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LINEAR ALGEBRA AND ITS APPLICATIONS

ELSEVIER Linear Algebra and its Applications 412 (2006) 132–153

www.elsevier.com/locate/laa

On Perron–Frobenius property of matrices having some negative entries ☆

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Received 27 July 2004; accepted 9 June 2005
Available online 16 September 2005
Submitted by M. Neumann

Abstract

We extend the theory of nonnegative matrices to the matrices that have some negative entries. We present and prove some properties which give us information, when a matrix possesses a Perron–Frobenius eigenpair. We apply also this theory by proposing the Perron–Frobenius splitting for the solution of the linear system $Ax = b$ by classical iterative methods. Perron–Frobenius splittings constitute an extension of the well known regular splittings, weak regular splittings and nonnegative splittings. Convergence and comparison properties are given and proved.

AMS classification: Primary 65F10

Keywords: Perron–Frobenius theorem; Nonnegative matrices; Perron–Frobenius splitting

1. Introduction

In 1907, Perron [11] proved that the dominant eigenvalue of a matrix with positive entries is positive and the corresponding eigenvector could be chosen to be positive. With the term *dominant eigenvalue* we mean the eigenvalue which corresponds to

☆ This research was partially funded by a grant of the framework PYTHAGORAS from the Greek Ministry of Education.
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doi:10.1016/j.laa.2005.06.037

the spectral radius. Later in 1912, Frobenius [6] extended this result to irreducible nonnegative matrices. Since then the well known *Perron–Frobenius* theory has been developed, for nonnegative matrices and the well known *Regular, Weak Regular* and *Nonnegative Splittings* have been introduced and developed for the solution of large sparse linear systems by iterative methods [13,16,2,1,14,5,9,8,7,15,4]. (An excellent account of all sorts of splittings can be found in Nteirmentzidis [10]).

Recently, Tarazaga et al. [12], have given a sufficient condition that guarantees the existence of the Perron–Frobenius eigenpair, for the class of real symmetric matrices which have some negative entries. Their result was obtained by studying some convex and closed cones of matrices.

Very recently Zaslavsky and McDonald [17] and Carnochan Naqvi and McDonald [3] have studied combinatorial properties of eventually nonnegative matrices. They have proved that many of the combinatorial properties of nonnegative matrices carry over to the eventually nonnegative matrices. As we will see later, the Perron–Frobenius theory is valid for the latter class of matrices.

It is obvious, from the continuity of the eigenvalues and the entries of the eigenvectors, as functions of the entries of matrices, that the Perron–Frobenius theory may hold also in the case where the matrix has some absolutely small negative entries. This observation brings up some questions. For example: How small could these entries be? What is their distribution? When does a change in a matrix with the property of having a Perron eigenvector implies a loss of this property? These questions are very difficult to answer. Tarazaga et al. in [12] gave a partial answer to the first question by providing a sufficient condition for the symmetric case.

In this paper we study matrices that have some negative entries. Specifically, sufficient and necessary conditions for a matrix to have a Perron eigenvector are presented, together with some monotonicity relations. So, we answer implicitly the above questions by extending the Perron–Frobenius theory of nonnegative matrices to the class of matrices that possess the Perron–Frobenius property. Finally, we apply this theory by introducing the *Perron–Frobenius splitting* for the solution of linear systems by classical iterative methods. This splitting is an extension and a generalization of the well known regular, weak regular and nonnegative splittings. We also present and prove convergence and comparison properties for the proposed splitting.

This work is organized as follows: In Section 2 the main results of the extension of the Perron–Frobenius theory are stated and proved. In Section 3 we propose the Perron–Frobenius splitting and give convergence and comparison properties based on it. As the theory is being developed, various numerical examples are given in the text to illustrate it.

2. Extension of the Perron–Frobenius theory

We begin with our theory by giving three definitions:

Definition 2.1. A matrix $A \in \mathbb{R}^{n,n}$ possesses the Perron–Frobenius property if its dominant eigenvalue λ_1 is positive and the corresponding eigenvector $x^{(1)}$ is nonnegative.

Definition 2.2. A matrix $A \in \mathbb{R}^{n,n}$ possesses the strong Perron–Frobenius property if its dominant eigenvalue λ_1 is positive, the only one in the circle $|\lambda_1|$ ($\lambda_1 > |\lambda_i|$, $i = 2, 3, \ldots, n$) and the corresponding eigenvector $x^{(1)}$ is positive.

Definition 2.3 [3]. A matrix $A \in \mathbb{R}^{n,n}$ is said to be *eventually positive* (*eventually nonnegative*) if there exists a positive integer k_0 such that $A^k > 0$ ($A^k \geqslant 0$) for all $k \geqslant k_0$.

It is noted that Definition 2.1 is the most general of the relevant ones given so far. The analogous definition in the well known Perron–Frobenius theory is that for nonnegative matrices. On the other hand, in Definition 2.2 a subset of matrices of Definition 2.1 is defined, which is analogous to that of irreducible and primitive nonnegative matrices. The next two theorems give relationships between the second and third classes of matrices.

Theorem 2.1. *For a symmetric matrix $A \in \mathbb{R}^{n,n}$ the following properties are equivalent*:

(i) *A possesses the strong Perron–Frobenius property.*
(ii) *A is an eventually positive matrix.*

Proof. (i) $\Rightarrow$ (ii): Since A possesses the strong Perron–Frobenius property, its eigenvalues can be ordered as follows:

$$\lambda_1 = \rho(A) > |\lambda_2| \geqslant |\lambda_3| \geqslant \cdots \geqslant |\lambda_n|,$$

where λ_1 is a simple eigenvalue with the corresponding eigenvector $x^{(1)} \in \mathbb{R}^n$ being positive. We choose an arbitrary nonnegative vector $x^{(0)} \in \mathbb{R}^n$ with $\|x^{(0)}\|_2 = 1$. We expand $x^{(0)}$ as a linear combination of the eigenvectors of A: $x^{(0)} = \sum_{i=1}^{n} c_i x^{(i)}$. Since A is symmetric the eigenvectors constitute an orthogonal basis. So, the coefficients c_i are the inner products $c_i = (x^{(0)}, x^{(i)})$, $i = 1, 2, \ldots, n$, which means that $c_1 > 0$. We apply now the theorem of the Power method. So, the limit of $A^k x^{(0)}$ tends to the eigenvector $x^{(1)}$ as k tends to infinity. This means that for a certain $x^{(0)} \geqslant 0$ there exists an m such that $A^k x^{(0)} > 0$ for all $k \geqslant m$. If we choose the largest of all m's over all initial choices $x^{(0)} \geqslant 0$, specifically

$$k_0 = \max_{0 \leqslant x^{(0)} \in \mathbb{R}^n,\ ||x^{(0)}||_2 = 1} \{m | Ax^k > 0\ \forall k \geqslant m\},$$

we take that for all $x^{(0)} \geqslant 0$, $A^k x^{(0)} > 0$ for all $k \geqslant k_0$. So, A is an eventually positive matrix.

(ii) $\Rightarrow$ (i): From the Perron–Frobenius theory of nonnegative matrices, the assumption $A^k > 0$ means that the dominant eigenvalue of A^k is positive and the only one in the circle while the corresponding eigenvector is positive. It is well known that the matrix A has as eigenvalues the kth roots of those of A^k with the same eigenvectors. Since this happens $\forall k \geqslant k_0$, A possesses the strong Perron–Frobenius property. $\square$

The case where A is a nonsymmetric matrix is covered in the following theorem.

