Minimal surface area position of a convex body is not always an M-position

Christos Saroglou

Abstract

Milman proved that there exists an absolute constant $C > 0$ such that, for every convex body K in $\mathbb{R}^n$ there exists a linear image TK of K with volume 1, such that $|TK + D_n|^{1/n} \leq C$, where D_n is the Euclidean ball of volume 1. TK is then said to be in M-position. Giannopoulos and Milman asked if every convex body that has minimal surface area among all its affine images of volume 1 is also in M-position. We prove that the answer to this question is negative, even in the 1-unconditional case.

1 Introduction

Let $C > 0$. We say that a convex body K of volume 1 in $\mathbb{R}^n$ is in M-position with constant C if

$$|K + D_n|^{1/n} \leq C, \tag{1}$$

where D_n is the Euclidean ball of volume 1. Here, $|\cdot| = |\cdot|_n$ denotes n-dimensional volume and $A + B = \{x + y : x \in A, y \in B\}$ is the Minkowski sum of the sets A and B.

The starting point of this paper is a famous result of Milman [7] stating that there exists an absolute positive constant C such that every convex body has a linear image of volume 1 which is in M-position with constant C. In what follows, we will refer to the M-position without specifying some precise value for this numerical constant. It follows from the Brunn–Minkowski inequality

$$|K_1 + K_2|^{1/n} \geq |K_1|^{1/n} + |K_2|^{1/n}$$

for non-empty compact subsets K_1, K_2 of $\mathbb{R}^n$, that if K is in M-position then

$$|K + D_n|^{1/n} \sim \min\left\{|TK + D_n|^{1/n} : T \in SL(n)\right\}.$$

Thus, (1) may be viewed as an inverse form of the Brunn–Minkowski inequality. The notation $a \sim b$ means that the ratio a/b is bounded by absolute constants (from above and from below).

We say that K is in minimal surface area position if the surface area $\partial(K)$ of K is minimal among those of its affine images of the same volume. Petty [9] gave a characterization of the minimal surface area position: K has minimal surface area if and only if the function

$$x \mapsto \int_{S^{n-1}} \langle x, y \rangle^2 dS_K(y)$$

is constant on S^{n-1}, where S_K denotes the surface area measure of K. It is also well-known that the minimal surface area position is unique up to orthogonal transformations.

Giannopoulos and Milman [4] gave characterizations of the same type for problems that involve other quermassintegrals. In the same paper, they asked if the minimal surface area position is also an M-position. Our main result is the following.

Theorem 1.1. *There exists an absolute constant $c_0 > 0$ with the following property: for every positive integer n, there exists a 1-unconditional convex body K of volume 1 in $\mathbb{R}^n$ which is in minimal surface area position and, at the same time,*

$$|K + D_n|^{1/n} \geq c_0 n^{1/8}.$$

Theorem 1.1 shows that, if n is large enough then the minimal surface area position may be far from being an M-position, even in the 1-unconditional case. Recall that K is called 1-unconditional if it is symmetric with respect to all coordinate hyperplanes.

Let us also note that, in the case $n = 2$, it was proved in [4] that every convex body K with minimal surface area has the property

$$|K + tD_n| = \min\{|TK + tD_n| : T \in SL(n)\}$$

for every $t > 0$. Theorem 1.1 shows that this is no longer true in higher dimensions, at least when $\frac{1}{cn^{1/8}} \leq t \leq cn^{1/8}$, where $c > 0$ is a small enough absolute positive constant.

2 Background

Let K and L be two convex bodies in $\mathbb{R}^n$. It is well known by a classical result of Minkowski that, for $t > 0$, the volume of the convex body $L + tK$ is a polynomial of degree n, as a function of t. More precisely, one can write

$$|L + tK| = \sum_{j=1}^{n} t^j \binom{n}{j} V(K[j], L[n-j]) ,$$

where $V(K[j], L[n-j])$ are non-negative quantities, called in common mixed volumes of K and L. In this paper we are mostly interested in the case $j = 1$. We will write for simplicity $V(K, L) := V(K, L, \dots, L) = V(K[1], L[n-1])$. Note that $V(K, K) = |K|$. Also, if B_2^n is the Euclidean unit ball in $\mathbb{R}^n$ then $V(K, B_2^n)$ and $V(B_2^n, K)$ are (up to a constant depending only on n) the mean width and the surface area of K respectively.

