Acoustic Scattering from Two Eccentric Spheroids. 
Theory and Numerical Investigation

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Abstract. In this paper, the direct acoustic scattering problem of a point source field by a penetrable spheroidal scatterer hosting an impenetrable spheroidal body of arbitrary position, size and orientation, is considered. The application background corresponds to the near field measurement of the acoustic field, scattered by a soft tissue organ including a hard inhomogeneity. The methodology incorporates two independent techniques which are modified appropriately to fit together and are combined for the first time: First, the Vekua method, which is based on the well known Vekua transformation, providing with fully analytic solutions of Helmholtz equation and second, the method of auxiliary sources in order to represent the net wave contribution of the inhomogeneity. The satisfaction of transmission and boundary conditions is accomplished via the collocation method while the wave character of the fields and the outwards propagating property of the exterior wave are implicitly guaranteed in exact form through the analytic nature of the method. Special effort has been devoted to the self-evaluation of the method by constructing and calculating an indicative error function representing the failure of satisfaction of the boundary conditions on a rich grid over the interfaces, much larger than the set of collocation points, where the error is by construction negligible. This numerical approach leads to very reliable results. The determination of the near scattered field as well as of the far-field pattern are the final outcomes of the present work, providing a thorough solution of the direct scattering problem and giving insight to the corresponding inverse problem.

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1. Introduction

The investigation of boundary value problems incorporating spheroidal boundaries plays a fundamental role in simulating very interesting structures and systems emerging in several physical and technological applications. We focus here on scattering processes by spheroidal structures. The large range of applicability of methods concerning the interference of waves with spheroidal bodies is apparent in the literature. For example in [1, 2], the human head is modeled as a prolate spheroidal body, while being considered interacting with a cellular phone through electromagnetic waves. In addition, raindrops can be modeled as oblate spheroids, when we study the rainfall attenuation of microwave signals in satellite telecommunication systems. Rockets, aircraft noses and guided missiles are generally assumed to be of spheroidal shape. Electromagnetic scattering by buried spheroidal bodies can be found in [3, 4, 5]. The implication of separation of variables techniques, gives birth to the spheroidal wave functions [6, 7] which play an important role to the solution of boundary value problems with spheroidal interfaces [8, 9].

Special attention is paid on scattering processes by multi-layered spheroidal structures as the investigation of the acoustic illumination of the kidney-stone system considered in [10, 11, 12]. Specially, this last group of works belongs to a specific biomedical engineering framework, characterized mainly by the effort to identify how the scattering mechanism is affected when an interior inhomogeneity is hosted undesirably in most cases, by a human organ, disposing approximately a spheroidal shape. In these works, the authors exploit the spheroidal wave functions framework to face the boundary value problem that emerges. The exterior and trapped waves are expanded in terms of the spheroidal wave functions and the implication of the boundary and transmission condition leads to the determination of the aforementioned expansion coefficients. We notice here the complexity of this process especially in [12], where the eccentricity of the two spheroids involves the necessity to adapt addition theorem contributing to the formation of a very demanding analytical framework, which, though rigorous, renders the numerical treatment of the problem complex enough.

To avoid the numerical implications due to spherical wave function, a new approach has been presented in [13] to provide with closed form scattering problem solutions, in the exterior space of a soft impenetrable spheroidal body. The methodology has been based on the very powerful Vekua transformation, helping in acquiring Helmholtz equation solutions by mapping suitably the much easier kernel space of Laplace operator [14]. This alternative approach has been adapted appropriately to face the case of a penetrable spheroidal body, simulating a soft tissue organ [15]. Again, closed form solutions for the exterior outwards propagating wave as well as the interior standing fields have been obtained as superpositions of the novel set of dynamic eigensolutions constructed via the implication of Vekua transformation. The transmission condition is fulfilled through the collocation method and very rigorous results have been obtained there, based on monitoring the error function calculated on a dense grid point set on
the spheroidal surface. The purpose of the present work is to extend the methodologies followed in \cite{13, 15} to the case of a double-layered spheroidal structure, where the exterior spheroid simulates the soft-tissue organ while the interior one, of arbitrary location and orientation, stands for an impenetrable hard inhomogeneity. The authors now have decided to avoid the use of addition theorems applying this time to the closed form, though involved, Vekua eigensolutions. A first attempt to follow this technique has been proved much more intrinsic than the relevant effort made in \cite{12}. Consequently the authors have decided to combine their novel approach of the Vekua transformation with the well known technique of auxiliary sources met in several versions indicatively in \cite{16, 17, 18, 19, 20}. This approach is actually inspired by Kupradze method \cite{21, 22}, allowing to represent the involved fields as discrete superpositions of Green’s functions, i.e point source fields emanating by a suitably selected auxiliary point source grid. These point sources belong of course outside the physical region of interest, where the wave phenomena take place.

Consequently, the plan of the work leading to the analytical formulation of the problem met in section 2, has the following ingredients. We present first the formulation of the problem. The excitation of the double-layered spheroidal structure is accomplished via a point source located in the exterior space and disposing an appropriately selected amplitude to guarantee that when this point source is sent away from the scatterer, gives place to the corresponding plane wave. The outcome of our task in \cite{15} is exploited and adapted suitably to the purposes of the present work, in order to provide the scattered and interior fields in the absence of the inhomogeneity. This last field illuminates the interior body. This technique of separating the scattering process in two steps is well known and motivated by the methodology of the Green’s function construction. In other words, the boundary value problem of the Penetrable Exterior Spheroid Alone (PESA) constitutes the Green’s function of the human organ alone and this is the field, which impinges the hard interior body leading to a secondary field, totally due to the presence of the interior spheroid, a part of which oscillates in the organ while the rest lives outside propagating outwards.

The superposition of the (PESA)-field along with the secondary field originated by the IMPenetrable Interior Spheroid Alone ((IMPISA)-field) constitutes the total field of the process. The (IMPISA)-field is formulated via the method of the auxiliary point sources (located of course inside the interior body). It is considered to constitute the discrete superposition of these artificial point sources fields. We mention here that the method of Discrete Sources Method (DSM) and and Modified Rayleigh Conjecture (MRC) met correspondingly in refs \cite{20, 19} are broader, since they allow to include in this superposition, point source fields of higher order (not only monopoles). However the numerical analysis of our approach has proved the sufficiency of using the simplified version of auxiliary sources, which actually constitutes the very simple concept of the discrete analogue of the single layer potential met in the integral representation theory of Helmholtz equation \cite{23}. The method of the auxiliary sources has of course some sensitive points as the decision concerning the location of every point source along with
its amplitude. The optimal decision of the location is a difficult generally problem \cite{17}, especially when several additional geometrical and physical parameters are involved. This is why we preferred to use the auxiliary sources method only for the (IMPISA)-field, while we have used the robust and very accurate numerically Vekua method to represent the (PESA)-field. Otherwise we should have introduced two more sets of auxiliary points (close to the interior and exterior side of the penetrable spheroidal surface), fact which would be expected to increase drastically the parameters to be optimized.