Theorem 2.2. *For a matrix $A \in \mathbb{R}^{n,n}$ the following properties are equivalent*:

(i) *Both matrices A and A^{T} possess the strong Perron–Frobenius property.*
(ii) *A is an eventually positive matrix.*
(iii) *A^{T} is an eventually positive matrix.*

Proof. (i) $\Rightarrow$(ii): Let $A = XDX^{-1}$ be the Jordan canonical form of the matrix A. We assume that the eigenvalue $\lambda_1 = \rho(A)$ is the first diagonal entry of D. So the Jordan canonical form can be written as

$$A = [x^{(1)}|X_{n,n-1}]\left[\begin{array}{c|c}\lambda_1 & 0\\ \hline 0 & D_{n-1,n-1}\end{array}\right]\left[\begin{array}{c} {y^{(1)}}^{\mathrm{T}} \\ \hline Y_{n-1,n}\end{array}\right], \tag{2.1}$$

where ${y^{(1)}}^{\mathrm{T}}$ and $Y_{n-1,n}$ are the first row and the matrix formed by the last $n-1$ rows of X^{-1}, respectively. Since A possesses the strong Perron–Frobenius property, the eigenvector $x^{(1)}$ is positive. From (2.1), the block form of A^{T} is

$$A^{\mathrm{T}} = \left[y^{(1)}|Y^{\mathrm{T}}_{n-1,n}\right]\left[\begin{array}{c|c}\lambda_1 & 0\\ \hline 0 & D^{\mathrm{T}}_{n-1,n-1}\end{array}\right]\left[\begin{array}{c} {x^{(1)}}^{\mathrm{T}} \\ \hline X^{\mathrm{T}}_{n,n-1}\end{array}\right]. \tag{2.2}$$

The matrix $D^{\mathrm{T}}_{n-1,n-1}$ is the block diagonal matrix formed by the transposes of all Jordan blocks except λ_1. It is obvious that there exists a permutation matrix $P \in \mathbb{R}^{n-1,n-1}$ such that the associated permutation transformation on the matrix $D^{\mathrm{T}}_{n-1,n-1}$ transposes all the Jordan blocks. So, $D_{n-1,n-1} = P^{\mathrm{T}} D^{\mathrm{T}}_{n-1,n-1} P$ and relation (2.2) takes the form

$$\begin{aligned} A^{\mathrm{T}} &= \left[y^{(1)}|Y^{\mathrm{T}}_{n-1,n}\right]\left[\begin{array}{c|c}1 & 0\\ \hline 0 & P\end{array}\right]\left[\begin{array}{c|c}1 & 0\\ \hline 0 & P^{\mathrm{T}}\end{array}\right]\left[\begin{array}{c|c}\lambda_1 & 0\\ \hline 0 & D^{\mathrm{T}}_{n-1,n-1}\end{array}\right] \\ &\quad \times \left[\begin{array}{c|c}1 & 0\\ \hline 0 & P\end{array}\right]\left[\begin{array}{c|c}1 & 0\\ \hline 0 & P^{\mathrm{T}}\end{array}\right]\left[\begin{array}{c} {x^{(1)}}^{\mathrm{T}} \\ \hline X^{\mathrm{T}}_{n,n-1}\end{array}\right] \\ &= \left[y^{(1)}|Y'^{\mathrm{T}}_{n-1,n}\right]\left[\begin{array}{c|c}\lambda_1 & 0\\ \hline 0 & D_{n-1,n-1}\end{array}\right]\left[\begin{array}{c} {x^{(1)}}^{\mathrm{T}} \\ \hline X'^{\mathrm{T}}_{n,n-1}\end{array}\right], \end{aligned} \tag{2.3}$$

where $Y'^{\mathrm{T}}_{n-1,n} = Y^{\mathrm{T}}_{n-1,n} P$ and $X'^{\mathrm{T}}_{n,n-1} = P^{\mathrm{T}} X^{\mathrm{T}}_{n,n-1}$. The last relation is the Jordan canonical form of A^{T} which means that $y^{(1)}$ is the eigenvector corresponding to the dominant eigenvalue λ_1. Since A^{T} possesses the strong Perron–Frobenius property, $y^{(1)}$ is a positive vector or a negative one. Since $y^{(1)^{\mathrm{T}}}$ is the first row of X^{-1} we have that $(y^{(1)}, x^{(1)}) = 1$ implying that $y^{(1)}$ is a positive vector.

We return now to the Jordan canonical form (2.1) of A and form the power A^k

$$A^k = [x^{(1)}|X_{n,n-1}]\left[\begin{array}{c|c}\lambda_1^k & 0\\ \hline 0 & D^k_{n-1,n-1}\end{array}\right]\left[\begin{array}{c}y^{(1)^{\mathrm{T}}}\\ \hline Y_{n-1,n}\end{array}\right]$$

or

$$\frac{1}{\lambda_1^k}A^k = [x^{(1)}|X_{n,n-1}]\left[\begin{array}{c|c}1 & 0\\ \hline 0 & \frac{1}{\lambda_1^k}D^k_{n-1,n-1}\end{array}\right]\left[\begin{array}{c}y^{(1)^{\mathrm{T}}}\\ \hline Y_{n-1,n}\end{array}\right].$$

Since λ_1 is the dominant eigenvalue, the only one in the circle $|\lambda_1|$, we get that

$$\lim_{k\to\infty}\frac{1}{\lambda_1^k}D^k_{n-1,n-1} = 0.$$

So,

$$\lim_{k\to\infty}\frac{1}{\lambda_1^k}A^k = x^{(1)}y^{(1)^{\mathrm{T}}} > 0.$$

The last relation means that there exists an integer $k_0 > 0$ such that $A^k > 0$ for all $k \geqslant k_0$. So, A is an eventually positive matrix and the first part of theorem is proved.

(ii) $\Leftrightarrow$ (iii): Obvious from Definition 2.3.

(ii) $\Rightarrow$ (i): The proof is the same as that of Theorem 2.1, by considering that A and A^{T} are both eventually positive matrices. □

We observe that Theorem 2.1 is a special case of Theorem 2.2. Nevertheless, it is stated and proved since the proof is quite different and easier than that of Theorem 2.2.

We have to remark here that an analogous equivalence concerning the class of the eventually nonnegative matrices and the class of matrices possessing the Perron–Frobenius property, does not hold. Instead, we give the following two “one way” theorems:

Theorem 2.3. *Let that $A \in \mathbb{R}^{n,n}$ is an eventually nonnegative matrix. Then, both matrices A and A^{T} possess the Perron–Frobenius property.*

Proof. Analogous to the proof of the part (ii) $\Rightarrow$ (i) of Theorem 2.1. □

Theorem 2.4. *Let that both the matrices $A \in \mathbb{R}^{n,n}$ and A^{T} possess the Perron–Frobenius property, with the dominant eigenvalue $\lambda_1 = \rho(A)$ being the only one in the circle $|\lambda_1|$ ($\lambda_1 > |\lambda_i|, i = 2, 3, \ldots, n$). Then,*

$$\lim_{k\to\infty}\frac{1}{\lambda_1^k}A^k = x^{(1)}y^{(1)^{\mathrm{T}}} \geqslant 0. \tag{2.4}$$

Proof. Analogous to the proof of part (i)$\Rightarrow$(ii) of Theorem 2.2. Finally, we observe that we cannot conclude that A is an eventually nonnegative matrix, since $x^{(1)}y^{(1)^{\mathrm{T}}} \geqslant 0$. In conclusion, some entries of the powers of A may tend to zero from negative values. □

In concluding, it is noted here, based on the above theorems, that the class of the eventually positive matrices is a subclass of the class of matrices possessing the strong Perron–Frobenius property, while the class of the eventually nonnegative matrices is a subclass of the class of matrices possessing the Perron–Frobenius property. This is shown by the following example.

Example 2.1. The matrix $A = \begin{pmatrix} 1 & 2 & 1 \\ -.4 & 1 & 1 \\ -.4 & 5 & 8 \end{pmatrix}$ has dominant eigenvalue 8.5523 and corresponding eigenvector $(0.1618\ \ 0.1211\ \ 0.9794)^{\mathrm{T}}$. The matrix A^{T} has the same dominant eigenvalue and corresponding eigenvector $(0.07308\ \ -0.5371\ \ -0.8404)^{\mathrm{T}}$. As one can readily see, A possesses the strong Perron–Frobenius property while A^{T} does *not*. According to Theorem 2.2, A is not an eventualy positive matrix. This is easily checked by seeing that the first column vector of A^k, $k \geqslant 2$ is negative.

In the sequel some statements with only necessary conditions follow.

Theorem 2.5. *If* $A^{\mathrm{T}} \in \mathbb{R}^{n,n}$ *possesses the Perron–Frobenius property, then either*

$$\sum_{j=1}^{n} a_{ij} = \rho(A) \quad \forall i = 1(1)n \tag{2.5}$$

or

$$\min_i \left(\sum_{j=1}^{n} a_{ij} \right) \leqslant \rho(A) \leqslant \max_i \left(\sum_{j=1}^{n} a_{ij} \right). \tag{2.6}$$

Moreover, if A^{T} *possesses the strong Perron–Frobenius property, then both inequalities in* (2.6) *are strict.*

Proof. Let that $(\rho(A), y)$ is the Perron–Frobenius eigenpair of the matrix A^{T} and $\xi \in \mathbb{R}^n$ is the vector of ones ($\xi = (1\ 1\ \cdots\ 1)^{\mathrm{T}}$). We form the product $y^{\mathrm{T}}A\xi$:

$$y^{\mathrm{T}}A\xi = y^{\mathrm{T}} \begin{pmatrix} \sum_{j=1}^{n} a_{1j} \\ \sum_{j=1}^{n} a_{2j} \\ \vdots \\ \sum_{j=1}^{n} a_{nj} \end{pmatrix} = \sum_{i=1}^{n} \left(y_i \sum_{j=1}^{n} a_{ij} \right) \leqslant \max_i \left(\sum_{j=1}^{n} a_{ij} \right) \sum_{i=1}^{n} y_i. \tag{2.7}$$

