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ON DIFFERENT CLASSES OF MONOPARAMETRIC STATIONARY ITERATIVE METHODS FOR THE SOLUTION OF LINEAR SYSTEMS

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The class of monoparametric k-step methods

$$x^{(m)} = \omega T x^{(m-1)} + (1-\omega) x^{(m-k)} + \omega c \tag{1}$$

used for the solution of the linear system $(I-T)x = c$ is studied. Under certain conditions the spectrum $\sigma(T)$ of T must satisfy, for $\omega > 1$ and given $k(\geqslant 2)$ and $\rho \in (0, 1)$ (a quantity defined in the paper), (optimum) convergent methods (1) are derived. Next, an equivalence between (optimum) convergent methods (1) and a class of Successive Overrelation (SOR) ones is established. Then, based on (1), a class of new monoparametric methods, called $k/2$-step block iterative methods faster than those in (1), is introduced and studied. Finally, various applications and numerical examples in support of the theory developed in this paper are provided.

1. Introduction

For the solution of the nonsingular linear system of equations

$$Ax = b, \tag{1.1}$$

with $A \in \mathbb{C}^{n,n}$ and $x,\ b \in \mathbb{C}^n$, de Pillis and Neumann [8] considered the splitting

$$A = A_0 - A_1 - \cdots - A_k, \tag{1.2}$$

where $A_j \in \mathbb{C}^{n,n}$, $j = 0(1)k$ and $\det(A_0) \neq 0$. (1.2) generates the linear stationary k-step iterative method

$$x^{(m)} = B_1 x^{(m-1)} + B_2 x^{(m-2)} + \cdots + B_k x^{(m-k)} + c, \quad m = k, k+1, \cdots, \tag{1.3}$$

with $x^{(j)} \in \mathbb{C}^n$, $j = 0(1)k-1$ arbitrary, $B_j = A_0^{-1} A_j$, $j = 1(1)k$ and $c = A_0^{-1} b$. For a suitable choice of the A_j's based on a knowledge of the spectrum σ(A) of A, which is assumed to satisfy certain conditions, scheme (1.3) converges to the solution of (1.1). If we consider the new splitting $A = A_0' - A_1' - \cdots - A_k'$ with

$$A_0' = \frac{1}{\mu_0} A_0, \qquad A_1' = (\mu_0 + \mu_1) A_0' - a, \qquad A_j' = \mu_j A_0', \quad j = 2(1)k, \tag{1.4}$$

where $\mu_j \in \mathbb{C}$, $j = 0(1)k$ satisfy

$$\mu_0 \neq 0, \qquad \mu_0 + \mu_1 + \cdots + \mu_k = 1 \tag{1.5}$$

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and put $T \equiv I - A_0'^{-1}A$, then the k-step iterative method of Niethammer and Varga [19] is derived:

$$x^{(m)} = (\mu_0 T + \mu_1 I)x^{(m-1)} + \mu_2 x^{(m-2)} + \cdots + \mu_k x^{(m-k)} + \mu_0 c,$$
$$m = k,\ k+1,\ \cdots. \tag{1.6}$$

If the spectrum $\sigma(T)$ or its convex hull $H(T)$ (i.e. the smallest convex polygon containing $\sigma(T)$ in the closure of its interior) or, more generally, a closed region $R_k \supset \sigma(T)$ (and $1 \notin H(T)$ or R_k respectively) is known, the problem of determining the optimal values of the parameters μ_j, $j = 0(1)k$, except in the special cases of the next paragraph, is still an open one. In the following we refer to some of the works for the cases $k = 1$ and 2.

For the case $k = 1$, $\mu_0 \in \mathbb{R}$ ($\mu_1 = 1 - \mu_0$) and $H(T) \subset (-\infty, 1)$ or $H(T) \subset (1, +\infty)$ the optimal solution can be found in Isaacson and Keller [15, Theorem 1, pp. 73–78]. For other configurations of $\sigma(T)$ optimal solutions were given in [8,12] etc. In case $H(T)$ lies in the half complex plane Re $z < 1$ (or Re $z > 1$) the problem was solved by Hughes Hallett [13] (see also [14]) and Hadjidimos [10]. For $\mu_0 \in \mathbb{C}$ ($\mu_1 = 1 - \mu_0$) and $1 \notin H(T)$ a first result was obtained by Buoni and Varga [6] and the complete solution was given by Hadjidimos [11]. For $k = 2$ and μ_0, $\mu_1 \in \mathbb{R}$ ($\mu_2 = 1 - \mu_0 - \mu_1$) the very first optimum result was obtained by Golub and Varga [9]. Later came the works by Manteuffel [17–18], de Pillis and Neumann [8], Avdelas and Hadjidimos [4], Avdelas [1] Leontitsis [16] and the one by Niethammer and Varga [19]. The latter seems to be the only one in which an optimal solution for complex μ_j's (and $H(T)$ a line segment) is given.

Some researchers considered monoparametric stationary k-step ($k \geqslant 2$) schemes of type (1.6) with $\mu_j = 0$, $j = 1(1)k-1$, $\mu_0 \in \mathbb{R}$. In view of (1.5), if we put $\mu = \mu_0$ (1.6) is then simplified to

$$x^{(m)} = \omega T x^{(m-1)} + (1-\omega)x^{(m-k)} + \omega c, \quad m = k,\ k+1, \cdots. \tag{1.7}$$

Apart from the works above for $k = 2$, for special cases of which schemes (1.6) become monoparametric ones, we mention the works by de Pillis [7], Avdelas et al. [3] and Avdelas et al. [2], who studied the case $k = 2$ and Niethammer and Varga [19], who treated as an example the case $k = 4$. As far as the authors know nothing has been done for schemes (1.7) or their variants for any $k \geqslant 3$.

The remaining of this paper is organized as follows. In Section 2 scheme (1.7) for $\omega > 1$ and given $k(\geqslant 3)$ and $\rho \in (0, 1)$ (a quantity to be defined later) is studied by means of Euler functions (see [19]) and its optimum is derived. In Section 3 the extension of an idea in Varga [21, pp. 141–144] enables us, under certain assumptions concerning $\sigma(T)$, to establish an equivalence between a type of Successive Overrelaxation (SOR) method and the method (1.7) such that the optimum $\omega(\hat{\omega}_k)$ for both methods is the same. In Section 4 we exploit the equivalence established to introduce what we call monoparametric $k/2$-step block iterative methods and prove that when they apply they are faster than the corresponding k-step ones. Finally in Section 5 applications referring to Section 2 are given and a comparison of the $k/2$-step block methods and the k-step methods is made by giving selected numerical examples.