A fundamental fact concerning $V(K, L)$ is Minkowski's inequality

$$V(K, L) \geq |K|^{\frac{1}{n}} |L|^{\frac{n-1}{n}},$$

with equality if and only if K and L are homothetic.

Assume, now, that both K and L have the origin as an interior point. The quantity V(K, L) can be expressed as:

$$V(K, L) = \frac{1}{n} \int_{S^{n-1}} h_K(x)\, dS_L(x),$$

where $h_K(x) = \max_{y \in K} \langle x, y \rangle$ is the support function of K and S_L is the surface area measure of L (S_L is a measure on S^{n-1}). Recall the definition of S_L: If ω is a Borel subset of S^{n-1}, then

$$S_L(\omega) = \left| \left\{ x \in \mathrm{bd}(L) : \exists\ (u, t) \in \omega \times \mathbb{R},\ \text{so that } (tu + u^{\perp}) \cap L = \{x\} \right\} \right|_{n-1} .$$

Here, $\mathrm{bd}(L)$ denotes the boundary of L.

Let K be a convex body in $\mathbb{R}^n$ with the origin as an interior point. The polar body K^* of K is defined by

$$K^* = \{x \in \mathbb{R}^n : \langle x, y \rangle \leq 1,\, y \in K\}.$$

We will use the asymptotic formula

$$\frac{1}{b_1 n} \leq (|K|\, |K^*|)^{1/n} \leq \omega_n^{2/n} \leq \frac{b_1}{n},$$

which holds true if the centroid of K is at the origin. Here, ω_n denotes the volume of the Euclidean unit ball B_2^n and $b_1 > 0$ is an absolute constant. The right hand side inequality is the classical Blaschke–Santaló inequality and it is sharp; equality holds if and only if K is an ellipsoid centered at the origin. The left hand side inequality was proved much later by Bourgain and Milman (see [2] and [7]) and it is often called the inverse Blaschke–Santaló inequality.

We close this Section with three facts that will be needed in the sequel. The first one, roughly speaking, states that K is in M-position if and only if K^* is in M-position. This follows from the proof of Milman's theorem (see [7, Section 4, Remark 3]) on the existence of the M-position (see also [3, Theorem 5.3]).

Fact I. *There exists an absolute constant $b_2 > 0$ such that, if K is a convex body with centroid at the origin in $\mathbb{R}^n$, then*

$$|\overline{K} + D_n|^{1/n} \leq b_2 |\overline{K^*} + D_n|^{1/n},$$

where $\overline{A} = |A|^{-1/n} A$.

For the next two, we need the definition of the covering number of K by L. If K and L are compact subsets of $\mathbb{R}^n$ with non-empty interiors, we define

$$N(K, L) = \min \left\{ k \in \mathbb{N} : K \subseteq \bigcup_{i=1}^{k} (x_i + L) \text{ for some } x_1, \dots, x_k \in \mathbb{R}^n \right\}.$$

Fact II. *If K is compact and L is a centrally symmetric convex body in $\mathbb{R}^n$, then*

$$2^{-n} \frac{|K + L|}{|L|} \leq N(K, L) \leq 2^n \frac{|K + L|}{|L|}.$$

Fact III. *There exists an absolute constant $b_3 > 0$ such that if K and L are convex bodies in $\mathbb{R}^n$, then*

$$|L + D_n| \leq b_3^n \frac{|K + L|}{|K|} |K + D_n|.$$

Fact II is an easy consequence of the definitions. The inequality on the left is trivial while a short proof for the inequality on the right is given in [7, section 5] (actually, this is stated in [7] in the special case where K is convex and L is a ball; however the same proof works for our purposes as well). Fact III follows immediately from Fact II in the case where L is centrally symmetric. The general case can be deduced from the Rogers–Shephard inequality [10].

3 Curvature Images

The notion of the curvature image of a star-body was introduced by Lutwak [6] and will play an important role in the proof of our main result. If K is a star-body with centroid at 0, the curvature image $C(K)$ of K is the unique convex body with centroid at the origin and surface area measure

$$dS_{C(K)} = \frac{1}{n+1} \rho_K^{n+1} d\lambda =: f_{C(K)} d\lambda,$$

where ρ_K is the radial function of K and λ is the Lebesgue measure on S^{n-1}. Existence and uniqueness of $C(K)$ are guaranteed by Minkowski's existence theorem (see e.g. [11], pp. 389–393). It can be proved that $C(K)$ is affinely associated with K. One can check that

$$C(TK) = (T^*)^{-1} C(K) \tag{2}$$

for every volume preserving linear transformation T of $\mathbb{R}^n$.

