{Cone(f')}.$$

By Remark 3.1, the right $\mathcal{Y}$-approximations $Y_{C'}, Y_{B'}$ of C', B' and $Y_{C''}, Y_{B''}$ of $Y_{C''}, Y_{B''}$ can be chosen to be the right $\mathcal{Y}$-approximations of $Cone(f'), Cone(f'')$. Then the first of the above isomorphisms take the following form:

$$C' \oplus Y_{Cone(f'')} \cong C'' \oplus Y_{Cone(f')}, \qquad B' \oplus Y_{Cone(f'')} \cong B'' \oplus Y_{Cone(f')}.$$

Hence in $\mathrm{K}_0(\mathcal{D}, \oplus)$ we have:

$$|C'| + |Y_{Cone(f'')}| = |C''| + |Y_{Cone(f')}|,$$

$$|B'| + |Y_{Cone(f'')}| = |B''| + |Y_{Cone(f')}|,$$

$$|Cone(f')| + |Y_{Cone(f'')}| = |Cone(f'')| + |Y_{Cone(f')}|.$$

It follows that $|Cyl(f'')| - |Cyl(f')| = |B'' \oplus Y_{B''}| - |B' \oplus Y_{B'}| = |B''| + |Y_{B''}| - |B'| - |Y_{B'}| = |B''| + |Y_{Cone(f'')}| - |B'| - |Y_{Cone(f')}|$. By the above equalities we infer easily that $\alpha(f') = \alpha(f'')$, so $\mathbf{F}_2$ is well-defined. Since $\mathsf{A}(\mathcal{X})$ is free on all elements $|A| - |B| + |C|$ of $\mathrm{K}_0(\mathcal{C}, \oplus)$, for any $\mathcal{X}$-exact sequence $0 \to A \overset{g}{\to} B \overset{f}{\to} C$ in $\mathcal{C}$, it follows that $\mathbf{F}_2$ extends to a morphism $\mathbf{F}_2 : \mathsf{A}(\mathcal{X}) \to \mathsf{A}(\mathcal{Y})$ and dually the quasi-inverse $\mathbf{G}$ of $\mathbf{F}$, induces a morphism $\mathbf{G}_2 : \mathsf{A}(\mathcal{Y}) \to \mathsf{A}(\mathcal{X})$. We claim that $\mathbf{G}_2\mathbf{F}_2$ acts identically on the generator $|A| - |B| + |C|$ of $\mathsf{A}(\mathcal{X})$. Indeed it is easy to see that $\mathbf{G}_2\mathbf{F}_2(|A| - |B| + |C|) = |Cone(f)| - |Cyl(f)| + |C| = \alpha(f)$, which comes from the $\mathcal{X}$-exact sequence $0 \to Cone(f) \to Cyl(f) \overset{\iota(f,\chi_C)}{\to} C$. Since f is an $\mathcal{X}$-epic, by Remark 3.1, we have $Cone(f) \cong A \oplus X_C$. Then $\alpha(f) = |A| + |X_C| - |B| - |X_C| + |C| = |A| - |B| + |C|$. Hence $\mathbf{G}_2\mathbf{F}_2(|A| - |B| + |C|) = |A| - |B| + |C|$ in $\mathsf{A}(\mathcal{X})$. Interchanging the roles of $\mathbf{F}_2, \mathbf{G}_2$, we have $\mathbf{F}_2\mathbf{G}_2(|A| - |B| + |C|) = |A| - |B| + |C|$ in $\mathsf{A}(\mathcal{Y})$. We conclude that $\mathbf{G}_2$ is an inverse of $\mathbf{F}_2$. □

3.4. Krull-Schmidt Categories

Assume throughout this subsection that in the left homotopy pair $(\mathcal{C}, \mathcal{X})$, $\mathcal{C}$ is a skeletally small Krull-Schmidt additive category with weak kernels and split idempotents. Since $\mathcal{X}$ is contravariantly finite, it follows easily that $\mathcal{X}$ admits weak kernels. Since the stable category $\mathcal{C}/\mathcal{X}$ is left triangulated, $\mathcal{C}/\mathcal{X}$ also admits weak kernels. It follows that the categories $\mathrm{mod}(\mathcal{C})$, $\mathrm{mod}(\mathcal{X})$, $\mathrm{mod}(\mathcal{C}/\mathcal{X})$ are abelian with enough projectives.

We denote by $\mathrm{mod}_{\mathcal{X}}(\mathcal{C})$ the full subcategory of $\mathrm{mod}(\mathcal{C})$ consisting of all finitely presented functors of the form F_f, where f is an $\mathcal{X}$-epic in $\mathcal{C}$, i.e., $F \in \mathrm{mod}_{\mathcal{X}}(\mathcal{C})$ iff there exists a presentation $\mathbb{Y}(B) \xrightarrow{\mathbb{Y}(f)} \mathbb{Y}(C) \to F \to 0$, where $f : B \to C$ is an $\mathcal{X}$-epic in $\mathcal{C}$. It is easy to see that the functor $\mathsf{H}_{\mathcal{X}} : \mathrm{mod}_{\mathcal{X}}(\mathcal{C}) \to \mathrm{mod}(\mathcal{C}/\mathcal{X})$ defined by $\mathsf{H}_{\mathcal{X}}(F_f) = F_{\underline{f}}$ is an equivalence of categories with inverse the functor $\mathsf{K}_{\mathcal{X}} : \mathrm{mod}(\mathcal{C}/\mathcal{X}) \to \mathrm{mod}_{\mathcal{X}}(\mathcal{C})$ defined as follows. If $\mathcal{C}/\mathcal{X}(-, \underline{B}) \xrightarrow{\mathbb{Y}(\underline{f})} \mathcal{C}/\mathcal{X}(-, \underline{C}) \to F_{\underline{f}} \to 0$ is a finite presentation of $F_{\underline{f}}$ in $\mathrm{mod}(\mathcal{C}/\mathcal{X})$, then $\mathsf{K}_{\mathcal{X}}(F_{\underline{f}}) = F_{f^*}$, where $f^* := {}^t(\chi_C, f) : Cyl(f) = X_C \oplus B \to C$ and χ_C is a right $\mathcal{X}$-approximation of C. Usually we view the above equivalences as identifications.

Since $\mathcal{X}$ is contravariantly finite in $\mathcal{C}$, the restriction functor $\mathsf{R}_{\mathcal{X}} : \mathcal{C} \to \mathrm{Mod}(\mathcal{X})$ defined by $\mathsf{R}_{\mathcal{X}} = \mathcal{C}(-, A)|_{\mathcal{X}}$, has its image in $\mathrm{mod}(\mathcal{X})$. Moreover $\mathsf{R}_{\mathcal{X}}$ induces an equivalence between $\mathcal{X}$ and $\mathrm{Proj}(\mathrm{mod}(\mathcal{X}))$ and sends an $\mathcal{X}$-exact sequence $0 \to A \to B \to C$ in $\mathcal{C}$ to an exact sequence $0 \to \mathsf{R}_{\mathcal{X}}(A) \to \mathsf{R}_{\mathcal{X}}(B) \to \mathsf{R}_{\mathcal{X}}(C) \to 0$ in $\mathrm{mod}(\mathcal{X})$. Hence it induces a morphism $\mathbf{s}_{\mathcal{X}} : \mathrm{K}_0(\mathcal{C}, \mathcal{X}) \to \mathrm{K}_0(\mathrm{mod}(\mathcal{X}))$. The functor $\mathsf{R}_{\mathcal{X}}$ extends uniquely, via the Yoneda embedding $\mathbb{Y} : \mathcal{C} \hookrightarrow \mathrm{mod}(\mathcal{C})$, to a right exact functor $\mathcal{R}_{\mathcal{X}} : \mathrm{mod}(\mathcal{C}) \to \mathrm{mod}(\mathcal{X})$. Obviously $\mathcal{R}_{\mathcal{X}}$ coincides with the restriction functor $\mathcal{R}_{\mathcal{X}}(\mathcal{F}) = F|_{\mathcal{X}}$. In particular $\mathcal{R}_{\mathcal{X}}$ is exact and induces a morphism $\mathbf{r}_{\mathcal{X}} : \mathrm{K}_0(\mathrm{mod}(\mathcal{C})) \to \mathrm{K}_0(\mathrm{mod}(\mathcal{X}))$. Using the identification $\mathrm{mod}_{\mathcal{X}}(\mathcal{C}) = \mathrm{mod}(\mathcal{C}/\mathcal{X})$, it is not difficult to see that $\mathcal{R}_{\mathcal{X}}$ induces a short exact sequence of abelian categories

$$0 \to \mathrm{mod}(\mathcal{C}/\mathcal{X}) \to \mathrm{mod}(\mathcal{C}) \xrightarrow{\mathcal{R}_{\mathcal{X}}} \mathrm{mod}(\mathcal{X}) \to 0.$$

By well-known results of Quillen [18, 4], there exist a long exact sequence

$$\cdots \to \mathrm{K}_1(\mathrm{mod}(\mathcal{X})) \to \mathrm{K}_0(\mathrm{mod}(\mathcal{C}/\mathcal{X})) \xrightarrow{\mathbf{j}_{\mathcal{X}}} \mathrm{K}_0(\mathrm{mod}(\mathcal{C})) \xrightarrow{\mathbf{r}_{\mathcal{X}}} \mathrm{K}_0(\mathrm{mod}(\mathcal{X})) \to 0$$

The morphisms $\mathbf{s}_{\mathcal{X}}, \mathbf{j}_{\mathcal{X}}, \mathbf{r}_{\mathcal{X}}$ are embedded in the exact commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{E}(\mathcal{X}) & \xrightarrow{\mathbf{i}_{\mathcal{X}}} & \mathrm{K}_0(\mathcal{C}, \oplus) & \xrightarrow{\varpi_{\mathcal{X}}} & \mathrm{K}_0(\mathcal{C}, \mathcal{X}) & \longrightarrow & 0 \\
 & & \Big\downarrow {\scriptstyle \exists !\, \omega^*_{\mathcal{X}}} & & \Big\downarrow {\scriptstyle c} & & \Big\downarrow {\scriptstyle \mathbf{s}_{\mathcal{X}}} & & \\
\cdots & \longrightarrow & \mathrm{K}_0(\mathrm{mod}(\mathcal{C}/\mathcal{X})) & \xrightarrow{\mathbf{j}_{\mathcal{X}}} & \mathrm{K}_0(\mathrm{mod}(\mathcal{C})) & \xrightarrow{\mathbf{r}_{\mathcal{X}}} & \mathrm{K}_0(\mathrm{mod}(\mathcal{X})) & \longrightarrow & 0
\end{array}$$

where $\mathbf{c} : \mathrm{K}_0(\mathcal{C}, \oplus) \to \mathrm{K}_0(\mathrm{mod}(\mathcal{C}))$ is the Cartan map induced by the Yoneda embedding $\mathbb{Y} : \mathcal{C} \hookrightarrow \mathrm{mod}(\mathcal{C})$, i.e. it is defined by sending $|C|$ to $[\mathbb{Y}(C)]$. The morphism $\omega^*_{\mathcal{X}}$ is constructed as follows. Let $\mathbf{j}_{\mathcal{X}} = \kappa \circ \lambda : \mathrm{K}_0(\mathrm{mod}(\mathcal{C}/\mathcal{X})) \xrightarrow{\kappa} \mathrm{Im}(\mathbf{j}_{\mathcal{X}}) \xrightarrow{\lambda} \mathrm{K}_0(\mathrm{mod}(\mathcal{C}))$ be the canonical factorization of $\mathbf{j}_{\mathcal{X}}$. The above diagram induces a unique morphism $\omega_{\mathcal{X}} : \mathrm{E}(\mathcal{X}) \to \mathrm{Im}(\mathbf{j}_{\mathcal{X}})$ such that $\omega_{\mathcal{X}} \circ \lambda = \mathbf{i}_{\mathcal{X}} \circ \mathbf{c}$. Since $\mathcal{C}$ is Krull-Schmidt, the Grothendieck group $\mathrm{K}_0(\mathcal{C}, \oplus)$ is free with basis the set $\mathrm{Ind}(\mathcal{C})$ of isoclasses of indecomposable objects of $\mathcal{C}$. Hence $\mathrm{E}(\mathcal{X})$ is free. It follows that there exists a unique morphism $\omega^*_{\mathcal{X}} : \mathrm{E}(\mathcal{X}) \to \mathrm{K}_0(\mathrm{mod}(\mathcal{C}/\mathcal{X}))$ such that $\omega^*_{\mathcal{X}} \circ \kappa = \omega_{\mathcal{X}}$.