This strategy is justified actually in section 3, which includes the numerical treatment of the problem. In this section, the error analysis and the multi-parametric investigation of the problem are exposed. More precisely, as explained in section 2, the constructed representations of the (PESA)-field (via Vekua eigensolutions) and of the (IMPISA)-field (via expansion in point sources fields) must be forced to satisfy the transmission and boundary condition on the interfaces. As a matter of fact, the (IMPISA)-field, by construction, satisfies automatically the homogeneous transmission conditions on the exterior scatterer. The non-homogeneous transmission conditions concerning the (PESA)-field together with the boundary interior condition concerning the (IMPISA)-field are demanded to be fulfilled on two collocation points grids, each one on the corresponding surface. The linear systems that emerge, are solved and lead to the determination of the expansion coefficients of the aforementioned field representations. To evaluate the correctness of the solution, and to decide about the truncation level, two error functions are formed on two dense grids, located again on the two spheroidal surfaces. These error functions represent the failure of the satisfaction of the necessary conditions on the interfaces, except the collocation points, where the accuracy is guaranteed.

These error functions depend strongly on the geometrical and physical parameters of the problem as well as on the grid of the auxiliary point sources. Although the scattering process under examination is strongly multi-parametric, we have accomplished a thorough analysis, revealing the role of every specific ingredient of the problem.

In addition, in section 3, the outcome of the solution of direct scattering problem is presented. This consists mainly of 3D plots of near scattering fields, revealing the consequences of the existence of the inner spheroidal body and giving hints towards the very complex problem of the identification of hidden structures inside soft tissues.

Finally, in Appendix special features of the matrices participating in the crucial linear systems under investigation, are presented.

2. Formulation of the problem - Analytical Approach

We consider the double layer structure presented in Fig. 1, and confined by two spheroidal surfaces \(S\) and \(S_1\). The surface \(S\) is characterized by the equation \(\mu = \mu_0\) where the triple \((\mu, \theta, \phi)\) stands for the spheroidal coordinates introduced by the
Scattering from Two Eccentric Spheroids. Theory and Numerical Investigation

Figure 1. The problem geometry.

Spheroidal surface $S$, related with the Cartesian coordinates through the relation

$$
\begin{align*}
 x &= \frac{\alpha}{2} \sinh \mu \sin \theta \cos \phi & 0 \leq \mu < \infty \\
 y &= \frac{\alpha}{2} \sinh \mu \sin \theta \sin \phi & 0 \leq \theta \leq \pi \\
 z &= \frac{\alpha}{2} \cosh \mu \cos \theta & 0 \leq \phi < 2\pi
\end{align*}
$$

where $\alpha$ stands for the focal distance. The surface $S_1$ introduces a new coordinate system with origin $O'$, with respect to which every observation point disposes Cartesian coordinates $(x', y', z')$ and spheroidal coordinates $(\mu', \theta', \phi')$ connected through the relation

$$
\begin{align*}
 x' &= \frac{\alpha'}{2} \sinh \mu' \sin \theta' \cos \phi' & 0 \leq \mu' < \infty \\
 y' &= \frac{\alpha'}{2} \sinh \mu' \sin \theta' \sin \phi' & 0 \leq \theta' \leq \pi \\
 z' &= \frac{\alpha'}{2} \cosh \mu' \cos \theta' & 0 \leq \phi' < 2\pi
\end{align*}
$$

where $\alpha'$ stands for the focal distance of the primed system. In this system, $S_1$ is represented by the relation $\mu' = \mu_1$. The region between the two surfaces simulates the soft tissue under consideration and is characterized by the density $\rho_{int}$ and bulk modules $K_{int}$. The exterior region $V^{ext}$ is filled with a different material with corresponding physical parameters $\rho_{ext}$ and $K_{ext}$. Usually the exterior space is filled with air or water. We consider an acoustic point source located in the exterior space $V^{ext}$, emitting the spherical time harmonic acoustic wave

$$
u^{inc}(r) = \frac{e^{ik_{ext}|r-r_0|}}{4\pi|r-r_0|} e^{i\omega t}$$
where $r_0$ is the source location point and $k_{ext}$ stands for the wavenumber in the exterior domain, given by $k_{ext} = \frac{\omega}{c_{ext}}$ where $\omega$ is the frequency and $c_{ext}$ is the sound velocity in the host environment ($c_{ext} = \sqrt{\frac{K_{ext}}{\rho_{ext}}}$). As it is well known the time factor $e^{i\omega t}$ appeared in Eq.(3) is inherited to all the forthcoming secondary fields, defining precisely their time dependence and can of course be suppressed for simplicity. So we will omit this factor and deal with time suppressed components of the wave fields. Usually, a multiplicative renormalization factor is incorporated in expression (3) in such a way that the point source field transforms uniformly to a plane wave when the location of the point source is remoted far from the scatterer [24]. This renormalized incident field has the form

$$u^{inc}(r) = r_0 \frac{e^{i k_{ext} (|r - r_0|)}}{4\pi |r - r_0|}. \quad (4)$$

This assumption helps in formulating the scattering problem in a generalized form, such that the plane wave excitation presented in [15] is a particular case. The interference of the incident wave with the spheroidal layered structure leads to the creation of secondary waves: the scattered one $u^{sc}$, propagating outwards in region $V^{ext}$ and the interior field $u^{int}$ trapped in $V$. Following techniques based on the superposition principle used frequently in buried objects problems, the scattering process described above can be reformulated in two consecutive steps as follows: First, remove the interior spheroid and consider the full interior space confined by surface $S$, to be filled with the material with parameters $\rho_{int}$ and $K_{int}$. Then the scattering mechanism leads to the creation of the scattered field $\tilde{u}^{sc}$, living in $V^{ext}$, and of the trapped interior field $\tilde{u}^{int}$ inside $S$, vibrating with wave number equal to $k_{int} = \frac{\omega}{c_{int}}$ ($c_{int} = \sqrt{\frac{K_{int}}{\rho_{int}}}$). Secondly we consider separately the interior spheroid being "illuminated" by the wave $\tilde{u}^{int}$ generated at first stage and restricted in region $V$, which actually represents the "total" incident wave incorporating the existence of the exterior interface. So the first problem consists in scattering of a spherical incident wave by a simple penetrable spheroidal structure while the second problem stands for scattering of a complicated incident field by a simple again but impenetrable scatterer. The solution of the second scattering problem is denoted by

$$\tilde{u}(r) = \begin{cases} \tilde{u}^{sc}(r), & r \in V^{ext} \\ \tilde{u}^{int}(r), & r \in V \end{cases} \quad (5)$$

and the following superposition relations hold:

$$u^{sc}(r) = \tilde{u}^{sc}(r) + \tilde{u}^{sc}(r)$$

$$u^{int}(r) = \tilde{u}^{int}(r) + \tilde{u}^{int}(r). \quad (6)$$

The total field and the secondary field of the penetrable scattering problem from the exterior scatterer alone, satisfy the same transmission boundary conditions [15], i.e

$$\rho_{ext} \left( u^{sc}(r) + u^{inc}(r) \right) = \rho_{int} u^{int}(r), \quad r \in S$$

$$\frac{\partial u^{sc}(r)}{\partial n} + \frac{\partial u^{inc}(r)}{\partial n} = \frac{\partial u^{int}(r)}{\partial n}, \quad r \in S \quad (7)$$
along with
\[ \rho_{ext} (\hat{u}^{sc}(\mathbf{r}) + u^{inc}(\mathbf{r})) = \rho_{int} \hat{u}^{int}(\mathbf{r}), \quad \mathbf{r} \in S \]
\[ \frac{\partial \hat{u}^{sc}(\mathbf{r})}{\partial n} + \frac{\partial u^{inc}(\mathbf{r})}{\partial n} = \frac{\partial \hat{u}^{int}(\mathbf{r})}{\partial n}, \quad \mathbf{r} \in S. \]

