Spectral equivalence and matrix algebra preconditioners for multilevel Toeplitz systems: a negative result

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ABSTRACT. In the last two decades a lot of matrix algebra optimal and superlinear preconditioners (those assuring a strong clustering at the unity) have been proposed for the solution of polynomially ill-conditioned Toeplitz linear systems. The corresponding generalizations to multilevel structures do not preserve optimality neither superlinearity (see e.g. [11]). Regarding the notion of superlinearity, it has been recently shown that this is simply impossible (see [15, 17, 18]). Here we propose some ideas and a proof technique for demonstrating that also the spectral equivalence and the essential spectral equivalence (up to a constant number of diverging eigenvalues) are impossible and therefore the search for optimal matrix algebra preconditioners in the multilevel setting cannot be successful.

1. Introduction

In the past few years a lot of attention has been paid to the solution of multilevel Toeplitz systems due to the great variety of applications in which these structures occur such as signal processing, image restoration, PDEs, time series (see e.g. [2]). If $f$ is a complex-valued function of $d$ variables, integrable on the $d$-cube $Q_d := (0, 2\pi)^d$, the symbol $\mathcal{F}_{Q_d}$ stands for $(2\pi)^{-d} \int_{Q_d} f(x) e^{-i \langle j, x \rangle} \, dx$, $i^2 = -1$, $j \in \mathbb{Z}^d$, are used for building up $d$-level Toeplitz matrices generated by $f$. More precisely, if $n = (n_1, \ldots, n_d)$ is a $d$-index with positive entries, then the symbol $T_n(f)$ denotes the $d$-level Toeplitz matrix of order $N(n)$ (throughout, we let $N(n) := \prod_{i=1}^d n_i$) constructed according to the rule

$$T_n(f) = \sum_{|j|<n} \mathcal{F}_{Q_d} f^{(j)} = \sum_{|j_1|<n_1} \cdots \sum_{|j_d|<n_d} \mathcal{F}_{Q_d} (j_1, \ldots, j_d) \otimes \cdots \otimes \mathcal{F}_{Q_d} (j_d).$$

In the above equation, $\otimes$ denotes tensor product, $J_n^{(j)}$ denotes the matrix of order $m$ whose $(i, j)$ entry equals $1$ if $j - i = l$ and equals zero otherwise, while $J_n^{(j)}$, where $j$ and $n$ are multiindices, is the tensor products of all $J_{n_i}^{(j_i)}$ for $i = 1, \ldots, d$. In other
words, the $2m - 1$ matrices $\{J_m^{(l)}\}, l = 0, \pm 1, \ldots, \pm(m - 1)$ are the canonical basis of the linear space of $m \times m$ Toeplitz matrices, and the tensor notation emphasizes the $d$-level Toeplitz structure of $T_n(f)$ and, indeed, the set $\{J_n^{(j)}\}$ is the canonical basis of the linear space of the $N(n) \times N(n)$ $d$-level Toeplitz matrices.

On the other hand, multilevel Toeplitz matrices are not only interesting from the point of view of the applications [2] or from a "pure mathematics" point of view [1, 21], but also from the viewpoint of the complexity theory since the costs of determining the vector $u = T_n(f)v$ for an arbitrary vector $v$ is of $O(N(n) \log N(n))$ arithmetic operations that is the cost of applying a constant number of multilevel Fast Trigonometric/Fourier transforms.

Now let us come back to the applications and let us recall that the main problem is to solve linear systems of the form $T_n(f)u = v$ for a given vector $v$ and for a given $L^1$ symbol $f$. Since the matrix vector multiplication can be performed efficiently, a simple but good idea is to solve the considered linear systems by using iterative solvers in which the involved matrices preserve a Toeplitz structure. Some possibilities are the following: conjugate gradient methods, Chebyshev iterations, Jacobi or Richardson methods with or without polynomial or matrix algebra preconditioning (see [7]). Under these assumptions, the total cost for computing $u$ within a preassigned accuracy $\epsilon$, is $O(k_n(\epsilon)N(n) \log N(n))$ where $k_n(\epsilon)$ is the required number of iterations. If $f$ is strictly positive and bounded or if the closed convex hull of the range of $f$ is bounded and does not contain the complex zero, then many of the cited iterations are optimal and we have $k_n(\epsilon) = O(1)$ [14]. The same is true in the case where $f$ is continuous, nonnegative, with a finite number of zeros of even orders, the number $d$ of levels equals 1 and we use a preconditioned conjugate gradient (PCG) method [3, 5, 4, 13, 10].