2. Study of the scheme (1.7)

We begin our study with the following lemma.

Lemma 1. (a) *The circle* $\varphi=\eta e^{i\theta}$, $\eta>0$, $\theta\in[0, 2\pi)$ *in the complex plane is transformed through*

$$z:=\tilde{p}(\varphi)=1/p(\varphi)=\frac{1-(1-\omega)\varphi^k}{\omega\varphi}, \quad \omega>1,\ k\geqslant 3 \tag{2.1}$$

into a closed curve C_k *consisting of k consecutive arcs (let the first one correspond to* $\theta[0, 2\pi/k]$*) each of which is symmetric with respect to (w.r.t.) the line passing through the origin and its midpoint. Also rotations of the complex plane about the origin through angles of* $2\pi/k$*, but through no smaller angles, carry* C_k *into itself.*

(b) *For a given* $\rho=\tilde{p}(\hat{\eta})\in(0, 1)$ *and for every* $\eta\in(0, \hat{\eta}\ ]$, *with* $\hat{\eta}=((k-1)(\hat{\omega}_k-1))^{-1/k}$, *and only for these values of* η, C_k *is a simple curve and* $p(\varphi)$ *in (2.1) for* $\eta\in(1, \hat{\eta}\ ]$ *is an Euler function, where* $\hat{\omega}_k\in(1, k/(k-1))$ *is the unique positive real root of the equation*

$$(\omega\rho)^k=k^k(k-1)^{1-k}(\omega-1). \tag{2.2}$$

Proof. We write (2.1) in polar coordinates and obtain

$$z:=\tilde{p}(\varphi)=\frac{1}{p(\varphi)}=\frac{1}{\omega}\left[\left(\frac{1}{\eta}\cos\theta-(1-\omega)\eta^{k-1}\cos(k-1)\theta\right)\right.$$
$$\left.-i\left(\frac{1}{\eta}\sin\theta+(1-\omega)\eta^{k-1}\sin(k-1)\theta\right)\right]. \tag{2.3}$$

From (2.3) it is found out that the graph C_k of $z=\tilde{p}(\varphi)$ has polar radius

$$r(\eta, \omega, \theta)=\frac{1}{\omega}\left[\frac{1}{\eta^2}+(1-\omega)^2\eta^{2(k-1)}-2(1-\omega)\eta^{k-2}\cos k\theta\right]^{1/2}. \tag{2.4}$$

From the latter expression the properties of C_k in part (a) follow (see also [20]).

For the part (b) we have that $p(0)=0$ and $p(1)=1$, so C_k is a simple curve for all $\eta\in(0, \hat{\eta}]$ for which $\operatorname{Im}(d\tilde{p}(\varphi)/d\theta)\,|_{\theta=0}$ does not change sign and $p(\varphi)$ is an Euler function for all $\eta\in(1, \hat{\eta}]$ for which $\tilde{p}(\varphi)$ is univalent in $D_\eta\supset\bar{D}_1$ ($D_\eta:=\{\varphi\in\mathbb{C}: |\varphi|<\eta,\ \eta>0\}$, with $\bar{D}_\eta$ its closure). We follow [19, Example 4, pp. 202–204], differentiate (2.3) w.r.t. θ and obtain

$$\frac{dz}{d\theta}=\frac{1}{\omega}\left[\left(-\frac{1}{\eta}\sin\theta+(k-1)(1-\omega)\eta^{k-1}\sin(k-1)\theta\right)\right.$$
$$\left.-i\left(\frac{1}{\eta}\cos\theta+(k-1)(1-\omega)\eta^{k-1}\cos(k-1)\theta\right)\right]. \tag{2.5}$$

Since $\operatorname{Im}(dz/d\theta)\,|_{\theta=0}\to-\infty$ as $\eta\to0^+$, $\operatorname{Im}(dz/d\theta)\,|_{\theta=0}$ remains nonpositive iff $1/\eta+(k-1)(1-\omega)\eta^{k-1}\geqslant0$ or equivalently $\eta\in(0, ((k-1)(\omega-1))^{-1/k}]$. The maximum value of $\eta(=\hat{\eta})>1$ for which the quantity of interest is nonpositive (and $p(\varphi)$ in (2.1) is an Euler function for $\eta\in(1, \hat{\eta}]$) is that for which $r(\hat{\eta}, \hat{\omega}_k, 0)=\rho$. Then $\hat{\eta}=((k-1)(\hat{\omega}_k-1))^{-1/k}$ and from (2.4) $\hat{\omega}_k$ must be the (unique) positive real root of (2.2). □

Note: The graph of $C_5(\equiv\hat{C}_5)$ for a given $\rho\in(0, 1)$ is shown in Fig. 1.

By virtue of Lemma 1 and the theory in [19] we state below a theorem and a corollary whose proof is easy.