Similarly, we have that

$$y^{\mathrm{T}}A\xi = \sum_{i=1}^{n}\left(y_i \sum_{j=1}^{n} a_{ij}\right) \geqslant \min_i \left(\sum_{j=1}^{n} a_{ij}\right)\sum_{i=1}^{n} y_i. \tag{2.8}$$

On the other hand we get

$$y^{\mathrm{T}}A\xi = \xi^{\mathrm{T}}A^{\mathrm{T}}y = \rho(A)\xi^{\mathrm{T}}y = \rho(A)\sum_{i=1}^{n} y_i. \tag{2.9}$$

Relations (2.7)–(2.9) give us relation (2.6). It is obvious that the inequalities in (2.6) become equalities if $\max_i \left(\sum_{j=1}^{n} a_{ij}\right) = \min_i \left(\sum_{j=1}^{n} a_{ij}\right)$, which proves the equality (2.5). It is also obvious that the inequalities in (2.7) and (2.8) become strict if $y > 0$. So, the inequalities in (2.6) become strict if A^{T} possesses the strong Perron–Frobenius property. □

Note that it is necessary to have $\max_i \left(\sum_{j=1}^{n} a_{ij}\right) > 0$, otherwise Theorem 2.5 does not hold and so, A^{T} does not possess the Perron–Frobenius property. On the other hand, it is not necessary to have $\min_i \left(\sum_{j=1}^{n} a_{ij}\right) \geqslant 0$ as is shown in the following example.

Example 2.2. Let

$$A = \begin{pmatrix} 1 & 1 & -3 \\ -4 & 1 & 1 \\ 8 & 5 & 8 \end{pmatrix}.$$

The vector of the row sums of A is $(-1\ \ -2\ \ 21)^{\mathrm{T}}$, while A^{T} possesses the strong Perron–Frobenius property with the Perron–Frobenius eigenpair: $\left(6.868, (0.4492\ \ 0.6225\ \ 0.6408)^{\mathrm{T}}\right)$.

By interchanging the roles of A and A^{T}, Theorem 2.5 gives an analogous result for the column sums.

We define now the set $\mathscr{P}$ of all vectors $x \geqslant 0$ with at least one component being positive and its subset $\mathscr{P}^*$, the orthant of vectors $x > 0$. Then, the previous results are generalized as follows.

Theorem 2.6. *If $A^{\mathrm{T}} \in \mathbb{R}^{n,n}$ possesses the Perron–Frobenius property and $x \in \mathscr{P}^*$, then either*

$$\frac{\sum_{j=1}^{n} a_{ij}x_j}{x_i} = \rho(A) \quad \forall\, i = 1(1)n \tag{2.10}$$

or

$$\min_i \left(\frac{\sum_{j=1}^{n} a_{ij} x_j}{x_i} \right) \leqslant \rho(A) \leqslant \max_i \left(\frac{\sum_{j=1}^{n} a_{ij} x_j}{x_i} \right). \tag{2.11}$$

Moreover, if A^{T} possesses the strong Perron–Frobenius property, then both inequalities in (2.11) *are strict and*

$$\sup_{x \in \mathscr{P}^*} \left\{ \min_i \left(\frac{\sum_{j=1}^{n} a_{ij} x_j}{x_i} \right) \right\} = \rho(A) = \inf_{x \in \mathscr{P}^*} \left\{ \max_i \left(\frac{\sum_{j=1}^{n} a_{ij} x_j}{x_i} \right) \right\}. \tag{2.12}$$

Proof. Let $x \in \mathscr{P}^*$. We define the diagonal matrix $D = \operatorname{diag}(x_1, x_2, \ldots, x_n)$ and consider the similarity transformation $B = D^{-1}AD$ (see [13, Theorem 2.2]). Then the entries of B are $b_{ij} = \frac{a_{ij}x_j}{x_i}$. Since B is produced from A by a similarity transformation and D and D^{-1} are both nonnegative matrices, we obtain that B^{T} possesses also the Perron–Frobenius property. As a consequence we have

$$\sup_{x \in \mathscr{P}^*} \left\{ \min_i \left(\frac{\sum_{j=1}^{n} a_{ij} x_j}{x_i} \right) \right\} \leqslant \rho(A) \leqslant \inf_{x \in \mathscr{P}^*} \left\{ \max_i \left(\frac{\sum_{j=1}^{n} a_{ij} x_j}{x_i} \right) \right\}, \tag{2.13}$$

which implies (2.11). We choose now the Perron–Frobenius eigenvector y in the place of x. It is easily seen that inequalities (2.11) become equalities, which means that those in (2.13) become also equalities and the proof is complete. □

By interchanging the roles of A and A^{T}, Theorem 2.6 gives us analogous results for the column sums.

In the sequel we give some monotonicity properties concerning the dominant eigenvalue in the case where the matrices possess the Perron–Frobenius property. It is well known that the eigenvalues and the entries of the eigenvectors are continuous functions of the entries of a matrix A. So, if A possesses the strong Perron–Frobenius property, then a perturbation of A, $\widetilde{A} = A + E$ provided $\|E\|$ is small enough, possesses also the strong Perron–Frobenius property. It is also well known, from the theory of nonnegative matrices, that the dominant eigenvalue of a nonnegative matrix A is a nondecreasing function of the entries of A, when A is reducible, while if A is an irreducible matrix, it is a strictly increasing function. Then two questions come up: What happens to the monotonicity in case the matrices possess the Perron–Frobenius property? Does the property of "possessing the Perron–Frobenius property" still hold when the entries of A increase, as it does in the nonnegative case? Unfortunately, the answer to the second question is not positive. It depends on the direction in which we increase the entries, as we will see later. First we give some properties which provide an answer to the first question.

Theorem 2.7. *If the matrices $A, B \in \mathbb{R}^{n,n}$ are such that $A \leqslant B$, and both A and B^{T} possess the Perron–Frobenius property* (*or both A^{T} and B possess the Perron–Frobenius property*), *then*

$$\rho(A) \leqslant \rho(B). \tag{2.14}$$

Moreover, if the above matrices possess the strong Perron–Frobenius property and $A \neq B$, then the inequality in (2.14) *is strict.*

Proof. Let $x \geqslant 0$ be the Perron–Frobenius eigenvector of A associated with the dominant eigenvalue λ_A and let $y \geqslant 0$ be the Perron–Frobenius eigenvector of B^{T} associated with the dominant eigenvalue λ_B. Then the following equalities hold:

$$y^{\mathrm{T}}Ax = \lambda_A y^{\mathrm{T}}x, \quad y^{\mathrm{T}}Bx = \lambda_B y^{\mathrm{T}}x.$$

Since $A \leqslant B$, we can write $B = A + C$, where $C \geqslant 0$. So,

$$y^{\mathrm{T}}Bx = y^{\mathrm{T}}(A + C)x = y^{\mathrm{T}}Ax + y^{\mathrm{T}}Cx \geqslant y^{\mathrm{T}}Ax.$$

Assuming that $y^{\mathrm{T}}x > 0$, the above relations imply that $\lambda_B \geqslant \lambda_A$. The case where $y^{\mathrm{T}}x = 0$ is covered by using a continuity argument. For this we consider the matrices A' and B' which are small perturbations of the matrices A and B, respectively, such that for the corresponding perturbed eigenvectors we will have $y'^{\mathrm{T}}x' > 0$. The above inequality holds for the perturbed eigenvalues and because of the continuity the same property holds for the eigenvalues of A and B. It is obvious that if we follow the same reasoning we can obtain the same result in case both A^{T} and B possess the Perron–Frobenius property. It is also obvious that the inequality becomes strict in case the associated Perron–Frobenius properties are strong. □

We note that the above property does not guarantee the existence of the Perron–Frobenius property for an intermediate matrix C ($A \leqslant C \leqslant B$) and does not give any information about $\rho(C)$. We confirm this by the following example.