Here we want to consider the same case ($f$ nonnegative with a finite number of zeros) but in the multilevel setting i.e. $d > 1$. The reason of this attention relies on the importance of the considered case since the discretization of elliptic $d$-dimensional PDEs by Finite Differences on equispaced grids leads to sequences $\{T_n(p)\}$ where $p$ is positive except at $x = (0, \ldots, 0)$ and is a multivariate trigonometric polynomial. A similar situation occurs in the case of image restoration problems where the sequence $\{T_n(p)\}$ is associated to a polynomial $p$ which is positive everywhere with the exception of the point $x = (\pi, \pi)$.

Unfortunately, no optimal PCG methods are known in this case in the sense that the number of iterations $k_n(\epsilon)$ is a mildly diverging function of the dimensions $n$. In this paper we will show that the search for spectrally equivalent preconditioners cannot be successful in general and indeed we will illustrate a proof technique for obtaining such negative results.

2. Tools, definitions and main results

In the following we will restrict our attention to the simpler case where the generating function $f$ is nonnegative, bivariate ($d = 2$) and has isolated zeros so that the matrices $T_n(f)$ are positive definite and ill-conditioned. We will consider as case study two multilevel matrix algebras: the two-level $\tau$ algebra and the two-level circulant algebra.

The two level $\tau$ algebra is generated by the pair

$$\Theta_{n_1} \otimes I_{n_2}, \quad I_{n_1} \otimes \Theta_{n_2}$$
in the sense that each matrix of the algebra can be expressed as a bivariate polynomial in the variables $\Theta_{n_1} \otimes I_{n_2}$ and $I_{n_1} \otimes \Theta_{n_2}$; moreover every matrix of the algebra is simultaneously diagonalized by the orthogonal matrix $Q$ of size $N(n)$. Here $I_m$ denotes the $m$-sized identity matrix,

$$
\Theta_m = \begin{pmatrix}
0 & 1 & & \\
1 & 0 & \ddots & \\
\ddots & \ddots & \ddots & 1 \\
1 & 0 & & \\
\end{pmatrix}_m,
$$

and the columns of matrix $Q$ are given by $v_j^{(n_1)} \otimes v_k^{(n_2)}$ where

$$
v_s^{(m)} = \sqrt{\frac{2}{m+1}} \left( \sin \left( \frac{s j \pi}{m+1} \right) \right)_{j=1}^{j=m}.
$$

Similarly, the two level circulant algebra is generated (in the same sense as before) by the pair

$$
Z_{n_1} \otimes I_{n_2}, \ I_{n_1} \otimes Z_{n_2}
$$

and every matrix of the algebra is simultaneously diagonalized by the unitary Fourier matrix $F$ of size $N(n)$. Here

$$
Z_m = \begin{pmatrix}
0 & \cdots & 0 & 1 \\
1 & & 0 & \\
\ddots & \ddots & \ddots & \\
0 & 1 & 0 & \\
\end{pmatrix}_m,
$$

and the columns of matrix $F$ are given by $f_j^{(n_1)} \otimes f_k^{(n_2)}$ where

$$
f_s^{(m)} = \sqrt{\frac{1}{m}} \left( e^{i \frac{2(s-1)(j-1)\pi}{m}} \right)_{j=1}^{j=m}.
$$

The strong relationships between these algebras and Toeplitz structures, emphasized by the fact that the generators are of Toeplitz type, have been deeply studied. Here we mention a few examples: given $p$ bivariate complex polynomial we have

$$
T_n(p) = C_n(p) + \hat{T}_n(p)
$$

where $C_n(p)$ is the two level circulant matrix whose eigenvalues are

$$
p_{s,t}^C = p \left( \frac{2(s-1)\pi}{n_1}, \frac{2(t-1)\pi}{n_2} \right)
$$

and where $\hat{T}_n(p)$ is two level Toeplitz matrix of rank proportional to $n_1 + n_2$.