Theorem 3.4. *The morphism* $\omega^*_{\mathcal{X}} : \mathrm{E}(\mathcal{X}) \to \mathrm{K}_0(\mathrm{mod}(\mathcal{C}/\mathcal{X}))$ *is invertible. In particular* $\mathrm{K}_0(\mathrm{mod}(\mathcal{C}/\mathcal{X}))$ *is free.*

Proof. We examine first the action of the morphism $\omega^*_{\mathcal{X}}$ on the generators $|A| - |B| + |C|$ of $\mathrm{E}(\mathcal{X})$, where $0 \to A \xrightarrow{g} B \xrightarrow{f} C$ is an $\mathcal{X}$-exact sequence in $\mathcal{C}$. Since by construction $\omega_{\mathcal{X}}$ is the restriction of $\mathbf{c}$ on $\mathrm{E}(\mathcal{X})$, it follows that $\omega_{\mathcal{X}}(|A| - |B| + |C|) = \mathbf{c}(|A| - |B| + |C|) = [\mathbb{Y}(A)] - [\mathbb{Y}(B)] + [\mathbb{Y}(C)]$. Then we have a projective resolution $0 \to \mathbb{Y}(A) \to \mathbb{Y}(B) \to \mathbb{Y}(C) \to F_f \to 0$ of the functor $F_f = Coker(-,f)$ in $\mathrm{mod}(\mathcal{C})$. Then in $\mathrm{K}_0(\mathrm{mod}(\mathcal{C}))$ we have $[F_f] = [\mathbb{Y}(A)] - [\mathbb{Y}(B)] + [\mathbb{Y}(C)]$. It follows that $\omega_{\mathcal{X}}(|A| - |B| + |C|) = [F_f]$. Since f is $\mathcal{X}$-epic, the functor F_f lies in $\mathrm{mod}(\mathcal{C}/\mathcal{X})$. Then by the construction of $\omega^*_{\mathcal{X}}$ we have $\omega^*_{\mathcal{X}}(|A| - |B| + |C|) = [F_f] \in \mathrm{K}_0(\mathrm{mod}(\mathcal{C}/\mathcal{X}))$. We shall construct an inverse $\zeta^*_{\mathcal{X}}$ of $\omega^*_{\mathcal{X}}$. Let $\zeta_{\mathcal{X}} : \mathrm{Iso}(\mathrm{mod}(\mathcal{C}/\mathcal{X})) \to \mathrm{E}(\mathcal{X})$ be the map defined as follows. If $F \in \mathrm{mod}(\mathcal{C}/\mathcal{X})$, then there exists an $\mathcal{X}$-epic $f : B \to C$ in $\mathcal{C}$ such that $F = F_f = Coker(-,f)$. We denote the isomorphism class of F by (F). Consider the $\mathcal{X}$-exact sequence $0 \to A \to B \xrightarrow{f} C$ in $\mathcal{C}$. We set $\zeta_{\mathcal{X}}((F_f)) = |A| - |B| + |C| \in \mathrm{E}(\mathcal{X})$. Let $0 \to F_1 \to F_2 \to F_3 \to 0$ be a short exact sequence in $\mathrm{mod}(\mathcal{C}/\mathcal{X})$. Then there are $\mathcal{X}$-epics $f_i : B_i \to C_i$ in $\mathcal{C}$ such that $F_i = F_{f_i}$, $i = 1, 3$. Consider the $\mathcal{X}$-exact sequences $0 \to A_i \xrightarrow{g_i} B_1 \xrightarrow{f_i} C_i$, $i = 1, 3$, which induce projective resolutions $0 \to \mathbb{Y}(A_i) \xrightarrow{\mathbb{Y}(g_i)} \mathbb{Y}(B_i) \xrightarrow{\mathbb{Y}(f_i)} \mathbb{Y}(C_i) \to F_i \to 0$ in $\mathrm{mod}(\mathcal{C})$. Using Horseshoe Lemma, it follows that there exists a projective resolution $0 \to \mathbb{Y}(A_1 \oplus A_3) \xrightarrow{\mathbb{Y}(g_2)} \mathbb{Y}(B_1 \oplus B_3) \xrightarrow{\mathbb{Y}(f_2)} \mathbb{Y}(C_1 \oplus C_3) \to F_2 \to 0$ of F_2 in $\mathrm{mod}(\mathcal{C})$, i.e. $F_2 = F_{f_2}$. It is easy to check that the morphism f_2 is an $\mathcal{X}$-epic in $\mathcal{C}$, hence setting $A_2 = A_1 \oplus A_3$, $B_2 = B_1 \oplus B_3$, $C_2 = C_1 \oplus C_3$, we have an $\mathcal{X}$-exact sequence $0 \to A_2 \xrightarrow{g_2} B_2 \xrightarrow{f_2} C_2$ in $\mathcal{C}$. Then $\zeta_{\mathcal{X}}((F_{f_1})) - \zeta_{\mathcal{X}}((F_{f_2})) + \zeta_{\mathcal{X}}((F_{f_3})) = (|A_1| - |B_1| + |C_1|) - (|A_2| - |B_2| + |C_2|) + (|A_3| - |B_3| + |C_3|) = (|A_1| - |B_1| + |C_1|) - (|A_1| + |A_3| - |B_1| - |B_3| + |C_1| + |C_3|) + (|A_3| - |B_3| + |C_3|) = 0$. By the universal property of $\mathrm{K}_0(\mathrm{mod}(\mathcal{C}/\mathcal{X}))$, there exists a unique morphism $\zeta^*_{\mathcal{X}} : \mathrm{K}_0(\mathrm{mod}(\mathcal{C}/\mathcal{X})) \to \mathrm{E}(\mathcal{X})$ such that $\zeta^*_{\mathcal{X}}([F_f]) = \zeta_{\mathcal{X}}((F_f)) = |A| - |B| + |C|$, where $f : B \to C$ is an $\mathcal{X}$-epic with kernel A. By construction it follows easily that $\zeta^*_{\mathcal{X}}$ is an inverse of $\omega^*_{\mathcal{X}}$. □

Recall that an abelian category is called *regular*, if any object of $\mathcal{C}$ has finite projective dimension. For instance if $\mathcal{C}$ has kernels, then $\mathrm{mod}(\mathcal{C})$ has global dimension ≤ 2. If for an additive category $\mathcal{A}$, the category $\mathrm{mod}(\mathcal{A})$ is abelian, then the stable category modulo projectives of the latter is denoted by $\underline{\mathrm{mod}}(\mathcal{A})$.

Corollary 3.5. *Assume that* $\mathrm{mod}(\mathcal{C})$ *is regular. Then the morphism* $\mathbf{s}_{\mathcal{X}} : \mathrm{K}_0(\mathcal{C}, \mathcal{X}) \to \mathrm{K}_0(\mathrm{mod}(\mathcal{X}))$ *is invertible and we have a free presentation* $\mathrm{K}_0(\mathrm{mod}(\mathcal{C}/\mathcal{X})) \rightarrowtail \mathrm{K}_0(\mathrm{mod}(\mathcal{C})) \twoheadrightarrow \mathrm{K}_0(\mathrm{mod}(\mathcal{X}))$. *Moreover the functor* $\mathsf{R}_{\mathcal{X}} : \mathcal{C} \to \mathrm{mod}(\mathcal{X})$ *induces an isomorphism* $\mathsf{R}^*_{\mathcal{X}} : \mathrm{K}_0(\mathcal{C}/\mathcal{X}) \to \mathrm{K}_0(\underline{\mathrm{mod}}(\mathcal{X}))$.

Proof. Since $\mathrm{mod}(\mathcal{C})$ is regular, the Cartan map $\mathbf{c} : \mathrm{K}_0(\mathcal{C}, \oplus) \to \mathrm{K}_0(\mathrm{mod}(\mathcal{C}))$ is invertible, so by Theorem 3.4 the morphism $\mathbf{s}_{\mathcal{X}} : \mathrm{K}_0(\mathcal{C}, \mathcal{X}) \to \mathrm{K}_0(\mathrm{mod}(\mathcal{X}))$ is invertible. The last claim follows from the commutativity of the following diagram □

$$\begin{array}{ccccccc}
\mathrm{K}_0(\mathcal{X}, \oplus) & \xrightarrow{\mathbf{c}_{\mathcal{X}}} & \mathrm{K}_0(\mathcal{C}, \mathcal{X}) & \longrightarrow & \mathrm{K}_0(\mathcal{C}/\mathcal{X}) & \longrightarrow & 0 \\
\| \downarrow & & \mathbf{s}_{\mathcal{X}} \downarrow & & \mathsf{R}^*_{\mathcal{X}} \downarrow & & \\
\mathrm{K}_0(\mathcal{X}, \oplus) & \longrightarrow & \mathrm{K}_0(\mathrm{mod}(\mathcal{X})) & \longrightarrow & \mathrm{K}_0(\underline{\mathrm{mod}}(\mathcal{X})) & \longrightarrow & 0
\end{array}$$

Now let $(\mathcal{D}, \mathcal{Y})$ be another left homotopy pair, where $\mathcal{D}$ is skeletally small Krull-Schmidt category with weak kernels and split idempotents, and we fix a stable equivalence $\mathbf{F} : \mathcal{C}/\mathcal{X} \xrightarrow{\approx} \mathcal{D}/\mathcal{Y}$. The following consequence shows that the abelian group $\mathrm{E}(\mathcal{X})$ is invariant under stable equivalences.

Corrollary 3.6. *The functor* $\mathbf{F}$ *induces an isomorphism* $\mathbf{F}^! : \mathrm{E}(\mathcal{X}) \xrightarrow{\cong} \mathrm{E}(\mathcal{Y})$.

