Shadow systems: remarks and extensions

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Abstract

We study some problems from various aspects of convexity, concerning shadow systems. Namely, we extend an old result of Rogers and Shephard, we provide a new simple proof to a result of Reisner and we study a question related to Geometric Tomography, providing a characterization of central symmetry for convex bodies.

1 Introduction

We will denote by $x \cdot y$ the standard inner product of two vectors in $\mathbb{R}^d$. If ν is a vector in $\mathbb{R}^d$, the orthogonal to ν subspace of $\mathbb{R}^d$ is defined as usual,

$$\nu^{\perp} = \{x \in \mathbb{R}^d \mid x \cdot \nu = 0\} \ .$$

The projection of A onto the subspace $\nu^{\perp}$ will be denoted by $A \mid \nu^{\perp}$, where A stands for a set or a single vector.

Let K be a convex body in $\mathbb{R}^d$ (i.e convex, compact with non-empty interior). A shadow system along the direction $\nu \in S^{d-1}$ is a continuous transformation of convex bodies of the form:

$$K_t = \mathrm{conv}\{x + t\alpha(x)\nu \mid x \in K\} \ , \ t \in [t_1, t_2] \ ,$$

where "conv" stands for the convex hull, $\alpha : K \mapsto \mathbb{R}$ is any function and t_1, t_2 are real numbers. Shadow systems were introduced by Rogers and Shephard [10] [13], where a fundamental fact was proven: The volume function of a shadow system is a convex function of the parameter. Although the concept of shadow systems is rather old, it has attracted some attention in the recent years, mainly because of the work of Campi, Grochi and Colesanti (see e.g. [1] or [2]) on the study of Sylvester-type functionals (see also [3] or [15] for applications to other type of functionals). Let $p > 1$. The Sylvester functional is defined by:

$$S(K; m; p) = \frac{1}{|K|^{m(d+p)}} \int_{(x_1,\dots,x_m)\in K^m} |\mathrm{conv}(x_1, \dots x_m)|^p dx_1 \dots dx_m \ , \ m > d \ ,$$

where $|K| = |K|_d$ is the d-dimensional volume in $\mathbb{R}^d$. One can check that $S(K; m; p)$ is an affine invariant, thus it attains its extremal values. An outstanding problem in geometric convexity is the determination of those bodies for which the maximum of this quantity is attained (the case of the minimum was treated by Groemer [7]).

The planar case is well understood. Namely, it was proven in [1] that $S(K; m; p)$ is maximal among planar convex bodies if K is a triangle, improving a result of Dalla and Larman [4] (the problem of uniqueness was treated by the author [14], improving a result of Giannopoulos [6]).

The mentioned result in [1] was shown for polygons (the extension to the general case is straightforward), by proving that there always exists a shadow system $\{K_t\}_{t\in[-1,1]}$ with the following properties: (i) $|K_t|$ is constant, (ii) $S(K_t; m; p)$ is convex in t, (iii) $K_0 = K$ and K_{-1}, K_1 have strictly less number of vertices than K. Thus, one can eventually reduce K into a triangle without decreasing $S(K; m; p)$ (an application of this method to projection bodies of three dimensional cones can be found in [15, Section 5]).

Unfortunately, such a technique is not valid in larger dimensions, since shadow systems fail in general to reduce a convex polytope into a polytope with less vertices and simultaneously to have some control in the volume function of the movement (for example to be affine). See e.g. Section 3 for an example.

However, it is hoped that there might be other families of transformations that work sufficiently well for some functionals (as the one defined by Sylvester,), even in special cases. In Section 3, we establish a general criterion which allows to check if such a convexity property (that corresponds to propery (ii) discussed previously) holds or not. As a consequence, one recovers the Rogers-Shephard theorem for shadow systems.

The (parallel) X-ray function $X_\nu : K|\ \nu^\perp \to \mathbb{R}$ of a convex body K along the direction $\nu \in S^{d-1}$ is defined by $X_\nu(K)(x) = |K \cap (x + \mathbb{R}\nu)|_1$. Questions of the following form (roughly speaking) are common in Geometric Tomography: Can we deduce that K possess a certain property only by the knowledge of a family of its X-rays? If so, how large must this family be? We deal with such a question in Section 4. Geometric Tomography and especially this type of problems are important for applied sciences, for instance axial tomography. We refer to the book of Gardner [5] for an extensive discussion on this topic. There is a close connection of X-rays to a special family of shadow systems, the so called Steiner-Symmetrization. The definition of Steiner-symmetrization and its connection to X-rays will be clear in Paragragh 2.2.