For the case of two level $\tau$ matrices we have to restrict the attention to real valued even polynomials $p$. In that case, we have

$$
T_n(p) = \tau_n(p) + H_n(p)
$$

where $\tau_n(p)$ is the two level $\tau$ matrix whose eigenvalues are

$$
p_{s,t}^\tau = p \left( \frac{s\pi}{n_1 + 1}, \frac{t\pi}{n_2 + 1} \right)
$$

and where $H_n(p)$ is two level Hankel matrix of rank proportional to $n_1 + n_2$. To make these statements clear we give some examples that will be also useful in the following.
EXAMPLE 2.1. Let \( p_k(x, y) = (2 - 2 \cos(x))^k + (2 - 2 \cos(y))^k \) with \( k \geq 1 \). If \( k = 1 \) then \( T_n(p_1) = \tau_n(p_1) \) while \( T_n(p_1) = C_n(p_1) + \tilde{T}_n(p_1) \) where
\[
(2) \quad \tilde{T}_n(p_1) = (J_{n_1}^{(1-n_1)} + J_{n_1}^{(n_1-1)}) \otimes I_{n_2} + I_{n_1} \otimes (J_{n_2}^{(1-n_2)} + J_{n_2}^{(n_2-1)}).
\]
Moreover if \( k = 2 \) then \( T_n(p_2) = \tau_n(p_2) + H_n(p_2) \) where
\[
(3) \quad H_n(p_2) = (E_{1,1}^{(n_1)} + E_{n_1,n_1}^{(n_1)}) \otimes I_{n_2} + I_{n_1} \otimes (E_{1,1}^{(n_2)} + E_{n_2,n_2}^{(n_2)})
\]
with \( E_{j,k}^{(m)} \) being the \( m \) sized matrix having zero entries except for \( (E_{j,k}^{(m)})_{j,k} = 1 \).

A tool for proving that a PCG method is optimal when the coefficient matrix is \( A_n \) and the preconditioner is \( P_n \) is the spectral equivalence and the essential spectral equivalence.

DEFINITION 2.1. Given \( \{A_n\}_n \) and \( \{P_n\}_n \) two sequences of positive definite matrices of increasing size \( d_n \), we say that they are spectrally equivalent iff all the eigenvalues of \( \{P_n^{-1}A_n\}_n \) belong to a positive interval \([\alpha, \beta]\) independent of \( n \) with \( 0 < \alpha \leq \beta < \infty \). We say that the sequences \( \{A_n\}_n \) and \( \{P_n\}_n \) are essentially spectrally equivalent iff there is at most a constant number of outliers and they are all bigger than \( \beta \).

In practice, in terms of Reileigh quotients, the spectral equivalence means that for any nonzero \( v \in C^{d_n} \) we have
\[
\alpha \leq \frac{v^H A_n v}{v^H P_n v} \leq \beta
\]
while the essential spectral equivalence is equivalent to the following two conditions: for any nonzero \( v \in C^{d_n} \) we have
\[
\alpha \leq \frac{v^H A_n v}{v^H P_n v}
\]
and for any \( \theta_n \) going to infinity, for any subspace \( V \) of dimension \( \theta_n \) we have
\[
(4) \quad \min_{v \in V, v \neq 0} \frac{v^H A_n v}{v^H P_n v} \leq \beta.
\]

In other words, we can say that there exists a constant positive integer \( q \) independent of \( n \) such that, calling \( \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{d_n} \) the eigenvalues of \( P_n^{-1}A_n \), we have \( \lambda_{q+1} \leq \beta \) and possibly the first \( q \) eigenvalues diverging to infinity as \( n \) tends to infinity. In view of the min max characterization
\[
\lambda_{q+1} = \max_{\dim V = q+1} \min_{v \in V, v \neq 0} \frac{v^H A_n v}{v^H P_n v}
\]
it follows that for any subspace \( V \) of dimension \( q + 1 \) we must have (4).

