MINIMAL GRAPHS IN \( \mathbb{R}^4 \) WITH BOUNDED JACOBIANS

TH. HASANIS, A. SAVAS-HALILAJ, AND TH. VLACHOS

Abstract. In this paper we obtain a Bernstein type result for entire two dimensional minimal graphs in \( \mathbb{R}^4 \), which extends a previous one due to L. Ni. Moreover, we provide a characterization for complex analytic curves.

1. Introduction

The famous theorem of Bernstein states that the only entire minimal graphs in the Euclidean space \( \mathbb{R}^3 \) are the planes. More precisely, if \( f : \mathbb{R}^2 \to \mathbb{R} \) is an entire (i.e., defined over all of \( \mathbb{R}^2 \)) smooth function whose graph
\[
G_f := \{(x, y, f(x, y)) \in \mathbb{R}^3 : (x, y) \in \mathbb{R}^2\}
\]
is a minimal surface, then it is an affine function, and the graph is a plane.

This result has been generalized in higher dimension and codimension under various conditions. See [1], [3], [9] and the references therein for the codimension one case and [5], [10], [12] for the higher codimension case.

The aim of this paper is to study the following special case. Let \( M \) be a minimal surface in \( \mathbb{R}^4 \) that can be described as the graph of an entire and smooth vector valued function \( f : \mathbb{R}^2 \to \mathbb{R}^2, f = (f_1, f_2), \) that is
\[
M = G_f := \{(x, y, f_1(x, y), f_2(x, y)) \in \mathbb{R}^4 : (x, y) \in \mathbb{R}^2\}.
\]
The following question arises in a natural way: Is it true that the graph \( G_f \) of \( f \) is a plane in \( \mathbb{R}^4 \)? In general, the answer is negative. An easy counterexample is given by the function \( f(x, y) = (x^2 - y^2, 2xy) \).

1991 Mathematics Subject Classification. Primary 53C42.
Key words and phrases. Minimal surface, Bernstein type theorem, Jacobian.

The second author is supported financially from the "Foundation for Education and European Culture".
Actually, the graph of any holomorphic function \( \Phi : \mathbb{C} \to \mathbb{C} \) gives rise to a minimal surface. So, the problem of finding geometric conditions in order to have a result of Bernstein type is reasonable. R. Schoen [8] obtained a Bernstein type result by imposing the assumption that \( f : \mathbb{R}^2 \to \mathbb{R}^2 \) is a diffeomorphism. Moreover, L. Ni [6] by using the result of R. Schoen [8] and results due to J. Wolfson [11] on minimal Lagrangian surfaces has derived a result of Bernstein type under the assumption that \( f \) is an area-preserving map, that is the Jacobian \( J_f := \det (df) \) satisfies \( J_f = 1 \), where \( df \) denotes the differential of \( f \).

In this paper we prove, firstly, the following result of Bernstein type, which generalizes the result due to L. Ni.

**Theorem 1.1.** Let \( f : \mathbb{R}^2 \to \mathbb{R}^2 \) be an entire smooth vector valued function such that its graph \( G_f \) is a minimal surface in \( \mathbb{R}^4 \). If the Jacobian \( J_f \) of \( f \) is bounded, then \( G_f \) is a plane.

As a consequence we derive an easy alternative proof of the following well known result of Jörgens [4]:

**Jörgens’ Theorem.** The only entire solutions \( f : \mathbb{R}^2 \to \mathbb{R} \) of the Monge-Ampère equation \( f_{xx}f_{yy} - f_{xy}^2 = 1 \) are the quadratic polynomials.