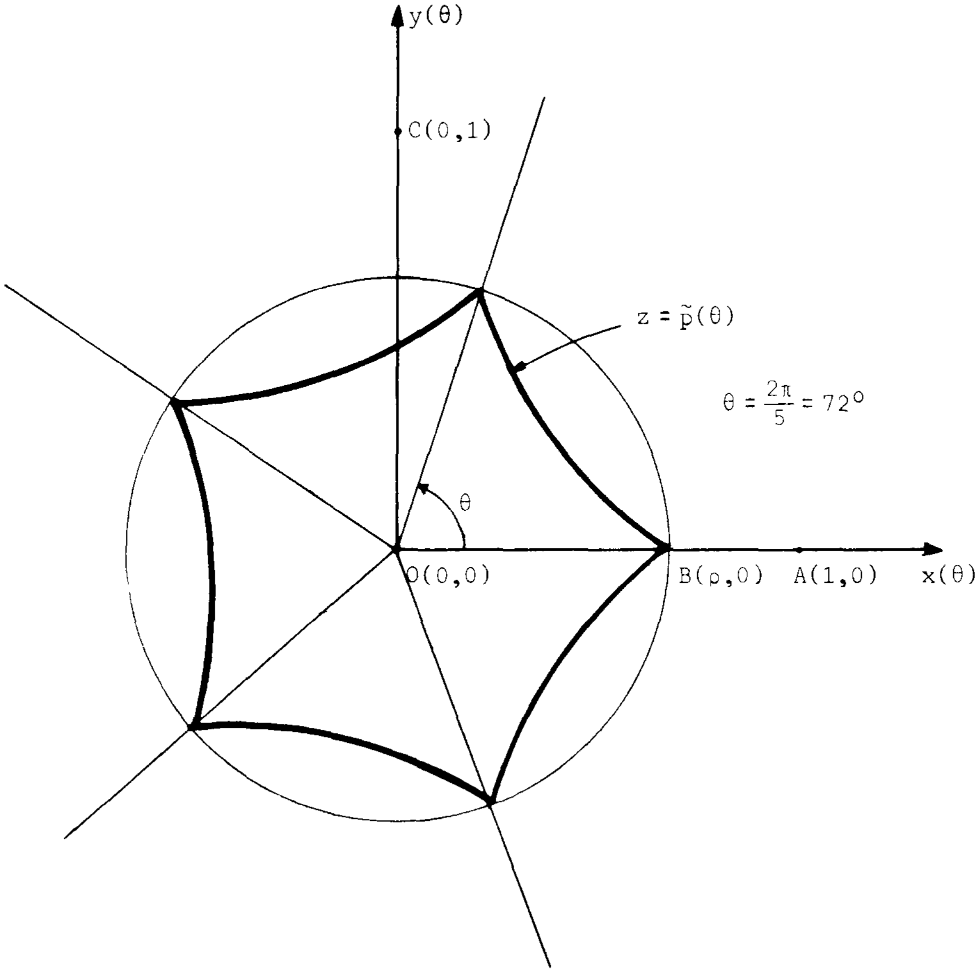

Fig. 1.

Theorem 2. *Let $\hat{R}_k$ be the closed interior of C_k ($\equiv \hat{C}_k$) of Lemma* 1 *corresponding to $\hat{\omega}_k$. If $\sigma(T) \subset \hat{R}_k$, then* (1.7), *under the initial conditions*

$$x^{(0)} = c, \qquad x^{(j)} = c + \omega T x^{(j-1)}, \quad j = 1(1)k-1 \tag{2.6}$$

and with $\omega = \hat{\omega}_k$, is a k-step iterative Euler method, converges and has an asymptotic convergence factor (a. c.f.) $\leqslant 1/\hat{\eta} = ((k-1)(\hat{\omega}_k - 1))^{1/k}$, with equality holding iff at least one element of $\sigma(T)$ lies on $\hat{C}_k$. In addition (1.7) *converges for any $\omega \in [\hat{\omega}_k, k/(k-1))$.*

Corollary 3. *If the elements of $\sigma(T^k)$ for $k \geqslant 3$ are real and nonegative with $\rho(T) \in (0, 1)$, then Lemma* 1 *and Theorem* 2 *hold with $\rho \equiv \rho(T)$.*

Note: The situation for $k = 5$ is shown in Fig. 2.

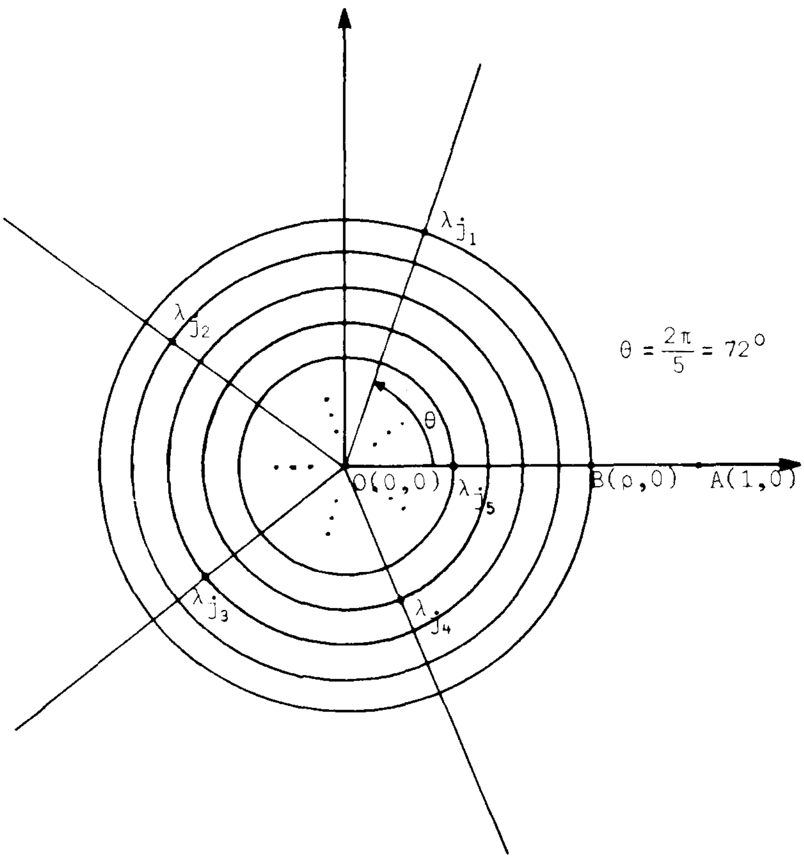

Fig. 2.

Remark. From what is known for $k = 2$ (see e.g. [19]), one can say that Lemma 1(b), Theorem 2 and Corollary 3 hold with C_2(and $\hat{C}_2$) degenerating into the double line segment $[-\rho, \rho]$, where only its end-points are simple points. □

It is noted that the optimum results for the scheme (1.7) bear a close resemblance with the corresponding ones for an SOR method which has a weakly cyclic of index k Jacobi matrix T (see [20]). This will be investigated further in the next section.

In view of what has been developed so far the following theorem can be stated and proved.