2.1. Negative results: the \( \tau \) case. We say that two nonnegative functions \( f \) and \( g \) are equivalent if \( cg(x) \leq f(x) \leq Cg(x) \) for some positive constants \( c \) and \( C \) independent of \( x \) and for any \( x \) belonging to the definition domain. We now state the main negative conjecture.
**Conjecture 2.1.** Let \( f \) be equivalent to \( p_k(x, y) = (2 - 2\cos(x))^k + (2 - 2\cos(y))^k \) with \( k \geq 2 \) and let \( \alpha \) be a fixed positive number independent of \( n \). Then for any sequence \( \{P_n\} \) with \( P_n \in \tau_n \) and such that

\[
\lambda_{\min}(P_n^{-1}T_n(f)) \geq \alpha
\]

uniformly with respect to \( n \), we have

(a) : the maximal eigenvalue of \( P_n^{-1}T_n(f) \) diverges to infinity (in other words \( \{T_n(f)\} \) does not possess spectrally equivalent preconditioners in the \( \tau_n \) algebra);

(b) : if \( \Sigma(X) \) denotes the complete set of the eigenvalues of \( X \), then the number

\[
\#\{\lambda(n) \in \Sigma(P_n^{-1}T_n(f)) : \lambda(n) \to \infty \}
\]

tends to infinity as \( n \) tends to infinity (in other words \( \{T_n(f)\} \) does not possess essentially spectrally equivalent preconditioners in the \( \tau_n \) algebra).

Since the spectral equivalence and the essential spectral equivalence are equivalence relationships, it is evident that the former conjecture holds in general if it is proved for \( f = p_k \) thus reducing the analysis to a multilevel banded case. In order to show the idea, we prove this negative result in the simplest case where \( k = 2 \).

**Proof of Conjecture 2.1:** items (a) and (b) in the specific case where \( k = 2 \).

Let \( \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{N(n)} \) be the eigenvalues of \( P_n^{-1}T_n(p_2) \) with \( P_n \) being a two level \( \tau \) matrix. For the sake of notational simplicity we assume that \( n_1 \sim n_2 \sim m \) i.e. the partial dimensions are proportional. We will prove (a) and (b) with the same argument. By contradiction we suppose that \( \{P_n\} \) and \( \{T_n(p_2)\} \) are essentially spectrally equivalent that is there exist positive constants \( q, \alpha \) and \( \beta \) independent of \( n \) such that

\[ \lambda_{N(n)} \geq \alpha \text{ and } \lambda_{q+1} \leq \beta. \]

Therefore, from the relation \( \lambda_{N(n)} \geq \alpha \) it follows that \( T_n(p_2) \geq \alpha P_n \) where the relation is in the sense of the partial ordering between Hermitian matrices. On the other hand, from wellknown results on the asymptotic spectra of Toeplitz matrices (see [21] and references therein), we infer that \( T_n(f), n_1 \sim n_2 \sim m \), has \( g(m) \) eigenvalues \( \lambda(m) \) going to zero asymptotically faster than \( m^{-3} \) i.e. \( \lambda(m) = o(m^{-3}) \) where \( g(m) \to \infty \) with the constraint that \( g(m) = o(m^{1/4}) \). Consequently, since \( P_n \leq \frac{1}{\alpha} T_n(f) \) it follows that also \( P_n \) has \( g(m) \) eigenvalues \( \lambda(m) \) going to zero asymptotically faster than \( m^{-3} \).