There are plenty of entire minimal graphs in \( \mathbb{R}^4 \), the so called complex analytic curves. More precisely, if \( \Phi : \mathbb{C} \to \mathbb{C} \) is any entire holomorphic or anti-holomorphic function, then the graph
\[
G_\Phi = \{ (z, \Phi(z)) : z \in \mathbb{C} \}
\]
of \( \Phi \) in \( \mathbb{C}^2 = \mathbb{R}^4 \) is a minimal surface and is called a complex analytic curve. Such surfaces are locally characterized (see for example L.P Eisenhart [2]) by the relation \( |K| = |K_N| \), where \( K \) and \( K_N \) stand for the Gauss and normal curvature of the surface, respectively. The following result is in valid.

**Theorem 1.2.** Suppose that \( G_f \) is the graph of an entire smooth vector valued function \( f : \mathbb{R}^2 \to \mathbb{R}^2 \) with Gaussian curvature \( K \) and normal curvature \( K_N \). Assume that \( G_f \) is minimal in \( \mathbb{R}^4 \). Then,
\[
\inf_{K < 0} \frac{|K_N|}{|K|} = 0,
\]
unless \( G_f \) is a complex analytic curve.

As a consequence of the above theorem, we obtain the following result,
Corollary 1.3. Suppose that $G_f$ is the graph of an entire smooth vector valued function $f : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ with Gaussian curvature $K$ and normal curvature $K_N$. If $K_N = cK$, where $c$ is a constant, then $G_f$ is a complex analytic curve. More precisely, $K_N = K = 0$ and $G_f$ is a plane or $|c| = 1$ and $G_f$ is a non-trivial complex analytic curve.

2. Basic Notation and Definitions

A surface $M$ in the Euclidean space $\mathbb{R}^n$ is represented, locally, by a transformation $X : D \subset \mathbb{R}^2 \rightarrow \mathbb{R}^n$ of rank 2, given by

$$X(x, y) = (f_1(x, y), f_2(x, y), \ldots, f_n(x, y)), \quad (x, y) \in D,$$

where $D$ is an open subset of $\mathbb{R}^2$ and $f_i : D \rightarrow \mathbb{R}$, $i \in \{1, \ldots, n\}$, are smooth functions. Following the standard notation of differential geometry, we denote by $\langle \cdot, \cdot \rangle$ the Euclidean inner product on $\mathbb{R}^n$ and by $E, F, G$ the coefficients of the first fundamental form, which are given by

$$E = \sum_{i=1}^{n} \left( \frac{\partial f_i}{\partial x} \right)^2, \quad F = \sum_{i=1}^{n} \frac{\partial f_i}{\partial x} \frac{\partial f_i}{\partial y}, \quad G = \sum_{i=1}^{n} \left( \frac{\partial f_i}{\partial y} \right)^2.$$

We recall that the parameters $(x, y)$ are called isothermal if and only if $E = G$ and $F = 0$, everywhere on $D$.

Consider a local orthonormal frame field $\{e_1, e_2; \xi_3, \ldots, \xi_n\}$ in $\mathbb{R}^n$ such that when restricted to $M$, the vectors $e_1, e_2$ are tangent to $M$ and, consequently, $\xi_3, \ldots, \xi_n$ are normal to $M$. Denote by $\nabla$ the usual linear connection on $\mathbb{R}^n$ and let

$$h_{ij}^\alpha = \langle \nabla_{e_i} \xi_\alpha, e_j \rangle, \quad i, j \in \{1, 2\}, \quad \alpha \in \{3, \ldots, n\},$$

be the coefficients of the second fundamental form.

The mean curvature vector $H$ and the Gauss curvature $K$ of $M$ are given, respectively, by

$$H = \frac{1}{2} \sum_{\alpha=3}^{n} (h_{11}^\alpha + h_{22}^\alpha) \xi_\alpha,$$

$$K = \sum_{\alpha=3}^{n} \left( h_{11}^\alpha h_{22}^\alpha - (h_{12}^\alpha)^2 \right).$$

Moreover, if

$$|h|^2 = \sum_{i,j=1}^{2} \sum_{\alpha=3}^{n} (h_{ij}^\alpha)^2$$
is the square of the length of the second fundamental form \( h \), then the Gauss equation implies

\[
2K = 4H^2 - |h|^2.
\]