We define the quantity

$$F_K = \frac{1}{\sqrt{n}|K|^{1+\frac{1}{n}}} \min\left\{\int_{TK} \|x\|_2 dx : T \in SL(n)\right\},$$

where $\|\cdot\|_2$ denotes Euclidean norm. Using integration in polar coordinates, we readily see that the surface area of $C(K)$ can be written in the form

$$\partial(C(K)) = \int_K \|x\|_2 dx.$$

Taking into account (2) we obtain the next Lemma.

Lemma 3.1. *Let K be a star-body with centroid at 0. Then, $C(K)$ has minimal surface area if and only if*

$$F_K = \frac{1}{\sqrt{n}|K|^{1+\frac{1}{n}}} \int_K \|x\|_2 dx.$$

The key step for the proof of Theorem 1.1 will be the following:

Theorem 3.2. *Let K be a star-body of volume 1 in $\mathbb{R}^n$, with centroid at 0. Let L be a convex body in $\mathbb{R}^n$, with centroid at 0, such that $K \subseteq L$ and $|L| \leq \alpha^n |K|$ for some $\alpha \geq 1$. Let $T \in SL(2n)$, such that*

$$F_{K\times D_n} = \frac{1}{\sqrt{2n}} \int_{T(K\times D_n)} \|x\|_2 dx.$$

If $\overline{C}(T(K \times D_n))$ is the homothet of $C(T(K \times D_n))$ of volume 1, then

$$|\overline{C}(T(K \times D_n)) + D_{2n}|^{1/2n} \geq c\sqrt[4]{F_K},$$

for some constant $c = c(\alpha) > 0$ which depends only on α.

For the proof of Theorem 3.2 we will modify an idea from [1, Proposition 1.4]; for our purposes, we have to deal with the L^1-case instead of the more convenient L^2-case.

The proof of Theorem 3.2 will be given in the next Section. We close this Section with some information on the relation of the M-position of K with the M-position of $C(K)$.

Proposition 3.3. *There exists an absolute constant $c_1 > 0$ such that, if K is as in Theorem 3.2, then*

$$\frac{1}{c_1} \leq |C(K)| \leq c_1 c(\alpha).$$

Proof. Minkowski's inequality implies that

$$\begin{aligned}
|L^*|^{\frac{1}{n}} |C(K)|^{\frac{n-1}{n}} &\leq V(L^*, C(K)) = \frac{1}{n}\int_{S^{n-1}} h_{L^*}(x) dS_{C(K)}(x) \\
&= \frac{1}{n}\int_{S^{n-1}} \frac{1}{\rho_L(x)} \frac{\rho_K^{n+1}(x)}{n+1} d\lambda(x) \leq \frac{1}{n}\int_{S^{n-1}} \frac{1}{\rho_K(x)} \frac{\rho_K^{n+1}(x)}{n+1} d\lambda(x) \\
&= \frac{1}{n+1}.
\end{aligned}$$

On the other hand, using the inverse Blaschke–Santaló inequality we get

$$|L^*|^{\frac{1}{n}} \geq \frac{1}{b_1 n}|L|^{-\frac{1}{n}} \geq \frac{1}{b_1 \alpha n}|K|^{-\frac{1}{n}} = \frac{1}{b_1 \alpha n}.$$

It follows that

$$|C(K)| \leq \left(\frac{b_1 \alpha n}{n+1}\right)^{\frac{n}{n-1}} \leq c_1(\alpha).$$