Theorem 4. *Let* $\{A^{(k)}\} \equiv \{I - T^{(k)}\}$, $k = 2, 3, 4, \ldots$ *be a sequence of matrix coefficients in* (1.1) *such that for all k and a given* $\rho \in (0, 1)$, $\sigma(T^{(k)}) \subset \hat{R}_k$, *with* $\hat{R}_k$ *and* $\hat{C}_k$ *being defined in Theorem* 2. *Let also that for each k at least one element of* $\sigma(T^{(k)})$ *lies on* $\hat{C}_k$. *Then the sequence of optimum a.c.f.'s of the corresponding schemes* (1.7), *let them be* $\rho_{\hat{\omega}_k}(= 1/\hat{\eta}_k)$, *is a strictly increasing one.*

Proof. Consider (2.2) with $\omega = \hat{\omega}_k$, put $y \equiv y(k) \equiv \rho_{\hat{\omega}_k} = ((k-1)(\hat{\omega}_k - 1))^{1/k} \in (0, 1)$ and solve for ρ to obtain

$$\rho = ky/(y^k + k - 1). \tag{2.7}$$

Assume that y is a continuous function of the real k (instead of an integer one), differentiate (2.7) w.r.t. k to take

$$\frac{1}{y}(1-y^k)(1-k)y' = y^k(\ln y^k + 1) - 1 \equiv \alpha. \tag{2.8}$$

By setting $y^k = \beta \in (0, 1)$ we have $\gamma = \alpha/\beta = 1 + \ln \beta - 1/\beta$ and $\partial\gamma/\partial\beta = (1+\beta)/\beta^2 > 0$. Hence γ increases with β and since $\gamma = 0$ at $\beta = 1$, γ is always negative. So is $\alpha = \beta\gamma$ and from (2.8) $y' > 0$, meaning that y increases with k. This proves our assertion. □

Theorem 4 states also that if for some values of k it so happens that its assumptions are satisfied, then among all optimum schemes (1.7) the one which corresponds to the smallest k is the best. As an application consider the case where $k = q \geqslant 2$ is the smallest integer for which the assumptions of Corollary 3 are satisfied. As is obvious the same assumptions will be satisfied for $k = 2q, 3q, 4q, \ldots$. However, according to Theorem 4 the best of all optimum schemes (1.7), corresponding to $k = q, 2q, 3q, \ldots$, will be the one for $k = q$.

Remark. If for $T^{(k)} \equiv T$ the assumptions of Theorem 4 are satisfied for some $k \geqslant 2$ or if there is no $k \geqslant 2$ satisfying them and the convex hull $H(T) \not\ni 1$ is known, then it is worth determining the optimum monoparametric 1- and 2-step schemes (1.7). The first one is determined by the algorithm in [13,14] or [10], while the second one is determined by the algorithm of Young and Eidson [23] (see also [22, pp. 194–200]) as this was proved in [3]. If and only if an optimum k-step scheme (1.7) ($k \geqslant 2$) exists (for the smallest possible value of k) and is superior to the two aforementioned ones, then it must be used in practice with no reservation.

3. Equivalence of the method (1.7) and a specific SOR method

Consider a splitting (1.2) for $k = 1$ so that (1.1) is rewritten as

$$x = Tx + c, \tag{3.1}$$

with $T \equiv B_1 = A_0^{-1}A_1$ and $C = A_0^{-1}b$. T in (3.1) is the generalized Jacobi matrix associated with the splitting $A = A_0 - A_1$ (see [5]). The analysis that follows is based on the extension of an idea in Varga [21, pp. 141–143]. For this we choose a $k(\geqslant 2)$ and form the linear system of kn equations with kn unknowns

$$\tilde{x} = \tilde{T}\tilde{x} + \tilde{c}, \tag{3.2}$$

where

$$\tilde{T} = \begin{bmatrix} 0 & 0 & \ldots & 0 & T \\ T & 0 & \ldots & 0 & 0 \\ 0 & T & \ldots & 0 & 0 \\ \vdots & & & & \\ 0 & 0 & \ldots & T & 0 \end{bmatrix} \tag{3.3a}$$

is of block order k and

$$\tilde{x} = [x^{\mathrm{T}}\ x^{\mathrm{T}}\ \ldots\ x^{\mathrm{T}}]^{\mathrm{T}}, \qquad \tilde{c} = [c^{\mathrm{T}}\ c^{\mathrm{T}}\ \ldots\ c^{\mathrm{T}}]^{\mathrm{T}}, \tag{3.3b}$$

with $\tilde{x}$, $\tilde{c} \in \mathbb{C}^{kn}$. As is obvious $\tilde{x}$ will be a unique solution of (3.2) iff $\det(\tilde{I} - \tilde{T}) \neq 0$, where $\tilde{I}$ is of order kn unit matrix, or iff $1 \notin \sigma(\tilde{T})$. Since

$$\tilde{T}^k = \begin{bmatrix} T^k & 0 & \dots & 0 \\ 0 & T^k & \dots & 0 \\ \vdots & & & \\ 0 & 0 & \dots & T^k \end{bmatrix}$$

implies $\sigma(\tilde{T}^k) = \sigma(T^k)$, the restriction $1 \notin \sigma(\tilde{T})$ is equivalent to $1 \notin \sigma(T^k)$. Thus we have proved.

Theorem 5. *Let λ_j be the eigenvalues of T satisfying for a given k, $\lambda_j^k \neq 1$, $j = 1(1)n$. Then* (3.2) *possesses a unique solution $\tilde{x}$, given in* (3.3b), *with k the unique solution of* (3.1).

Assume that for a given $k \geqslant 2$ and a given $\rho \in (0, 1)$, the spectrum of T in (3.1) satisfies $\sigma(T) \subset \hat{R}_k$ with $\hat{R}_k$ being defined in Theorem 2. Then consider the block SOR method associated with (3.2), namely

$$(\tilde{I} - \omega\tilde{L})\tilde{x}^{(l+1)} = \left[(1-\omega)\tilde{I} + \omega\tilde{U}\right]\tilde{x}^{(l)} + \omega\tilde{c}, \quad l = 0, 1, 2, \dots \tag{3.4}$$

where $-L$ and $-U$ are the strictly lower and upper triangular parts of $\tilde{I} - \tilde{T}$ and

$$\tilde{x}^{(l)} = \left[x_1^{(l)\mathrm{T}}\, x_2^{(l)\mathrm{T}} \dots x_k^{(l)\mathrm{T}}\right]^{\mathrm{T}}, \quad x_j^{(l)} \in \mathbb{C}^n,\ j = 1(1)k, \tag{3.5}$$

for any $\omega \in (1, 2)$ for which (3.4) converges. In view of (3.5), (3.4) gives

$$\begin{aligned} x_1^{(l+1)} &= \omega T x_k^{(l)} + (1-\omega)x_1^{(l)} + \omega c \\ x_2^{(l+1)} &= \omega T x_1^{(l+1)} + (1-\omega)x_2^{(l)} + \omega c, \quad l = 0, 1, 2, \dots. \\ &\vdots \\ x_k^{(l+1)} &= \omega T x_{k-1}^{(l+1)} + (1-\omega)x_k^{(l)} + \omega c \end{aligned} \tag{3.6}$$