Example 2.3. Let

$$A = \begin{pmatrix} 1 & 2 & 1 \\ -.4 & 1 & 1 \\ -.4 & 5 & 8 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 20 & 20 \\ -.4 & 1 & 1 \\ -.4 & 5 & 8 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 20 & 20 \\ -.4 & 1 & 1 \\ .3 & 5 & 8 \end{pmatrix}.$$

For the above matrices there hold

$$A \leqslant C \leqslant B.$$

It is easily checked that A possesses the Perron–Frobenius property, C and C^{T} do *not* possess the Perron–Frobenius property (the eigenvectors, corresponding to $\rho(C)$, of C and C^{T} are $(0.9623 \;\; -0.0220 \;\; 0.2709)^{\mathrm{T}}$ and $(-0.1036 \;\; 0.4547 \;\; 0.8846)^{\mathrm{T}}$, respectively), while B^{T} possesses the Perron–Frobenius property. For the spectral radii the following inequalities hold:

$$\rho(C) = 6.1723 < \rho(A) = 8.5523 < \rho(B) = 8.7530.$$

Theorem 2.8. *Let* (i) $A^{\mathrm{T}} \in \mathbb{R}^{n,n}$ *possesses the Perron–Frobenius property and* $x \geqslant 0$ $(x \neq 0)$ *be such that* $Ax - \alpha x \geqslant 0$ *for a constant* $\alpha > 0$ *or* (ii) $A \in \mathbb{R}^{n,n}$ *possesses the Perron–Frobenius property and* $x \geqslant 0$ $(x \neq 0)$ *be such that* $x^{\mathrm{T}}A - \alpha x^{\mathrm{T}} \geqslant 0$ *for a constant* $\alpha > 0$. *Then*

$$\alpha \leqslant \rho(A). \tag{2.15}$$

Moreover, if $Ax - \alpha x > 0$ *or* $x^{\mathrm{T}}A - \alpha x^{\mathrm{T}} > 0$, *then the inequality in* (2.15) *is strict.*

Proof. For hypothesis (i), let $y \geqslant 0$ be the Perron–Frobenius eigenvector of A^{T} associated with $\rho(A)$. Then, the following equivalence holds

$$y^{\mathrm{T}}(Ax - \alpha x) \geqslant 0 \Leftrightarrow (\rho(A) - \alpha)y^{\mathrm{T}}x \geqslant 0.$$

If $y^{\mathrm{T}}x > 0$, then the inequality (2.15) holds. In the case where $y^{\mathrm{T}}x = 0$ we recall the perturbation argument used in Theorem 2.7 to prove the validity of (2.15). If $Ax - \alpha x > 0$, the above inequalities become strict and therefore (2.15) becomes strict. For hypothesis (ii) the proof is similar. □

The above theorem is an extension of Corollary 3.2 given by Marek and Szyld in [7], for nonnegative matrices. The following theorem is also an extension of Lemma 3.3 of the same paper [7].

Theorem 2.9. *Let* (i) $A^{\mathrm{T}} \in \mathbb{R}^{n,n}$ *possesses the Perron–Frobenius property and* $x > 0$ *be such that* $\alpha x - Ax \geqslant 0$ *for a constant* $\alpha > 0$ *or* (ii) $A \in \mathbb{R}^{n,n}$ *possesses the Perron–Frobenius property and* $x > 0$ *be such that* $\alpha x^{\mathrm{T}} - x^{\mathrm{T}}A \geqslant 0$ *for a constant* $\alpha > 0$. *Then*

$$\rho(A) \leqslant \alpha. \tag{2.16}$$

Moreover, if $\alpha x - Ax > 0$ *or* $\alpha x^{\mathrm{T}} - x^{\mathrm{T}}A > 0$, *then the inequality in* (2.16) *becomes strict.*

Proof. As in the previous theorem we give the proof only for hypothesis (i). Let $y \geqslant 0$ be the Perron–Frobenius eigenvector of A^{T} associated with $\rho(A)$. Then, we have

$$y^{\mathrm{T}}(\alpha x - Ax) \geqslant 0 \Leftrightarrow (\alpha - \rho(A))y^{\mathrm{T}}x \geqslant 0.$$

Since $x > 0$ we have that $y^{\mathrm{T}}x > 0$ and the inequality (2.16) holds. If $\alpha x - Ax > 0$, the above inequalities become strict and therefore (2.16) becomes strict. □

We remark that the condition $x > 0$ is necessary. This is because for $x \geqslant 0$ such that $Ax = 0$, the condition $\alpha x - Ax \geqslant 0$ holds for any $\alpha \geqslant 0$, but the inequality (2.16) is not true for all $\alpha \geqslant 0$.

We give now the following monotonicity property depending on the direction in which the entries of a matrix increase.

Theorem 2.10. *Let $A \in \mathbb{R}^{n,n}$ possesses the Perron–Frobenius property with $x \geqslant 0$ the associated eigenvector and let $y \neq 0$ such that $y^{\mathrm{T}}x > 0$. Then, the matrix*

$$B = A + \epsilon x y^{\mathrm{T}}, \quad \epsilon > 0 \tag{2.17}$$

possesses the Perron–Frobenius property and for the spectral radii there holds

$$\rho(A) < \rho(B). \tag{2.18}$$

Moreover, if A possesses the strong Perron–Frobenius property, then so does B.

Proof. Let the eigenvalues of A be $\rho(A) = \lambda_1 \geqslant |\lambda_2| \geqslant |\lambda_3| \geqslant \cdots \geqslant |\lambda_n|$. To prove (2.18) we will prove that the dominant eigenvalue of B is $\tilde{\lambda}_1 = \rho(B) = \rho(A) + \epsilon y^{\mathrm{T}}x > \rho(A)$ while all the others remain unchanged ($\tilde{\lambda}_i = \lambda_i,\ i = 1, 2, \ldots, n$). For this we consider the following three cases:

(i) *The Jordan canonical form of A is diagonal.*

This means that there is a basis of n linearly independent eigenvectors $x^{(i)},\ i = 1, 2, \ldots, n$, of A corresponding to λ_i where $x^{(1)} = x$. By post-multiplying (2.17) by x we obtain

$$Bx = (A + \epsilon x y^{\mathrm{T}})x = (\rho(A) + \epsilon y^{\mathrm{T}}x)x$$

which means that $\rho(A) + \epsilon y^{\mathrm{T}}x$ is an eigenvalue of B corresponding to the same eigenvector x. So, $\rho(A) < \rho(B)$. We will prove now that the other eigenvalues are $\lambda_i,\ i = 2, 3, \ldots, n$, with eigenvectors

$$\tilde{x}^{(i)} = x - \alpha_i x^{(i)}, \quad \alpha_i = \frac{\lambda_1 - \lambda_i + \epsilon y^{\mathrm{T}}x}{\epsilon y^{\mathrm{T}}x^{(i)}}, \quad y^{\mathrm{T}}x^{(i)} \neq 0.$$

By post-multiplying (2.17) by $\tilde{x}^{(i)}$ we obtain

$$\begin{aligned} B\tilde{x}_i &= (A + \epsilon x y^{\mathrm{T}})(x - \alpha_i x^{(i)}) = (\lambda_1 + \epsilon y^{\mathrm{T}}x - \alpha_i \epsilon y^{\mathrm{T}}x^{(i)})x - \lambda_i \alpha_i x^{(i)} \\ &= \lambda_i(x - \alpha_i x^{(i)}) = \lambda_i \tilde{x}^{(i)}. \end{aligned}$$

In the case where $y^{\mathrm{T}}x^{(i)} = 0$ we get the same result by choosing $\tilde{x}^{(i)} = x^{(i)}$. So, $\tilde{\lambda}_1 = \rho(A) + \epsilon y^{\mathrm{T}}x$ is the dominant eigenvalue of B and the theorem has been proven. We have to remark here that the case where $\lambda_i = \lambda_1$ (λ_1 is a multiple eigenvalue) is covered by the above proof with $\alpha_i = \frac{y^{\mathrm{T}}x}{y^{\mathrm{T}}x^{(i)}},\ y^{\mathrm{T}}x^{(i)} \neq 0$.

(ii) *There is an $m \times m$ ($m > 1$) block in the Jordan canonical form of A corresponding to λ_k while the block corresponding to $\lambda_1 = \rho(A)$ is 1×1.*

In this case there is a set of vectors $\{x^{(k)}, x^{(k+1)} \cdots x^{(k+m-1)}\}$, where $x^{(k)}$ is the eigenvector of λ_k and $x^{(k+i)},\ i = 1, 2, \ldots, m-1$, are the generalized eigenvectors of λ_k. From case (i) we get that $\tilde{\lambda}_1 = \rho(A) + \epsilon y^{\mathrm{T}}x > \rho(A)$ with x the associated eigenvector while $\tilde{\lambda}_k = \lambda_k$ with associated eigenvector $\tilde{x}^{(k)} = x - \alpha_k x^{(k)}$. We do not obtain any other information from case (i) concerning the generalized eigenvectors. We will prove here that there is also an $m \times m$ block in the Jordan canonical form of B, corresponding to λ_k with the associated set of eigenvector and generalized eigenvectors $\{\tilde{x}^{(k)}, \tilde{x}^{(k+1)}, \ldots, \tilde{x}^{(k+m-1)}\}$, where

$$\tilde{x}^{(k+i)} = \beta_i x - \alpha_k x^{(k+i)}, \quad i = 1, 2, \ldots, m-1$$

with

$$\beta_i = \frac{1 + \alpha_k \epsilon y^{\mathrm{T}} x^{(k+1)}}{(\alpha_k \epsilon y^{\mathrm{T}} x^{(k)})^i} + \frac{\alpha_k \epsilon y^{\mathrm{T}} x^{(k+2)}}{(\alpha_k \epsilon y^{\mathrm{T}} x^{(k)})^{i-1}} + \cdots + \frac{\alpha_k \epsilon y^{\mathrm{T}} x^{(k+i)}}{\alpha_k \epsilon y^{\mathrm{T}} x^{(k)}}.$$

To prove this, we post-multiply (2.17) by $\tilde{x}^{(k+i)}$, $i = 1, 2, \ldots, m-1$. After some manipulations, taking into account the above considerations, we obtain

$$\begin{aligned} B\tilde{x}^{(k+i)} &= (A + \epsilon y^{\mathrm{T}} x)(\beta_i x - \alpha_k x^{(k+i)}) \\ &= \lambda_k \tilde{x}^{(k+i)} + \tilde{x}^{(k+i-1)}, \quad i = 1, 2, \ldots, m-1, \end{aligned}$$

which proves our assertion.