In the case where \( M \) is minimal, i.e., \( H = 0 \), the above become

\[
K = -\sum_{\alpha=3}^{n} \left\{ (h_{\alpha}^{11})^2 + (h_{\alpha}^{12})^2 \right\}, \quad (2.1)
\]

\[
2K = -|h|^2. \quad (2.2)
\]

Another geometric invariant which plays a very important role in the theory of surfaces in \( \mathbb{R}^4 \) is the normal curvature \( K_N \) of \( M \) which is given by

\[
K_N = \sum_{i=1}^{2} \left( h_{i1}^3 h_{i2}^4 - h_{i2}^3 h_{i1}^4 \right).
\]

In particular, for minimal surfaces we have

\[
K_N = 2 \left( h_{11}^3 h_{12}^4 - h_{12}^3 h_{11}^4 \right). \quad (2.3)
\]

One simple way to express a surface in \( \mathbb{R}^{n+2} \) is in non-parametric form, that is to say, as the graph

\[
G_f = \{(x, y, f_1(x, y), \ldots, f_n(x, y)) \in \mathbb{R}^{n+2} : (x, y) \in D\}
\]

of a vector valued map \( f : D \rightarrow \mathbb{R}^n, f(x, y) = (f_1(x, y), \ldots, f_n(x, y)) \), where \( D \) is an open subset of \( \mathbb{R}^2 \). Of course, any surface can be locally described in this manner. By computing the Euler-Lagrange equations for the area integral we see that the surface \( G_f \) is minimal if and only if \( f \) satisfies the following equation,

\[
(1 + |f_y|^2) f_{xx} - 2 \langle f_x, f_y \rangle f_{xy} + (1 + |f_x|^2) f_{yy} = 0. \quad (2.4)
\]

This is the classical non-parametric minimal surface equation.

The following result due to R. Osserman [7, Theorem 5.1] is the main tool for the proofs of our results.

**Theorem 2.1.** Let \( f : \mathbb{R}^2 \rightarrow \mathbb{R}^2 \) be an entire solution of the minimal surface equation. Then there exists real constants \( a, b \), with \( b > 0 \), and a non-singular linear transformation

\[
x = u, \quad y = au + bv,
\]

such that \( (u, v) \) are global isothermal parameters for the surface \( G_f \).

Moreover the following identity is useful in the proofs.
Lagrange’s Identity. For two real valued vectors $V = (v_1, \cdots, v_n)$ and $W = (\omega_1, \cdots, \omega_n)$ in $\mathbb{R}^n$ we have
\[
\left( \sum_{i=1}^{n} v_i^2 \right) \left( \sum_{i=1}^{n} \omega_i^2 \right) - \left( \sum_{i=1}^{n} v_i \omega_i \right)^2 = \sum_{i<j} (v_i \omega_j - v_j \omega_i)^2.
\] (2.5)