Consequently the secondary fields generated by scattering from the interior impenetrable body, satisfy homogeneous transmission condition on \( S \), i.e
\[ \rho_{ext} \hat{u}^{sc}(\mathbf{r}) = \rho_{int} \hat{u}^{int}(\mathbf{r}), \quad \mathbf{r} \in S \]
\[ \frac{\partial \hat{u}^{int}(\mathbf{r})}{\partial n} = - \frac{\partial \hat{u}^{int}(\mathbf{r})}{\partial n}, \quad \mathbf{r} \in S_1. \]

The investigation of the first scattering problem will be based on the research executed by the authors in [15]. More precisely, the implementation of the Vekua transform in [15] permitted the creation of new wave basis functions for scattering and standing waves. So the "hat" fields \( \hat{u}^{int} \) and \( \hat{u}^{sc} \) can be expanded in terms of the aforementioned basis. More precisely
\[ \hat{u}^{sc}(\mathbf{r}) = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} N_{nm} \hat{u}^{ext}_{nm}(\mathbf{r}), \quad \mathbf{r} \in V^{ext} \]
\[ \hat{u}^{int}(\mathbf{r}) = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} A_{nm} \hat{u}^{int}_{nm}(\mathbf{r}), \quad \mathbf{r} \in R^3 \setminus V^{ext} \]

where
\[ \hat{u}^{ext}_{nm}(\mathbf{r}) = \sum_{l=0}^{[n/2]} \sum_{p=0}^{[n-2p]} B_{n,m,p,l} \Gamma(n-2p+\frac{3}{2}) \frac{2}{\sqrt{\pi}} \frac{h^{(1)}_{n-2p}(kr)}{(\frac{k}{2})^{n-2p}} P^{m}_{n-2p-2l}(\cosh \mu) P^{m}_{n-2p-2l}(\cos \theta) e^{im\phi}, \quad n = 0, 1, 2, \ldots; |m| \leq n, \]
\[ \hat{u}^{int}_{nm}(\mathbf{r}) = \sum_{l=0}^{[n/2]} \sum_{p=0}^{[n-2p]} B_{n,m,p,l} \Gamma(n-2p+\frac{3}{2}) \frac{J^{(n-2p+\frac{3}{2})}_{n-2p-1/2}(kr)}{(\frac{k}{2})^{n-2p+1/2}} P^{m}_{n-2p-2l}(\cosh \mu) P^{m}_{n-2p-2l}(\cos \theta) e^{im\phi}, \quad n = 0, 1, 2, \ldots; |m| \leq n \]
are the elements of the Vekua wave basis. We recognize the Gamma function \( \Gamma(n-2p+\frac{3}{2}) \), the spherical Bessel function \( h^{(1)}_{n} \) and the cylindrical Bessel function \( J_{n} \). In addition we meet the Legendre associated function \( P^{m}_{n} \) while the coefficients \( B_{n,m,p,l} \) are given by [15]
\[ B_{n,m,p,l} = \begin{cases} \frac{(-1)^p(n+m)!(n-2p-2l-m)!(2n-2p)!}{(n-m)!(n-2p-2l+m)!(2n-2p)!} \times \frac{(n-2p-l)!(2n-4p-4l+1)!}{(n-p)!(2p+2)!} & |m| \leq \kappa \\
0 & |m| > \kappa \end{cases} \]
where $\kappa = n - 2p - 2l$. As explained in [13, 14], the "hybrid" forms (13) and (14) (in the sense that radial coordinate $r$ coexists with the spheroidal coordinates) are more dense than the full developed forms expressed exclusively in terms of spheroidal coordinates. Independently of this choice, the Vekua basis elements follow the behavior of the spheroidal wave functions. They constitute a complete set of independent functions representing, via expansions, the involving fields up to the boundary of the spheroidal surface $S$. The complex coefficients $N_{nm}$ and $A_{nm}$ of the expansions (11), (12) will be determined by forcing the waves $\hat{u}^{sc}$, $\hat{u}^{int}$ to satisfy the boundary conditions (8) on surface $S$. This task will be accomplished through the collocation method, as will be demonstrated later on in the numerical section. There it will be explained also how the infinite summation above is truncated to a specific degree of accuracy. The secondary field $\tilde{u}(r)$ is of course strongly dependent on the penetrating field $\hat{u}^{int}(r)$ via the boundary condition (10). However it is not possible to exploit now the same wave basis mentioned above, since the interior spheroid has arbitrary location, orientation and shape. So if we insisted on using the Vekua transformation, a new basis would be necessary to be introduced attached to the coordinate system referring to the interior body. This coexistence of basis functions of different kind would introduce an extreme complexity to the resulting analysis and numerical treatment. Instead of doing that, the secondary field $\tilde{u}(r)$ is selected to be represented as an expansion in terms of spherical waves generated by a number of equivalent point sources located inside the impenetrable scatterer. Actually these point sources are selected to be placed on a new spheroidal surface $S_2$, congruent to $S_1$, as presented in Fig. 1.

Every point source $r_j$ emanates the spherical wave $\frac{e^{ik_{int}|r-r_j|}}{4\pi|r-r_j|}$. The interaction of this spherical field with the surface $S$ gives a secondary field, which constitutes the Green’s function of the penetrable structure without the interior scatterer. This Green’s function is represented as

$$G(r, r_j) = \begin{cases} 
G^{int}(r, r_j) = \frac{e^{ik_{int}|r-r_j|}}{4\pi|r-r_j|} + G^{reg}(r, r_j), & r \in \mathbb{R}^3 \setminus \mathbb{V}^{ext} \\
G^{sc}(r, r_j), & r \in \mathbb{V}^{ext} 
\end{cases}$$

(16)

where $G^{reg}$, $G^{sc}$ are regular functions to be defined. The last functions satisfy actually transmission conditions (7) where in the place of $u^{inc}(r)$ we substitute the spherical wave originated by $r_j$ (instead of $r_0$) and disposing the interior wave number. Then applying the methodology followed for the "hat" fields, we are in position to express the Green’s function as

$$G^{sc}(r, r_j) = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} E^{j}_{nm} \hat{u}^{ext}_{nm}(r), \quad r \in \mathbb{V}^{ext}$$

$$G^{int}_{reg}(r, r_j) = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \Lambda^{j}_{nm} \hat{u}^{int}_{nm}(r), \quad r \in \mathbb{R}^3 \setminus \mathbb{V}^{ext}.$$  

(17)

The complex coefficients $E^{j}_{nm}$, $\Lambda^{j}_{nm}$ are produced via the same collocation algorithm just as the coefficients $N_{nm}$, $A_{nm}$ of expansions (11), (12). What differs is only the
non-homogeneous term of the transmission conditions, which refers now to the auxiliary interior point source instead of the real exterior one. Consequently, the wave \( \tilde{u}(r) \) is expanded in terms of the Green’s functions generated by \( N \) interior point sources as follows:

\[
\tilde{u}(r) = \sum_{j=1}^{N} \delta_j G(r, r_j) = \left\{
\begin{align*}
\tilde{u}^{int}(r) &= \sum_{j=1}^{N} \delta_j G^{int}(r, r_j), \quad r \in V \\
\tilde{u}^{sc}(r) &= \sum_{j=1}^{N} \delta_j G^{sc}(r, r_j), \quad r \in V_{ext}.
\end{align*}
\right.
\] (18)

The ability to establish representation (18) stems from the possibility to express \( \tilde{u}(r) \) as a single layer potential over the auxiliary surface \( S_2 \) via an unknown density function. So representation (18) is the discrete analogue of this potential and the coefficients are, roughly speaking, generated by the discretization of the density function. Normally, representation (18) is an approximation of the corresponding field, which becomes better as the number of the auxiliary sources increases. It is worth to be mentioned that this approximate solution satisfies exactly the differential equations whenever in space. Its approximate character consists of being condemned to satisfy approximately the boundary conditions in which it involves. The same approximate behavior is followed by the ”hat” fields (11,12) when a truncation to the infinite sums is selected.