On the other hand, from Hölder's inequality we have

$$\begin{aligned}
1 = |K| &= \frac{1}{n}\int_{S^{n-1}} \rho_K^n(x)\, d\lambda(x) = \frac{1}{n}\int_{S^{n-1}} h_{C(K)}^{-\frac{n}{n+1}}(x) \left[h_{C(K)}^{\frac{n}{n+1}}(x)\rho_K^n(x)\right] d\lambda(x) \\
&\leq \frac{1}{n}\left(n\int_{S^{n-1}} \frac{1}{n} h_{C(K)}^{-n}(x)\, d\lambda(x)\right)^{\frac{1}{n+1}} \left(n(n+1)\int_{S^{n-1}} \frac{1}{n(n+1)} h_{C(K)}(x)\rho_K^{n+1}(x)\, d\lambda(x)\right)^{\frac{n}{n+1}} \\
&= \frac{1}{n} n^{\frac{1}{n+1}} [n(n+1)]^{\frac{n}{n+1}} |C^*(K)|^{\frac{1}{n+1}} |C(K)|^{\frac{n}{n+1}} \\
&\leq c|C(K)|^{\frac{n-1}{n+1}},
\end{aligned}$$

where $c > 0$ is an absolute constant (in the last step, we have used the Blaschke–Santaló inequality for $C(K)$). □

Proposition 3.4. *There exists a constant $c(\alpha) > 0$, which depends only on α, such that if K is as in Theorem 3.2 and $\overline{C}(K) := \frac{1}{|C(K)|^{1/n}} C(K)$, then*

$$|K + D_n|^{1/n} \leq c(\alpha)|\overline{C}(K) + D_n|^{1/n}.$$

Proof. Using Minkowski's inequality, we have

$$\begin{aligned}
|nL^* + C(K)|^{\frac{1}{n}} |C(K)|^{\frac{n-1}{n}} &\leq V(nL^* + C(K), C(K)) \\
&= \frac{1}{n}\int_{S^{n-1}} (nh_{L^*}(x) + h_{C(K)}(x)) f_{C(K)}(x) d\lambda(x) \\
&= \frac{1}{n}\int_{S^{n-1}} nh_{L^*}(x) f_{C(K)}(x)\, d\lambda(x) + |C(K)| \\
&= \frac{1}{n}\int_{S^{n-1}} n\rho_L^{-1}(x) f_{C(K)}(x)\, d\lambda(x) + |C(K)| \\
&\leq \frac{1}{n}\int_{S^{n-1}} n\rho_K^{-1}(x) \frac{\rho_K^{n+1}(x)}{n+1}\, d\lambda(x) + |C(K)| \\
&\leq |K| + |C(K)| = 1 + |C(K)|.
\end{aligned}$$

We set $\overline{L^*} = |L^*|^{-1/n} L^*$. From the inverse Blaschke–Santaló inequality and Proposition 3.3 it follows that

$$|\overline{L^*} + \overline{C}(K)|^{1/n} \leq b_1 c_1 |nL^* + C(K)|^{1/n} \leq b_1 c_1 (1 + c_1 c(\alpha)). \tag{3}$$

Then, using Facts I and III for $\overline{L} = |L|^{-1/n} L$, we obtain

$$\begin{aligned}
|K + D_n|^{1/n} &\leq |L + D_n|^{1/n} \leq \alpha|\overline{L} + D_n|^{1/n} \\
&\leq b_2 \alpha |\overline{L^*} + D_n|^{1/n} \leq b_2 b_3 \alpha \frac{|\overline{C}(K) + \overline{L^*}|^{1/n}}{|\overline{C}(K)|^{1/n}} |\overline{C}(K) + D_n|^{1/n} \\
&\leq c(\alpha)|\overline{C}(K) + D_n|^{1/n}.
\end{aligned}$$

Remark 1. One can use (3) and Facts I and III as in the proof of Proposition 3.4 to derive an inverse form of the Proposition. Thus, if $|L|/|K| \leq \alpha^n$, then K is in M-position if and only if $C(K)$ is in M-position (with a constant that depends only on α).

4 Proof of Theorem 3.2

Lemma 4.1. *Let $F_1 \subseteq \mathbb{R}^{n_1}$ and $F_2 \subseteq \mathbb{R}^{n_2}$ be compact sets with $|F_1|_{n_1} = |F_2|_{n_2} = 1$. Then, for every $T \in SL(n_1+n_2)$,*

$$|TF_1|_{n_1}|TF_2|_{n_2} \geq 1.$$

Proof. Approximating F_1 and F_2 by unions of non-overlapping cubes, we may assume that F_1 and F_2 are cubes. In this case, for every $T \in SL(n_1+n_2)$ we have

$$|TF_1|_{n_1}|TF_2 \mid V^{\perp}|_{n_2} = 1,$$

where $V = \text{span}(T(F_1))$ and $TF_2 \mid V^{\perp}$ is the orthogonal projection of TF_2 onto $V^{\perp}$. The result follows. □

Lemma 4.2. *Let K be a star-body of volume 1 in $\mathbb{R}^n$ such that*

$$\int_K \|x\|_2 dx = \sqrt{n} F_K,$$

and let $T' \in SL(2n)$ satisfy

$$F_{K\times D_n} = \frac{1}{\sqrt{2n}} \int_{T'(K\times D_n)} \|x\|_2 dx.$$

Then,

$$|T'D_n|_n^{1/n} \geq c_1\sqrt{F_K}$$

for some absolute constant $c_1 > 0$.