3. PROOFS OF THE RESULTS

Let $f : \mathbb{R}^2 \to \mathbb{R}^2$, $f(x, y) = (f_1(x, y), f_2(x, y))$, $(x, y) \in \mathbb{R}^2$, be an entire solution of the minimal surface equation. Then, the graph
\[
G_f = \{(x, y, f_1(x, y), f_2(x, y)) \in \mathbb{R}^4 : (x, y) \in \mathbb{R}^2\}
\]
of $f$ is a minimal surface in $\mathbb{R}^4$. By virtue of the Theorem 2.1, we can introduce global isothermal parameters $(u, v)$, via a non-singular transformation
\[
x = u, \quad y = au + bv,
\]
where $a, b$ are real constants with $b > 0$. Now, the minimal surface $G_f$ is parametrized via the map
\[
X(u, v) = (u, au + bv, \varphi(u, v), \psi(u, v)),
\]
where $\varphi(u, v) := f_1(u, au + bv)$ and $\psi(u, v) := f_2(u, au + bv)$. Since $(u, v)$ are isothermal parameters, the vectors
\[
X_u = (1, a, \varphi_u, \psi_u), \quad X_v = (0, b, \varphi_v, \psi_v)
\] (3.1)
are orthogonal and of the same length, that is
\[
\varphi_u \varphi_v + \psi_u \psi_v = -ab,
\] (3.2)
\[
E = 1 + a^2 + \varphi_u^2 + \psi_u^2 = b^2 + \varphi_v^2 + \psi_v^2.
\]
Moreover, the fact that the surface $G_f$ is minimal, implies that the functions $\varphi$ and $\psi$ are harmonic, that is
\[
\varphi_{uu} + \varphi_{vv} = 0, \quad \psi_{uu} + \psi_{vv} = 0.
\] (3.3)
Appealing to the Lagrange’s Identity and taking the relations (3.2) into account, we obtain
\[
E^2 = b^2 + \varphi_v^2 + \psi_v^2 + (a\varphi_v - b\varphi_u)^2
+ (a\psi_v - b\psi_u)^2 + (\varphi_u \psi_v - \varphi_v \psi_u)^2,
\]
or equivalently,
\[
E^2 = (1 + a^2 + b^2) E - b^2 + (\varphi_u \psi_v - \varphi_v \psi_u)^2.
\] (3.4)
We set $\Phi(u,v) = (\varphi(u,v), \psi(u,v))$. Because of the relation

$$\frac{\partial (\varphi, \psi)}{\partial (u,v)} = \frac{\partial (f_1, f_2)}{\partial (x,y)} \frac{\partial (x,y)}{\partial (u,v)}$$

for the Jacobians, we have

$$J_\Phi = b J_f,$$

where $J_f, J_\Phi$ stand for the Jacobians of $f$ and $\Phi$, respectively. So (3.4) becomes

$$J_\Phi^2 = E^2 - (1 + a^2 + b^2) E + b^2,$$

(3.5)
a useful identity for us.

Now we are ready to give the proof of Theorem 1.1.

**Proof of Theorem 1.1.** Since the Jacobian $J_\Phi$ is bounded, we conclude from (3.5) that $E$, and thus $\log E$, is bounded from above. On the other hand, the Gaussian curvature $K$ of $G_f$ is given by

$$K = -\frac{\Delta \log E}{2E},$$

where $\Delta$ is the usual Laplacian operator on the $(u,v)$-plane. From (2.1), we deduce that the Gaussian curvature $K$ is non-positive. Thus, $\Delta \log E \geq 0$ and thus the function $\log E$ is a subharmonic function defined on the whole plane. Since $\log E$ is also bounded from above, we deduce that $E$ is constant and consequently $K$ is identically zero. Then it follows immediately from (2.2) that the graph $G_f$ of $f$ is totally geodesic and hence a plane. □

**Remark 3.1.** In a similar way, we can prove the following result: Let $f : \mathbb{R}^2 \to \mathbb{R}^n,$

$$f(x,y) = (f_1(x,y), f_2(x,y), \cdots, f_n(x,y)),$$

be a vector valued function, defined on the whole $\mathbb{R}^2$, which is a solution of the minimal surface equation. If the quantity

$$\sum_{i<j} \left( \frac{\partial f_i}{\partial x} \frac{\partial f_j}{\partial y} - \frac{\partial f_j}{\partial x} \frac{\partial f_i}{\partial y} \right)^2$$

is bounded, then the graph $G_f$ of $f$ is a plane in $\mathbb{R}^{n+2}$.

Now, we show that one can get the well known Jörgens’ result [4] as a consequence of Theorem 1.1.
Proof of Jörgens’ Theorem. Obviously $f_{xx} + f_{yy} \neq 0$ everywhere on $\mathbb{R}^2$.
We consider the function $\Theta : \mathbb{R}^2 \to \mathbb{R}$, given by

$$\Theta = \frac{f_{xx}f_{yy} - f_{xy}^2 - 1}{f_{xx} + f_{yy}}.$$ 

The function $\Theta$, thanks to our assumption, is identically zero, and so $\Theta_x = \Theta_y = 0$.