The coefficients \( \delta_j \) are to be determined by imposing on the expansion \( \tilde{u}^{int}(r) = \sum_{j=1}^{N} \delta_j G^{int}(r, r_j) \) to satisfy the boundary condition (10). Using again the collocation method, now explicitly, we select \( N \) grid points on the impenetrable spherical surface \( S_1 \), where the boundary condition is imposed giving the linear system:

\[
\sum_{j=1}^{N} \sum_{n=0}^{N_{pen}} \sum_{m=-n}^{n} N_{nm}^{pen} \frac{\partial}{\partial n} \hat{u}^{int}_{nm}(r_i) + \left[ \frac{\partial}{\partial n} \left( \frac{e^{ik_{nm} |r-r_j|}}{4\pi |r-r_j|} \right) \right]_{|r|=r_i} \delta_j = - \frac{\partial \hat{u}^{int}}{\partial n}(r_i), r_i \in S_1, \quad i = 0, 1, 2, \ldots , N
\] (19)

where \( N_{pen}^{tr} \) is the truncation level adopted for the solution of the proper penetrable scattering problem without the inclusion. The outcome of the direct scattering problem is the determination of the total scattered field along with the corresponding far-field pattern. The superposition principle implies that

\[
u^{sc}(r) = \hat{u}^{sc}(r) + \tilde{u}^{sc}(r), \quad r \in V_{ext}
\] (20)

or

\[
u^{sc}(r) = \sum_{n=0}^{N_{pen}} \sum_{m=-n}^{n} \left[ N_{nm} + \sum_{j=1}^{N} \delta_j E_j^{nm} \right] \hat{u}^{ext}_{nm}(r), \quad r \in V_{ext}
\]

for the near scattered field.
Imposing the scattered field to asymptotic analysis as \( r = |r| \to \infty \) and evoking the asymptotic forms of the basis functions \( \hat{u}_{\text{ext}} \) by [15], we infer that
\[
 u^{sc}(r) = \frac{e^{ikr}}{kr} f_\infty(\theta, \phi) + O\left(\frac{1}{r^2}\right), \quad r \to \infty
\] (21)
where the far-field pattern emerges as follows:
\[
f_\infty(\theta, \phi) = \sum_{n=0}^{N_{\text{pen}}^{\text{tr}}} \sum_{m=-n}^{n} \frac{2}{\sqrt{\pi}} e^{-i\frac{3}{4}} A_{nm} \left[ N_{nm} + \sum_{j=1}^{N} \delta_j E^j_{nm} \right] \\
\times \sum_{p=0}^{\left\lfloor \frac{n}{2} \right\rfloor} B_{n,m,p,0} \Gamma(n - 2p + \frac{3}{2}) e^{-i\frac{1}{2}(n-2p+\frac{1}{2})\pi} \\
(\frac{4}{k \alpha})^{n-2p} \frac{2^{2p-n}[2(n-2p)]!}{(n-2p)!} P_{n-2p}^m(\cos \theta) e^{im\phi}.
\] (22)
We notice though that in our approach, we are interested in the investigation of the near field close to the exterior side of the penetrable interface, giving rise to the well known boundary measurement technique [25].

3. Numerical Investigation - The solution of the scattering problem

The numerical investigation of the scattering problem under discussion has been proved very demanding. A lot of physical and geometrical parameters are involved and several numerical special features have emerged, rendering the numerical treatment very complicated. The main effort of the numerical investigation is the solution of the linear system (19). However this system is very involved incorporating a series of individual numerical steps.

It is clear that the corner stone of the linear system (19) is the appearance of the interior wave field \( \hat{u}_{\text{int}} \) produced in absence of the impenetrable interior spheroidal body. The accuracy in the numerical determination of this field influences drastically the numerical efficiency of the solution of system (19). In other words the selection of the truncation level \( N_{tr}^{\text{pen}} \) is the first decision step which should be made to guarantee the indispensable accuracy in the evaluation of the field illuminating the interior spheroidal scatterer. However this selection should be made on the parallel purpose not to augment drastically the numerical cost in the solution of the linear equation (19). The extended investigation followed in [15], offered very helpful insight in our study, although it was necessary to be refreshed due to the point source excitation in contrast to plane wave incidence encountered in [15]. It is mentioned here that the truncation level \( N_{tr}^{\text{pen}} \) is directly connected with the number of the collocation points located at surface \( S \) and then this number reflects the precision imposed to the satisfaction of the transmission boundary conditions on \( S \). This precision is controlled by the error function which is constructed, a posteriori, on a very dense grid on \( S \). The error function consists on the failure of the satisfaction of the boundary conditions on this grid. We found that selecting \( N_{tr}^{\text{pen}} = 11 \), in any case of wave number and spherical aspect ratio of interest.
Figure 2. The average error for the PESA scattering field.

(i.e. in the range of physical and geometrical parameters inherited by [15]), the average of error function on the grid points is less than 0.01 per cent.

A more precise dependence of the error function in terms of special parameters of the problem is presented in Fig. 2. We present the two limit cases as far as the aspect ratio of the exterior scatterer is concerned. The condition number of the linear system emerging from the satisfaction of the transmission boundary conditions (in the absence of \( S_1 \)) on the collocation points of \( S \), for the range of parameters given in Fig. 2, varies from \( 10^{58} \) to \( 10^{63} \). The arbitrary precision technique, which is necessary to face such kind of linear systems (or of much greater condition numbers) is borrowed from the methodology developed in [13].

The previous analysis gives a robust evaluation of the right hand side term along with the truncation level \( N_{pen} \) in the crucial linear system (19). In addition, this analysis is applied \( N \) times for the determination of the coefficients \( \Lambda_{nm}^j \) in (19)! In other words, there appears the necessity to solve \( N \) direct auxiliary scattering problems, excited from interior sources, with the exterior spheroid alone. All this problems have the same numerical characteristics described above.

The remaining strategy for the solution of the system (19) concerns the location and the number of the auxiliary artificial interior equivalent sources \( r_j, j = 1, 2, ..., N \). Actually the number of these sources coincides with the truncation level of the representation of "tilde" fields in terms of spherical waves but determines directly as well, the denseness of the collocation points scheme on the interior surface \( S_1 \). Actually the points \( r_i \) of Eq.(19) build this collocation points set.

The solution of (19) follows the same methodology in general terms. We select a truncation level \( N \), we solve the so constructed linear system, determining so the
coefficients $\delta_j, j = 1, 2, \ldots N$. We form then the new error function living on surface $S_1$, which coincides with the $l_2$-norm of the difference (l.h.s)-(r.h.s) of Eq.(19) evaluated on the positions of a dense grid set defined on $S_1$ built by $5 \cdot 10^3$ uniformly distributed points. To acquire the most possible reliable results we pay attention on the relative error function after dividing the terms participating in the aforementioned error with the r.h.s of Eq.(19). Taking the average on the grid points we obtain the average relative error function, which is our main criterion for the validation of the solution.