By our assumption (3.4) converges and therefore it converges to the unique solution $\tilde{x}$ of (3.2). Hence, by Theorem 5, all the sequences $\{x_j^{(l)}\}$, $l = 0, 1, 2, \dots$, $j = 1(1)k$, from (3.6), will converge to the unique solution x of (3.1). So if we put $x^{(lk+j-1)} = x_j^{(l)}$ and $m = lk + j - 1$, $j = 1(1)k$, $l = 0, 1, 2, \dots$, relations (3.6) can be rewritten as

$$x^{(m)} = \omega T x^{(m-1)} + (1-\omega)x^{(m-k)} + \omega c, \quad m = k, k+1, \dots,$$

which is nothing but the monoparametric k-step iterative scheme (1.7). Therefore one application of the method (3.4) will be equivalent to k applications of the method (1.7). But for the latter method, Lemma 1, Theorem 2 and Corollary 3 hold. So they will for the SOR method (3.4). The only difference will be that the (optimum) a.c.f. of the SOR method (3.4) will be the kth power of the (optimum) a.c.f. of the method (1.7) or the (optimum) a.c.f. of (1.7) will be the kth root of the (optimum) a.c.f. of (3.4). Hence $\rho(\mathscr{L}_{\hat{\omega}_k}) = 1/\hat{\eta}^k = (k-1)(\hat{\omega}_k - 1)$.

Assume now that together with the system (1.1) we also consider the system

$$x' = T'x' + c', \tag{3.7}$$

with $T' \in \mathbb{C}^{n',n'}$, x', $c' \in \mathbb{C}^{n'}$ and where T' is a weakly cyclic matrix of index $k(\geqslant 2)$, the spectrum $\sigma(T')$ of which, for a given $\rho \in (0, 1)$ and for $\hat{C}_k$ and $\hat{R}_k$ of Theorem 2, satisfies $\sigma(T) \setminus \{0\} \subseteq \sigma(T') \subset \hat{R}_k$, $\sigma(T^k) \setminus \{0\} \equiv \sigma(T'^k) \setminus \{0\}$ and at least one of the common elements of $\sigma(T)$ and $\sigma(T')$ lies on $\hat{C}_k$. It is obvious that the (optimum) convergence results of the block SOR method associated with (3.7) will coincide with those of the block SOR method (3.4) associated with (3.1) and in turn with those of the method (1.7). Thus one can have the optimum result of [20] via another root. More specifically.

Theorem 6. *Let that T in* (3.1) *is a weakly cyclic matrix of index k ($\geqslant 2$) and that, for a given $\rho \in (0, 1)$, $\sigma(T) \subset \hat{R}_k$. Then Theorem* 2 *and Corollary* 3 *hold with $\rho(\mathscr{L}_{\hat{\omega}_k}) \leqslant (k-1)(\hat{\omega}_k - 1)$ and equality holds iff at least one element of $\sigma(T)$ lies on $\hat{C}_k$.*

Based on Theorem 6 the following theorem can be proved.

Theorem 7. *Let that $\{A^{(k)}\} \equiv \{I - T^{(k)}\}$ $k = 2, 3, 4, \ldots$ is a sequence of consistently ordered k-cyclic matrices with unit diagonal submatrices. Let also that for the sequence $\{T^{(k)}\}$, $k = 2, 3, 4, \ldots$ of the associated block Jacobi matrices and for all k and a given $\rho \in (0, 1)$, $\sigma(T^{(k)}) \subset \hat{R}_k$ and at least one element (k elements since $A^{(k)}$ is k-cyclic) of $\sigma(T^{(k)})$ lies on $\hat{C}_k$. Then the sequence of the spectral radii of the associated SOR method $\{\rho(\mathscr{L}_{\hat{\omega}_k})\}$, $k = 2, 3, 4, \ldots$ is a strictly decreasing one.*

Proof. From (2.2), by putting $y \equiv y(k) \equiv \rho(\mathscr{L}_{\hat{\omega}_k}) = (k-1)(\hat{\omega}_k - 1) \in (0, 1)$ we obtain

$$\rho = k y^{1/k} / (y + k - 1). \tag{3.8}$$

From (3.8), if we work in the same way as in Theorem 4, we take

$$\frac{1}{y}(1-y)(1-k)y' = y - 1 + \frac{(1-y)}{k} \ln y - \ln y \geqslant y - 1 - \frac{(y+1)}{2} \ln y \equiv \alpha. \tag{3.9}$$

The sign of α is that of β, where

$$\beta \equiv 2\alpha/(y+1) \equiv 2(y-1)/(y+1) - \ln y. \tag{3.10}$$

From (3.10) we have $\partial\beta/\partial y = -(y-1)^2/(y(y+1)^2) < 0$, so β decreases continuously with y, and for $y \to 0^+$, $\beta \to +\infty$, while for $y \to 1^-$, $\beta \to 0^+$. Therefore $\beta > 0$ for any $y \in (0, 1)$. So, from (3.10), $\alpha > 0$ and from (3.9), $y' < 0$, implying that y strictly decreases with k. □

Remark 1. In [20], Theorem 6 and Corollary 4 (a special case of which are Theorem 4.4 and its corollary in [21, pp. 112–113]) and the discussion which follows are of great value since they essentially give the behavior of the optimum asymptotic rate of convergence, $R(\mathscr{L}_{\hat{\omega}_k}) = -\ln \rho(\mathscr{L}_{\hat{\omega}_k})$, for the same $\rho \in (0, 1)$, as $k(\geqslant 2) \to +\infty$. However, these very useful results were obtained under the assumption that $\rho \to 1^-$. What makes Theorem 7 be quite different from the results in [20] is that it holds for any $\rho \in (0, 1)$.