The polar body K^* of an origin symmetric convex body K is defined by

$$K^* = \{x \in \mathbb{R} \mid x \cdot y \leq 1 \ , \text{ for all } y \in K\} \ .$$

The quantity $|K| \cdot |K^*|$ (called "the volume product" of K) is invariant under non-singular linear maps. The famous conjecture of Mahler states that the minimum of this quantity is attained if K is a cube. This was established by Reisner [11] [12] (see also [3] [8] for proofs using shadow systems) for a special class of convex bodies called zonoids (see Paragraph 2.1). In section 5, we provide a new proof of this fact, using a special family of shadow systems. The interesting element of our proof (besides being remarkably simple) is the connection of the volume of the polar body K^* with the integral $M(K, y) := \int_{K^*} |x \cdot y| dy$. It should be noted that one of the major open problems in the area is to estimate $M(K, y)$, when $K^* = 1$ and $|y| = 1$. We refer to [9] for the connection of this problem to the problem of finding the maximum of $S(K; d + 1, 1)$, mentioned at the beginning of this note.

2 Background

2.1 Support functions and Minkowski addition

The support function h_k of K is defined by

$$h_K(x) = \max_{y \in K}(x \cdot y) \ , \ x \in \mathbb{R}^d \ .$$

Clearly, h_K is convex and positively homogeneous. The importance of the notion of support functions lies to the fact that h_k determines the convex body K. For instance, the support function of a (centered) line segment $[-y, y]$ is given by $h_{[-y-y]}(x) = |x \cdot y|$, $x \in \mathbb{R}^d$.

The Minkowski sum $K + L$ of two convex bodies $K,\ L$ is defined to be the set of vectors in $\mathbb{R}^d$ which can be written as the sum of a vector in K and a vector in L. Additivity under Minkowski sums is one the nice properties of support functions (i.e. $h_{K+L} = h_K + h_L$).

A finite sum of line segments is called a zonotope. One can easily check that a zonotope is always centrally symmetric. In this paper we deal only with zonotopes with center at the origin, so we only have to consider sums of centered line segments. Using the additivity property mentioned above, it is easy to compute the support function of the zonotope $Z = \sum_{i=1}^m [-x_i, x_i]$:

$$h_Z(x) = \sum_{i=1}^m |x_i \cdot x| \ , \ x \in \mathbb{R}^d \ .$$

Limits of zonotopes in the sense of the Hausdorff metric are called zonoids. Obviously, in volumetric computations it is enough to deal with zonotopes instead of general zonoids. Moreover, it follows immediately that a projection of a zonotope onto a subspace of $\mathbb{R}^d$ is also a zonotope.

2.2 Steiner-Symmetrization

The Steiner symmetrization $S_\nu(K)$ of the convex body K along the direction $\nu \in S^{d-1}$ is defined to be the convex body whose intersection with any line parallel to ν remains unchanged with respect to K

and, in addition, it is symmetric with respect to the hyperplane $\nu^{\perp}$. An analytic definition of $S_\nu(K)$ will be useful. It is well known that K can be written in the form

$$K = \{x + t\nu \mid x \in K \mid \nu^{\perp},\ f(x) \le t \le g(x)\} \ ,$$

where $f,\ -g : K \mid \nu^{\perp} \to \mathbb{R}$ are convex functions. Then,

$$S_\nu(K) = \{x + t\nu \mid x \in K \mid \nu^{\perp},\ |t| \le (1/2)[g(x) - f(x)]\ \} \tag{1}$$

It follows from (1) that $S_\nu(K) \mid \nu^{\perp} = K \mid \nu^{\perp}$. The connection of Steiner-symmetrization to the notion of X-rays is now clear and can be summarized in the following identity:

$$X_\nu(K) = X_\nu(S_\nu K) \ .$$

It should be noted here that the Steiner symmetrization of K can be represented as a shadow system, as noted in [10] (see also [2]). Indeed, consider the shadow system $\{K_t\}_{t\in[-1,1]}$, with

$$K_t = \left\{x + t\nu \;\middle|\; x \in K \mid \nu^{\perp},\ \frac{(1+t)f(x) - (1-t)g(x)}{2} \le t \le \frac{(1+t)g(x) - (1-t)f(x)}{2}\right\} \ .$$

Then, $K_0 = S_\nu(K)$, $K_1 = K$ and $K_{-1} = K^\nu$ (the reflection of K with respect to the hyperplane $\nu^{\perp}$).