The first remark to be mentioned is that the condition number of the matrix involved in the solution of (19) is much less than the corresponding one met in the solution of the penetrable problem alone. In addition, special attention has been devoted to the location of the interior sources. We followed the well known technique according to which the equivalent sources are distributed uniformly over a new spheroidal surface $S_2$ as presented in Fig. 1. This selection reflects the discrete analogue of the integral representation via a single layer potential.

We present now the geometrical and physical parameters that we have considered in our analysis. The exterior space is considered to be filled with water, assumption which reflects the usual medical treatment in such kind of measurements, involving use of suitable gels to avoid the total reflection of the energy. So we take $\rho_{ext} = 1000 kgr \cdot m^{-2}$, $c_{ext} = 1480 m \cdot s^{-1}$ while for the interior space, which is taken to simulate a muscle system, typical representative values are $\rho_{int} = 1080 kgr \cdot m^{-2}$ and $c_{ext} = 1580 m \cdot s^{-1}$.

A typical geometrical dimension of the muscle system is described by a circumscribing sphere of radius $0.1 m$. This implies the selection $C_{ext} = 0.1 m$ for the large semi-axis of the exterior spheroid.

Two representative cases concerning the size of the interior inhomogeneity are considered here in : We investigate the so called ”large” interior spheroid, whose large axis equals $2/3$ times the corresponding exterior spheroid axis $C_{ext}$ as well as the ”small” interior spheroid, disposing a large axis of magnitude equal to $2/5 C_{ext}$. The small axis of the interior spheroid is suitably adapted to fit with the specific aspect ratio $a_{R, int}$. Without loss of generality the center $O'$ of the interior spheroid is placed on $\theta = \frac{\pi}{2}$, $\phi = \pi$, with distance $OO' = 6 cm$, while the large axis is oriented to the direction $(\theta = \phi = \frac{\pi}{128})$ just to avoid parallelization of the two spheroids. Nevertheless, any other selection could be possible. We have avoided it since for the aspect ratios under consideration, no essential deviation is remarked if the orientation of the interior spheroid changes drastically. In addition we denote by c.f. the congruence factor of the spheroidal surface $S_2$ with respect to the spheroid surface $S_1$. The numerical experiments brought to notice two special cases (c.f.$= \frac{1}{6}$ or c.f.$= \frac{1}{3}$), with remarkable error analysis outcomes. Locating the sources closer or far away from these two bounds leads generally to augmentation of the error. To reveal the dependence of the scattering process on the imposed frequency we consider four cases of stimulation : $5, 6, 7, 8 \text{ kHz}$. Indicatively, the dimensionless product $k_{int}C_{int}$ varies from $1.32$ to $2.12$ (for the ”large” inhomogeneity) while $k_{ext}C_{ext}$ varies from $2.12$ to $3.39$, assuring that we are placed in the resonance region and step to the dawn of high frequencies.
All the numerical experiments are presented in Tables 1 - 5. More precisely, in Table 1, we consider the case of two spheroidal surfaces both of aspect ratio 0.999, which coincide with perturbed spheres. Then we expose the average relative error (for the boundary condition on surface $S_1$) for all frequencies of interest and for the two special cases concerning the location of the spheroidal surface $S_2$ hosting the auxiliary sources. The same structure is followed in Table 2 with the difference that the exterior surface is now a ”genuine” spheroid of aspect ratio $a_{R,ext} = 0.95$. In all the involved simulations, described in Tables 1, 2, the scattering incidence is realized via a point source located at the left of the system with angular position ($\theta = \frac{\pi}{2}, \phi = \pi$). In Table 3, the same results are tabulated for the case that both spheroids have aspect ratio equal to 0.95. The relative errors presented in Tables 1 - 3, prove that the behavior of the error is stable for the ”large” and ”small” inhomogeneities as the frequency increases towards the border of high frequency region, and this is true for several combinations of geometrical forms of the two spheroids. This last remark is strengthened by the results of table 5 for the case of the ”small” inhomogeneity. In particular the rather strange result of amelioration of the error function as the aspect ratio diminishes is revealed in Table 5. The explanation of this peculiar - at first sight - behavior, is involved in the essence of the collocation method suggested for the solution of system (19). When the exterior surface is a perturbed sphere, the r.h.s. of the crucial system (19) has a uniform behavior rendering more difficult the solvability of this system, since all the collocation points on $S_1$ are required to provide very adjacent non-homogeneous values, which is a fact that worsens the global behavior of the relevant error. In other words, when the r.h.s. of Eq.(19) obtain a very ”homogenised” form, interpreted in very similar row elements, a higher level of truncation is needed to provide with better error estimation. In addition, Tables 1 - 3 reveal, as expected, that generally, the ”small” spheroid behaves better as frequency increases, while the ”large” spheroid gives larger errors when frequency augments, due to the increment of the crucial dimensionless parameter $k_{int}C_{int}$, which provides oscillations and warns for the necessity of implication of higher order of truncation. In Table 5, we present the behavior of the relevant error, for several positions of the stimulating point source. The relative error presents a uniform behavior, as the stimulus changes location.

Finally, Table 6 indicates the usefulness of the present approach to the application point of view. We have tabulated the moduli of the (PESA), (IMPISA) and total field, for the case of $a_{R,ext} = 0.95$, $a_{R,int} = 0.999$ and all the frequencies of interest. It is clear that the (IMPISA)-field, which is generated by the interior inhomogeneity is much larger than the (PESA)-field, which is the background field in the absence of inclusion. So, the presence of the impenetrable inclusion is recovered, in the framework of an inverse problem approach.
Scattering from Two Eccentric Spheroids. Theory and Numerical Investigation

<table>
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<tr>
<td>6</td>
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<tr>
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<td>0.0145</td>
</tr>
<tr>
<td>8</td>
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Table 1. Average relative error for the boundary condition on the inclusion surface for the cases $a_{R,ext} = 0.999$, $a_{R,int} = 0.999$, P.S location: $(\theta = \frac{\pi}{2}, \phi = \pi)$ and c.f=$\frac{1}{6}$ (left), c.f=$\frac{1}{3}$ (right).

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<td>8</td>
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<td>8</td>
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Table 2. Average relative error for the boundary condition on the inclusion surface for the cases $a_{R,ext} = 0.95$, $a_{R,int} = 0.999$, P.S location: $(\theta = \frac{\pi}{2}, \phi = \pi)$ and c.f=$\frac{1}{6}$ (left), c.f=$\frac{1}{3}$ (right).

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Table 3. Average relative error for the boundary condition on the inclusion surface for the cases $a_{R,ext} = 0.95$, $a_{R,int} = 0.95$, P.S location: $(\theta = \frac{\pi}{2}, \phi = \pi)$ and c.f=$\frac{1}{3}$.

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<td>8</td>
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Table 4. Average relative error for the boundary condition on the inclusion surface for the cases $a_{R,ext} = 0.95$, $a_{R,int} = 0.999$, with f=8 kHz and c.f=$\frac{1}{3}$.