Proof. Obviously the stable equivalence $\mathbf{F}$ extends to an equivalence $\widehat{\mathbf{F}} : \mathrm{mod}(\mathcal{C}/\mathcal{X}) \xrightarrow{\approx} \mathrm{mod}(\mathcal{D}/\mathcal{Y})$. It follows that $\mathbf{F}$ induces an isomorphism $\mathrm{K}_0(\mathrm{mod}(\mathcal{C}/\mathcal{X})) \xrightarrow{\cong} \mathrm{K}_0(\mathrm{mod}(\mathcal{D}/\mathcal{Y}))$ and the assertion follows from Theorem 3.4. □

Corollary 3.7. *Assume that* $\mathrm{mod}(\mathcal{C})$ *and* $\mathrm{mod}(\mathcal{D})$ *are regular. Let* $\mathsf{F} : \mathcal{C} \to \mathcal{D}$ *be an additive functor which preserves weak kernels (this happens for instance if F admits a left adjoint) and such that* $\mathsf{F}(\mathcal{X}) \subseteq \mathcal{Y}$. *If the induced functor* $\widehat{\mathsf{F}} : \mathrm{mod}(\mathcal{X}) \to \mathrm{mod}(\mathcal{Y})$ *induces a stable equivalence* $\underline{\widehat{\mathbf{F}}} : \underline{\mathrm{mod}}(\mathcal{X}) \to \underline{\mathrm{mod}}(\mathcal{Y})$, *then the induced functor* $\underline{\mathsf{F}} : \mathcal{C}/\mathcal{X} \to \mathcal{D}/\mathcal{Y}$ *induces an isomorphism* $\mathbf{F}^* : \mathrm{K}_0(\mathcal{C}/\mathcal{X}) \xrightarrow{\cong} \mathrm{K}_0(\mathcal{D}/\mathcal{Y})$.

Proof. By our assumption on F it follows easily that the induced functor $\mathbf{F}_1$ is exact and preserves projectives. Hence the induced functor $\underline{\mathbf{F}}_1$ is a triangle equivalence. Then the assertion follows from Corollary 3.5. □

It is useful to indicate the action of the isomorphism $\mathbf{F}^! : \mathrm{E}(\mathcal{X}) \to \mathrm{E}(\mathcal{Y})$ constructed above, which is the composition of the isomorphisms

$$\mathrm{E}(\mathcal{X}) \xrightarrow{\omega^*_{\mathcal{X}}} \mathrm{K}_0(\mathrm{mod}_{\mathcal{X}}(\mathcal{C})) \xrightarrow{\mathrm{K}_0(\mathsf{H}_{\mathcal{X}})} \mathrm{K}_0(\mathrm{mod}(\mathcal{C}/\mathcal{X})) \xrightarrow{\mathrm{K}_0(\widehat{\mathbf{F}})}$$

$$\mathrm{K}_0(\mathrm{mod}(\mathcal{D}/\mathcal{Y})) \xrightarrow{\mathrm{K}_0(\mathsf{K}_{\mathcal{Y}})} \mathrm{K}_0(\mathrm{mod}_{\mathcal{Y}}(\mathcal{D})) \xrightarrow{(\omega^*_{\mathcal{Y}})^{-1}} \mathrm{E}(\mathcal{Y})$$

where the three middle isomorphism are induced by the equivalences $\mathsf{H}_{\mathcal{X}} : \mathrm{mod}_{\mathcal{X}}(\mathcal{C}) \stackrel{\approx}{\to} \mathrm{mod}(\mathcal{C}/\mathcal{X})$, $\widehat{\mathbf{F}} : \mathrm{mod}(\mathcal{C}/\mathcal{X}) \stackrel{\approx}{\to} \mathrm{mod}(\mathcal{D}/\mathcal{Y})$ and $\mathsf{K}_{\mathcal{Y}} : \mathrm{mod}(\mathcal{D}/\mathcal{Y}) \stackrel{\approx}{\to} \mathrm{mod}_{\mathcal{Y}}(\mathcal{D})$ respectively. If $|A| - |B| + |C| \in \mathrm{E}(\mathcal{X})$ is a generator, then $\mathbf{F}^!(|A| - |B| + |C|) = |C'| - |Cyl(f')| + |Cone(f')| \in \mathrm{E}(\mathcal{Y})$, where $C', B' \in \mathcal{D}$ such that $\underline{C'} = \mathbf{F}(\underline{C}), \underline{B'} = \mathbf{F}(\underline{B})$ and $f' : B' \to C'$ is a morphism in $\mathcal{D}$ such that $\mathbf{F}(\underline{f}) = \underline{f'}$.

The free presentation (3.4) for the left homotopy pairs $(\mathcal{C}, \mathcal{X})$ and $(\mathcal{D}, \mathcal{Y})$ of subsection 3.2, induces the following exact diagram with invertible vertical morphisms:

$$\begin{array}{ccccccccc}
0 \longrightarrow & \mathrm{Ker}(\mathbf{c}_{\mathcal{X}}) & \longrightarrow & \mathrm{E}(\mathcal{X}) & \xrightarrow{\mathbf{i}_{\mathcal{X}}} & \mathrm{K}_0(\mathcal{C}/\mathcal{X}, \oplus) & \xrightarrow{\pi_{\mathcal{X}}} & \mathrm{K}_0(\mathcal{C}/\mathcal{X}) & \longrightarrow 0 \\
 & & & {\scriptstyle \mathbf{F}^!}\downarrow & & {\scriptstyle \mathbf{F}_*}\downarrow & & & \\
0 \longrightarrow & \mathrm{Ker}(\mathbf{c}_{\mathcal{Y}}) & \longrightarrow & \mathrm{E}(\mathcal{Y}) & \xrightarrow{\mathbf{i}_{\mathcal{Y}}} & \mathrm{K}_0(\mathcal{D}/\mathcal{Y}, \oplus) & \xrightarrow{\pi_{\mathcal{Y}}} & \mathrm{K}_0(\mathcal{D}/\mathcal{Y}) & \longrightarrow 0
\end{array}$$

Lemma 3.8. *The central square of the above diagram commutes iff for any morphism f in $\mathcal{C}$, we have: $|\mathbf{F}(\underline{Cone(f)})| = |Cone(\mathbf{F}(\underline{f}))|$ in $\mathrm{K}_0(\mathcal{D}/\mathcal{Y}, \oplus)$.*

Proof. Let $f : B \to C$ be a morphism in $\mathcal{C}$ and consider the element $\alpha(f) = |C| - |Cyl(f)| + |Cone(f)| \in \mathrm{E}(\mathcal{X})$ induced by the $\mathcal{X}$-exact sequence $0 \to Cone(f) \to Cyl(f) \xrightarrow{f^*} C$ in $\mathcal{C}$. If the central square commutes, then: $\mathbf{i}_{\mathcal{Y}}\, \mathbf{F}^!(\alpha(f)) = \mathbf{i}_{\mathcal{Y}}(\alpha(f')) = |\underline{C'}| - |\underline{Y_{C'} \oplus B'}| + |\overline{Cone(f')}| = |\underline{C'}| - |\underline{B'}| + |\overline{Cone(f')}| = |\underline{C'}| - |\underline{B'}| + |Cone(\mathbf{F}(\underline{f}))|$, where $C', B' \in \mathcal{D}$ such that $\mathbf{F}(\underline{C}) = \underline{C'}$, $\mathbf{F}(\underline{B}) = \underline{B'}$, $\psi : Y_{C'} \to C'$ is a right $\mathcal{Y}$-approximation of C' and $f' : B' \to C'$ is a morphism in $\mathcal{D}$ such that $\underline{f'} = \mathbf{F}(\underline{f})$. On the other hand $\mathbf{F}_*\mathbf{i}_{\mathcal{X}}(\alpha(f)) = |\mathbf{F}(\underline{C})| - |\mathbf{F}(\overline{Cyl(f)})| + |\mathbf{F}(\overline{Cone(f)})| = |\mathbf{F}(\underline{C})| - |\mathbf{F}(\underline{B})| + |\mathbf{F}(\overline{Cone(f)})| = |\underline{C'}| - |\underline{B'}| + |\mathbf{F}(\overline{Cone(f)})|$. Since $\overline{Cone(f)} = \underline{Cone(f)}$, we have $|\underline{C'}| - |\underline{B'}| + |\mathbf{F}(\underline{Cone(f)})| = |\underline{C'}| - |\underline{B'}| + |Cone(\mathbf{F}(\underline{f}))| \Rightarrow |\mathbf{F}(\underline{Cone(f)})| = |Cone(\mathbf{F}(\underline{f}))|$. Conversely if the last relation holds for any morphism f in $\mathcal{C}$, then the above analysis shows that the central square in the above diagram commutes. □

We have the following consequence of Lemma 3.8 and Theorem 2.10.

Corollary 3.9. *Let* $\mathbf{F} : \mathcal{C}/\mathcal{X} \to \mathcal{D}/\mathcal{Y}$ *be a stable equivalence and assume that either:* (α) *for any morphism* f *in* $\mathcal{C}$ *we have:* $|\mathbf{F}(Cone(f))| = |Cone(\mathbf{F}(f))|$ *in* $\mathrm{K}_0(\mathcal{D}/\mathcal{Y}, \oplus)$ *or* (β) $\mathbf{F}$ *commutes functorially with the loop functors* $\Omega_{\mathcal{X}}, \Omega_{\mathcal{Y}}$. *Then the stable equivalence* $\mathbf{F}$ *induces isomorphisms:*

(i) $\mathbf{F}^* : \mathrm{K}_0(\mathcal{C}/\mathcal{X}) \xrightarrow{\cong} \mathrm{K}_0(\mathcal{D}/\mathcal{Y})$.
(ii) $\mathbf{F}_* : \mathrm{Ker}(\mathbf{c}_{\mathcal{X}}) \xrightarrow{\cong} \mathrm{Ker}(\mathbf{c}_{\mathcal{Y}})$.
(iii) $\mathrm{D}(\mathcal{X}) \xrightarrow{\cong} \mathrm{D}(\mathcal{Y})$.

Observe that a direct consequence of condition (α) above is that $\mathbf{F}$ commutes with the loop functors $\Omega_{\mathcal{X}}, \Omega_{\mathcal{Y}}$, not necessarily in a functorial way.

3.5. Abelian Categories

In this subsection we assume that $\mathcal{C}$ is a Krull-Schmidt abelian category with enough projective and injective objects. If $\mathcal{P}(\mathcal{C})$ is the full subcategory of projective objects of $\mathcal{C}$, then $(\mathcal{C}, \mathcal{P}(\mathcal{C}))$ is a left homotopy pair and the Grothendieck Group $\mathrm{K}_0(\mathcal{C}, \mathcal{P}(\mathcal{C}))$ is identified with the usual Grothendieck group $\mathrm{K}_0(\mathcal{C})$. The stable Grothendieck group $\mathrm{K}_0(\mathcal{C}/\mathcal{P}(\mathcal{C}))$ of the left triangulated category $\mathcal{C}/\mathcal{P}(\mathcal{C})$ is identified with $Coker(\mathbf{c}_{\mathcal{C}})$, where $\mathbf{c}_{\mathcal{C}} : \mathrm{K}_0(\mathcal{P}(\mathcal{C}), \oplus) \to \mathrm{K}_0(\mathcal{C})$ is the induced Cartan map, and as before admits the free presentation $0 \to \mathsf{D}(\mathcal{P}(\mathcal{C})) \to \mathrm{K}_0(\mathcal{C}/\mathcal{P}(\mathcal{C}), \oplus) \to \mathrm{K}_0(\mathcal{C}/\mathcal{P}(\mathcal{C})) \to 0$. The loop functor of $\mathcal{C}/\mathcal{P}(\mathcal{C})$ is denoted by $\Omega_{\mathcal{C}}$.

Let $\mathbf{F} : \mathcal{C}/\mathcal{P}(\mathcal{C}) \to \mathcal{D}/\mathcal{P}(\mathcal{D})$ be a stable equivalence, where $\mathcal{D}$ is a Krull-Schmidt abelian category with enough projective and injective objects. The following shows that in this case we can remove the functoriallity of the isomorphism $\mathbf{F}\Omega_{\mathcal{C}}(\underline{C}) \cong \Omega_{\mathcal{D}}\mathbf{F}(\underline{C})$ in Corollary 3.9.