3 An extension of the Rogers-Shephard Theorem

As mentioned above, the idea described in the introduction for proving planar geometric inequalities is not applicable to any dimension. This is because for each $d \ge 3$, there is a polytope K in $\mathbb{R}^d$, whose vertices cannot move linearly in order to produce a non-trivial shadow system $\{K_t\}_{t\in[t_0,t_1]}$ with affine volume function in $[t_0, t_1]$, where $0 \in (t_0, t_1)$. Such an example in three dimensions is a polytope of six quadrilateral facets, with any two of its edges not being parallel. Similar examples can be constructed in the symmetric case as well. The extension to larger dimensions is straightforward, by taking cartesian products with cubes (for instance) of appropriate dimension. One might say here that such examples are the rule and not the exception among convex bodies.

Now, we state the general theorem, mentioned in the introduction.

Theorem 3.1. *Let n, d be positive integers, μ a Borel measure in $(\mathbb{R}^d)^n$, an absolutely continuous measure with respect to the Lebesque measure, with positive density and K be a convex body in $\mathbb{R}^d$. Suppose X is a linear normed space and consider a family of continuous maps $\alpha_i : X \to \mathbb{R}^d$, $i \in I$, where I is a set of indexes. Define*

$$K_x = \mathrm{conv}\{x_i + \alpha_i(x) : x_i \in \mathrm{Ext}(K),\ i \in I\} \ .$$

If the function

$$X \ni x \mapsto \mu(K_{1,x} \times \cdots \times K_{n,x})$$

is convex for any n-tuple of simplices $(K_1, \ldots, K_n)$ with vertices from $\mathrm{Ext}(K)$ (the set of the extremal points of K), then the function $X \ni x \mapsto \mu(K_x^n)$ is also convex.

Remark 3.2. *The Sylvester functional is also of the form $\mu(K^n)$. This is a motivation for our formulation of Theorem 3.1 for measures in $\mathbb{R}^{nd}$ instead of measures in $\mathbb{R}^d$.*

Remark 3.3. *One can easily construct examples to which Theorem 3.1 applies. Take for instance $n = 1$, $d = 2$ and K_t to be the family of pentagons $\mathrm{conv}\{(1,0), (-1,0), (0,1), (-1/2 + t, -1/2 + (1/20)t), (-1/2 + t, 1/2 - (1/20)t)\}$, $t \in [-\varepsilon, \varepsilon]$, for sufficiently small ε. The family K_t is not a shadow system or a family of affine transformations of K_0, but satisfies the assumptions of Theorem 3.1. However, we do not know if there exists a measure μ and a family of maps which are good for every convex body.*

Remark 3.4. *As mentioned in the Introduction, Theorem 3.1 is clearly a generalization of the theorem of Rogers and Shephard for shadow systems. Indeed, let K be a convex body and $K_t = \mathrm{conv}\{x+t\alpha(x)\nu \mid x \in K\}$ be a shadow system in $\mathbb{R}^d$. Take $n = 1$, $X = \mathbb{R}$, $\mu(\cdot) = |\cdot|$, $K_1 = \Delta = \mathrm{conv}\{u_1, \ldots, u_{d+1}\}$, for some affinely independent points from $I =: \mathrm{Ext}(K)$ and $\alpha_i(t) = t\alpha(i)\nu$ for any i, $t \in I$. Then, $|\Delta_t| = |\det(x_1 + t\alpha_1, \ldots, x_{d+1} + t\alpha_{d+1}, (1, \ldots, 1))|$. It is obvious that the function $t \mapsto |\Delta_t|$ is convex and the assertion follows.*

The following lemma is easy and well known.