Remark 2. From Theorem 7 the optimum relaxation factor $\hat{\omega}_k$ strictly decreases with $k(\geqslant 2)$ and is such that $\hat{\omega}_k \in (1, \min\{\hat{\omega}_{k-1}, k/(k-1)\})$, $k = 3, 4, 5, \ldots$. (The same results hold for the $\hat{\omega}_k$'s of Theorem 4.) In addition $\lim_{k\to\infty}\hat{\omega}_k = 1$ so that the limiting optimum SOR method is the Gauss–Seidel one. (The limiting optimum method (1.7) is the Jacobi method.)

4. $k/2$-step block iterative methods

Let us return to the solution of system (3.2), where (3.1) and (3.3) hold and assume that there exist $k \geqslant 3$ and $\rho \in (0, 1)$ such that $\sigma(T) \subset \hat{R}_k$ with at least one element of $\sigma(T)$ on $\hat{C}_k$ ($\hat{C}_k$ and $\hat{R}_k$ are those of Theorem 2). Assume also that k is an even integer and consider the matrix $\tilde{A} = \tilde{I} - \tilde{T}$ of (3.2) as a consistently ordered $k/2$-cyclic one with diagonal matrix $\tilde{D}$, which is the direct sum of the 2×2 block element $\left[\begin{smallmatrix} I & 0 \\ -T & I \end{smallmatrix}\right]$. The block Jacobi matrix is given by $\tilde{T} = \tilde{I} - \tilde{D}^{-1}\tilde{A}$ and simple calculations show that

$$\tilde{T}^{k/2} = \begin{bmatrix} 0 & T^{k-1} & 0 & 0 & \dots & 0 & 0 \\ 0 & T^k & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & T^{K-1} & \dots & 0 & 0 \\ 0 & 0 & 0 & T^k & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & & & \\ 0 & 0 & 0 & 0 & \dots & 0 & T^{k-1} \\ 0 & 0 & 0 & 0 & \dots & 0 & T^k \end{bmatrix}, \tag{4.1}$$

so $\tilde{T}^{k/2}$ has as eigenvalues the numbers λ_j^k (λ_j, $j = 1(1)n$ are the eigenvalues of T) with multiplicity $k/2$ each and the number zero with multiplicity $(k/2)n$. For the matrix $\tilde{A}$ we consider the splitting $\tilde{A} = \tilde{D} - \tilde{L} - \tilde{U}$, where $-\tilde{L}$ and $-\tilde{U}$ are the corresponding block strictly lower and strictly upper triangular parts of it. Then the associated SOR method will be

$$(\tilde{D} - \omega\tilde{L})\tilde{x}^{(l+1)} = \left[(1-\omega)\tilde{D} + \omega\tilde{U}\right]\tilde{x}^{(l)} + \omega\tilde{c}, \quad l = 0, 1, 2, \dots \tag{4.2}$$

If from (4.2) we compute the block vector components $x_j^{(l+1)}$, $j = 1(1)k$ of $\tilde{x}^{(l+1)}$ we have an algorithm analogous to (3.6). Since Theorem 5 holds, we follow the same reasoning as in the corresponding part of the analysis of Section 3 and we end up with the two-sweep scheme below

$$x^{(m)} = \omega T x^{(m-1)} + (1-\omega)x^{(m-k)} + \omega c, \tag{4.3a}$$

$$x^{(m+1)} = \omega T x^{(m)} + (1-\omega)x^{(m-k+1)} + (1-\omega)T(x^{(m)} - x^{(m-k)}) + \omega c, \tag{4.3b}$$

$$m = k,\ k+2,\ \dots$$

(4.3a) is exactly the same as the k-step scheme (1.7), while (4.3b) corresponds to a $(k+1)$-step scheme. In view of (4.3a), (4.3b) can be written as

$$x^{(m+1)} = \omega T^2 x^{(m-1)} + (1-\omega)x^{(m-k+1)} + \omega(I+T)c,$$

which, if it is combined with (4.3a), gives

$$\begin{bmatrix} x^{(m)} \\ x^{(m+1)} \end{bmatrix} = \omega \begin{bmatrix} 0 & T \\ 0 & T^2 \end{bmatrix} \begin{bmatrix} x^{(m-2)} \\ x^{(m-1)} \end{bmatrix} + (1-\omega) \begin{bmatrix} x^{(m-k)} \\ x^{(m-k+1)} \end{bmatrix} + \omega \begin{bmatrix} I & 0 \\ T & I \end{bmatrix} \begin{bmatrix} c \\ c \end{bmatrix},$$

$$m = k,\ k+2, \dots. \tag{4.4}$$

However, (4.4) is a $k/2$-step block iterative method and has the form (1.7) with iteration matrix $T' \equiv \left[\begin{smallmatrix} 0 & T \\ 0 & T^2 \end{smallmatrix}\right]$ and its (optimum) convergence results are known from the theory of Section 2. So are those of the two-sweep scheme (4.3) and of the SOR method (4.2), having in mind that $k/2$ applications of either (4.4) or (4.3) are equivalent to one of (4.2). Let that for given $k \geqslant 4$ even and $\rho' = \rho^2 \in (0, 1)$, $C'_{k/2}$ and $\hat{R}'_{k/2}$ are defined by Lemma 1 and Theorem 2 for (4.4). Since $\sigma(T') \setminus \{0\} \equiv \sigma(T^2) \setminus \{0\}$, for (optimum) convergence we must have $\sigma(T^2) \subset \hat{R}_k^2$, with $\hat{C}_k$ and $\hat{R}_k$ corresponding to the same $k \geqslant 4$ and $\rho \in (0, 1)$, and $\hat{R}_k^2$ defining the region consisting of the images of the squares of the points of $\hat{R}_k$. Since it can be proved that $\hat{R}'_{k/2} \subset \hat{R}_k^2$ and that $\hat{C}'_{k/2}$ and $\hat{C}_k^2$ (the boundary of $\hat{R}_k^2$) share only the points $\rho^2 e^{4\pi i j'/k}$, $j' = 0(1)k/2 - 1$ in order to be able to compare (4.4) (or (4.3)) with (4.2) instead of $\sigma(T) \subset \hat{R}_k$, we must assume that the stronger condition $\sigma(T^2) \subset \hat{R}'_{k/2}$ holds. This is not unrealistic because such cases arise in practice as for example in the case of Corollary 3. Under the assumption $\sigma(T^2) \subset \hat{R}'_{k/2}$ the equation corresponding to (2.2) and giving $\hat{\omega}'_k \equiv \hat{\omega}'_{k/2} \in (1, k/(k-2))$ will be

$$(\omega'\rho^2)^{k/2} = (k/2)^{k/2}(k/2 - 1)^{1-k/2}(\omega' - 1). \tag{4.5}$$

Also the average optimum a.c.f. for the sequence $\{x^{(m)}\}$, $m = k, k+1, k+2, \ldots$ in (4.4) (or (4.3)) will be $\rho_{\hat{\omega}'_k} = 1/\hat{\eta}'_k = ((k/2 - 1)(\hat{\omega}'_k - 1))^{1/k}$, while that of the block SOR method (4.2) will be $\rho(\mathscr{L}_{\hat{\omega}'_k}) = (k/2 - 1)(\hat{\omega}'_k - 1)$. For these to hold at least one element of $\sigma(T^2)$ must lie on $\hat{C}'_{k/2}$.