Proposition 3.10. *If* $\forall C \in \mathcal{C}$: $\mathbf{F}\Omega_{\mathcal{C}}(\underline{C}) \cong \Omega_{\mathcal{D}}\mathbf{F}(\underline{C})$, *then* $\mathbf{F}$ *induces isomorphisms* $\mathbf{F}^* : \mathrm{K}_0(\mathcal{C}/\mathcal{P}(\mathcal{C})) \to \mathrm{K}_0(\mathcal{D}/\mathcal{P}(\mathcal{D}))$ *and* $\mathbf{F}_* : \mathrm{Ker}(\mathbf{c}_{\mathcal{C}}) \to \mathrm{Ker}(\mathbf{c}_{\mathcal{D}})$.

Proof. By our previous results it suffices to show that for any generator $|\underline{A}| - |\underline{B}| + |\underline{C}|$ of $\mathsf{D}(\mathcal{P}(\mathcal{C}))$ which comes from a left triangle $\Omega_{\mathcal{C}}(\underline{C}) \xrightarrow{h} \underline{A} \xrightarrow{g} \underline{B} \xrightarrow{f} \underline{C}$ in $\mathcal{C}/\mathcal{P}(\mathcal{C})$, we have: $[\mathbf{F}(\underline{A})] - [\mathbf{F}(\underline{B})] + [\mathbf{F}(\underline{C})] = 0$ in $\mathrm{K}_0(\mathcal{D}/\mathcal{P}(\mathcal{D}))$.

First we assume that $\underline{g} = 0$. Then the triangle $\Omega_{\mathcal{C}}(\underline{A}) \to \Omega_{\mathcal{C}}(\underline{B}) \to \Omega_{\mathcal{C}}(\underline{C}) \to \underline{A}$ is split, hence $\Omega_{\mathcal{C}}(\underline{C}) \cong \underline{A} \oplus \Omega_{\mathcal{C}}(\underline{B})$. Then $\mathbf{F}\Omega_{\mathcal{C}}(\underline{C}) \cong \mathbf{F}(\underline{A}) \oplus \mathbf{F}\,\Omega_{\mathcal{C}}(\underline{B}) \Rightarrow \Omega_{\mathcal{D}}\,\mathbf{F}(\underline{C})) \cong \mathbf{F}(\underline{A}) \oplus \Omega_{\mathcal{D}}\mathbf{F}(\underline{B})$. Hence $[\Omega_{\mathcal{D}}\mathbf{F}(\underline{C}))] = [\mathbf{F}(\underline{A})] + [\Omega_{\mathcal{D}}\mathbf{F}(\underline{B})] \Rightarrow -[\mathbf{F}(\underline{C})] = [\mathbf{F}(\underline{A})] - [\mathbf{F}(\underline{B})] \Rightarrow [\mathbf{F}(\underline{A})] - [\mathbf{F}(\underline{B})] + [\mathbf{F}(\underline{C})] = 0$.

Assume now that $\underline{g} \neq 0$. We fix a short exact sequence $(E) : 0 \to A' \to B' \to C' \to 0$ in $\mathcal{C}$ which induces the given triangle in $\mathcal{C}/\mathcal{P}(\mathcal{C})$. If

$\underline{A} = \underline{A}_1 \oplus \underline{A}_2$, then since $\underline{A} \cong \underline{A}'$, there are projective objects P, Q and an isomorphism $A' \oplus Q \cong A_1 \oplus A_2 \oplus P$. Consider the push-out diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & A' \oplus Q & \longrightarrow & B' \oplus Q & \longrightarrow & C' \oplus Q & \longrightarrow & 0 \\
 & & \big\downarrow & & \big\downarrow & & \Vert\big\downarrow & & \\
0 & \longrightarrow & A_2 \oplus P & \longrightarrow & E & \longrightarrow & C' \oplus Q & \longrightarrow & 0
\end{array}$$

which induces a triangle $\Omega_{\mathcal{C}}(\underline{E}) \to \underline{A}_1 \to \underline{B} \to \underline{E}$ and a split triangle $\Omega_{\mathcal{C}}(\underline{A}_2) \to \underline{A}_1 \to \underline{A} \to \underline{A}_2$. Then $|\underline{A}| - |\underline{B}| + |\underline{C}| = (|\underline{A}_1| - |\underline{B}| + |\underline{E}|) + (|\underline{A}_2| - |\underline{E}| + |\underline{C}|)$. Since $\mathcal{C}$, $\mathcal{C}/\mathcal{P}(\mathcal{C})$ are Krull-Schmidt, using induction it follows directly that any element $|\underline{A}| - |\underline{B}| + |\underline{C}|$ as above is a sum of elements of the form $|\underline{A}'| - |\underline{B}'| + |\underline{C}'|$, where $\Omega(\underline{C}') \to \underline{A}' \to \underline{B}' \to \underline{C}'$ is a triangle in $\mathcal{C}/\mathcal{P}(\mathcal{C})$ and A' is indecomposable. Hence it suffices to show that $[\mathbf{F}(\underline{A})] - [\mathbf{F}(\underline{B})] + [\mathbf{F}(\underline{C})] = 0$, for any triangle $(T) : \Omega(\underline{C}) \xrightarrow{\underline{h}} \underline{A} \xrightarrow{\underline{g}} \underline{B} \xrightarrow{\underline{f}} \underline{C}$ in $\mathcal{C}$, where A, hence $\underline{A}$, is indecomposable and $\underline{g} \neq 0$. Since $\mathbf{F}$ preserves split triangles, we can assume that the sequence (E) which induces (T) has no split exact summands. Then since A is indecomposable, the sequence $\mathbb{Y}(\underline{A}) \xrightarrow{\mathbb{Y}(\underline{g})} \mathbb{Y}(\underline{B}) \xrightarrow{\mathbb{Y}(\underline{f})} \mathbb{Y}(C) \to F_{\underline{f}} \to 0$ is the start of a minimal projective resolution of $F_{\underline{f}}$ in $\text{mod}(\mathcal{C}/\mathcal{P}(\mathcal{C}))$. Since $\mathbf{F}$ is an equivalence, $\mathbb{Y}\mathbf{F}(\underline{A}) \xrightarrow{\mathbb{Y}\mathbf{F}(\underline{g})} \mathbb{Y}\mathbf{F}(\underline{B}) \xrightarrow{\mathbb{Y}\mathbf{F}(\underline{f})} \mathbb{Y}\mathbf{F}(C) \to F_{\mathbf{F}(\underline{f})} \to 0$ is the start of a minimal projective resolution of $F_{\mathbf{F}(\underline{f})}$ in $\mathcal{D}/\mathcal{P}(\mathcal{D})$. Since $\mathcal{C}$ has injective envelopes and A is indecomposable, the functor $\mathcal{E}xt^1(-, A)$ lies in $\text{mod}(\mathcal{C}/\mathcal{P}(\mathcal{C}))$ and is indecomposable. Following the methods of Auslander-Reiten in [3], we infer that there exists an exact sequence $0 \to A'' \to B'' \xrightarrow{f''} C'' \to 0$ wihout split exact summands, such that $\underline{A}'' \cong \mathbf{F}(\underline{A})$, $\underline{B}'' \cong \mathbf{F}(\underline{B})$, $\underline{C}'' \cong \mathbf{F}(\underline{C})$ and $\underline{f}'' = \mathbf{F}(\underline{f})$. Then $[\mathbf{F}(\underline{A})] - [\mathbf{F}(B)] + [\mathbf{F}(C)] = [\underline{A}''] - [\mathbf{F}(B)] + [\mathbf{F}(C)]$ which is zero since the above short exact sequence induces a triangle $\Omega_{\mathcal{D}}\mathbf{F}(\underline{C}) \to \underline{A}'' \to \mathbf{F}(B) \xrightarrow{\mathbf{F}(\underline{f})} \mathbf{F}(C)$ in $\mathcal{D}/\mathcal{P}(\mathcal{D})$. We conclude that $[\mathbf{F}(\underline{A})] - [\mathbf{F}(\underline{B})] + [\mathbf{F}(\underline{C})] = 0$. □

4. STABLE GROTHENDIECK GROUPS AS WALDHAUSEN GROUPS

We recall that an (additive) *Waldhausen category* is a triple $(\mathcal{C}, \mathsf{Cof}, \mathsf{Weq})$, where $\mathcal{C}$ is an additive category and Cof, resp. Weq, is a class of morphisms in $\mathcal{C}$, called *cofibrations*, resp. *weak equivalences*, subject to certain axioms, see [24] for details. An easy consequence of the axioms is that any cofibration $f : A \to B$ admits a cokernel g, and then the resulting sequence $A \xrightarrow{f} B \xrightarrow{g} C$ is called a *cofibration sequence*. A morphism f is a *trivial* *cofibration*, if f is a cofibration and a weak equivalence. An object C is called *cofibrant*, resp. *acyclic*, if $0 \to C$ is a cofibration, resp. weak equivalence. An object C is called *fibrant*, if any trivial cofibration $f: C \to R$ is split monic. The full subcategory of acyclic objects is denoted by $\mathcal{A}c(\mathcal{C})$. A Waldhausen category is called *strict*, if any of its objects is fibrant and cofibrant. A Waldhausen category *satisfies the factorization property*, if any morphism $f: A \to B$ admits a factorization $f = g \circ h$, where g is a cofibration and h is a weak equivalence. Finally a Waldhausen category is called *saturated*, if whenever $f \circ g$ is a weak equivalence, f is a weak equivalence iff g is.

Now let $(\mathcal{C}, \mathcal{X})$ be a right homotopy pair. We denote by $\mathsf{Cof}_{\mathcal{X}}(\mathcal{C})$ the class of $\mathcal{X}$-monics in $\mathcal{C}$ and by $\mathsf{Weq}_{\mathcal{X}}(\mathcal{C})$ the class of stable equivalences, i.e., the morphisms f in $\mathcal{C}$ such that $\underline{f}$ is invertible in $\mathcal{C}/\mathcal{X}$. In [7] it is shown that (left and right) homotopy pairs correspond bijectively with special closed model categories in the sense of Quillen. The following result is a one-sided version of this correspondence.

Proposition 4.1. *The map* $(\mathcal{C}, \mathcal{X}) \mapsto (\mathcal{C}, \mathsf{Cof}_{\mathcal{X}}(\mathcal{C}), \mathsf{Weq}_{\mathcal{X}}(\mathcal{C}))$ *is a bijection between right homotopy pairs and strict saturated Waldhausen categories satisfying the factorization property. The inverse map is given by* $(\mathcal{C}, \mathsf{Cof}, \mathsf{Weq}) \mapsto (\mathcal{C}, \mathcal{A}c(\mathcal{C}))$. *Moreover the homotopy category* $\mathrm{Ho}(\mathcal{C})$, *defined by formally inverting the weak equivalences, is equivalent to the stable category* $\mathcal{C}/\mathcal{A}c(\mathcal{C})$.