Lemma 3.5. *Let P, Q be two polytopes in $\mathbb{R}^d$, having the same number of vertices say $x_1, \ldots, x_k$ and $y_1, \ldots, y_k$ respectively, $k \geq d + 2$. Assume, furthermore, that for any subset $\{i_1, \ldots, i_{d+2}\}$ of $\{1, \ldots, k\}$, the following hold:*
i) The vertices $x_{i_1}, \ldots, x_{i_{d+2}}$ are contained in the same hyperplane if and only if $y_{i_1}, \ldots, y_{i_{d+2}}$ are contained in the same hyperplane.
ii) If the sets $\{x_{i_1}, \ldots, x_{i_d}, x_{i_{d+1}}\}$, $\{x_{i_1}, \ldots, x_{i_d}, x_{i_{d+2}}\}$ are affinely independent (hence the same is true for $y_{i_1}, \ldots, y_{i_{d+2}}$), then:
The vertices $x_{i_{d+1}}$, $x_{i_{d+2}}$ belong to the same open half-space spanned by $x_{i_1}, \ldots, x_{i_d}$, if and only if $y_{i_{d+1}}$, $y_{i_{d+2}}$ belong to the same open half-space spanned by $y_{i_1}, \ldots, y_{i_d}$.
Then the following is true: The set $\{x_{j_1}, \ldots, x_{j_m}\}$ defines a facet of P if and only if the set $\{y_{j_1}, \ldots, y_{j_m}\}$ defines a facet of Q. In other words, there exists a combinatorial equivalence between P and Q that respects the index order.

Proof of Theorem 3.1:

The term "triangulation" will be used to denote a subdivision of a polytope K into non-overlapping simplices, whose vertices are vertices of K. First note that it suffices to show that the restriction of our function in any line of X is convex, thus we may assume that $\dim X = 1$, i.e. $X = \mathbb{R}$. Also, since the (pointwise) limit of convex functions is a convex function and μ is absolutely continuous, we may assume that K is a polytope. Also, it suffices to show that for each real number t_0, there exists a $\varepsilon > 0$, such that the function $\mathbb{R} \ni t \mapsto \mu(K_t^n)$ is convex in the open interval $(t_0 - \varepsilon, t_0 + \varepsilon)$. Finally, we may assume that $t_0 = 0$ and $\alpha_i(0) = 0$, i in I.

Let $x_1, \ldots, x_m$ be the vertices of K. By the continuity of α_i, there exists a small interval around 0, so that the points $x_i + \alpha_i(t)$, $i = 1, \ldots, m$, are exactly the vertices of K_t, for all t in this interval. We will shoe that there exists some $\delta > 0$, such that for any t_1, t_2 in $(0, \delta)$, the polytopes $P = K_{t_1}$, $Q = K_{t_2}$, satisfy the assumptions of Lemma 3.5. Since there is only a finite number of vertices, it suffices to find such a δ for a single $(d+2)$-tuple of indices. In particular, it suffices to show that there exists a $\delta > 0$, such that for every t_1, t_2 in $(0, \delta)$ the following hold:
i) If the points $x_1+\alpha_1(t_1), \ldots, x_{d+1}+\alpha_{d+1}(t_1)$ are affinely dependent, then the points $x_1+\alpha_1(t_2), \ldots, x_{d+1} + \alpha_{d+1}(t_2)$ are affinely dependent.
ii) If the sets $\{x_1+\alpha_1(t_1), \ldots, x_d+\alpha_d(t_1), x_{d+1}+\alpha_{d+1}(t_1)\}$, $\{x_1+\alpha_1(t_1), \ldots, x_d+\alpha_d(t_1), x_{d+2}+\alpha_{d+2}(t_1)\}$ are affinely dependent the points $x_{d+1} + \alpha_{d+1}(t_1)$, $x_{d+2} + \alpha_{d+2}(t_1)$ belong to the same open half-space spanned by $x_1 + \alpha_1(t_1), \ldots, x_d + \alpha_d(t_1)$, then the points $x_{d+1} + \alpha_{d+1}(t_2)$, $x_{d+2} + \alpha_{d+2}(t_2)$ belong to the same open half-space spanned by $x_1 + \alpha_1(t_2), \ldots, x_d + \alpha_d(t_2)$.