As in Section 3 the present optimum results for (4.2) will also hold for a block SOR method associated with the solution of a system $x' = T'x' + c'$, with Jacobi matrix T' weakly cyclic of index $k/2$ for which $\sigma(T^2) \subset \hat{R}'_{k/2}$. Thus a theorem, call it Theorem 8, which is not given here, can be stated.

Remark. If $k(\geqslant 3)$ is an odd integer, then we construct a $(k+1)/2$-step block iterative method (for convenience we will call it again $k/2$-step method), with the only difference being that the last block of $\tilde{D}$ will be the unit matrix I. This time $\tilde{T}^{(k+1)/2}$ looks like (4.1) except that its last block is simply T^k instead of being $\left[\begin{smallmatrix} 0 & T^{k-1} \\ 0 & T^k \end{smallmatrix}\right]$. A similar but more complicated analysis leads to the same conclusions, except that $\rho' = \rho^{2k/(k+1)} \in (0, 1)$ and in order to have (optimum) convergence there must hold $\sigma(T^2) \subset \hat{R}'^{(k+1)/k}_k$. We note that although $\hat{R}'^{(k+1)/k}_k$ shares with $\hat{R}_k^2$ the point ρ^2 it does not satisfy the restriction $\hat{R}'^{(k+1)/k}_k \subset \hat{R}_k^2$. It is pointed out that for every cycle of k simple iterations the two-sweep scheme (4.3) is applied $(k-1)/2$ times, while in the $(k+1)/2$st block iteration only (4.3a) is applied.

Having reached the above conclusions we can state and prove the following two theorems.

Theorem 9. *Let $k(\geqslant 4)$ be an even integer. If from the k-step iterative scheme* (1.7) *for the solution of the system* (3.1) *we construct the $k/2$-step block iterative scheme* (4.4) (*or* (4.3)) *and it so happens that for a given $\rho \in (0, 1)$, $\sigma(T^2) \subset \hat{R}'_{k/2}$, with at least the element of $\sigma(T^2)$ on $\hat{C}'_{k/2}$ ($\hat{C}'_{k/2}$, $\hat{R}'_{k/2}$ are defined from Theorem* 2 *for ρ^2 and $k/2$ in the places of ρ and k), then the average optimum a.c.f., $\rho_{\hat{\omega}'_k}$, of* (4.3) *is less than the optimum a.c.f., $\rho_{\hat{\omega}_k}$, of* (1.7).

Proof. From (2.2) and (4.5) we have that

$$\text{(a)} \quad \rho^2 = \frac{k^2}{\left(\rho_{\hat{\omega}_k}^k + k - 1\right)^2}\rho_{\hat{\omega}_k}^2 \quad \text{and} \quad \text{(b)} \quad \rho^2 = \frac{k}{2\rho_{\hat{\omega}'_k}^k + k - 2}\rho_{\hat{\omega}'_k}^2 \tag{4.6}$$

In order to determine which of $\rho_{\hat{\omega}_k}$ and $\rho_{\hat{\omega}'_k}$ is the largest of the two we work the other way round. We assume that $\rho_{\hat{\omega}_k} = \rho_{\hat{\omega}'_k} = y \in (0, 1)$ and examine which of ρ and ρ' of the left-hand sides of (4.6a) and (4.6b) is the largest. For this we form the function

$$f(y) = \rho^2/\rho'^2 = k(2y^k + k - 2)/(y^k + k - 1)^2.$$

Then, $f'(y) = 2k^2y^{k-1}(1 - y^k)/(y^k + k - 1)^3 > 0$ for every $y \in (0, 1)$. Consequently $f(y)$ strictly increases in (0, 1) and because $f'(1) = 0$ there exists a local maximum at $y = 1$, $f(1) = k(2 + k - 2)/(1 + k - 1)^2 = 1$. Therefore $f(y) = \rho^2/\rho' < 1$ in (0, 1) implying that $\rho < \rho'$. From this conclusion and the fact that the functions $\rho_{\hat{\omega}_k}$ and $\rho_{\hat{\omega}'_k}$, turning back to our original notation in (4.6), are strictly increasing ones w.r.t. ρ (as is easily checked) the theorem is proved. □

In case $k(\geqslant 3)$ is odd the equation corresponding to (4.5) giving $\hat{\omega}'_k \equiv \hat{\omega}'_{(k+1)/2} \in (1, (k+1)/(k-1))$ is

$$\left(\omega'\rho^{2k/(k+1)}\right)^{(k+1)/2} = \left(\frac{k+1}{2}\right)^{(k+1)/2}\left(\frac{k+1}{2} - 1\right)^{1-(k+1)/2}(\omega' - 1). \tag{4.7}$$

A theorem completely analogous to Theorem 9, where the obvious assumptions are omitted, can be stated and proved.

Theorem 10. *Let $k(\geqslant 3)$ be an odd integer. If from the k-step iterative scheme* (1.7) *for the solution of the system* (3.1) *we construct the $(k+1)/2$-step block iterative scheme* (4.3), *then the average optimum a.c.f.,* $\rho_{\hat{\omega}'_k}$, *of* (4.3) *is less than the optimum a.c.f.,* $\rho_{\hat{\omega}_k}$, *of* (1.7).