<table>
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<th>$a_{R,ext}$</th>
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<td>0.95</td>
<td>0.0070</td>
</tr>
<tr>
<td>0.96</td>
<td>0.0080</td>
</tr>
<tr>
<td>0.97</td>
<td>0.0114</td>
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<tr>
<td>0.98</td>
<td>0.0236</td>
</tr>
<tr>
<td>0.99</td>
<td>0.0817</td>
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Table 5. Average relative error for the boundary condition on the inclusion surface for the cases $a_{R,int} = 0.999$, with f=8 kHz, P.S location: $(\theta = \frac{\pi}{2}, \phi = \pi)$ and c.f=$\frac{1}{3}$. 
Figure 3. Total Near Field for $a_{R,\text{ext}} = 0.95, a_{R,\text{int}} = 0.999$, P.S. location: $\theta = \frac{\pi}{2}, \phi = \pi$, f= 5 kHz (left), 6 kHz (right), ”Small” inclusion, c.f.= $\frac{1}{3}$.

The special features of the normalized total near field (very close to the exterior side of $S_1$) are demonstrated in Figs. 3 - 8. In any case, we remark that the scattered energy has a strong back-scattering preference. This behavior is totally accounted to the presence of the inhomogeneity. Indeed, in Figs. 9 - 10 we focus on the behavior of the (PESA)-field in all frequencies of interest. We remark that, due to the fact that the physical parameters of the water and the muscle are practically adjacent, the (PESA)-field tends to reveal a forward redirection of the energy of the incident spherical wave, at least in the vicinity of the exterior scatterer. Consequently, the inhomogeneity is totally responsible for the appearance of the back-scattered lobes in Figs. 3 - 8. All these remarks concern the near field redistribution of energy, which encodes the information about the presence of the inhomogeneity. If someone detours the connection of this approach to the boundary measurements technique (useful in medical applications) and pays attention theoretically to the far-field pattern $f_\infty(\theta, \phi)$ (Figs. 11 - 12), the re-orientation of the energy to the back-scattering direction is doubtless.

| f(kHz) | Average of |Total Field| |Average of |PESA Field| |Average of |IMPISA Field|
|--------|----------------|----------------|----------------|----------------|----------------|----------------|
| 5      | 0.0254          | 0.0085          | 0.0219          |                |                |                |
| 6      | 0.0379          | 0.0089          | 0.0336          |                |                |                |
| 7      | 0.0325          | 0.0091          | 0.0351          |                |                |                |
| 8      | 0.0415          | 0.0099          | 0.0431          |                |                |                |

Table 6. Average of modulus of the fields for the cases $a_{R,\text{ext}} = 0.95, a_{R,\text{int}} = 0.999$, P.S location: $(\theta = \frac{\pi}{2}, \phi = \pi)$ and c.f.= $\frac{1}{3}$.  

Scattering from Two Eccentric Spheroids. Theory and Numerical Investigation
Figure 4. Total Near Field for $a_{R,ext} = 0.95, a_{R,int} = 0.999$, P.S. location: $\theta = \frac{\pi}{2}$, $\phi = \pi$, f= 7 kHz (left), 8 kHz (right), "Small" inclusion, c.f.= $\frac{1}{3}$.

Figure 5. Total Near Field for $a_{R,ext} = 0.95, a_{R,int} = 0.999$, P.S. location: $\theta = \frac{\pi}{2}$, $\phi = \pi$, f= 5 kHz (left), 6 kHz (right), "Large" inclusion, c.f.= $\frac{1}{3}$.

Figure 6. Total Near Field for $a_{R,ext} = 0.95, a_{R,int} = 0.999$, P.S. location: $\theta = \frac{\pi}{2}$, $\phi = \pi$, f= 7 kHz (left), 8 kHz (right), "Large" inclusion, c.f.= $\frac{1}{3}$.
Figure 7. Total Near Field for $a_{R, ext} = 0.95$, $a_{R, int} = 0.999$, $f = 8$ kHz, P.S. location: $\theta = \frac{\pi}{2}$, $\phi = \pi$ (left), $\theta = \frac{\pi}{2}$, $\phi = 0$ (right), "Small" inclusion, c.f. = $\frac{1}{3}$.

Figure 8. Total Near Field for $a_{R, ext} = 0.95$, $a_{R, int} = 0.999$, $f = 8$ kHz, P.S. location: $\theta = \pi$, $\phi = 0$ (left), $\theta = 0$, $\phi = 0$ (right), "Small" inclusion, c.f. = $\frac{1}{3}$.

Figure 9. (PESA)-Near Field for $a_{R, ext} = 0.95$, $a_{R, int} = 0.999$, P.S. location: $\theta = \frac{\pi}{2}$, $\phi = \pi$, $f = 5$ kHz (left), 6 kHz (right), "Small" inclusion, c.f. = $\frac{1}{3}$. 

Scattering from Two Eccentric Spheroids. Theory and Numerical Investigation
Figure 10. (PESA)-Near Field for $a_{R,ext} = 0.95$, $a_{R,int} = 0.999$, P.S. location: $\theta = \frac{\pi}{2}$, $\phi = \pi$, $f$= 7 kHz (left), 8 kHz (right), "Small" inclusion, c.f.= $\frac{1}{3}$.

Figure 11. Total Far Field for $a_{R,ext} = 0.95$, $a_{R,int} = 0.999$, P.S. location: $\theta = \frac{\pi}{2}$, $\phi = \pi$, $f$= 5 kHz (left), 6 kHz (right), "Small" inclusion, c.f.= $\frac{1}{3}$.

Figure 12. Total Far Field for $a_{R,ext} = 0.95$, $a_{R,int} = 0.999$, P.S. location: $\theta = \frac{\pi}{2}$, $\phi = \pi$, $f$= 7 kHz (left), 8 kHz (right), "Small" inclusion, c.f.= $\frac{1}{3}$. 
Acknowlegments

We would like to thank Dr. Drosos Kourounis for providing the visualization software “OPSIS”. Computations have been performed mainly at the Laboratory of Mathematical Modeling and Scientific Computing of the Materials Science Department. Additional computer resources have been provided by the Research Center for Scientific Simulations (RCSS) of the University of Ioannina.

Appendix A.

The determination of the matrices that build the two hand sides of system (19) is a demanding process. The main difficulty lies on the fact that, the Vekua representations of the (IMPISA)-field, which are calculated through the coordinate system that is connected with the exterior spheroid, are forced to participate (along with their normal derivatives) on the boundary conditions on the interior surface, which introduces a new translated and rotated coordinate system (centered at $O'$). This coexistence of two different systems necessitates the implication of the suitable coordinate transformation relations. We denote by $r_1$ the displacement vector $OO'$ and so

$$r = r_1 + r',$$

where prime indicates reference w.r.t. the interior spheroidal system. So, the normal derivatives (and every relevant quantity) appeared in (19) in their general notation - under an unprimed symbolism - will be denoted, locally in Appendix - as $\frac{\partial}{\partial \mu'}$ to be firmly reminiscent of their connection with the interior spheroidal system. It is so essential to make distinction between the unit normal vector

$$\hat{n} = \sinh \mu_0 \cos \theta \hat{z} + \cosh \mu_0 \sin \theta \hat{\rho}$$

(A.2)

on the exterior spheroid $\mu = \mu_0$, and

$$\hat{n}' = \sinh \mu_1 \cos \theta' \hat{z}' + \cosh \mu_1 \sin \theta' \hat{\rho}'$$