If there exists an open interval $(0, \delta)$, so that the points $x_i + \alpha_i(t)$, $i = 1, \ldots, d + 1$, are affinely independent, δ is as required for (i). In the opposite case, there exists a sequence of positive numbers $\{t_j\}$, with $t_j \to 0$, such that the points $x_i + \alpha_i(t_j)$, $i = 1, \ldots, d+1$ are affinely dependent for all j. Thus,

$$\left|\Big(\mathrm{conv}\{x_i + \alpha_i(t_j) : i = 1, \ldots, d+1\}\Big)^n\right| = 0 \ , \ j \in \mathbb{N} \ ,$$

so by the fact that μ is absolutely continuous and the convexity assumption, we have:

$$\mu\Big((\mathrm{conv}\{x_i + \alpha_i(t) : i = 1, \ldots, d+1\})^n\Big) = 0 \ , \ t \in [0, t_1] \ .$$

Therefore, in any case, there exists such a δ for (i).

Moreover, because of (i), there exists a $\delta' > 0$, so that either for all t in $(0, \delta')$, the sets $\{x_i + \alpha_i(t) : i = 1, \ldots, d, d+1\}$, $\{x_i + \alpha_i(t) : i = 1, \ldots, d, d+2\}$ are affinely independent or for all t in $(0, \delta')$ at least one of them is not. In the first case, the points $x_{d+1} + \alpha_{d+1}(t)$, $x_{d+2} + \alpha_{d+2}(t)$ are contained for all $t \in (0,\delta')$ in the same subspace or contained for all $t \in (0,\delta')$ in opposite subspaces defined by the hyperplane $\mathrm{aff}\{x_i + \alpha_i(t) : i = 1,\ldots,d\}$. If not (again by continuity), we would find a $t_0 \in (0,\delta')$, such that the one of the points $x_{d+1} + \alpha_{d+1}(t_0)$, $x_{d+2} + \alpha_{d+2}(t_0)$ is contained in $\mathrm{aff}\{x_i + \alpha_i(t_0) : i = 1,\ldots,d\}$ which is impossible. Thus, the required δ exists for (ii) as well.

Consequently, by Lemma 3.5, there exists a combinatorial equivalence between K_{t_1} and K_{t_2} that respects the ordering of indices, for all t_1, t_2 in $(0,\delta)$. Replacing t with $-t$, one gets the same in some interval $(-\delta_1, 0)$. Set for simplicity, $\varepsilon = \min\{\delta,\delta_1\}$. Suppose now that $S = \{K_1,\ldots,K_q\}$ is a triangulation of K. Then, the bodies $K_{i,t}$, $i = 1,\ldots,q$, are all non-overlapping and of positive volume in $(-\varepsilon,\varepsilon)$, since from what we showed, the relative position of vertices and hyperplanes, spanned by vertices of K_t, remains unchanged.

For arbitrary $t_1 \in (0,\varepsilon)$, the set $\{K_{1,t_1},\ldots,K_{q,t_1}\}$ can be extended to a triangulation

$$\{K_{1,t_1},\ldots,K_{q,t_1},K_1^+,\ldots,K_r^+\}$$

of K_{t_1}. Because of the mentioned combinatorial equivalence in $(0,\varepsilon)$, the set

$$\{K_{1,t},\ldots,K_{q,t},K_{1,t}^+,\ldots,K_{r,t}^+\}$$

is a triangulation of K_t, for all t in $(0,\varepsilon)$, where $K_{i,t}^+ := \mathrm{conv}\{x_j + \alpha_j(t) : x_j + \alpha_j(t_1) \text{ is a vertex of } K_i^+\}$.

In addition, since

$$\lim_{t\to 0}\mu(K_t^n) = \mu(K_0^n) = \sum_{(i_1,\ldots,i_n)\in\{1,\ldots,q\}^n} \mu(K_{i_1,0}\times \ldots \times K_{i_n,0}) \ ,$$

it is clear that $\mu(T_{1,0}\times\cdots\times T_{n,0}) = 0$, where at least one of the $T_{i,0}$'s equals $K_{j,0}^+$, for some $j \in \{1,\ldots,r\}$. Working similarly for $t < 0$, we can construct simplices $K_{1,t}^-,\ldots,K_{s,t}^-$, with the same properties as $K_{1,t}^+,\ldots,K_{r,t}^+$.