Proof. For every $T \in SL(2n)$ we can write

$$\begin{aligned}
\int_{T(K\times D_n)} \|z\|_2 dz &= \int_{K\times D_n} \|Tz\|_2 dz = \int_K\int_{D_n} \|Tx+Ty\|_2 dy\, dx \\
&\leq \int_K\int_{D_n} (\|Tx\|_2 + \|Ty\|_2)\ dy\, dx.
\end{aligned}$$

Since

$$\int_K\int_{D_n} \|Tx+Ty\|_2 dy\, dx = \int_K\int_{D_n} \|Tx-Ty\|_2 dy\, dx,$$

we can also write

$$\int_{T(K\times D_n)} \|z\|_2 = \int_K\int_{D_n} \frac{\|Tx+Ty\|_2 + \|Tx-Ty\|_2}{2}\, dy\, dx \geq \frac{1}{2}\int_K\int_{D_n} (\|Tx\|_2+\|Ty\|_2)\ dy\, dx.$$

In other words,

$$\begin{aligned}
\int_{T(K\times D_n)} \|z\|_2 dz &\sim \int_K\int_{D_n} (\|Tx\|_2 + \|Ty\|_2)\ dy\, dx \\
&= |D_n| \int_K \|Tx\|_2 dx + |K| \int_{D_n} \|Ty\|_2 dy \\
&= \int_K \|Tx\|_2 dx + \int_{D_n} \|Ty\|_2 dy.
\end{aligned}$$

Moreover, there exist orthogonal transformations $U_1, U_2 \in O(2n)$ such that

$$U_1(\text{span}(TK)) = \text{span}(K) \ \text{ and } \ U_2(\text{span}(TD_n)) = \text{span}(D_n).$$

We set $T_1 := U_1 T\mid_{\mathrm{span}(K)}$ and $U_2 := U_2 T\mid_{\mathrm{span}(D_n)}$. Then,

$$\begin{aligned}\int_K \|Tx\|_2 dx &= \int_K \|U_1 Tx\|_2 dx = \int_K \|T_1 x\|_2 dx\\ &= |\det T_1|^{1/n} \int_K \left\| \frac{1}{|\det T_1|^{1/n}} T_1 x \right\|_2 dx\\ &= |\det T_1|^{1/n} \int_{S_1 K} \|x\|_2 dx,\end{aligned}$$

where $S_1 := T_1 / |\det T_1|^{1/n}$ (note that $S_1 \in SL(n)$). Similar calculations give

$$\int_{D_n} \|Tx\|_2 dx = |\det T_2|^{1/n} \int_{S_2 D_n} \|x\|_2 dx,$$

for some $S_2 \in SL(n)$. Since

$$|\det T_1|^{1/n} = \left(\frac{|T_1 K|_n}{|K|_n}\right)^{1/n} = |TK|_n^{1/n},$$

Lemma 4.1 shows that

$$|\det T_1|^{1/n} |TD_n|_n^{1/n} \geq 1. \tag{4}$$

If

$$T := \mathrm{diag}\left(\frac{1}{\sqrt{F_K}}, \dots, \frac{1}{\sqrt{F_K}}, \sqrt{F_K}, \dots, \sqrt{F_K}\right).$$