(A.3)

on the interior surface $\mu' = \mu_1$. We recognize here the radial unit vectors $\hat{\rho}$, $\hat{\rho}'$ of the cylindrical coordinate systems connected with surfaces $S$ and $S_1$ respectively. The emerged normal derivatives are resulted as follows:

$$\frac{\partial}{\partial n} = \hat{n} \cdot \nabla = \frac{2}{\alpha} \frac{1}{(\cosh^2 \mu_0 - \cos^2 \theta) \frac{1}{2}} \frac{\partial}{\partial \mu}$$

(A.4)
on $S$ and
\[
\frac{\partial}{\partial n'} = \hat{n}' \cdot \nabla = \frac{2}{\sqrt{\cosh^2 \mu_1 - \cos^2 \theta}} \left\{ \frac{1}{\sqrt{\cosh^2 \mu_0 - \cos^2 \theta}} \left[ \sinh \mu_1 \cos \theta' (\hat{z}' \cdot \hat{\mu}) \frac{\partial}{\partial \mu} + \sinh \mu_1 \cos \theta' (\hat{z}' \cdot \hat{\theta}) \frac{\partial}{\partial \theta} + \cosh \mu_1 \sin \theta' (\hat{z}' \cdot \hat{\phi}) \frac{\partial}{\partial \phi} \right] \right\} 
+ \frac{1}{\sinh \mu_0 \sin \theta} \left[ \sinh \mu_1 \cos \theta' (\hat{z}' \cdot \hat{\phi}) + \cosh \mu_1 \sin \theta' (\hat{z}' \cdot \hat{\phi}) \right] \frac{\partial}{\partial \phi} \right\} \quad (A.5)
\]
on the interior surface $S_1$. Here $(\hat{\mu}, \hat{\theta}, \hat{\phi})$ and $(\hat{\mu}', \hat{\theta}', \hat{\phi}')$ are the spheroidal unit vectors on the spheroidal surfaces. Special mention must be paid on the form of $\frac{\partial}{\partial n'}$, which has been constructed so that the involved partial derivatives to be expressed in terms of the un-primed coordinate system. This is necessary since this operator will be applied on the Vekua eigensolutions (system 19), which dispose a "heavy" representation in terms of the exterior coordinate system. The inner products encountered in (A.5) are expressed in terms of the orientation cosines of the two systems as follows:

\[
\hat{z}' \cdot \hat{\mu} = \frac{1}{\sqrt{\cosh^2 \mu_0 - \cos^2 \theta}} \left[ \sinh \mu_0 \cos \theta (\hat{z}' \cdot \hat{z}) \right. 
+ \cosh \mu_0 \sin \theta \left[ \cos \phi (\hat{z}' \cdot \hat{x}) + \sin \phi (\hat{z}' \cdot \hat{y}) \right] \right], \quad (A.6)
\]

\[
\hat{z}' \cdot \hat{\theta} = \frac{\sinh \mu_0 \cosh \mu_0}{(\cosh^2 \mu_0 - \cos^2 \theta)^{3/2}} \left[ - \cosh \mu_0 \sin \theta (\hat{z}' \cdot \hat{z}) 
+ \sinh \mu_0 \cos \theta \left[ \cos \phi (\hat{z}' \cdot \hat{x}) + \sin \phi (\hat{z}' \cdot \hat{y}) \right] \right], \quad (A.7)
\]

\[
\hat{z}' \cdot \hat{\phi} = - \sin \phi (\hat{z}' \cdot \hat{x}) + \cos \phi (\hat{z}' \cdot \hat{y}), \quad (A.8)
\]

\[
\hat{p}' \cdot \hat{\mu} = (\cos \phi' \hat{x}' + \sin \phi' \hat{y}') \cdot \hat{\mu} = \frac{1}{\sqrt{\cosh^2 \mu_0 - \cos^2 \theta}} \left[ \sinh \mu_0 \cos \theta \cos \phi'(\hat{x}' \cdot \hat{z}) + \sinh \mu_0 \cos \theta \sin \phi'(\hat{y}' \cdot \hat{z}) 
+ \cosh \mu_0 \sin \theta \left[ \cos \phi [\cos \phi'(\hat{x}' \cdot \hat{x}) + \sin \phi'(\hat{y}' \cdot \hat{x})] 
+ \sin \phi [\cos \phi'(\hat{x}' \cdot \hat{y}) + \sin \phi'(\hat{y}' \cdot \hat{y})] \right] \right], \quad (A.9)
\]
Scattering from Two Eccentric Spheroids. Theory and Numerical Investigation

\begin{equation}
\dot{\rho}' \cdot \dot{\theta} = \frac{\sinh \mu_0 \cosh \mu_0}{(\cosh^2 \mu_0 - \cos^2 \theta)^{3/2}} \left[ - \cosh \mu_0 \sin \theta \left[ \cos \phi' (\dot{x}' \cdot \dot{z}) + \sin \phi' (\dot{y}' \cdot \dot{z}) \right] \\
+ \sinh \mu_0 \cos \theta \left[ \cos \phi [\cos \phi' (\dot{x}' \cdot \dot{x}) + \sin \phi' (\dot{y}' \cdot \dot{y})] \\
+ \sin \phi [\cos \phi' (\dot{x}' \cdot \dot{x}) + \sin \phi' (\dot{y}' \cdot \dot{y})] \right] \right],
\end{equation}

\begin{equation}
\dot{\rho}' \cdot \dot{\phi} = - \sin \phi \left[ \cos \phi' (\dot{x}' \cdot \dot{x}) + \sin \phi' (\dot{y}' \cdot \dot{x}) \right] + \cos \phi \left[ \cos \phi' (\dot{x}' \cdot \dot{y}) + \sin \phi' (\dot{y}' \cdot \dot{y}) \right].
\end{equation}

The crucial term \( \frac{\partial}{\partial n^*} \tilde{u}_{int}^\mu (r) \) of the system (19) (it is reminded that the prime is suppressed in the general notation leading to Eq.19), which explicitly appears in the r.h.s. (due to Eq.12), is then determined as follows:

\begin{equation}
\frac{\partial \tilde{u}_{int}^\mu (r)}{\partial n^*} = \frac{(2 \pi)}{\sqrt{\cosh^2 \mu_1 - \cos^2 \theta'}} \left[ \frac{1}{\cosh^2 \mu_0 - \cos^2 \theta} (\sinh \mu_1 \cos \theta' (\dot{z}' \cdot \dot{\mu}) \frac{\partial \tilde{u}_{int}^\mu (r)}{\partial \mu} \\
+ \sinh \mu_1 \cos \theta' (\dot{z}' \cdot \dot{\mu}) \frac{\partial \tilde{u}_{int}^\mu (r)}{\partial \theta} + \cosh \mu_1 \sin \theta' (\dot{\rho}' \cdot \dot{\mu}) \frac{\partial \tilde{u}_{int}^\mu (r)}{\partial \phi} \\
+ \frac{1}{\sinh \mu_0 \sin \theta} \left[ \sinh \mu_1 \cos \theta' (\dot{z}' \cdot \dot{\phi}) + \cosh \mu_1 \sin \theta' (\dot{\rho}' \cdot \dot{\phi}) \right] \frac{\partial \tilde{u}_{int}^\mu (r)}{\partial \phi} \right]. \quad (A.12)
\end{equation}