(iii) *The block in the Jordan canonical form of A corresponding to* $\lambda_1 = \rho(A)$ *is an* $m \times m$ $(m > 1)$ *matrix.*

This means that there is a set of vectors $\{x, x^{(2)}, \ldots, x^{(m)}\}$, where $x^{(i)}$, $i = 2, 3, \ldots, m$, are the generalized eigenvectors of λ_1. It is easily seen from the proofs of the above two cases that $\tilde{\lambda}_1 = \rho(A) + \epsilon y^{\mathrm{T}} x > \rho(A)$ with corresponding eigenvector x while $\tilde{\lambda}_k = \lambda_k$ with eigenvectors as has been described in the previous cases, for all λ_k that are associated with the other Jordan blocks. We have to study the behavior of the generalized eigenvectors of the above set. We will prove that $\tilde{\lambda}_1$ becomes a simple eigenvalue of B while the set $\{\tilde{x}^{(2)}, \tilde{x}^{(3)}, \ldots, \tilde{x}^{(m)}\}$ corresponds to an $m-1 \times m-1$ Jordan block of B with eigenvalue λ_1. In this set $\tilde{x}^{(2)}$ is an eigenvector while $\tilde{x}^{(3)}, \ldots, \tilde{x}^{(m)}$ are generalized eigenvectors where

$$\tilde{x}^{(2)} = x - \alpha x^{(2)}, \quad \alpha = \frac{\epsilon y^{\mathrm{T}} x}{1 + \epsilon y^{\mathrm{T}} x^{(2)}}$$

and

$$\tilde{x}^{(i)} = \beta_i x - \alpha x^{(i)}, \quad i = 3, 4, \ldots, m,$$

where

$$\beta_i = \frac{1 + \alpha \epsilon y^{\mathrm{T}} x^{(3)}}{(\epsilon y^{\mathrm{T}} x)^{(i-2)}} + \frac{\alpha \epsilon y^{\mathrm{T}} x^{(4)}}{(\epsilon y^{\mathrm{T}} x)^{i-3}} + \cdots + \frac{\alpha \epsilon y^{\mathrm{T}} x^{(i)}}{\epsilon y^{\mathrm{T}} x}, \quad i = 3, 4, \ldots, m.$$

The proof will be completed by post-multiplying (2.17) by $\tilde{x}^{(i)}$, $i = 2, 3, \ldots, m$, as in case (ii). $\square$

It is obvious that an analogous property could be given by considering that A^{T} possesses the Perron–Frobenius property.

Based on continuity arguments we can conclude that the last result is valid also for $\hat{x}$ belonging to a cone of directions around x while y is chosen such that $y^{\mathrm{T}} \hat{x} > 0$.

3. Convergence theory of Perron–Frobenius splittings

In this section we define first the Perron–Frobenius splittings analogous to regular, weak regular and nonnegative splittings.

Definition 3.1. Let $A \in \mathbb{R}^{n,n}$ be a nonsingular matrix. The splitting $A = M - N$ is

(i) A Perron–Frobenius splitting of the first kind if $M^{-1}N$ possesses the Perron–Frobenius property.
(ii) A Perron–Frobenius splitting of the second kind if NM^{-1} possesses the Perron–Frobenius property.

In the sequel, for simplicity, by the term *Perron–Frobenius splitting* we mean Perron–Frobenius splitting of the first kind. It is obvious from the above definition that the classes of regular splittings, weak regular splittings and nonnegative splittings belong to the class of Perron–Frobenius splittings. So, the class of Perron–Frobenius splittings is an extension of the well known, previously defined, classes. In the following, we state and prove convergence and comparison statements about this new class of splittings.

3.1. Convergence theorems

The following theorem is an extension of the one given by Climent and Perea [4].

Theorem 3.1. *Let $A \in \mathbb{R}^{n,n}$ be a nonsingular matrix and the splitting $A = M - N$ be a Perron–Frobenius splitting, with x the Perron–Frobenius eigenvector of $M^{-1}N$. Then the following properties are equivalent*:

(i) $\rho(M^{-1}N) < 1$.
(ii) $A^{-1}N$ *possesses the Perron–Frobenius property.*
(iii) $\rho(M^{-1}N) = \frac{\rho(A^{-1}N)}{1+\rho(A^{-1}N)}$.
(iv) $A^{-1}Mx \geqslant x$.
(v) $A^{-1}Nx \geqslant M^{-1}Nx$.

Proof. It can be readily found out that the matrices $A^{-1}N$ and $M^{-1}N$ are connected via the relations yielded below

$$A^{-1}N = (M - N)^{-1}N = (I - M^{-1}N)^{-1}M^{-1}N \tag{3.19}$$

or

$$M^{-1}N = (A + N)^{-1}N = (I + A^{-1}N)^{-1}A^{-1}N. \tag{3.20}$$

The above relations imply that the matrices $A^{-1}N$ and $M^{-1}N$ have the same sets of eigenvectors with their eigenvalues being connected by

$$\mu_i = \frac{\lambda_i}{1-\lambda_i}, \quad i = 1, 2, \ldots, n, \tag{3.21}$$

where $\lambda_i, \mu_i,\ i = 1, 2, \ldots, n$, are the eigenvalues of $M^{-1}N$ and $A^{-1}N$, respectively.

(i) $\Rightarrow$ (ii): From $\rho(M^{-1}N) < 1$ and (3.21), there is an eigenvalue $\mu = \frac{\rho(M^{-1}N)}{1-\rho(M^{-1}N)} > 0$ of $A^{-1}N$ corresponding to the eigenvector x. Looking for a contradiction, assume that there is another eigenvalue $\mu' = \frac{\lambda'}{1-\lambda'}$ corresponding to $\rho(A^{-1}N)$. So,

$$\rho(A^{-1}N) = |\mu'| = \frac{|\lambda'|}{|1-\lambda'|} > \frac{\rho(M^{-1}N)}{1-\rho(M^{-1}N)} = |\mu|.$$

The eigenvalue λ' belongs to the disc $|z| \leqslant \rho(M^{-1}N)$ and $1-\rho(M^{-1}N)$ is the distance of the point 1 from this disc. So, $|1-\lambda'| \geqslant 1-\rho(M^{-1}N)$ which constitutes a contradiction.

(ii) $\Rightarrow$ (iii): Since $A^{-1}N$ has the Perron–Frobenius eigenpair $(\rho(A^{-1}N), x)$, property (iii) follows from (3.20) by a post-multiplication by x.

(iii) $\Rightarrow$ (i): It holds because $\rho(A^{-1}N) > 0$.

(i) $\Leftrightarrow$ (iv): It is obvious that

$$A^{-1}Mx = (M-N)^{-1}Mx = (I-M^{-1}N)^{-1}x = \frac{1}{1-\rho(M^{-1}N)}x.$$

Since $x \geqslant 0,\ x \neq 0$,

$$\frac{1}{1-\rho(M^{-1}N)}x \geqslant x \Leftrightarrow 0 < 1-\rho(M^{-1}N) < 1 \Leftrightarrow 0 < \rho(M^{-1}N) < 1.$$

(i) $\Leftrightarrow$ (v): Considering relation (3.19) and the fact that $x \geqslant 0$, $x \neq 0$, we get

$$\begin{aligned} A^{-1}Nx \geqslant M^{-1}Nx &\Leftrightarrow \frac{\rho(M^{-1}N)}{1-\rho(M^{-1}N)}x \geqslant \rho(M^{-1}N)x \\ &\Leftrightarrow \rho(M^{-1}N) < 1. \qquad \square \end{aligned}$$

We can also state an analogous theorem for the convergence properties of the Perron–Frobenius splittings of the second kind. The proof follows the same lines as before and is omitted.