Define the functions $f,\ g,\ h : (-\varepsilon,\varepsilon)\to\mathbb{R}$, with

$$f(t) = \sum_{(i_1,\ldots,i_n)\in\{1,\ldots,q\}^n} \mu(K_{i_1,t}\times\cdots\times K_{i_n,t}) \ ,$$

$$g(t) = \sum \mu(T_{1,t}\times\cdots\times T_{n,t})\cdot \mathbf{1}(t>0) \ ,$$

where the sum runs over all n-tuples $(T_{1,t},\ldots,T_{n,t})$ from the set $\{K_{1,t},\ldots,K_{n,t},K_{1,t}^+,\ldots,K_{r,t}^+\}$ and at least one of $T_{i,t}$ is equal to some $K_{j,t}^+$ and

$$h(t) = \sum \mu(T'_{1,t}\times\cdots\times T'_{n,t})\cdot \mathbf{1}(t<0) \ ,$$

where the sum runs over all n-tuples $(T'_{1,t},\ldots,T'_{n,t})$ from the set $\{K_{1,t},\ldots,K_{n,t},K_{1,t}^-,\ldots,K_{r,t}^-\}$ and at least one of $T'_{i,t}$ is equal to some $K_{j,t}^-$.

The functions f,g,h are obviously convex and, since $\mu(K_t^n) = f(t)+g(t)+h(t)$, $t\in(-\varepsilon,\varepsilon)$, it follows that the function $t\mapsto\mu(K_t^n)$ is convex in $(-\varepsilon,\varepsilon)$, completing the proof. □

4 Characterizations of central symmetry

In this section we deal with a geometric problem that concerns parallel X-rays. To be more specific, assume that K is a convex body in $\mathbb{R}^d$ and U is a certain non-empty subset of S^{d-1}, for which it is known that for every direction ν in U, the X-ray function of K along ν is even. Does it then follow that K is centrally symmetric? It seems more comfortable to use the equivalent formulation that involves Steiner-symmetrizations: If for all $\nu\in U$, $S_\nu(K)$ is centrally symmetric, is K centrally symmetric as well? To see that this is indeed a reformulation of the original problem, note that $S_\nu(K)$ is centrally symmetric if and only if $X_\nu(K)$ is even.

Let k be a positive integer. It is easy to construct a finite set U with k-elements for which the answer to the previous question is negative, as the examples of regular polygons with odd number of vertices show. We do not know, however, if there exist at most countable sets U (with only finite number of points of accumulation) that characterize central symmetry. More generally it is natural to pose the following:

Problem. Determine the sets U that characterize central symmetry.

We provide an answer to this question in the case of U being open.

Theorem 4.1. *Let K be a convex body in $\mathbb{R}^d$, and U be an open non-empty subset of S^{d-1}, such that for each $\nu \in U$, the Steiner-symmetrization $S_\nu(K)$ of K, along the direction ν is centrally symmetric. Then K is itself centrally symmetric.*

Proof. Suppose that we have shown the assertion for $d = 2$. We will prove that this implies Theorem 4.1 for any $d \geq 3$.

Let $\nu \in U$. Consider a 2-dimensional subspace H of $\mathbb{R}^d$, such that $\dim(K \cap H) = 2$ and $\nu \in H$. The set $U \cap H$ is an open non-empty subset of the unit sphere of H. The Steiner symmetrization $S_{\nu,H}(K \cap H)$ (regarded as a set in H) of $K \cap H$ equals $S_\nu(K) \cap H$, thus by assumption, the set $S_{\nu,H}(K \cap H)$ has central symmetry. However, the same is true for any direction in $U \cap H$, so since it is supposed that the assertion is true for $d = 2$, the set $K \cap H$ has some center of symmetry, say Σ. It follows that $S_{\nu,H}(K \cap H)$ has also some center of symmetry Σ' with $\Sigma' = \Sigma \mid \nu^\perp$. Thus, Σ is the midpoint of the chord of K, which is parallel to ν and contains Σ'. On the other hand Σ' is the center of $S_\nu(K) \cap H$, hence it is also the center of $S_\nu(K)$. Consequently, Σ' depends only on ν and not by the choice of H. Evidently, the same is true for Σ and since K is the union of all intersections of K with 2-dimensional subspaces that satisfy the assumptions made for H, Σ is the center of symmetry for K. Thus it is enough to deal with the two dimensional case.