Proof. This is similar to the proof of Theorem 4.5 in [7] and is left to the reader. □

We recall [24] that the Waldhausen group $\mathrm{K}_0(\mathcal{C})$ of a skeletally small additive Waldhausen category $(\mathcal{C}, \mathsf{Cof}, \mathsf{Weq})$ is defined as follows. $\mathrm{K}_0(\mathcal{C})$ has one generator $[C]$ for each object of $\mathcal{C}$, subject to the relations: $[A] = [C]$, if there exists a weak equivalence $A \to C$ and $[B] = [A] + [C]$, if there exists a cofibration sequence $A \to B \to C$. Now let $(\mathcal{C}, \mathcal{X})$ be a right homotopy pair, where $\mathcal{C}$ is skeletally small. It is clear that the relative Grothendieck group $\mathrm{K}_0(\mathcal{C}, \mathcal{X})$ is isomorphic to the Waldhausen group of the Waldhausen category $(\mathcal{C}, \mathsf{Cof}_{\mathcal{X}}(\mathcal{C}), \mathsf{Iso})$, where Iso is the class of isomorphisms in $\mathcal{C}$ and the split group $\mathrm{K}_0(\mathcal{X}, \oplus)$ is the Waldhausen group of the Waldhausen subcategory $(\mathcal{X}, \mathsf{Cof}_{\mathcal{X}}, \mathsf{Iso})$.

Corollary 4.2. *The Grothendieck group* $\mathrm{K}_0(\mathcal{C}/\mathcal{X})$ *of the stable left triangulated category* $\mathcal{C}/\mathcal{X}$ *is isomorphic to the Waldhausen group* $\mathrm{K}_0(\mathcal{C})$ *of the Waldhausen category* $(\mathcal{C}, \mathsf{Cof}_{\mathcal{X}}(\mathcal{C}), \mathsf{Weq}_{\mathcal{X}}(\mathcal{C}))$.

Proof. Define a morphism $\alpha : \mathrm{K}_0(\mathcal{C}) \to \mathrm{K}_0(\mathcal{C}/\mathcal{X})$ by $\alpha([C]) = [\underline{C}]$. If $[C] = [A]$, then there exists a weak equivalence $C \to A$, i.e., an isomorphism

$\underline{C} \to \underline{A}$, hence $[\underline{C}] = [\underline{A}]$. If $[B] = [A] + [C]$ in $\mathrm{K}_0(\mathcal{C})$, then there exists an $\mathcal{X}$-exact (or cofibration) sequence $A \to B \to C \to 0$ in $\mathcal{C}$, hence a triangle (T): $\underline{A} \to \underline{B} \to \underline{C} \to \Sigma_{\mathcal{X}}(\underline{A})$ in $\mathcal{C}/\mathcal{X}$, where $\Sigma_{\mathcal{X}}$ is the suspension functor of $\mathcal{C}/\mathcal{X}$ [5]. Then $[\underline{B}] = [\underline{A}] + [\underline{C}]$. Hence α is well-defined. Define a morphism $\beta : \mathrm{K}_0(\mathcal{C}/\mathcal{X}) \to \mathrm{K}_0(\mathcal{C})$ as follows. If $[\underline{C}] \in \mathrm{K}_0(\mathcal{C}/\mathcal{X})$, then $\beta([\underline{C}] = [C]$. If $[\underline{C}] = [\underline{A}]$, then by the dual of Remark 3.1, $C \oplus X_A \cong A \oplus X_C$, where X_A, X_C are left $\mathcal{X}$-approximations of A, C. Then $[C] + [X_A] = [A] + [X_C]$. Since the objects of $\mathcal{X}$ are weakly equivalent to 0, it follows that $[C] = [A]$ in $\mathrm{K}_0(\mathcal{C})$. If $[\underline{B}] = [\underline{A}] + [\underline{C}]$, then there exists a triangle (T) as above. By [5], there exists an $\mathcal{X}$-exact sequence $A' \to B' \to C' \to 0$ in $\mathcal{C}$ such that the induced right triangle $\underline{A'} \to \underline{B'} \to \underline{C'} \to \Sigma_{\mathcal{X}}(\underline{A'})$ is isomorphic to (T) in $\mathcal{C}/\mathcal{X}$. This implies that $[B] = [A] + [C]$ in $\mathrm{K}_0(\mathcal{C})$. It follows that β is a well–defined morphism which plainly is the inverse of α. □

Note that the exact sequence $\mathrm{K}_0(\mathcal{X}, \oplus) \xrightarrow{\mathbf{c}_{\mathcal{X}}} \mathrm{K}_0\ (\mathcal{C}, \mathcal{X}) \to \mathrm{K}_0(\mathcal{C}/\mathcal{X}) \to 0$, where $\mathbf{c}_{\mathcal{X}}$ is the Cartan map, coincides with the exact sequence induced by the exact functors $(\mathcal{X}, \mathsf{Cof}_{\mathcal{X}}(\mathcal{C}), \mathsf{Iso}) \to (\mathcal{C}, \mathsf{Cof}_{\mathcal{X}}(\mathcal{C}), \mathsf{Iso}) \to (\mathcal{C}, \mathsf{Cof}_{\mathcal{X}}(\mathcal{C}), \mathsf{Weq}_{\mathcal{X}}(\mathcal{C}))$, using for instance the Localization Theorem [24].

Proposition 4.3. *Let $\mathsf{F} : \mathcal{C} \to \mathcal{D}$ be an additive functor, where $(\mathcal{C}, \mathcal{X})$ and $(\mathcal{D}, \mathcal{Y})$ are right homotopy pairs, such that $\mathsf{F}(\mathcal{X}) \subseteq \mathcal{Y}$. Assume that F satisfies the following:*

(i) F sends $\mathcal{X}$-monics to $\mathcal{Y}$-monics and F preserves cokernels of $\mathcal{X}$-monics.
(ii) If $\underline{\mathsf{F}(f)}$ is invertible, then so is $\underline{f}$.
(iii) For any morphism $f : \mathsf{F}(A) \to B$ in $\mathcal{D}$, there exists an $\mathcal{X}$-monic $g : A \to A'$ and a stable equivalence $f' : \mathsf{F}(A') \to B$ such that $f = \mathsf{F}(g) \circ f'$.

Then F induces an exact functor $\mathbf{F} : \mathcal{C}/\mathcal{X} \to \mathcal{D}/\mathcal{Y}$ and an isomorphism

$$\mathrm{K}_0(\mathbf{F}) : \mathrm{K}_0(\mathcal{C}/\mathcal{X}) \xrightarrow{\cong} \mathrm{K}_0(\mathcal{D}/\mathcal{Y}).$$

Proof. The imposed assumptions imply that F is an exact functor between the associated Waldhausen categories $(\mathcal{C}, \mathsf{Cof}_{\mathcal{X}}(\mathcal{C}), \mathsf{Weq}_{\mathcal{X}}(\mathcal{C}))$, $(\mathcal{D}, \mathsf{Cof}_{\mathcal{Y}}(\mathcal{D}), \mathsf{Weq}_{\mathcal{Y}}(\mathcal{D}))$. Since by [5] any triangle in the stable category comes from an $\mathcal{X}$-exact sequence, it follows that F induces an exact functor $\mathbf{F} : \mathcal{C}/\mathcal{X} \to \mathcal{D}/\mathcal{Y}$ between the associated right triangulated categories. Since trivially F satisfies the conditions of the Approximation Theorem [24], it follows that $\mathrm{K}_0(\mathbf{F})$ is invertible. □

Given a skeletally small Waldhausen category $\mathcal{C}$, Waldhausen generalizing work of Quillen [18], defines in [24] a sequence of Waldhausen groups $\mathrm{K}_n(\mathcal{C})$, $\forall n \geq 0$, satisfying most of the nice properties which are known to be true for the Quillen K-groups. If $(\mathcal{C}, \mathcal{X})$ is a right homotopy pair, where $\mathcal{C}$ is skeletally small, then since $\mathrm{K}_0(\mathcal{C}) \cong \mathrm{K}_0(\mathcal{C}/\mathcal{X})$, it is reasonable to define the higher stable Grothendieck groups $\mathrm{K}_n(\mathcal{C}/\mathcal{X})$, $\forall n \geq 0$, of the right triangulated category $\mathcal{C}/\mathcal{X}$ as the Waldhausen groups $\mathrm{K}_n(\mathcal{C})$, of the associated Waldhausen category $(\mathcal{C}, \mathsf{Cof}_{\mathcal{X}}(\mathcal{C}), \mathsf{Weq}_{\mathcal{X}}(\mathcal{C}))$. Then most of the above results are true in the higher theory. We don't know if it is possible for a general right triangulated category $\mathcal{R}$, to define a higher K-theory $\mathrm{K}'_*(\mathcal{R})$, for instance using the methods of Neeman [16], such that $\mathrm{K}'_*(\mathcal{R}) = \mathrm{K}_*(\mathcal{C}/\mathcal{X})$, if $\mathcal{R}$ is the stable category $\mathcal{C}/\mathcal{X}$ induced by a right homotopy pair $(\mathcal{C}, \mathcal{X})$.

5. OTHER TYPES OF EQUIVALENCES

5.1. Stabilizations

We recall from [14, 8], that for any left triangulated category $\mathcal{C} = (\mathcal{C}, \Omega, \Delta)$, there exists a triangulated category $\mathcal{S}(\mathcal{C})$, the *stabilization* of $\mathcal{C}$, and an exact functor $\mathbf{S} : \mathcal{C} \to \mathcal{S}(\mathcal{C})$, the *stabilization functor*, which is universal for exact functors out of $\mathcal{C}$ to triangulated categories. We need the following useful result which shows that the Grothendieck group is invariant under stabilization.

Proposition 5.1. *[8] The stabilization functor* $\mathbf{S} : \mathcal{C} \to \mathcal{S}(\mathcal{C})$ *induces an isomorphism* $\mathrm{K}_0(\mathbf{S}) : \mathrm{K}_0(\mathcal{C}) \xrightarrow{\cong} \mathrm{K}_0(\mathcal{S}(\mathcal{C}))$.

5.2. Complexes

Throughout this subsection we fix an additive category $\mathcal{D}$ with split idempotents. Let $\mathsf{C}(\mathcal{D})$ be the category of complexes over $\mathcal{D}$. The full subcategory of $\mathsf{C}(\mathcal{D})$ consisting of right bounded, resp. bounded, complexes is denoted by $\mathsf{C}^-(\mathcal{D})$, resp. $\mathsf{C}^b(\mathcal{D})$. Let $\mathcal{X}$ be the full subcategory of $\mathsf{C}(\mathcal{D})$ consisting of all contractible complexes. It is easy to see that the pair $(\mathsf{C}(\mathcal{D}), \mathcal{X})$ is a homotopy pair and the stable category $\mathsf{C}(\mathcal{D})/\mathcal{X}$ coincides with the triangulated homotopy category $\mathcal{H}(\mathcal{D})$. Similarly we obtain the triangulated homotopy categories $\mathcal{H}^-(\mathcal{D})$, $\mathcal{H}^b(\mathcal{D})$.

We denote by $\mathsf{S} : \mathcal{D} \to \mathcal{H}(\mathcal{D})$ the functor sending an object D to the stalk complex $D^\bullet[0]$ concentrated in degree zero. The following is well-known.

Lemma 5.2. *The functor* S *induces an isomorphism* $\mathsf{S}_* : \mathrm{K}_0(\mathcal{D}, \oplus) \to \mathrm{K}_0(\mathcal{H}^b(\mathcal{D}))$ *defined by* $\mathsf{S}_*(|D|) = [\mathsf{S}(D)]$.