Proof. This time from (2.2) and (4.7) we have

$$\rho^{2k/(k+1)} = \frac{k^{2k/(k+1)}}{\left(\rho^k_{\hat{\omega}_k} + k - 1\right)^{2k/(k+1)}}\rho^{2k/(k+1)}_{\hat{\omega}_k}, \tag{4.8a}$$

$$\rho^{2k/(k+1)} = \frac{(k+1)}{\left(2\rho^k_{\hat{\omega}'_k} + k - 1\right)}\rho^{2k/(k+1)}_{\hat{\omega}'_k}. \tag{4.8b}$$

The proof goes on as in Theorem 9 and the conclusion follows. □

The following Lemmas 11 and 12, which can be proved in the same way as Theorem 9, help in the statement of Theorem 13, which gives the behavior of the optimum a.c.f. for a $k/2$-step block iterative scheme (4.3) as $k(\geqslant 3) \to +\infty$. It is assumed that $\sigma(T)$ passes all the criteria for which an optimum $k/2$-step block scheme for a given $\rho \in (0, 1)$ (and a given k) exists. For example for $k = 5$ we must have $\sigma(T^3) \subset \hat{R}'_3$, with $\hat{R}'_3$ being defined in Theorem 2, where instead of (2.2), (4.7) with $k = 5$ is used.

Lemma 11. *For $k(>3)$ even, the optimum a.c.f. of the $k/2$-step block iterative scheme, obtained from the $(k-1)$-step iterative scheme* (1.7), *is less than the optimum a.c.f. of the $k/2$-step block iterative scheme, which is obtained from the k-step iterative scheme* (1.7). (*Note*: A similar statement holds if k is odd).

Lemma 12. *For k (> 3) even, the optimum a.c.f. of $k/2$-step block iterative scheme, obtained from the k-step iterative scheme* (1.7), *is less than the optimum a.c.f. of* $(k/2+1)$*-step block iterative scheme obtained from the* $(k+1)$*-step iterative scheme* (1.7).(*Note*: A similar statement hold if k is odd.)

A theorem, call it Theorem 13, quite analogous to Theorem 4, suitably adjusted to the present $k/2$-step block iterative schemes, which is not to be given, can be stated and proved. The conclusion is the same, namely that the optimum a.c.f.'s for a given $\rho \in (0, 1)$ strictly increase with k. So the best out of all possible optimum schemes is the one corresponding to the smallest k.

5. Applications and numerical examples

(a) First we give two simple applications of the remark made at the end of Section 2.

(i) Let $\sigma(T) \subset [-\rho, 0]$ with $\rho \in (0, 1)$ and $-\rho, 0 \in \sigma(T)$. It is readily checked by Corollary 3 that for $k = 2, 4, 6, \ldots$ the optimum k-step methods (1.7) can be used and the best out of all them is, by virtue of Theorem 4, the one for which $k = 2$. This gives $\rho_{\hat{\omega}_2} = \rho/(1 + (1-\rho^2)^{1/2})$. On the other hand, for $k = 1$, the optimum extrapolated Jacobi method gives (see [15]) $\hat{\omega}_1 = 2/(2+\rho)$ and $\rho_{\hat{\omega}_1} = \rho/(2+\rho)$. It is readily checked that $\rho_{\hat{\omega}_1} < \rho_{\hat{\omega}_2}$, so that the best optimum scheme (1.7) is obtained for $k = 1$.

(ii) Let $\sigma(T) \subset R$, where R is the rectangle with vertices $\pm 0.6 \pm 1.2\mathrm{i}$, and $\pm 0.6 \pm 1.2\mathrm{i} \in \sigma(T)$. It is obvious that the theory of Section 2 cannot be applied since all regions $\hat{R}_k$ for $k = 2, 3, 4, \ldots$ and any $\rho \in (0, 1)$ will not contain the vertices of R which are elements of $\sigma(T)$. It is also obvious that for $k = 1$, because of the symmetry of R w.r.t. the origin, the optimum scheme (see [8]) corresponds to $\hat{\omega}_1 = 1$, which gives a divergent scheme. However, following [3], by applying

Table 1 ($k = 3$)

ρ	$\hat{\omega}_3$	$\hat{\omega}'_3$	$\rho_{\hat{\omega}_3}$	$\rho_{\hat{\omega}'_3}$
0.680711	1.05485	1.09445	0.478697	0.455416
0.790230	1.09634	1.16842	0.577572	0.552247
0.897083	1.17232	1.30948	0.701111	0.676414
0.977898	1.31511	1.59406	0.857363	0.840639
0.998978	1.45402	1.89512	0.968356	0.963741

Table 2 ($k = 4$)

ρ	$\hat{\omega}_4$	$\hat{\omega}'_4$	$\rho_{\hat{\omega}_4}$	$\rho_{\hat{\omega}'_4}$
0.680711	1.02500	1.06035	0.523294	0.495647
0.790230	1.05000	1.12293	0.622300	0.592127
0.897083	1.10000	1.25502	0.740100	0.710628
0.977898	1.20000	1.54746	0.880100	0.860179
0.998978	1.30000	1.87990	0.974000	0.968518

Table 3 ($k = 5$)

ρ	$\hat{\omega}_5$	$\hat{\omega}'_5$	$\rho_{\hat{\omega}_5}$	$\rho_{\hat{\omega}'_5}$
0.680711	1.01276	1.02319	0.551515	0.541107
0.790230	1.02914	1.05336	0.650608	0.639211
0.897083	1.06530	1.12137	0.764530	0.753402
0.977898	1.14278	1.27386	0.894019	0.886568
0.998978	1.22298	1.44136	0.977385	0.975361

the algorithm by Young and Eidson we can find an optimum scheme for $k = 2$, with $\hat{\omega}_2 = 0.59589$ and $\rho_{\hat{\omega}2} = 0.8813$ (see [22, Table 4.1, pr. 198–199]).

(b) Finally we give some numerical examples to compare the k-step and the $k/2$-step block iterative methods for $k = 3$, 4 and 5. The basic numerical example, corresponding to $k = 4$ and a 4-step method, was taken from [19]. Tables 1, 2 and 3 give a comparison of the k-step and $k/2$-step block methods as regards their optimum a.c.f.'s for a given $\rho \in (0, 1)$. It is assumed that $\sigma(T)$ satisfies the requirement for the corresponding optimum method to exist. From the Tables 1, 2 and 3 it is seen that the $k/2$-step block methods converge faster than the corresponding k-step ones. Thus, it is shown by numerical examples that the former methods are better than the latter ones. It is also noted that for the same ρ as k decreases the corresponding method becomes better, something which was expected from the theory developed in this paper.

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