Theorem 3.2. *Let $A \in \mathbb{R}^{n,n}$ be a nonsigular matrix and the splitting $A = M - N$ be a Perron–Frobenius splitting of the second kind, with x the Perron–Frobenius eigenvector of NM^{-1}. Then the following properties are equivalent*:

(i) $\rho(M^{-1}N) = \rho(NM^{-1}) < 1$.
(ii) NA^{-1} *possesses the Perron–Frobenius property.*

(iii) $\rho(M^{-1}N) = \frac{\rho(A^{-1}N)}{1+\rho(A^{-1}N)}$.
(iv) $MA^{-1}x \geqslant x$.
(v) $NA^{-1}x \geqslant NM^{-1}x$.

Theorems 3.1 and 3.2 give sufficient and necessary conditions for a Perron–Frobenius splitting to be convergent. The following two theorems give only sufficient convergence conditions and constitute also extensions of the ones given by Climent and Perea [4].

Theorem 3.3. *Let $A \in \mathbb{R}^{n,n}$ be a nonsingular matrix and the splitting $A = M - N$ is a Perron–Frobenius splitting, with x the Perron–Frobenius eigenvector of $M^{-1}N$. If one of the following properties holds*:

(i) *There exists $y \in \mathbb{R}^n$ such that $A^{\mathrm{T}}y \geqslant 0$, $N^{\mathrm{T}}y \geqslant 0$ and $y^{\mathrm{T}}Ax > 0$,*
(ii) *There exists $y \in \mathbb{R}^n$ such that $A^{\mathrm{T}}y \geqslant 0$, $M^{\mathrm{T}}y \geqslant 0$ and $y^{\mathrm{T}}Ax > 0$, then $\rho(M^{-1}N) < 1$.*

Proof. We consider the vector z such that $y = (A^{\mathrm{T}})^{-1}z$, then the above properties are modified as follows:

(i) There exists $z \geqslant 0$ such that $z^{\mathrm{T}}(A^{-1}N) \geqslant 0$, $z^{\mathrm{T}}x > 0$, and
(ii) There exists $z \geqslant 0$ such that $z^{\mathrm{T}}(A^{-1}M) \geqslant 0$, $z^{\mathrm{T}}x > 0$,

respectively. We suppose that property (i) holds. By post-multiplying by x we get

$$z^{\mathrm{T}}(A^{-1}N)x = \mu z^{\mathrm{T}}x \geqslant 0,$$

where μ is the eigenvalue of $A^{-1}N$ corresponding to the eigenvector x. So, $\mu = \frac{\rho(M^{-1}N)}{1-\rho(M^{-1}N)}$. Since $z^{\mathrm{T}}x > 0$ we get that $\mu \geqslant 0$, which means that $\rho(M^{-1}N) < 1$.

Let that property (ii) holds, then by following the same steps we get

$$z^{\mathrm{T}}(A^{-1}M)x = \mu' z^{\mathrm{T}}x > 0,$$

where $\mu' = \frac{1}{1-\rho(M^{-1}N)} > 0$ which leads to the same result. $\square$

Moreover, we can prove that property (ii) is stronger than property (i), which means that the validity of (i) implies the validity of (ii) but the converse is not true. For this let that property (i) holds. Then

$$A^{\mathrm{T}}y \geqslant 0 \Rightarrow M^{\mathrm{T}}y - N^{\mathrm{T}}y \geqslant 0 \Rightarrow M^{\mathrm{T}}y \geqslant N^{\mathrm{T}}y \geqslant 0.$$

it is obvious that the converse cannot hold.

For the Perron–Frobenius splittings of the second kind, the following theorem is stated.

Theorem 3.4. *Let $A \in \mathbb{R}^{n,n}$ be a nonsigular matrix and the splitting $A^{\mathrm{T}} = M^{\mathrm{T}} - N^{\mathrm{T}}$ is a Perron–Frobenius splitting of the second kind, with x the Perron–Frobenius eigenvector of $(M^{-1}N)^{\mathrm{T}}$. If one of the following properties holds*:

(i) *There exists $y \in \mathbb{R}^n$ such that $Ay \geqslant 0$, $Ny \geqslant 0$ and $y^{\mathrm{T}}A^{\mathrm{T}}x > 0$.*
(ii) *There exists $y \in \mathbb{R}^n$ such that $Ay \geqslant 0$, $My \geqslant 0$ and $y^{\mathrm{T}}A^{\mathrm{T}}x > 0$, then $\rho(M^{-1}N) < 1$.*

We have to remark here that because of the sufficient conditions only, in Theorems 3.3 and 3.4, we cannot have any information about the convergence unless such a y vector exists. We show this by the following three examples.

Example 3.1

(i)
$$A = \begin{pmatrix} 1 & -2 \\ 3 & -4 \end{pmatrix}, \quad N = \begin{pmatrix} -2 & 3 \\ -7 & 7 \end{pmatrix},$$
$$M = \begin{pmatrix} -1 & 1 \\ -4 & 3 \end{pmatrix}, \quad T = \begin{pmatrix} 1 & 2 \\ -1 & 5 \end{pmatrix},$$
$$A^{-1}N = \begin{pmatrix} -3 & 1 \\ -0.5 & -1 \end{pmatrix}, \quad A^{-1}M = \begin{pmatrix} -2 & 1 \\ -0.5 & 0 \end{pmatrix},$$
$$\rho(T) = 4.4142, \quad x = \begin{pmatrix} 0.5054 \\ 0.8629 \end{pmatrix},$$

where $T = M^{-1}N$. A vector $z \geqslant 0$ $(z \neq 0)$ such that either $z^{\mathrm{T}}(A^{-1}N) \geqslant 0$ or $z^{\mathrm{T}}(A^{-1}M) \geqslant 0$ does not exist and so the splitting is *not* convergent.

(ii)
$$A = \begin{pmatrix} 1 & -2 \\ 3 & -4 \end{pmatrix}, \quad N = \begin{pmatrix} -1 & 0 \\ 5 & -1 \end{pmatrix},$$
$$M = \begin{pmatrix} 0 & -2 \\ 8 & -5 \end{pmatrix}, \quad T = \begin{pmatrix} 0.9375 & -0.125 \\ 0.5 & 0 \end{pmatrix},$$
$$A^{-1}N = \begin{pmatrix} 7 & -1 \\ 4 & -0.5 \end{pmatrix}, \quad A^{-1}M = \begin{pmatrix} 8 & -1 \\ 4 & 0.5 \end{pmatrix},$$
$$\rho(T) = 0.8653, \quad x = \begin{pmatrix} 0.8658 \\ 0.5003 \end{pmatrix}.$$

There exists no $z \geqslant 0$ $(z \neq 0)$ such that $z^{\mathrm{T}}(A^{-1}N) \geqslant 0$ but for $z^{\mathrm{T}} = (1\ 3)$ we have $z^{\mathrm{T}}(A^{-1}M) \geqslant 0$, so the splitting *is* convergent.

(iii) $$A = \begin{pmatrix} 1 & -2 \\ 3 & -4 \end{pmatrix}, \quad N = \begin{pmatrix} -1 & 0 \\ 5 & -3 \end{pmatrix},$$

$$M = \begin{pmatrix} 0 & -2 \\ 8 & -7 \end{pmatrix}, \quad T = \begin{pmatrix} 1.0625 & -0.3750 \\ 0.5 & 0 \end{pmatrix},$$

$$A^{-1}N = \begin{pmatrix} 7 & -3 \\ 4 & -1.5 \end{pmatrix}, \quad A^{-1}M = \begin{pmatrix} 8 & -3 \\ 4 & -0.5 \end{pmatrix},$$

$$\rho(T) = 0.8390, \quad x = \begin{pmatrix} 0.8590 \\ 0.5119 \end{pmatrix}.$$

There exists no $z \geqslant 0$ ($z \neq 0$) such that either $z^{\mathrm{T}}(A^{-1}N) \geqslant 0$ or $z^{\mathrm{T}}(A^{-1}M) \geqslant 0$ but the splitting *is* convergent.

We have also to remark that the strict condition $y^{\mathrm{T}}Ax > 0$ is necessary. This is shown in the following example.