On the other hand, one can readily verify that the equations

$$\Theta_x = \Theta_y = 0$$

are equivalent to the minimal surface equation for the vector valued function $g : \mathbb{R}^2 \to \mathbb{R}^2$, defined by $g = (f_x, f_y)$. Moreover, we have $J_g = 1$. So, according to Theorem 1.1, the graph $G_g$ of $g$ is a plane and the result is immediate. 

For the proof of Theorem 1.2, we need the following auxiliary result.

Lemma 3.2. Let $\Phi : \mathbb{R}^2 \to \mathbb{R}^2$, $\Phi(u,v) = (\varphi(u,v), \psi(u,v))$, be a map, where $\varphi$ and $\psi$ are harmonic functions on $\mathbb{R}^2$, i.e., a harmonic map. Then $\inf |J_\Phi| = 0$, unless $\Phi$ is an affine map.

Proof. Suppose in the contrary that $\Phi$ is not affine and $\inf |J_\Phi| = c > 0$. Hence $|J_\Phi| \geq c > 0$. Assume at first that $J_\Phi \geq c > 0$. We view $\Phi$ as a complex valued function $\Phi : \mathbb{C} \to \mathbb{C}$, $\Phi = \varphi + i\psi$. Then, for $z = u + iv$, we have

$$\Phi_z = \frac{1}{2} (\varphi_u + \psi_v) + \frac{i}{2} (\psi_u - \varphi_v)$$

and

$$\Phi_{\overline{z}} = \frac{1}{2} (\varphi_u - \psi_v) + \frac{i}{2} (\psi_u + \varphi_v).$$

A simple calculation shows that

$$J_\Phi = |\Phi_z|^2 - |\Phi_{\overline{z}}|^2.$$ 

Furthermore, since $\varphi$ and $\psi$ are harmonic functions it follows that the function $\Phi_z$ is holomorphic. From our assumption and (3.6) we get

$$|\Phi_z|^2 \geq |\Phi_{\overline{z}}|^2 + c \geq c > 0.$$ 

Since $\Phi_z$ is an entire holomorphic function, Picard’s Theorem implies that $\Phi_z$ must be constant. Therefore, there are real constants $\kappa, \lambda$ such that

$$\varphi_u + \psi_v = 2\kappa \text{ and } \psi_u - \varphi_v = 2\lambda.$$
Then from (3.7) we deduce that
\[(\psi_v^2 - 2\kappa)^2 + (\psi_u^2 - 2\lambda)^2 \leq \kappa^2 + \lambda^2 - c.\]
By the harmonicity of the real functions \(\psi_v - 2\kappa, \psi_u - 2\lambda\) and the Liouville’s Theorem, we deduce that \(\varphi\) and \(\psi\) are affine functions, which contradicts our assumptions.

Assume now that \(J_\Phi \leq -c < 0\). In this case, we consider the complex valued function \(\tilde{\Phi} = \psi + i\varphi\). Since \(J_{\tilde{\Phi}} = -J_\Phi \geq c > 0\), proceeding as above we deduce that \(\tilde{\Phi}\) is affine, and consequently \(\Phi\) is affine. This is again a contradiction. Thus \(\inf |J_\Phi| = 0\), and the proof is concluded. \(\square\)