Proof. Any triangle in $\mathcal{H}^b(\mathcal{D})$ is induced by a sequence $A^\bullet \rightarrowtail B^\bullet \twoheadrightarrow C^\bullet$ of bounded complexes such that $A^n \rightarrowtail B^n \twoheadrightarrow C^n$ is split exact in $\mathcal{D}$, $\forall n \in \mathbb{Z}$. It is easy to see that defining $\mathbf{j} : \mathrm{K}_0(\mathcal{H}^b(\mathcal{D})) \to \mathrm{K}_0(\mathcal{D}, \oplus)$ by $\mathbf{j}([D^\bullet]) = \sum_{i \in \mathbb{Z}} (-1)^i |D^i|$, we obtain a well-defined morphism which plainly is the inverse of S_*. □

Assume now that $\mathcal{D}$ is an exact category with enough projectives and let $\mathcal{P}$ be the full subcategory of projectives. Then $(\mathcal{D}, \mathcal{P})$ is a left homotopy pair, and the Grothendieck group $\mathrm{K}_0(\mathcal{D}, \mathcal{P})$ coincides with the group $\mathrm{K}_0(\mathcal{D})$ of the exact category $\mathcal{D}$ as defined by Quillen [18]. The stable category $\mathcal{D}/\mathcal{P}$ is left triangulated and we have the exact sequence, where $\mathbf{c}$ is the induced Cartan morphism:

$$0 \to \mathrm{Ker}(\mathbf{c}) \to \mathrm{K}_0(\mathcal{P}, \oplus) \xrightarrow{\mathbf{c}} \mathrm{K}_0(\mathcal{D}) \to \mathrm{K}_0(\mathcal{D}/\mathcal{P}) \to 0$$

Let $\mathcal{H}^{-,b}(\mathcal{P})$ be the full subcategory of $\mathcal{H}^-(\mathcal{P})$ consisting of all complexes which are acyclic [14] everywhere, except of a finite number of degrees. Then $\mathcal{H}^{-,b}(\mathcal{P})$ is a full triangulated subcategory of $\mathcal{H}^-(\mathcal{P})$ and contains as a thick subcategory, the homotopy category $\mathcal{H}^b(\mathcal{P})$ of bounded complexes over $\mathcal{P}$. As in [12] the Verdier quotient $\mathcal{H}^{-,b}(\mathcal{P})/\mathcal{H}^b(\mathcal{P})$ is denoted by $\mathcal{D}_\mathcal{P}$. By a well-known result of Grothendieck [10] the short exact sequence of triangulated categories $\mathcal{H}^b(\mathcal{P}) \hookrightarrow \mathcal{H}^{-,b}(\mathcal{P}) \twoheadrightarrow \mathcal{D}_\mathcal{P}$ induces an exact sequence $\mathrm{K}_0(\mathcal{H}^b(\mathcal{P})) \xrightarrow{\mathbf{c}^\bullet} \mathrm{K}_0\,(\mathcal{H}^{-,b}(\mathcal{P})) \to \mathrm{K}_0(\mathcal{D}_\mathcal{P}) \to 0$, first considered by Happel [12]. Note that if $\mathcal{C}$ is abelian then as is well-known, $\mathcal{H}^{-,b}(\mathcal{P})$ is identified with the bounded derived category $\mathbf{D}^b(\mathcal{C})$ of $\mathcal{C}$.

Let $\mathsf{P} : \mathcal{D} \to \mathcal{H}^{-,b}(\mathcal{P})$ be the functor sending an object $D \in \mathcal{D}$ to its deleted projective resolution. The composite functor $\mathcal{D} \xrightarrow{\mathsf{P}} \mathcal{H}^{-,b}(\mathcal{P}) \xrightarrow{\mathsf{Q}} \mathcal{D}_\mathcal{P}$, where Q is the quotient functor, kills projective objects and it induces in a natural way an exact functor $\mathsf{T} : \mathcal{D}/\mathcal{P} \to \mathcal{D}_\mathcal{P}$.

Proposition 5.3. *The functor* T *induce an isomorphism*

$$\mathsf{T}_* : \mathrm{K}_0(\mathcal{D}/\mathcal{P}) \xrightarrow{\cong} \mathrm{K}_0(\mathcal{D}_\mathcal{P})$$

such that the following diagram commutes:

$$\begin{array}{ccccccc}
\mathrm{K}_0(\mathcal{P},\oplus) & \xrightarrow{\mathrm{c}} & \mathrm{K}_0(\mathcal{D}) & \longrightarrow & \mathrm{K}_0(\mathcal{D}/\mathcal{P}) & \longrightarrow & 0 \\
{\scriptstyle \mathsf{S}_*}\big\downarrow & & {\scriptstyle \mathsf{P}_*}\big\downarrow & & {\scriptstyle \mathsf{T}_*}\big\downarrow & & \\
\mathrm{K}_0(\mathcal{H}^b(\mathcal{P})) & \xrightarrow{\mathrm{c}'} & \mathrm{K}_0(\mathcal{H}^{-,b}(\mathcal{P})) & \longrightarrow & \mathrm{K}_0(\mathcal{D}_{\mathcal{P}}) & \longrightarrow & 0.
\end{array}$$

If $\mathcal{D}$ is abelian, then $\mathsf{P}_* : \mathrm{K}_0(\mathcal{D}) \xrightarrow{\cong} \mathrm{K}_0(\mathbf{D}^b(\mathcal{D}))$.

Proof. Trivially the left square of the above diagram commutes and by Lemma 5.2, S_* is invertible. By [14, 8], T is isomorphic to the stabilization functor. Hence by Proposition 5.1, the induced morphism T_* is invertible. Then by the diagram, P_* is surjective. If $\mathcal{D}$ is abelian, then $\mathcal{H}^{-,b}(\mathcal{P}) \approx \mathbf{D}^b(\mathcal{D})$ and the last assertion follows by a well-known result of Grothendieck. For completeness we include the argument. Define a morphism $\mathsf{H}_* : \mathrm{K}_0(\mathbf{D}^b(\mathcal{D})) \to \mathrm{K}_0(\mathcal{D})$ by $\mathsf{H}_*([A^\bullet]) = \sum_{i\in\mathbb{Z}}(-1)^i[\mathsf{H}^i(A^\bullet)]$, where $\mathsf{H}^i(A^\bullet)$ is the i-th cohomology object. Plainly $\mathsf{H}_*\mathsf{P}_*([A]) = [A]$ and this shows that P_* is injective. Hence P_* is invertible. □

5.3. Resolving Subcategories

Let $\mathcal{C}$ be an abelian category with enough projectives and let $\mathcal{P}$ be the full subcategory of projective objects of $\mathcal{C}$. We recall that a full additive subcategory $\mathcal{A} \subseteq \mathcal{C}$ is called *resolving*, if $\mathcal{A}$ is closed under extensions, kernels of epimorphisms and contains the projectives. If $\mathcal{A}$ is resolving in $\mathcal{C}$, then trivially $(\mathcal{C},\mathcal{P})$, $(\mathcal{A},\mathcal{P})$ are left homotopy pairs. Let $\mathrm{K}_0(\mathcal{C}/\mathcal{P}), \mathrm{K}_0(\mathcal{A}/\mathcal{P})$ be the Grothendieck groups of the stable left triangulated categories $\mathcal{C}/\mathcal{P}, \mathcal{A}/\mathcal{P}$. The relative Grothendieck groups $\mathrm{K}_0(\mathcal{C},\mathcal{P}), \mathrm{K}_0(\mathcal{A},\mathcal{P})$ are identified with the Grothendick groups $\mathrm{K}_0(\mathcal{C}), \mathrm{K}_0(\mathcal{A})$, the latter viewing $\mathcal{A}$ as an exact category. Trivially the exact inclusions $\mathcal{A} \hookrightarrow \mathcal{C}$ and $\mathcal{A}/\mathcal{P} \hookrightarrow \mathcal{C}/\mathcal{P}$ induce morphisms $\alpha : \mathrm{K}_0(\mathcal{A}) \to \mathrm{K}_0(\mathcal{C})$ and $\beta : \mathrm{K}_0(\mathcal{A}/\mathcal{P}) \to \mathrm{K}_0(\mathcal{C}/\mathcal{P})$ making the following exact diagram commutative:

$$\begin{array}{ccccccc}
\mathrm{K}(\mathcal{P},\oplus) & \longrightarrow & \mathrm{K}_0(\mathcal{A}) & \longrightarrow & \mathrm{K}_0(\mathcal{A}/\mathcal{P}) & \longrightarrow & 0 \\
\big\| & & {\scriptstyle \alpha}\big\downarrow & & {\scriptstyle \beta}\big\downarrow & & \\
\mathrm{K}(\mathcal{P},\oplus) & \xrightarrow{\mathrm{c}} & \mathrm{K}_0(\mathcal{C}) & \longrightarrow & \mathrm{K}_0(\mathcal{C}/\mathcal{P}) & \longrightarrow & 0
\end{array}$$

Corollary 5.4. *If $\widehat{\mathcal{A}} = \mathcal{C}$, i.e., if any object of $\mathcal{C}$ admits a finite resolution by objects from $\mathcal{A}$, then the morphisms α, β above are invertible.*

Proof. Since $\mathcal{A}$ is closed under kernels of epimorphisms, and any object of $\mathcal{C}$ admits a resolution by objects from $\mathcal{A}$, by the resolution Theorem [21], it follows that α is invertible. Then the above diagram implies that β is also invertible. □

If $\widehat{\mathcal{A}} = \mathcal{C}$ and $\mathcal{E}xt^t_{\mathcal{C}}(\mathcal{A}, \mathcal{P}) = 0, \forall t \geq 1$, then it can be shown that the inclusion $\mathcal{A}/\mathcal{P} \hookrightarrow \mathcal{C}/\mathcal{P}$ admits a right adjoint $\mathbf{S}$ which induces the inverse of β in K_0. Actually $\mathbf{S}$ is the stabilization functor of $\mathcal{C}/\mathcal{P}$, in case $\mathcal{P}$ is a cogenerator of $\mathcal{A}$, see [8].

5.4. Derived Equivalences and Gorenstein Rings

If Λ is an associative ring, then we denote by $\mathrm{Mod}(\Lambda)$ the category of right Λ-modules and by $\mathrm{mod}(\Lambda)$ the category of finitely presented right Λ-modules. The category of finitely generated projective right Λ-modules is denoted by $\mathcal{P}_\Lambda$. If Λ is right coherent, then $\mathrm{mod}(\Lambda)$ is abelian, $(\mathrm{mod}(\Lambda), \mathcal{P}_\Lambda)$ is a left homotopy pair and the stable category $\mathrm{mod}(\Lambda)/\mathcal{P}_\Lambda := \underline{\mathrm{mod}}(\Lambda)$ is left triangulated.

Recall that two associative rings Λ, Γ are *derived equivalent*, if there exists a triangle equivalence $\mathbf{D}^b(\mathrm{Mod}(\Lambda)) \xrightarrow{\approx} \mathbf{D}^b(\mathrm{Mod}(\Gamma))$. The following shows that for right coherent rings the stable Grothendieck group is invariant under derived equivalence.

Corollary 5.5. *Let Λ, Γ be derived equivalent right coherent rings. Then the stable Grothendieck groups $\mathrm{K}_0(\underline{\mathrm{mod}}(\Lambda))$, $\mathrm{K}_0(\underline{\mathrm{mod}}(\Gamma))$ are isomorphic.*

Proof. By [20], a given derived equivalence restricts to triangle equivalences $\mathcal{H}^b(\mathcal{P}_\Lambda) \approx \mathcal{H}^b(\mathcal{P}_\Gamma)$ and $\mathcal{H}^{-,b}(\mathcal{P}_\Lambda) \approx \mathcal{H}^{-,b}(\mathcal{P}_\Gamma)$. Hence it induces a triangle equivalence $\mathcal{D}_{\mathcal{P}_\Lambda} \approx \mathcal{D}_{\mathcal{P}_\Gamma}$. Then the assertion follows from Proposition 5.3. □

Recall that a Noetherian ring Λ is called *Gorenstein*, if Λ has finite self-injective dimension from both sides. If Λ is a Gorenstein ring, then the category $\mathrm{CM}(\Lambda)$ of *Cohen-Macaulay modules* is defined by $\mathrm{CM}(\Lambda) = \{C \in \mathrm{mod}(\Lambda) | \mathcal{E}xt^n_\Lambda(C, \Lambda) = 0, \forall n \geq 1\}$. Trivially $\mathrm{CM}(\Lambda)$ is a resolving subcategory of $\mathrm{mod}(\Lambda)$, so its stable category modulo projectives, denoted by $\underline{\mathrm{CM}}(\Lambda)$, is left triangulated.