Then, we may choose $U_1 = U_2 = I_{2n}$, and hence, $S_1 = S_2 = I_n$, $|\det T_1|^{1/n} = \frac{1}{\sqrt{F_K}}$ and $|\det T_2|^{1/n} = \sqrt{F_K}$. Since $\int_K \|x\|_2 dx = \sqrt{n} F_K$, we have

$$\begin{aligned}\frac{1}{\sqrt{2n}} \int_{T(K\times D_n)} \|z\|_2 dz &\sim \frac{1}{\sqrt{F_K}} \frac{1}{\sqrt{n}} \int_K \|x\|_2 dx + \sqrt{F_K} \frac{1}{\sqrt{n}} \int_{D_n} \|x\|_2 dx\\ &= \frac{1}{\sqrt{F_K}} F_K + \sqrt{F_K} F_{D_N} \sim \sqrt{F_K},\end{aligned}$$

because $F_{D_n} \sim 1$. Therefore, we have shown that

$$\frac{1}{\sqrt{2n}} \int_{T'(K\times D_n)} \|z\|_2 dz \leq \frac{1}{\sqrt{2n}} \int_{T(K\times D_n)} \|z\|_2 dz \sim c\sqrt{F_K},$$

which implies that

$$\begin{aligned}c'\sqrt{F_K} &\geq |\det T_1'|^{1/n} \frac{1}{\sqrt{n}} \int_{S_1' K} \|x\|_2 dx + |\det T_2'|^{1/n} \frac{1}{\sqrt{n}} \int_{S_2' D_n} \|x\|_2 dx\\ &\geq |\det T_1'|^{1/n} \frac{1}{\sqrt{n}} \int_{S_1' K} \|x\|_2 dx\\ &\geq |\det T_1'|^{1/n} F_K,\end{aligned}$$

where $c, c' > 0$ are absolute constants. It follows from (4) that

$$c'\sqrt{F_K} \geq |T' D_n|_n^{-1/n} F_K,$$

and this completes the proof of the Lemma. $\square$

Lemma 4.3. *There exists an absolute constant $c_4 > 0$ such that, for every star-body K of volume 1 in $\mathbb{R}^n$ and $T \in SL(2n)$ so that*

$$F_{K\times D_n} = \frac{1}{\sqrt{2n}} \int_{T(K\times D_n)} \|x\|_2 dx,$$

one has

$$c_4 \sqrt[4]{F_K} \leq |T(K\times D_n) + D_{2n}|_{2n}^{1/2n}.$$

Proof. We set $t = |T(K\times D_n) + D_{2n}|_{2n}^{1/2n}$. Fact II implies that

$$N(T(K\times D_n), D_{2n})^{1/2n} \leq 2t.$$

Consequently, there exist $k \leq (2t)^{2n}$ and $x_1, \dots, x_k \in \mathbb{R}^{2n}$ such that

$$T(K\times D_n) \subseteq \bigcup_{i=1}^{k} (x_i + D_{2n}),$$

which gives

$$TD_n \subseteq \bigcup_{i=1}^{k} (x_i + D_{2n}),$$

and hence,

$$TD_n \subseteq \bigcup_{i=1}^{k} \left[(x_i + D_{2n}) \mid V\right],$$

where $V = \mathrm{span}(TD_n)$. It follows that

$$|TD_n|_n \leq \sum_{i=1}^{k} |(x_i + D_{2n}) \mid V|_n = k|D_{2n} \mid V|_n \leq (2Ct)^{2n},$$

where $C > 0$ is an absolute constant. Now, we have

$$c_3 \sqrt{F_K} \leq |TD_n|_n^{1/n} \leq (2Ct)^2,$$

from Lemma 4.2. □

Proof of Theorem 3.2. From Proposition 3.4 and Lemma 4.3, we have

$$|\overline{C}(T(K\times D_n)) + D_{2n}|_{2n}^{1/2n} \geq c_2^{-1} |T(K\times D_n) + D_{2n}|_{2n}^{1/2n} \geq c(\alpha)^{-1} c_4 \sqrt[4]{F_K}.$$ □

5 Proof of the main result

In order to deduce Theorem 1.1 from Theorem 3.2, we need to construct 1-unconditional star-bodies which have large F_K and, in addition, are "almost convex".