Example 3.2

$$A = \begin{pmatrix} 1 & -2 & -1 \\ 3 & -4 & 1 \\ -1 & 1 & 1 \end{pmatrix}, \quad N = \begin{pmatrix} -2 & 3 & 1 \\ -7 & 7 & 1 \\ 2.5 & -2 & 1 \end{pmatrix},$$

$$M = \begin{pmatrix} -1 & 1 & 0 \\ -4 & 3 & 2 \\ 1.5 & -1 & 2 \end{pmatrix}, \quad T = \begin{pmatrix} 1 & 2 & \frac{8}{3} \\ -1 & 5 & \frac{11}{3} \\ 0 & 0 & \frac{1}{3} \end{pmatrix},$$

$$A^{-1}N = \begin{pmatrix} -3 & 1 & -2.5 \\ -0.5 & -1 & -2 \\ 0 & 0 & 0.5 \end{pmatrix}, \quad \rho(T) = 4.4142, \quad x = \begin{pmatrix} 0.5054 \\ 0.8629 \\ 0 \end{pmatrix}.$$

For the vector $z^{\mathrm{T}} = (0\ \ 0\ \ 1)$ all but one of the conditions of Theorem 3.3(i) hold. However, since $z^{\mathrm{T}}x = 0$ the splitting is *not* convergent.

From Theorem 3.3 the corollary below follows.

Corollary 3.1 *Let $A \in \mathbb{R}^{n,n}$ be a nonsigular matrix and the splitting $A = M - N$ be a Perron–Frobenius splitting, with x the Perron–Frobenius eigenvector of $M^{-1}N$. If one of the matrices $(A^{-1}N)^{\mathrm{T}}$ or $(A^{-1}M)^{\mathrm{T}}$ possesses also the Perron–Frobenius property with y the associated Perron–Frobenius eigenvector, such that $y^{\mathrm{T}}x > 0$, then $\rho(M^{-1}N) < 1$.*

An analogous property holds by considering the splitting of A^{T}.

3.2. Comparison theorems

The following theorem is an extension of the one given by Marek and Szyld [7] for nonnegative splittings.

Theorem 3.5. *Let $A \in \mathbb{R}^{n,n}$ be a nonsingular matrix. If one of the following properties holds*:

(i) *$A = M_1 - N_1$ and $A^{\mathrm{T}} = M_2^{\mathrm{T}} - N_2^{\mathrm{T}}$ are two convergent Perron–Frobenius splittings of the first kind and of the second kind, respectively, with $T_1 := M_1^{-1}N_1$, $T_2^{\mathrm{T}} := (M_2^{-1}N_2)^{\mathrm{T}}$ and $x \geqslant 0$, $y \geqslant 0$ the associated Perron–Frobenius eigenvectors, respectively, such that*

$$y^{\mathrm{T}}A^{-1} \geqslant 0, \quad y^{\mathrm{T}}x > 0 \quad and \quad N_2x \geqslant N_1x, \tag{3.22}$$

(ii) *$A^{\mathrm{T}} = M_1^{\mathrm{T}} - N_1^{\mathrm{T}}$ and $A = M_2 - N_2$ are two convergent Perron–Frobenius splittings of the second kind and of the first kind, respectively, with $T_1^{\mathrm{T}} := (M_1^{-1}N_1)^{\mathrm{T}}$, $T_2 := M_2^{-1}N_2$ and $w \geqslant 0$, $z \geqslant 0$ the associated Perron–Frobenius eigenvectors, respectively, such that*

$$w^{\mathrm{T}}A^{-1} \geqslant 0, \quad w^{\mathrm{T}}z > 0 \quad and \quad N_2z \geqslant N_1z, \tag{3.23}$$

then

$$\rho(T_1) \leqslant \rho(T_2). \tag{3.24}$$

Moreover, if $y^{\mathrm{T}}A^{-1} > 0$ and $N_2x \neq N_1x$ for property (i) *or $w^{\mathrm{T}}A^{-1} > 0$ and $N_2z \neq N_1z$ for property* (ii), *then*

$$\rho(T_1) < \rho(T_2). \tag{3.25}$$

Proof. Let that property (i) holds. Then from the first and the last inequalities of (3.22) we get

$$y^{\mathrm{T}}A^{-1}N_2x \geqslant y^{\mathrm{T}}A^{-1}N_1x.$$

Since the above splittings are convergent, from Theorem 3.1 property (ii), we get that the matrix $A^{-1}N_1$ possesses the Perron–Frobenius property and from Theorem 3.2 property (ii), we get that the matrix $(A^{-1}N_2)^{\mathrm{T}}$ possesses the Perron–Frobenius property, with x and y the Perron–Frobenius eigenvectors, respectively. So,

$$\rho(A^{-1}N_2)y^{\mathrm{T}}x - \rho(A^{-1}N_1)y^{\mathrm{T}}x \geqslant 0.$$

Since $\rho(A^{-1}N_1) = \frac{\rho(T_1)}{1-\rho(T_1)}$, $\rho((A^{-1}N_2)^{\mathrm{T}}) = \rho(A^{-1}N_2) = \frac{\rho(T_2)}{1-\rho(T_2)}$ and the fact that the function $\frac{\rho}{1-\rho}$ is an increasing function of $\rho \in (0, 1)$, we obtain $\rho(T_1) \leqslant \rho(T_2)$. The strict inequality (3.25) is obvious. The proof in case property (ii) holds is similar. □

We have to remark here that a particular statement could be given from the above theorem by replacing the assumptions $y^{\mathrm{T}}A^{-1} \geqslant 0$, $y^{\mathrm{T}}x > 0$ (or $w^{\mathrm{T}}A^{-1} \geqslant 0$, $w^{\mathrm{T}}z > 0$) by $A^{-1} \geqslant 0$.

We illustrate the validity of this theorem by the following example.

Example 3.3. We consider the splittings $A = M_1 - N_1 = M_2 - N_2 = M_3 - N_3$ where

$$A = \begin{pmatrix} 4 & -1 & -1 & -1 \\ -1 & 4 & -1 & -1 \\ -1 & -1 & 4 & -1 \\ -1 & -1 & -1 & 4 \end{pmatrix}, \quad M_1 = \begin{pmatrix} 4 & -1 & 0 & 0 \\ -1 & 4 & -2 & 0 \\ 0 & -2 & 4 & -1 \\ 0 & 0 & -1 & 4 \end{pmatrix},$$

$$M_2 = \begin{pmatrix} 4 & -1.1 & 0.2 & 0 \\ -1.1 & 4 & -1 & 0 \\ 0.2 & -1 & 4 & -1 \\ 0 & 0 & -1 & 4 \end{pmatrix}, \quad M_3 = \begin{pmatrix} 4 & -1 & 0 & 0 \\ -1 & 4 & 0 & 0 \\ 0 & 0 & 4 & -1 \\ 0 & 0 & -1 & 4 \end{pmatrix}.$$

The splitting $A = M_1 - N_1$ is a Perron–Frobenius splitting with the Perron–Frobenius eigenpair being $(\rho(T_1),\ x_1) = \big(0.5345, (0.5680\ \ 0.4212\ \ 0.4212\ \ 0.5680)^{\mathrm{T}}\big)$. The splitting $A^{\mathrm{T}} = M_2^{\mathrm{T}} - N_2^{\mathrm{T}}$ is a Perron–Frobenius splitting of the second kind with the Perron–Frobenius eigenpair being $(\rho(T_2),\ y_2) = \big(0.6126,\ (0.6388\ 0.2855\ 0.3871\ 0.6005)^{\mathrm{T}}\big)$. Although $N_2 - N_1$ is not a nonnegative matrix, we have $(N_2 - N_1)x_1 = (0.0421\ \ 0.3644\ \ 0.5348\ \ 0)^{\mathrm{T}} \geqslant 0$. Moreover, $A^{-1} > 0$ and $N_2x_1 \neq N_1x_1$. So, property (i) of Theorem 3.5 holds and the inequality $\rho(T_1) < \rho(T_2)$ is confirmed. We can check that for the first two splittings, property (ii) of Theorem 3.5 also holds.

To compare the last two splittings we observe that the splitting $A = M_2 - N_2$ is a Perron–Frobenius splitting while $A = M_3 - N_3$ is a regular splitting, but properties (i) and (ii) of Theorem 3.5 do not hold. So, Theorem 3.5 does not give any information.

We have to observe here that both properties (i) and (ii) of Theorem 3.5 hold for the comparison of the first splitting with the last one, since $N_3 - N_1 \geqslant 0$. So, $\rho(T_1) = 0.5345 < \rho(T_3) = 0.6667$ is confirmed.