Corollary 5.6. *If Λ is a Gorenstein ring, then there are isomorphisms $\mathrm{K}_0(\mathrm{CM}(\Lambda)) \xrightarrow{\cong} \mathrm{K}_0(\mathrm{mod}(\Lambda))$ and $\mathrm{K}_0(\underline{\mathrm{CM}}(\Lambda)) \xrightarrow{\cong} \mathrm{K}_0(\underline{\mathrm{mod}}(\Lambda))$. Moreover if Γ is a Noetherian ring derived equivalent to Λ, then there are isomorphisms $\mathrm{K}_0(\mathrm{CM}(\Lambda)) \xrightarrow{\cong} \mathrm{K}_0(\mathrm{CM}(\Gamma))$ and $\mathrm{K}_0(\underline{\mathrm{CM}}(\Lambda)) \xrightarrow{\cong} \mathrm{K}_0(\underline{\mathrm{CM}}(\Gamma))$.*

Proof. By [8, 12], the stable category $\underline{\mathrm{CM}}(\Lambda)$ is the stabilization of $\underline{\mathrm{mod}}(\Lambda)$, so the first claim follows by Propositions 5.1, 5.3. If Γ is a Noetherian ring derived equivalent to Λ, then by [8], Γ is also Gorenstein and the second claim follows. □

5.5. Artin Algebras

Throughout we fix an Artin algebra Λ and consider the exact sequence

$$0 \to \mathrm{Ker}(\mathbf{c}_\Lambda) \to \mathrm{K}_0(\mathcal{P}_\Lambda, \oplus) \xrightarrow{\mathbf{c}_\Lambda} \mathrm{K}_0(\mathrm{mod}(\Lambda)) \to \mathrm{K}_0(\underline{\mathrm{mod}}(\Lambda)) \to 0 \tag{5.1}$$

induced by the Cartan map $\mathbf{c}_\Lambda$. We recall that the abelian groups $\mathrm{K}_0(\mathcal{P}_\Lambda, \oplus)$, $\mathrm{K}_0(\mathrm{mod}(\Lambda))$ are free of rank n, where n is the number of non-isomorphic simple Λ-modules. In particular $\mathrm{Ker}(\mathbf{c}_\Lambda)$ is free of rank $s \leq n$.

Lemma 5.7. *(i)* $\mathrm{K}_0(\underline{\mathrm{mod}}(\Lambda)) = 0$ *iff the Cartan map* $\mathbf{c}_\Lambda$ *is invertible iff* $\mathrm{Det}(\mathbf{c}_\Lambda) = \pm 1$. *This happens if* $\mathrm{gl.dim}\,\Lambda < \infty$.

(ii) $\mathrm{K}_0(\underline{\mathrm{mod}}(\Lambda))$ *is finite iff the Cartan map* $\mathbf{c}_\Lambda$ *is a monomorphism iff* $\mathrm{Det}(\mathbf{c}_\Lambda) \neq 0$. *In this case:* $|\mathrm{K}_0(\underline{\mathrm{mod}}(\Lambda))| = |\mathrm{Det}(\mathbf{c}_\Lambda)|$.

(iii) If $\mathrm{K}_0(\underline{\mathrm{mod}}(\Lambda))$ *is free, then* $\mathrm{Det}(\mathbf{c}_\Lambda) = 0$.

Proof. (i) Obviously $\mathbf{c}_\Lambda$ is invertible $\Leftrightarrow \mathrm{Det}(\mathbf{c}_\Lambda) = \pm 1 \Rightarrow \mathrm{K}_0(\underline{\mathrm{mod}}(\Lambda)) = 0$. If the last condition holds, the above exact sequence induces an isomorphism $\mathrm{K}_0(\mathcal{P}_\Lambda, \oplus) \cong \mathrm{K}_0(\mathrm{mod}(\Lambda)) \oplus \mathrm{Ker}(\mathbf{c}_\Lambda)$. Since $\mathrm{K}_0(\mathrm{mod}(\Lambda))$, $\mathrm{K}_0(\mathcal{P}_\Lambda, \oplus)$ are free of the same rank, it follows that $\mathrm{Ker}(\mathbf{c}_\Lambda) = 0$. Hence $\mathbf{c}_\Lambda$ is invertible. If $\mathrm{gl.dim}\,\Lambda = n < \infty$, then $\forall C \in \mathrm{mod}(\Lambda)$, $\Omega^n(\underline{C}) = 0$. Hence $(-1)^n[\underline{C}] = [\Omega^n(\underline{C})] = 0 \Rightarrow \mathrm{K}_0(\mathrm{mod}(\Lambda)) = 0$.

(ii) If $\mathrm{Det}(\mathbf{c}_\Lambda) \neq 0, \pm 1$, then it is well-known that $|\mathit{Coker}(\mathbf{c}_\Lambda)| = |\mathrm{K}_0(\underline{\mathrm{mod}}(\Lambda))| = |\mathrm{Det}(\mathbf{c}_\Lambda)|$. Hence in this case $\mathrm{K}_0(\underline{\mathrm{mod}}(\Lambda))$ is finite. If this is true, tensoring with $\mathbb{Q}$ the exact sequence (5.1), we have an exact sequence $0 \to \mathrm{Ker}(\mathbf{c}_\Lambda) \otimes_{\mathbb{Z}} \mathbb{Q} \to \mathbb{Q}^n \to \mathbb{Q}^n \to 0$. This implies that $\mathrm{Ker}(\mathbf{c}_\Lambda) \otimes_{\mathbb{Z}} \mathbb{Q} = 0$. Since $\mathrm{Ker}(\mathbf{c}_\Lambda) \cong \mathbb{Z}^s$, this implies that $s = 0$, hence $\mathrm{Ker}(\mathbf{c}_\Lambda) = 0$, so $\mathbf{c}_\Lambda$ is a monomorphism. Finally if $\mathbf{c}_\Lambda$ is a monomorphism, then arguing as above we infer that $\mathrm{K}_0(\underline{\mathrm{mod}}(\Lambda)) \otimes_{\mathbb{Z}} \mathbb{Q} = 0$, i.e., $\mathbf{c}_\Lambda \otimes_{\mathbb{Z}} \mathbb{Q}$ is invertible. This implies trivially that $\mathrm{Det}(\mathbf{c}_\Lambda) \neq 0$.

(iii) If $\mathrm{K}_0(\underline{\mathrm{mod}}(\Lambda))$ is free, then by (ii), $\mathrm{Det}(\mathbf{c}_\Lambda) = 0$. □

Example 5.8. Let Λ be an algebra given by a Brauer tree with e edges and multiplicity m of the exceptional vertex. It is well-known that $\mathrm{Det}(\mathbf{c}_\Lambda) = e^2$

$m+1$. Hence $|\mathrm{K}_0(\underline{\mathrm{mod}}(\Lambda))| = e^2m+1$. By a classical result of Brauer, for any finite group G and any field k of characteristic p, $|\mathrm{Det}(\mathbf{c}_{kG})|$, hence the order of $\mathrm{K}_0(\underline{\mathrm{mod}}(kG))$, is a power of p. On the other hand there are algebras of infinite global dimension, even self-injective, with trivial stable Grothendieck group, see [12].

Trivially the stable Grothendieck group is invariant under Morita equivalence. The next result summarizes our knowledge about the invariance of the stable Grothendieck group under stable, derived or tilting equivalence. First we need the following consequence of Corollary 3.6, first observed by Geigle [9] with a different proof, which shows that, with the notation of sec. 3, the group $\mathrm{E}(\Lambda) := \mathrm{E}(\mathcal{P}_\Lambda)$ which measures the complexity of the structure of the short exact sequences in $\mathrm{mod}(\Lambda)$, is always invariant under a stable equivalence.

Proposition 5.9. *Any stable equivalence* $\mathbf{F} : \underline{\mathrm{mod}}(\Lambda) \xrightarrow{\approx} \underline{\mathrm{mod}}(\Gamma)$, *induces an isomorphism* $\mathbf{F}^! : \mathrm{E}(\Lambda) \xrightarrow{\cong} \mathrm{E}(\Gamma)$.

Theorem 5.10. *Assume that one of the conditions in* (i) *holds or the condition* (ii) *holds.*

(i) There is a stable equivalence $\mathbf{F} : \underline{\mathrm{mod}}(\Lambda) \xrightarrow{\approx} \underline{\mathrm{mod}}(\Gamma)$, *such that:*
(a) $\mathbf{F}$ *commutes (not necessarily functorially) with the loop functors* $\Omega_\Lambda, \Omega_\Gamma$.
(b) Λ *or* Γ *is representation-finite and* $\mathbf{F}$ *commutes with the equivalences* $\mathrm{DTr}_\Lambda, \mathrm{DTr}_\Gamma$.
(c) Λ, Γ *are symmetric, e.g. group algebras.*
(d) Λ, Γ *are self-injective with Lowey length* ≥ 3
(e) Λ, Γ *are indecomposable self-injective.*
(f) Λ, Γ *are without nodes in the sense of [15].*

(ii) Λ, Γ *are derived equivalent, in particular if* Λ, Γ *are tilting-cotilting equivalent.*

Then in any of the above cases the equivalence functor induces the following isomorphisms and equality:

$$\mathrm{Ker}(\mathbf{c}_\Lambda) \xrightarrow{\cong} \mathrm{Ker}(\mathbf{c}_\Gamma), \quad \mathrm{K}_0(\underline{\mathrm{mod}}(\Lambda)) \xrightarrow{\cong} \mathrm{K}_0(\underline{\mathrm{mod}}(\Gamma)) \quad \text{and} \quad |\mathrm{Det}(\mathbf{c}_\Lambda)| = |\mathrm{Det}(\mathbf{c}_\Gamma)|.$$

Proof. (i) If $\mathbf{F}$ commutes with the loop functors $\Omega_\Lambda, \Omega_\Gamma$, then the assertion follows from Proposition 3.10. Assume now that Λ is representation-finite and $\mathbf{F}$ commutes with the equivalences $\mathrm{DTr}_\Lambda, \mathrm{DTr}_\Gamma$. By Butler-Auslander Theorem [1], the set of elements $|\mathrm{DTr}_\Lambda(C)| - |E| + |C|$, where $0 \to \mathrm{DTr}(C)$