Proposition 5.1. *There exists an absolute constant $c_5 > 0$ such that, for every positive integer n, there exists an 1-unconditional star-body K with $|\mathrm{conv}(K)|/|K| \leq c_5^n$ and $F_K \geq c_5^{-1}\sqrt{n}$.*

Proof. Let $\{e_1, \dots, e_n\}$ be an orthonormal basis of $\mathbb{R}^n$. We define $v_i = ne_i$ and $v_{n+i} = -ne_i$, $i = 1, \dots, n$. Since the radius of D_n is of order $\sqrt{n}$, if n is large enough we have that the balls $v_i + D_n$, $i = 1, \dots, 2n$, are disjoint. We set

$$K_i = \mathrm{conv}\left(\{0\} \cup (v_i + D_n)\right)$$

and $K = \bigcup_{i=1}^{2n} K_i$. It is clear that K is centrally symmetric and 1-symmetric (hence, 1-unconditional), that is

$$\rho_K(|x_1|e_1 + \cdots + |x_n|e_n) = \rho_K(x_{\sigma(1)}e_1 + \cdots + x_{\sigma(n)}e_n)$$

for all real numbers $x_1, \ldots, x_n$ and every permutation σ of the indices $1, \ldots, n$. It is also clear that $C(K)$ is 1-symmetric, and so, by Petty's characterization, $C(K)$ must have minimal surface area. Then, by Lemma 3.1,

$$\int_K \|x\|_2 dx = |K|^{1+\frac{1}{n}}\sqrt{n}F_K.$$

We estimate F_K: we have

$$\begin{aligned} \frac{1}{\sqrt{n}}\int_{K_i} \|x\|_2 dx &\geq \frac{1}{\sqrt{n}}\int_{D_n} \|x+v_i\|_2 dx \\ &\geq \frac{1}{\sqrt{n}}\int_{D_n} (\|v_i\|_2 - \|x\|_2) dx \ . \end{aligned}$$

Using the fact that $\|v_i\|_2 - \|x\|_2 \geq n - c\sqrt{n}$ for all $x \in D_n$, we easily conclude:

$$\frac{1}{\sqrt{n}}\int_{K_i} \|x\|_2 dx \geq c_5'\sqrt{n} \ .$$

On the other hand,

$$\begin{aligned} |K_i| &\leq |v_i + D_n| + \left|\mathrm{conv}\left[\{0\} \cup \left((D_n|v_i^{\perp}) + v_i\right)\right]\right| \\ &= |D_n| + \frac{1}{n}|D_n \mid v_i^{\perp}|_{n-1}\|v_i\| \leq c_5'', \end{aligned}$$

where c_5', c_5'' are absolute constants. Thus,

$$F_K \geq 2n \cdot c_5\sqrt{n}/(2c_5'n)^{1+1/n}.$$

It remains to prove that the ratio $|\mathrm{conv}(K)|/|K|$ is small. Set $C = \mathrm{conv}(\{v_1, \ldots, v_{2n}\})$. It is well-known that $|C|^{1/n} \sim 1$ and also that C is in M-position. Since $v_1 + D_n \subseteq K \subseteq C + D_n$, it follows that

$$(|\mathrm{conv}(K)|/|K|)^{1/n} \leq |C + D_n|^{1/n} \sim 1. \qquad \square$$

Remark. By a classical theorem of F. John, the order $\sqrt{n}$ for F_K is the maximal possible.

Lemma 5.2. *Let M be a convex body in $\mathbb{R}^n$. If M is in minimal surface position, then the $(n+1)$-dimensional body*

$$M' = \left(\frac{\partial(M)^{\frac{1}{n+1}}}{(2n)^{\frac{1}{n+1}}}M\right) \times \left(\frac{(2n)^{\frac{n}{n+1}}}{\partial(M)^{\frac{n}{n+1}}}[-1/2, 1/2]\right)$$

is also in minimal surface area position and has the same volume as M.

Lemma 4.4 is a simple consequence of Petty's characterization of the minimal surface area position and we omit its proof.

Proof of Theorem 1.1. Given a positive integer n, we consider the star-body K of Proposition 4.1. We may apply Theorem 3.2 to construct an 1-unconditional body M_{2n} in $\mathbb{R}^{2n}$, which has volume 1 and is in minimal surface area position, and at the same time,

$$|M_{2n} + D_{2n}|_{2n}^{1/2n} \geq c\sqrt[4]{F_K} \geq cc_5 n^{1/8}.$$

We also set $M_{2n+1} = M'_{2n}$, where M'_{2n} is the convex body defined in Lemma 4.4, with $M = M_{2n}$. Then,

$$M_{2n+1} + D_{2n+1} \supseteq \left(\frac{\partial(M)^{\frac{1}{2n+1}}}{(4n)^{\frac{1}{2n+1}}} M + (D_{2n+1} \mid e_{2n+1}^{\perp}) \right) \times \left(\frac{(4n)^{\frac{2n}{2n+1}}}{\partial(M)^{\frac{2n}{2n+1}}} [-1/2, 1/2] \right).$$