Theorem 3.6. *Let $A \in \mathbb{R}^{n,n}$ be a nonsingular matrix. If one of the following holds*:

(i) $A = M_1 - N_1$ *and* $A^{\mathrm{T}} = M_2^{\mathrm{T}} - N_2^{\mathrm{T}}$ *are two convergent Perron–Frobenius splittings of the first kind and of the second kind*, *respectively*, *with* $T_1 := M_1^{-1}N_1$, $T_2^{\mathrm{T}} := (M_2^{-1}N_2)^{\mathrm{T}}$ *and* $x \geqslant 0$, $y \geqslant 0$ *the associated Perron–Frobenius eigenvectors*, *respectively*, $N_1x \geqslant 0$ *and* $y^{\mathrm{T}}M_1^{-1} \geqslant y^{\mathrm{T}}M_2^{-1}$, $y^{\mathrm{T}}x > 0$,

(ii) $A^{\mathrm{T}} = M_1^{\mathrm{T}} - N_1^{\mathrm{T}}$ *and* $A = M_2 - N_2$ *are two convergent Perron–Frobenius splittings of the second kind and of the first kind*, *respectively*, *with* $T_1^{\mathrm{T}} := (M_1^{-1}N_1)^{\mathrm{T}}$, $T_2 := M_2^{-1}N_2$ *and* $w \geqslant 0$, $z \geqslant 0$ *the associated Perron–Frobenius eigenvectors*, *respectively*, $N_2z \geqslant 0$ *and* $w^{\mathrm{T}}M_1^{-1} \geqslant w^{\mathrm{T}}M_2^{-1}$, $w^{\mathrm{T}}z > 0$,

then

$$\rho(T_1) \leqslant \rho(T_2). \tag{3.26}$$

Moreover, if $y^{\mathrm{T}}M_1^{-1} > y^{\mathrm{T}}M_2^{-1}$ *and* $N_1x \neq 0$ *or* $w^{\mathrm{T}}M_1^{-1} > w^{\mathrm{T}}M_2^{-1}$ *and* $N_2z \neq 0$, *respectively, then the inequality* (3.26) *is strict, while if* $y^{\mathrm{T}}M_1^{-1} = y^{\mathrm{T}}M_2^{-1}$ *or* $w^{\mathrm{T}}M_1^{-1} = w^{\mathrm{T}}M_2^{-1}$, *respectively, then the inequality* (3.26) *becomes an equality.*

Proof. We assume that property (i) holds. Then

$$M_1x = \frac{1}{\rho(T_1)}N_1x \geqslant 0$$

and

$$Ax = M_1(I - T_1)x = \frac{1-\rho(T_1)}{\rho(T_1)}N_1x \geqslant 0.$$

By pre-multiplying by $y^{\mathrm{T}}M_1^{-1} - y^{\mathrm{T}}M_2^{-1} \geqslant 0$ we get

$$\begin{aligned} y^{\mathrm{T}}(M_1^{-1} - M_2^{-1})Ax &= y^{\mathrm{T}}(I - T_1)x - y^{\mathrm{T}}(I - T_2)x \\ &= \rho(T_2)y^{\mathrm{T}}x - \rho(T_1)y^{\mathrm{T}}x \geqslant 0, \end{aligned}$$

which proves the result (3.26). The strict inequality or equality, under the certain assumptions, are obvious and the proof in case property (ii) holds is quite analogous. □

We observe that Theorem 3.6 provides an answer to Example 3.3 where Theorem 3.5 failed. Especially, we have $M_2^{-1} - M_3^{-1} > 0$ and $N_2x_2 \geqslant 0$, $N_2x_2 \neq 0$. So the strict inequality $\rho(T_2) = 0.6126 < \rho(T_3) = 0.6667$ is confirmed. It is easily checked that property (ii) of Theorem 3.6 also holds. We also observe that both properties (i) and (ii) of Theorem 3.6 hold for the comparison of the first with the second splitting as well as the first with the last one.

We point out that if instead of the assumptions $y^{\mathrm{T}}M_1^{-1} \geqslant y^{\mathrm{T}}M_2^{-1}$, $y^{\mathrm{T}}x > 0$ (or $w^{\mathrm{T}}M_1^{-1} \geqslant w^{\mathrm{T}}M_2^{-1}$, $w^{\mathrm{T}}x > 0$) we assume $M_1^{-1} \geqslant M_2^{-1}$ then we obtain a particular case of the theorem just proved.

In the following example we illustrate how the previous theorems work.

Example 3.4. We consider the splittings $A = M_1 - N_1 = M_2 - N_2 = M_3 - N_3 = M_4 - N_4 = M_5 - N_5$ where

$$A = \begin{pmatrix} 3 & -1 & -1 \\ -1 & 3 & -1 \\ 1 & -1 & 3 \end{pmatrix}, \quad M_1 = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix},$$

$$M_2 = \begin{pmatrix} 3 & -1 & 0 \\ -1 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix}, \quad M_3 = \begin{pmatrix} 3 & 0 & 0 \\ -1 & 3 & 0 \\ 1 & 0 & 3 \end{pmatrix},$$

$$M_4 = \begin{pmatrix} 3 & -1 & -1 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix}, \quad M_5 = \begin{pmatrix} 3 & 0 & -1 \\ 0 & 3 & 0 \\ 1 & 0 & 3 \end{pmatrix}.$$

It is easily checked that all the above splittings are convergent with

$$\rho(T_2) = 0 < \rho(T_1) = \rho(T_3) = \rho(T_4) = \frac{1}{3} < \rho(T_5) = 0.4472.$$

The first four splittings are Perron–Frobenius splittings while the last one is a nonnegative splitting. The splittings $A^{\mathrm{T}} = M_1^{\mathrm{T}} - N_1^{\mathrm{T}} = M_3^{\mathrm{T}} - N_3^{\mathrm{T}} = M_4^{\mathrm{T}} - N_4^{\mathrm{T}}$ are also Perron–Frobenius splittings of the second kind while the splitting $A^{\mathrm{T}} = M_5^{\mathrm{T}} - N_5^{\mathrm{T}}$ is a nonnegative splitting. The associated Perron–Frobenius eigenvectors are

$$x_1 = x_2 = \begin{pmatrix} 0.7071 \\ 0.7071 \\ 0 \end{pmatrix}, \quad x_3 = \begin{pmatrix} 0.8018 \\ 0.5345 \\ 0.2773 \end{pmatrix}, \quad x_4 = \begin{pmatrix} 0.4082 \\ 0.8165 \\ 0.4082 \end{pmatrix},$$

$$x_5 = \begin{pmatrix} 0.6325 \\ 0.7071 \\ 0.3162 \end{pmatrix}, \quad y_1 = y_3 = y_4 = \begin{pmatrix} 0 \\ 0.7071 \\ 0.7071 \end{pmatrix}, \quad y_5 = \begin{pmatrix} 05130 \\ 0.6882 \\ 0.5130 \end{pmatrix},$$

where by x_i and y_i we have denoted the associated Perron–Frobenius eigenvectors of the first kind and of the second kind, respectively. We use the symbol $i \leftrightarrow j$ to denote the comparison of the ith splitting with the jth one:

$1 \leftrightarrow 2$: It is easily checked that assumptions (i) of Theorems 3.5 and 3.6 hold, where the roles of T_1 and T_2 have been interchanged, to obtain $\rho(T_2) \leqslant \rho(T_1)$. Note that the strict inequality cannot be obtained from any of the above theorems.

$1 \leftrightarrow 3$: Theorem 3.5 cannot be applied while both assumptions (i) and (ii) of Theorem 3.6 hold with the corresponding inequalities $y_3^{\mathrm{T}} M_1^{-1} \geqslant y_3^{\mathrm{T}} M_3^{-1}$ and $y_1^{\mathrm{T}} M_1^{-1} \geqslant y_1^{\mathrm{T}} M_3^{-1}$ being equalities. So, we obtain $\rho(T_1) = \rho(T_3)$.

$3 \leftrightarrow 2$: The same properties, as in the case $1 \leftrightarrow 2$, hold. Therefore, $\rho(T_2) \leqslant \rho(T_3)$.

$3 \leftrightarrow 4$: The same properties, as in the case $1 \leftrightarrow 3$, hold. So, $\rho(T_3) = \rho(T_4)$.

$4 \leftrightarrow 2$: The same properties, as in the case $1 \leftrightarrow 2$, hold. Consequently, $\rho(T_2) \leqslant \rho(T_4)$.

$4 \leftrightarrow 5$: Both properties of Theorems 3.5 and 3.6 are applied to give the inequality $\rho(T_4) \leqslant \rho(T_5)$. Moreover, we have that $y_5^{\mathrm{T}} A^{-1} > 0$ and $y_5^{\mathrm{T}} M_4^{-1} > y_5^{\mathrm{T}} M_5^{-1}$, which gives by Theorems 3.5 and 3.6, respectively, the strict inequality $\rho(T_4) < \rho(T_5)$.

$5 \leftrightarrow 2$: From property (i) of Theorem 3.5 and the fact that $y_5^{\mathrm{T}} A^{-1} > 0$ we obtain the strict inequality $\rho(T_2) < \rho(T_5)$.

We conclude this work by pointing out that the most general extensions and generalizations of the Perron–Frobenius theory for nonnegative matrices, have been introduced, stated and proved.

Acknowledgments

Warm thanks are due to the referee for the pertinent remarks and the useful suggestions that helped the author to improve the quality of the paper.

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