$\xrightarrow{g} E \xrightarrow{f} C \to 0$ is an almost split sequence in $\mathrm{mod}(\Lambda)$ is a basis of $\mathrm{E}(\Lambda)$. Since trivially Γ is also representation-finite, the same is true for $\mathrm{E}(\Gamma)$. By Proposition 5.9, $\mathbf{F}$ induces an isomorphism $\mathbf{F}^! : \mathrm{E}(\Lambda) \xrightarrow{\cong} \mathrm{E}(\Gamma)$, so by Proposition 2.9, to prove the assertion, it is sufficient to prove that $Cone(\mathbf{F}(\underline{f})) \cong \mathbf{F}(Cone(\underline{f}))$ holds for any minimal right almost split morphism f, with non-projective target. Let $|\mathrm{DTr}_\Lambda(C)| - |E| + |C|$ be a basis element of $\mathrm{E}(\Lambda)$ as above. If $\underline{f} \neq 0$, then by [1], there exists an almost split sequence $0 \to \mathrm{DTr}_\Gamma(C') \to E' \xrightarrow{f'} C' \to 0$ in $\mathrm{mod}(\Gamma)$, where $\underline{C'} = \mathbf{F}(\underline{C})$, $\underline{E'} = \mathbf{F}(C)$, $\mathbf{F}(\underline{f}) = \underline{f'}$ and then $\mathrm{DTr}_\Gamma(\underline{C'}) = \mathrm{DTr}_\Gamma\mathbf{F}(\underline{C}) = \mathbf{F}\mathrm{DTr}_\Lambda(\underline{C})$. Since $\mathrm{DTr}_\Lambda(\underline{C'}) = Cone(\mathbf{F}(\underline{f}))$ and $\mathrm{DTr}_\Lambda(\underline{C}) = Cone(\underline{f})$, we infer that $\mathbf{F}(Cone(\underline{f})) = Cone(\mathbf{F}(\underline{f}))$. If $\underline{g} = 0$, then we have a triangle $\Omega_\Lambda(\underline{C}) \xrightarrow{\underline{h}} \mathrm{DTr}_\Lambda(\underline{C}) \xrightarrow{0} \underline{E} \xrightarrow{\underline{f}} \underline{C}$ and a weak kernel sequence $\mathbf{F}\Omega_\Lambda(\underline{C}) \xrightarrow{\mathbf{F}(\underline{h})} \mathbf{F}\mathrm{DTr}_\Lambda(\underline{C}) \xrightarrow{0} \mathbf{F}(\underline{E}) \xrightarrow{\mathbf{F}(\underline{f})} \mathbf{F}(\underline{C})$, so $\mathbf{F}(\underline{f})$ is monic. Using that $\underline{f}$ is minimal right almost split and $\mathbf{F}$ preserves such maps, it is not difficult to see that $Cone(\mathbf{F}(\underline{f})) \cong \mathrm{DTr}_\Gamma(\mathbf{F}(\underline{C}))$. We infer again that $Cone(\mathbf{F}(\underline{f})) \cong \mathbf{F}(Cone(\underline{f}))$.

If (c), (d) or (f) holds, then by [1] and a simple modification of the arguments of [15] respectively, we infer that $\mathbf{F}$ commutes with the loop functors, so the assertion follows from part (a).

Assume now that Λ, Γ are indecomposable selfinjective. If LL denotes Loewy length, then it is easy to see that $LL(\Lambda) \leq 2$ iff $LL(\Gamma) \leq 2$ in which case $LL(\Lambda) = LL(\Gamma)$. If $LL(\Lambda) = 1$, then the assertion is trivial since $\underline{\mathrm{mod}}(\Lambda) = 0 = \underline{\mathrm{mod}}(\Gamma)$. If $LL(\Lambda) = 2 = LL(\Gamma)$, then Λ, Γ are Nakayama algebras and it is easy to see that the stable equivalence can be lifted to an equivalence $\mathrm{mod}(\Lambda) \approx \mathrm{mod}(\Gamma)$. We infer that in both cases the stable Grothendieck groups are isomorphic. If $LL(\Lambda), LL(\Gamma) \geq 3$, then the assertion follows by part (d).

(ii) Follows by Corollary 5.5. □

Remark 5.11. The above result contains some results of [4, 15, 17] and can be applied to more general situations. For instance using Proposition 3.10, if the condition (i)(a) holds, we obtain the results of Solberg [22]. Let R and S be commutative 2-dimensional integrally closed domains with an algebraically closed residue field. If R and S are stably equivalent by an equivalence which commutes with the loop functors, then R and S have isomorphic ideal class groups: $C(R) \cong C(S)$. This follows from the fact that in this situation, the stable group $\mathrm{K}_0(\underline{\mathrm{mod}}(R))$ is isomorphic to the ideal class group $C(R)$ of R, see [4]. We refer to [19] for further examples, mainly of blocks of group algebras, of stable equivalences commuting with the loop functors.

The following example shows that in general a stable equivalence $\mathbf{F}$ does not induces an isomorphism on the level of stable Grothedieck groups, even if $\mathbf{F}$ is induced by a functor $\mathsf{F} : \mathrm{mod}(\Lambda) \to \mathrm{mod}(\Gamma)$ on the level of module categories.

Example 5.12. Let Λ be an Artin Algebra with radical square zero and with infinite global dimension, e.g., $\Lambda = k[\mathbf{x}]/(\mathbf{x}^2)$, where k is a field. It is well-known [1] that Λ is stably equivalent to a hereditary Artin algebra Γ. More precisely there exists a functor $\mathsf{F} : \mathrm{mod}(\Lambda) \to \mathrm{mod}(\Gamma)$ which induces a stable equivalence $\mathbf{F} : \underline{\mathrm{mod}}(\Lambda) \xrightarrow{\approx} \underline{\mathrm{mod}}(\Gamma)$. Since Γ is hereditary, it follows that $\mathrm{K}_0(\underline{\mathrm{mod}}(\Gamma)) = 0$. However $\mathrm{K}_0(\underline{\mathrm{mod}}(\Lambda)) \neq 0$ in general. For instance if $\Lambda = k[\mathbf{x}]/(\mathbf{x}^2)$, then it is easy to see that $\mathrm{K}_0(\underline{\mathrm{mod}}(\Lambda)) \cong \mathbb{Z}_2$. Of course in this example we have that $\mathbf{F}$ does not commutes with the loop functors Ω_Λ, Ω_Γ or the equivalences DTr_Λ, DTr_Γ.

Example 5.13. Let $T \in \mathrm{mod}(\Lambda)$ and let $\Gamma := \mathrm{End}_\Lambda(T)$. Since $\mathcal{X} := \mathrm{add}(T)$ is contravariantly finite, the stable category $\mathrm{mod}(\Lambda)/\mathcal{X}$, henceforth denoted by $\underline{\mathrm{mod}}_T(\Lambda)$, is left triangulated. Then by Corollary 3.5 we have $\mathrm{K}_0(\mathrm{mod}(\Lambda), \mathcal{X}) \cong \mathrm{K}_0(\mathrm{mod}(\Gamma))$ and $\mathrm{K}_0(\underline{\mathrm{mod}}_T(\Lambda)) \cong \mathrm{K}_0(\underline{\mathrm{mod}}(\Gamma))$. If T is a (generalized) tilting module, then by [11] Λ, Γ are derived equivalent. Hence by Corollary 5.5 we infer that: $\mathrm{K}_0(\mathrm{mod}(\Lambda), \mathcal{X}) \cong \mathrm{K}_0(\mathrm{mod}(\Lambda))$ and $\mathrm{K}_0(\underline{\mathrm{mod}}_T(\Lambda)) \cong \mathrm{K}_0(\underline{\mathrm{mod}}(\Lambda))$. If T is a generator, then the isomorphism $\mathrm{K}_0(\mathrm{mod}(\Lambda), \mathcal{X}) \cong \mathrm{K}_0(\mathrm{mod}(\Gamma))$ above shows that $\mathrm{K}_0(\mathrm{mod}(\Lambda), \mathcal{X})$ is free with rank equal to the number of non-isomorphic indecomposable direct summands of T. It is an open question (equivalent to the generalized Nakayama conjecture [13]) whether the canonical epimorphism $\mathrm{K}_0(\mathrm{mod}(\Lambda), \mathcal{X}) \twoheadrightarrow \mathrm{K}_0(\mathrm{mod}(\Lambda))$ is invertible, if T is a generator without self-extensions.

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REFERENCES

1. Auslander, M.; Reiten, I.; Smalø, S. Representation Theory of Artin Algebras. Cambridge Studies in Advanced Mathematics **1995**, *36*.
2. Auslander, M.; Smalø, S. Preprojective Modules over Artin Algebras. J. Algebra **1980**, *66*, 61–122.
3. Auslander, M.; Reiten, I. Representation Theory of Artin Algebras VI: A Functorial Approach to Almost Split Sequences. Comm. Algebra **1977**, *11*, 257–300.

4. Auslander, M.; Reiten, I. Grothendieck Groups of Algebras and Orders. J. Pure and Appl. Algebra **1986**, *39*, 1–51.
5. Beligiannis, A.; Marmaridis, N. Left Triangulated Categories Arising from Contravariantly Finite Subcategories. Comm. Algebra **1994**, *22*, 5021–5036.
6. Beligiannis, A.; Marmaridis, N. Grothendieck Groups Arising from Contravariantly Finite Subcategories. Comm. Algebra **1996**, *24*, 4415–4438.
7. Beligiannis, A. Homotopy Theory of Modules and Gorenstein Rings. Math. Scand. to appear.
8. Beligiannis, A. The Homological Theory of Contravariantly Finite Subcategories: Auslander-Buchweitz Contexts, Gorenstein Categories and (Co-)Stabilization. Comm. Algebra **2000**, *28* (10), 4547–4596.
9. Geigle, D. Grothendieck Groups and Exact Sequences for Hereditary Artin Algebras. J. London Math. Soc. **1985**, *31*, 231–236.
10. Grothendieck, A. Groupes de Classes des Catégories Abeliennes et Triangulés. Springer Lecture Notes in Math. **1977**, *589*, 351–371.
11. Happel, D. *Triangulated Categories in the Representation Theory of Finite-Dimensional Algebras*; LMS Lecture Notes: Cambridge University Press, Vol. 119, 1988.
12. Happel, D. On Gorenstein Algebras. Progress in Mathematics **1991**, *95*, 389–404.
13. Happel, D. Reduction Techniques for Homological Conjectures. Tsukuba J. Math. **1993**, *17*, 115–130.
14. Keller, B.; Vossieck, D. Sous les Catégories Dérivées. C. R. Acad. Sci. Paris, 305 (Série I), **1987**, 225–228.
15. Martinez Villa, R. Properties that are Left Invariant under Stable Equivalence. Comm. Algebra **1991**, *18*, 4141–4169.
16. Neeman, A. *K*-Theory of Triangulated Categories I(A): Homological Functors. Asian J. Math. **1997**, *1*, 330–417.
17. Pogorzaly, Z. On Stable Grothendieck Groups. CMS Conf. Proc. **1993**, *14*, 393–406.
18. Quillen, D. Higher Algebraic *K*-Theory I. Springer Lecture Notes in Math. **1973**, *341*, 85–147.
19. Reiten, I. Generalized Stable Equivalences and Group Algebras. J. Algebra **1982**, *79*, 319–340.
20. Rickard, J. Morita Theory for Derived Categories. J. London Math. Soc. **1989**, *39*, 436–456.
21. Rosenberg, J. *Algebraic K-Theory and Its Applications; Graduate Texts in Math.*; Springer: Berlin, 1994; Vol. 147.
22. Solberg, O. Grothendieck Groups for Hypersurfaces Rings. Proc. Amer. Math. Soc. **1990**, *109*, 943–950.

23. Verdier, J.L. Catégories Derivées, état 0. In Springer Lecture Notes in Math. **1977**, *569*, 262–311.
24. Waldhausen, F. Algebraic *K*-Theory of Spaces. Springer Lecture Notes in Math. **1985**, *1126*, 318–419.

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