Since M_{2n} has minimal surface area, one can easily check that $\partial(M)^{1/2n+1} \sim 1$. It easily follows that

$$|M_{2n+1} + D_{2n+1}|_{2n+1}^{1/2n+1} \geq c_6 (n+1)^{1/8},$$

where $c_6 > 0$ is an absolute constant. $\square$

Let K be a convex body which contains the origin. It is well-known that there exists a volume preserving linear map T such that the quantity

$$L_{TK}^2 := \frac{1}{|K|^{\frac{n+2}{n}}} \int_{TK} \langle x, y \rangle^2 dx$$

is constant as a function of $y \in S^{n-1}$. Then, TK is said to be isotropic and the number L_{TK} is called isotropic constant of K (see [8] for basic results on this concept). It has been conjectured that the isotropic constants of all centrally symmetric convex bodies are uniformly bounded (this would actually imply that the isotropic constant of any convex body which contains the origin is uniformly bounded; see [5] for the proof of this result).

The class of bodies of elliptic type (the terminology is due to Lutwak) is defined to be the family of all curvature images of convex bodies. As a final remark, we would like to describe the connection of the problem we study in this paper with the problem of bounding the isotropic constant.

Proposition 5.3. *The isotropic constants of all centrally symmetric convex bodies are uniformly bounded if and only if, in the class of centrally symmetric convex bodies of elliptic type, the minimal surface area position is also an M-position.*

The "if" part of Proposition 4.2 can be deduced from Lemma 3.1, Theorem 3.2 and the well-known fact that $L_K \sim F_K$ for any centrally symmetric convex body K (one may alternatively use Proposition 3.2 and Proposition 1.4 from [1]). The other direction follows from Remark 1 and Section 2.4 from [8].

Acknowledgement. I would like to thank Professor Souzanna Papadopoulou for many helpful discussions concerning this paper. I also thank the referee for improving significantly the form of this paper.

References

[1] J. Bourgain, B. Klartag and V. D. Milman, *Symmetrization and isotropic constants of convex bodies*, in Geometric Aspects of Functional Analysis, Lecture Notes in Mathematics **1850** (2004), 101–116.

[2] J. Bourgain and V. D. Milman, *New volume ratio properties for convex symmetric bodies in* $\mathbb{R}^d$, Invent. Math. **88** (1987), 319–340.

[3] A. Giannopoulos and V.D. Milman, *Asymptotic Convex Geometry: a short overview*, Different faces of geometry (eds S.K. Donaldson, Ya. Eliashberg, M. Gromov). International Mathematical Series, Kluwer/Plenum Publishers **3** (2004), 87–162.

[4] A. Giannopoulos and V. D. Milman, *Extremal problems and isotropic positions of convex bodies*, Israel J. Math. **117** (2000), 29–60.

[5] B. Klartag, *An isomorphic version of the slicing problem*, J. Funct. Anal. **218** (2005), 372–394.

[6] E. Lutwak, *Selected affine isoperimetric inequalities*, in Handbook of Convex Geometry (P. M. Gruber and J. M. Wills eds.), Vol. A, Elsevier Science Publishers, North–Holland (1993), 151–176.

[7] V. D. Milman, *Isomorphic symmetrization and geometric inequalities*, in Geometric Aspects of Functional Analysis, Lecture Notes in Mathematics **1376** (1989), 64–104.

[8] V. D. Milman and A. Pajor, *Isotropic position and inertia ellipsoids and zonoids of the unit ball of a normed n-dimensional space*, in Geometric Aspects of Functional Analysis, Lecture Notes in Mathematics **1376** (1989), 64–104.

[9] C. M. Petty, *Surface area of a convex body under affine transformations*, Proc. Amer. Math. Soc. **12** (1961), 824–828.

[10] C. A. Rogers and G. C. Shephard, *The difference body of a convex body*, Arch. Math. **8** (1957), 220–233.

[11] R. Schneider, *Convex Bodies: The Brunn–Minkowski theory*, Encyclopedia of Mathematics and its Applications **44**, Cambridge University Press, Cambridge, 1993.

Ch. Saroglou: Department of Mathematics, University of Crete, 714-09 Heraklion, Crete, Greece.

E-mail: saroglou@math.uoc.gr