Take the standard coordinate system x, y in the plane. We may assume that ν is parallel to the y-axis. Set $A' = (\alpha, 0)$ for the center of $S_\nu(K)$. If A is the barycenter of K, it is clear that $A' = A \mid \nu^\perp$. Moreover, if

$$K = \{(x, y) : x \in [\gamma, \delta],\ f(x) \leq y \leq g(x)\}\ ,$$

for some (convex) functions $f, -g : [\gamma, \delta] \to \mathbb{R}$, then

$$S_\nu(K) = \{(x, y) : x \in [\gamma, \delta], (f(x) - g(x))/2 \leq y \leq (g(x) - f(x))/2\}\ .$$

By the central symmetry of $S_\nu(K)$, we have $[\gamma, \delta] = [-\theta + \alpha, \theta + \alpha]$, for some $\theta > 0$ while $g(x + \alpha) - f(x + \alpha) = g(-x + \alpha) - f(-x + \alpha)$, for each x in $[-\theta, \theta]$. Set l for the line which is parallel to ν and contains A. We have shown that any two chords of K, which are parallel to l and have the same distance from l, have equal lengths.

Define the set

$$J := \{x \in (-\theta, \theta) : A \text{ is the midpoint of } [(-x + \alpha, f(-x + \alpha)), (-x + \alpha, g(-x + \alpha))]\}.$$

It suffices to show that $J = (-\theta, \theta)$. The set J is clearly non-empty and closed in in the set $(-\theta, \theta)$. It remains to show that J is open as well. Since U is open and ν is parallel to the y-axis, there exists an open interval I around x, such that for every y in I, a parallel direction to the segment $[(-y + \alpha, f(-y + \alpha)), (x + \alpha, g(x + \alpha))]$ is contained in U. Let w be a point in I. Set: $B = (-x + \alpha, f(-x + \alpha))$, $\Gamma = (-x + \alpha, f(-x + \alpha))$, $\Delta = (x + \alpha, f(x + \alpha))$, $E = (x + \alpha, g(x + \alpha))$, $B' = (-w + \alpha, f(-w + \alpha))$. The quadrangle $B\Gamma E\Delta$ is a rectangle with $BA = AE$, thus we have $A\Gamma = \Delta A$. Next, take a boundary point E' of K, such that the line segment $\Delta E'$ is parallel to $\Gamma B'$. Since $A\Gamma = A\Delta$, it follows that the chords $B'\Gamma$ and $E'\Delta$ have the same distance from the line which is parallel to them and contains A. Since $w \in I$, the Steiner-symmetrization along the direction parallel to the chord $B'\Gamma$, is a centrally symmetric body, thus the segments $B'\Gamma$, $E'\Delta$ have equal lengths. Again, since $A\Gamma = A\Delta$ and $B'\Gamma E'\Delta$ is a rectangle, A is the midpoint of $B'E'$, hence the point $A' = (\alpha, 0) = A \mid \nu^\perp$ is the midpoint of the segment $[B' \mid \nu^\perp, E' \mid \nu^\perp]$. However, $B' \mid \nu^\perp = (-w + \alpha, 0)$ and consequently $E' \mid \nu^\perp = (w + \alpha, 0)$, which means that the point A is the midpoint of the chord $[(-w + \alpha, f(-w + \alpha)), (-w + \alpha, g(-w + \alpha))]$. In other words, w is contained in J, so I is contained in J, hence J is open. $\square$.

5 Polars of zonoids

As mentioned in the introduction, this chapter is devoted in providing an alternative proof to an inequality due to Reisner and Meyer, concerning the volume product of zonoids. The volume of the polar of a convex body K that contains 0 is explicitly expressed by means of its support function. More generally, φ is a homogeneous function in $\mathbb{R}^d$ of degree r, $r > -d$, integration in polar coordinates yields

$$\int_{K^*} \varphi(x)dx = \frac{1}{d+r}\int_{S^{d-1}} \frac{\varphi(x)}{h(x)^{d+r}}dx \ . \tag{2}$$

Now we are ready to state and prove the promised inequality.