The involved derivatives are expanded as follows:

\begin{equation}
\frac{\partial \tilde{u}_{int}^\mu (r)}{\partial \mu} = \sum_{p=0}^{[\frac{n-2p}{2}]} \sum_{l=0}^{[\frac{n-2p}{2}]} B(n, m, p, l) \Gamma(n - 2p + \frac{3}{2}) \left[ \frac{\partial}{\partial \mu} \left( \frac{J_{(n-2p+\frac{1}{2})}(k_{int}r)}{\left( \frac{k_{int}r}{2} \right)^{n-2p+1/2}} \right) \right] \\
\times P_{n-2p-2l}^m (\cosh \mu) P_{n-2p-2l}^m (\cos \theta) e^{im\phi} \\
+ \sum_{p=0}^{[\frac{n-2p}{2}]} \sum_{l=0}^{[\frac{n-2p}{2}]} B(n, m, p, l) \Gamma(n - 2p + \frac{3}{2}) \left( \frac{J_{(n-2p+\frac{1}{2})}(k_{int}r)}{\left( \frac{k_{int}r}{2} \right)^{n-2p+1/2}} \right) \\
\times \left( \frac{\partial}{\partial \mu} \right) P_{n-2p-2l}^m (\cosh \mu) P_{n-2p-2l}^m (\cos \theta) e^{im\phi}, \quad (A.13)
\end{equation}

\begin{equation}
\frac{\partial \tilde{u}_{int}^\mu (r)}{\partial \theta} = \sum_{p=0}^{[\frac{n-2p}{2}]} \sum_{l=0}^{[\frac{n-2p}{2}]} B(n, m, p, l) \Gamma(n - 2p + \frac{3}{2}) \left[ \frac{\partial}{\partial \theta} \left( \frac{J_{(n-2p+\frac{1}{2})}(k_{int}r)}{\left( \frac{k_{int}r}{2} \right)^{n-2p+1/2}} \right) \right] \\
\times P_{n-2p-2l}^m (\cosh \mu) P_{n-2p-2l}^m (\cos \theta) e^{im\phi} \\
+ \sum_{p=0}^{[\frac{n-2p}{2}]} \sum_{l=0}^{[\frac{n-2p}{2}]} B(n, m, p, l) \Gamma(n - 2p + \frac{3}{2}) \left( \frac{J_{(n-2p+\frac{1}{2})}(k_{int}r)}{\left( \frac{k_{int}r}{2} \right)^{n-2p+1/2}} \right) \\
\times \left( \frac{\partial}{\partial \theta} \right) P_{n-2p-2l}^m (\cosh \mu) P_{n-2p-2l}^m (\cos \theta) e^{im\phi}, \quad (A.14)
\end{equation}
In the relations above, the derivatives of the Legendre polynomials \( \frac{\partial}{\partial \mu} P_{n-2p-2l}^m (\cosh \mu) \), \( \frac{\partial}{\partial \theta} P_{n-2p-2l}^m (\cosh \theta) \) are easily calculated via the well known recurrence relations of Legendre polynomials. As far as the \( \mu \)- and \( \theta \)- derivatives of the kernel \( J_{(n-2p+\frac{1}{2})} \) are concerned, their derivation is based on chain-rule differentiation through the basic relation \( r = \frac{\alpha}{2} \sqrt{\sinh^2 \mu + \cos^2 \theta} \). We find, after extended manipulations, and using recurrence relations for Bessel functions that

\[
\frac{\partial}{\partial \mu} \left[ J_{(n-2p+\frac{1}{2})} (k_{int} r) \right] = -J_{(n-2p+\frac{3}{2})} (k_{int} r) \frac{\alpha^2}{4r} \sinh \mu \cosh \mu \quad (A.16)
\]

and

\[
\frac{\partial}{\partial \theta} \left[ J_{(n-2p+\frac{1}{2})} (k_{int} r) \right] = J_{(n-2p+\frac{3}{2})} (k_{int} r) \frac{\alpha^2}{8r} \sin 2\theta \quad (A.17)
\]

Finally, the remaining term of Eq. (19) involves the normal derivative of the point source spherical field, calculated easily as:

\[
\frac{\partial}{\partial n'} \left[ \frac{e^{ik_{int}|r'-r|}}{4\pi |r'-r'|} \right] = \frac{1}{4\pi} \frac{e^{ik_{int}|r'-r|}}{|r'-r'|} \left[ \hat{n}' \cdot (r'-r) \right] \left[ ik_{int} - \frac{1}{|r'-r'|} \right], \quad (A.18)
\]

with

\[
\hat{n}' \cdot (r'-r) = \hat{n}' \cdot (r - r_j)
\]

\[
= \frac{\sinh \mu_1 \cos \theta' \hat{z}' + \cosh \mu_1 \sin \theta' \hat{\rho}'}{(\cosh^2 \mu_1 - \cos^2 \theta')^{\frac{1}{2}}} \cdot \left[ (x - x_j) \hat{x} + (y - y_j) \hat{y} + (z - z_j) \hat{z} \right]
\]

\[
= \frac{1}{(\cosh^2 \mu_1 - \cos^2 \theta')^{\frac{1}{2}}} \left[ \sinh \mu_1 \cos \theta' (x - x_j) (\hat{z}' \cdot \hat{x}) + \sinh \mu_1 \cos \theta' (y - y_j) (\hat{z}' \cdot \hat{y}) + \sinh \mu_1 \cos \theta' (z - z_j) (\hat{z}' \cdot \hat{z}) + \cosh \mu_1 \sin \theta' (x - x_j) (\cos \phi' (\hat{x}' \cdot \hat{x}) + \sin \phi' (\hat{y}' \cdot \hat{z})) + \cosh \mu_1 \sin \theta' (y - y_j) (\cos \phi' (\hat{x}' \cdot \hat{y}) + \sin \phi' (\hat{y}' \cdot \hat{y})) + \cosh \mu_1 \sin \theta' (z - z_j) (\cos \phi' (\hat{x}' \cdot \hat{z}) + \sin \phi' (\hat{y}' \cdot \hat{z})) \right], \quad (A.19)
\]

again expressed in terms of orientation cosines and coordinates of the un-primed system (just notice that \( r' - r_j = r - r_j \)). It is useful to mention that the treatment of the stimulating point source incident field \( u_{inc}(r) \) (and its normal derivative) when facing the transmission conditions (7) do not need the implication of the interior spheroidal system. In that case

\[
\frac{\partial}{\partial n} \left[ \frac{e^{ik_{ext}|r-r_0|}}{4\pi |r-r_0|} \right] = \frac{e^{ik_{ext}|r-r_0|}}{4\pi |r-r_0|^2} [\hat{n} \cdot (r - r_0)] [ik_{ext} - \frac{1}{|r-r_0|}] \quad (A.20)
\]
with

\[ \hat{n} \cdot (r - r_0) = \left[ \frac{\sinh \mu_0 \cos \theta \hat{z} + \cosh \mu_0 \sin \theta \hat{\rho}}{(\cosh \mu_0^2 - \cos \theta^2)^{\frac{1}{2}}} \right] \cdot \left[ (x - x_0)\hat{x} + (y - y_0)\hat{y} + (z - z_0)\hat{z} \right] \]

\[ = \frac{1}{(\cosh \mu_0^2 - \cos \theta^2)^{\frac{1}{2}}} \left[ \sinh \mu_0 \cos \theta (z - z_0) + \cosh \mu_0 \sin \theta (x - x_0) \cos \phi \right. \]

\[ + \cosh \mu_0 \sin \theta (y - y_0) \sin \phi \left. \right]. \]  

(A.21)

References