Proof of Theorem 1.2. Assume that \(G_f\) is not a plane and that
\[
\inf_{K<0} \left|\frac{K_N}{|K|}\right| > 0.
\]
We introduce global isothermal parameters \((u, v)\) such that the minimal surface \(G_f\) is parametrized via the map
\[X (u, v) = (u, au + bv, \varphi (u, v), \psi (u, v)),\]
where \(a, b\) are real constants with \(b > 0\).
We claim that \((a, b) = (0, 1)\). Arguing indirectly, we assume that \((a, b) \neq (0, 1)\). Differentiating (3.2) with respect to \(u, v\) and taking (3.3) into account, we find
\[
\varphi_{uu} \varphi_v + \varphi_u \varphi_{uv} = -\psi_{uu} \psi_v - \psi_u \psi_{uv},
\]
\[
\varphi_{uu} \varphi_u - \varphi_v \varphi_{uv} = -\psi_{uu} \psi_v + \psi_u \psi_{uv}.
\]  
Squaring both of them and summing we obtain
\[
(\varphi_u^2 + \varphi_v^2) (\varphi_{uu}^2 + \varphi_{uv}^2) = (\psi_u^2 + \psi_v^2) (\psi_{uu}^2 + \psi_{uv}^2).
\]  
Consider the following subset of \(\mathbb{R}^2\)
\[M_0 = \{(u, v) \in \mathbb{R}^2 : \omega (u, v) = 0\},\]
where
\[
\omega (u, v) := (\varphi_u^2 + \varphi_v^2) (\varphi_{uu}^2 + \varphi_{uv}^2),
\]
or, equivalently, in view of (3.9)
\[
\omega (u, v) = (\psi_u^2 + \psi_v^2) (\psi_{uu}^2 + \psi_{uv}^2).
\]
We claim that the complement \(M_1 = \mathbb{R}^2 - M_0\) is dense in \(\mathbb{R}^2\). To this purpose it is enough to show that the interior, \(\text{int} (M_0)\), of \(M_0\)
is empty. Assume in the contrary that \( \text{int}(M_0) \neq \emptyset \) and let \( U \) be a connected component of \( \text{int}(M_0) \). Then it follows easily that the analytic functions \( \varphi \) and \( \psi \) are affine. Thus, by analyticity, \( G_f \) is a plane, which is a contradiction.

In the sequel, we work on \( M_1 \). By virtue of (3.8), we get

\[
\varphi_{uv} = -\frac{(\varphi_u \psi_u + \varphi_v \psi_v) \psi_{uv} - J_\Phi \psi_{uu}}{\varphi_u^2 + \varphi_v^2},
\]

\[
\varphi_{uu} = \frac{J_\Phi \psi_{uu} - (\varphi_u \psi_u + \varphi_v \psi_v) \psi_{uu}}{\varphi_u^2 + \varphi_v^2}.
\]

(3.10)

The vector fields

\[
\xi = (-b\varphi_u + a\varphi_v, -\varphi_v, b, 0), \quad \eta = (-b\psi_u + a\psi_v, -\psi_v, 0, b)
\]

are normal to \( G_f \) and satisfy

\[
|\xi|^2 |\eta|^2 - \langle \xi, \eta \rangle^2 = b^2 E^2.
\]

We, easily, check that the vector fields \( \{e_1, e_2; \xi_3, \xi_4\} \) given by

\[
e_1 = \frac{1}{\sqrt{E}} X_u, \quad e_2 = \frac{1}{\sqrt{E}} X_v,
\]

\[
\xi_3 = \frac{\xi}{|\xi|}, \quad \xi_4 = \frac{1}{b|\xi|} \left( |\xi|^2 \eta - \langle \xi, \eta \rangle \xi \right),
\]

constitute an orthonormal frame field along \( G_f \). Moreover, \( \xi_3 \) and \( \xi_4 \) are normal to \( G_f \). Then a straightforward computation shows that the coefficients of the second fundamental form are given by

\[
h_{31}^3 = -\frac{b\varphi_{uu}}{E |\xi|}, \quad h_{11}^4 = \frac{\langle \xi, \eta \rangle \varphi_{uu} - |\xi|^2 \psi_{uu}}{E^2 |\xi|},
\]

\[
h_{32}^3 = -\frac{b\varphi_{uv}}{E |\xi|}, \quad h_{12}^4 = \frac{\langle \xi, \eta \rangle \varphi_{uv} - |\xi|^2 \psi_{uv}}{E^2 |\xi|}.
\]