Theorem 5.1. *Let Z be a centrally symmetric full-dimensional zonoid in $\mathbb{R}^d$. Set C_d for the unit cube. Then,*

$$P_d(Z) := |Z| \cdot |Z^*| \geq |C_d| \cdot |C_d^*| = \frac{4^d}{d!} \ . \tag{3}$$

Proof. We will use induction in the dimension d. The assertion is trivially true for $d = 1$. Assume that it holds for $d-1$. It suffices to prove (3) in dimension d. By approximation and an appropriate normalization, we may assume that $Z = [-\nu, \nu] + Z'$, for some unit vector ν another zonotope Z', so that Z is not a cylinder whose axis is parallel to ν and Z' has no edge parallel to ν. Consider the shadow system $Z_t = (1+t)[-\nu, \nu] + Z',\ t \in [-1, \infty)$. It is true that $Z_{-1} = Z'$ is a zonotope spanned by less line segments than Z. On the other hand, it is easily verified that the body $Z_\infty := \lim_{t\to\infty} Z_t$ (this limit is taken in the sense of the Hausdorff metric) is a cylinder whose basis is the body $Z'|\nu^\perp$ (which is also a zonotope as mentioned in Paragraph 2.1). It is then well known (and can be checked easily) that

$$|Z_\infty|_d \cdot |Z_\infty^*|_d = \frac{4}{d}|Z \mid \nu^\perp|_{d-1} \cdot |(Z \mid \nu^\perp)^*|_{d-1} \ .$$

By the inductive hypothesis, we have

$$P_{d-1}(Z \mid \nu^\perp) = |Z \mid \nu^\perp|_{d-1} \cdot |(Z \mid \nu^\perp)^*|_{d-1} \geq \frac{4^{d-1}}{(d-1)!} \ . \tag{4}$$

We intend to show that if $|Z|$ does not satisfy (3), then the function $f(t) = p(t)q(t)$ does not attain a minimal value in $(-1, \infty)$, where $p(t) = |Z_t|$, $q(t) = |Z_t^*|$. Actually, it is enough to show this for $t = 0$.

By definition, $p'(0) = 2|Z \mid \nu^\perp|$, thus $p''(0) = 0$. Also, by (2) and the additivity property of support function, we have

$$|Z_t^*| = \frac{1}{d}\int_{S^{d-1}} \frac{1}{[h_Z(x) + tx\cdot\nu]^d}dx \ .$$

Thus,

$$q'(0) = -(d+1)\int_{Z^*} |x\cdot\nu|dx \ , \quad q''(0) = (d+1)(d+2)\int_{Z^*} |x\cdot\nu|^2dx \ .$$

Suppose that $f'(0) = 0$. It is enough to prove that $f''(0) < 0$. An elementary calculation reveals:

$$f'(0) = 2|Z \mid \nu^\perp| \cdot |Z^*| - (d+1)\int_{Z^*} |x\cdot\nu|dx \cdot |Z| \ ,$$

so

$$\begin{aligned} f''(0) &= p(0)q''(0) + 2p'(0)q'(0) \\ &= (d+1)(d+2)|Z|\int_{Z^*} |x\cdot\nu|^2dx - 4(d+1)|Z \mid \nu^\perp|\int_{Z^*} |x\cdot\nu|dx \\ &= (d+1)(d+2)|Z|\int_{Z^*} |x\cdot\nu|^2dx - \frac{8|Z \mid \nu^\perp|^2 \cdot |Z^*|}{|Z|} \ . \end{aligned}$$

Recall our assumption, $|Z| \cdot |Z^*| < 4^d/d!$. Using (4) and the elementary fact that $|Z \mid \nu^\perp| = |Z^* \cap \nu^\perp|$, we deduce:

$$f''(0) < \frac{4^d(d+1)(d+2)}{d!}\left[\frac{\int_{Z^*} |x\cdot\nu|^2dx}{|Z^*|} - \frac{d^2}{2(d+1)(d+2)}\frac{|Z^*|^2}{|Z^* \cap \nu^\perp|^2}\right] \leq 0 \ .$$

We refer to [9] for the fact that the last quantity is indeed non-positive.

Suppose, now, that Z is the sum of m-line segments ($m \geq d$) and that

$$P_d(Z) = \min \{P_d(W) \mid \text{W is the sum of at most } m - \text{line segments}\} .$$

Then, from what we have shown, if Z does not satisfy (3), then $P_d(Z) \geq \min \{P_d(Z_{-1}), P_d(Z_\infty)\} = P_d(Z_{-1})$ (since $P_d(Z_\infty)$ satisfies (3), by the inductive hypothesis), where Z_{-1} is spanned by at most $(m-1)$-line segments. Iterations of the same argument lead to the inequality $P_d(Z) \geq P_d(C) = 4^d/d!$ where C is a parallelepiped (i.e. the sum of d-line segments). This completes the proof. □

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