So using (2.1) and (2.3) and (3.10), we find

\[
K = \frac{1}{E^3} \frac{\psi_{uu}^2 + \psi_{uv}^2}{\varphi_u^2 + \varphi_v^2} \left( 2b^2 - (1 + a^2 + b^2) E \right)
\]

(3.11)

and

\[
K_N = \frac{2b \psi_{uu}^2 + \psi_{uv}^2}{E^3} \frac{J_\Phi}{\varphi_u^2 + \varphi_v^2}.
\]

(3.12)

The second equation of (3.2), yields

\[
E \geq \frac{1 + a^2 + b^2}{2}.
\]
Hence, 
\[ 2b^2 - (1 + a^2 + b^2) E \leq -\frac{1}{2} \left( a^2 + (b - 1)^2 \right) \left( a^2 + (b + 1)^2 \right) < 0. \]
This shows that 
\[ M_1 \subset \{(u, v) \in \mathbb{R}^2 : K(u, v) < 0\}. \]
Moreover, 
\[ \frac{K_N^2}{K_2^2} = 4b^2 W(E), \]
where \( W(t) \) is the increasing real valued function 
\[ W(t) := \frac{t^2 - (1 + a^2 + b^2) t + b^2}{((1 + a^2 + b^2) t - 2b^2)^2}, \quad t \geq 1. \]
From our assumption \( \inf_{K<0} \frac{|K_N|}{|K|} > 0 \), we get 
\[ \inf_{M_1} \frac{|K_N|}{|K|} > 0. \]
Since \( W(t) \) is increasing, we have 
\[ \inf_{M_1} \frac{K_N^2}{K_2^2} = 4b^2 W \left( \inf_{M_1} E \right). \]
Hence \( W \left( \inf_{M_1} E \right) > 0 \) or, equivalently, 
\[ \left( \inf_{M_1} E \right)^2 - (1 + a^2 + b^2) \inf_{M_1} E + b^2 > 0. \]
Appealing to the identity (3.5), we deduce that \( \inf_{M_1} |J_\Phi| > 0 \). By continuity, and bearing in mind the fact that \( M_1 \) is dense in \( \mathbb{R}^2 \), we infer that \( |J_\Phi| \) is bounded from below away from zero. On the other hand \( \Phi(u, v) = (\varphi(u, v), \psi(u, v)) \) is a harmonic map. Therefore, according to Lemma 3.2, \( G_f \) is a plane which contradicts our assumptions. Thus \( (a, b) = (0, 1) \) and the equations (3.2) become 
\[ \varphi_u \varphi_v + \psi_u \psi_v = 0, \]
\[ \varphi_u^2 + \psi_u^2 = \varphi_v^2 + \psi_v^2. \]
So, \( \varphi_u = \pm \psi_v, \varphi_v = \mp \psi_u \) and \( G_f \) is a complex analytic curve. \( \square \)

Proof of Corollary 1.3. In the case where \( K \equiv 0 \), the graph \( G_f \) is a plane. Let consider the case where \( K \) is not identically zero. According to Theorem 1.2 we have \( c = 0 \), unless \( G_f \) is a complex analytic curve. We claim that the case \( c = 0 \) does not occur. Indeed, arguing indirectly
suppose that $c = 0$. As in the proof of Theorem 1.2, the set $M_1$ is dense in $\mathbb{R}^2$. From the assumption $K_N = cK$, we conclude that $K_N = 0$ in $M_1$. Furthermore, the relation (3.12) yields that $J_\Phi = 0$. Taking into account the identity (3.5), we get that $E$ is constant, which implies that $K$ is identically zero, a contradiction. Therefore, $G_f$ is a complex analytic curve. □

REFERENCES