In addition, if \( P_n^{-1}T_n(p_2) \) has at most \( q \) eigenvalues bigger than \( \beta \), since \( T_n(p_2) \geq \tau_n(p_2) \) (see relation (3)), it follows that \( P_n^{-1}\tau_n(p_2) \) has at most \( q \) outliers and then calling \( \lambda_{s,t} \) the eigenvalue of \( P_n \) related to the eigenvector \( v_s^{(n_1)} \otimes v_t^{(n_2)} \), it follows that

\[
\lambda_{s,t} \geq \frac{1}{\beta} p_{s,t}^r
\]

with \( p_{s,t}^r = p\left(\frac{\pi}{n_1+1}, \frac{t\pi}{n_2+1}\right) \) (see the beginning of this section) and with at most the exception of \( q \) indices. Therefore the eigenvalues of \( P_n \) that are \( o(m^{-3}) \) impose the relation \( o(m^{-3}) = \beta \lambda_{s,t} \geq p_{s,t}^r \) and finally the pairs \((s,t)\) must be such that
\[(s/m)^4 + (t/m)^4 = o(m^{-3})\] i.e.

\[s^4 + t^4 = o(m) .\]

This means that the subspace \(W_n\) spanned by the \(\tau\) eigenvectors related to \(o(m^{-3})\) eigenvalues of \(P_n\) has to be contained in

\[\text{span} \left( v_{s}^{(n_1)} \otimes v_{t}^{(n_2)} : s^4 + t^4 = o(m) \right).\]

Define now the following set of indices \(T_q = \{(s, t) : i = 0, \ldots, q, t_0 < t_1 \cdots < t_q \} \subset \{1, \ldots, n_1\} \times \{1, \ldots, n_2\}\), \(\hat{T}_q = \{(s, t) : i = 0, \ldots, q, s_0 < s_1 \cdots < s_q \} \subset \{1, \ldots, n_1\} \times \{1, \ldots, n_2\}\) and the corresponding subspaces of dimension \(q + 1\):

\[V[T_q] = \text{span} \left( v_{s}^{(n_1)} \otimes v_{t}^{(n_2)} : (s, t) \in T_q \right),\]

\(T_q = \hat{T}_q\) either \(T_q = \hat{T}_q\).

Now we look for the contradiction. By using relation (3) we infer the following chain of inequalities

\[
\beta \geq \lambda_{q+1} = \max_{\text{dim}V=q+1} \min_{v \in V, v \neq 0} \frac{v^HT_n(p_2)v}{v^HP_nv} = \max_{\text{dim}V=q+1} \min_{v \in V, v \neq 0} \frac{v^H(\tau_n(p_2) + H_n(p_2))v}{v^HP_nv} \\
\geq \max_{\text{dim}V=q+1} \min_{v \in V, v \neq 0} \frac{v^HN_n(p_2)v}{v^HP_nv} = \max_{\text{dim}V=q+1} \min_{v \in V, v \neq 0} \frac{v^HE_{n_1,n_1} + E_{n_1,n_1} \otimes I_{n_2}v}{v^HP_nv} \\
= \max_{\text{dim}V=q+1} \min_{v \in V, v \neq 0} \frac{v^H \text{diag}(I_{n_2}, 0, \ldots, 0)v}{v^HP_nv}.
\]

Moreover, from similar arguments, we also have

\[
\beta \geq \lambda_{q+1} = \max_{\text{dim}V=q+1} \min_{v \in V, v \neq 0} \frac{v^HT_n(p_2)v}{v^HP_nv} = \max_{\text{dim}V=q+1} \min_{v \in V, v \neq 0} \frac{v^HI_{n_1} \otimes E_{1,1}^{(n_2)}v}{v^HP_nv} \\
= \max_{\text{dim}V=q+1} \min_{v \in V, v \neq 0} \frac{v^H \text{diag}(E_{1,1}^{(n_2)}, E_{1,1}^{(n_2)}, \ldots, E_{1,1}^{(n_2)})v}{v^HP_nv},
\]

As a consequence, setting \(V = V[T_q]\) either \(V = V[\hat{T}_q]\) with the constraint that \(V\) is contained in the subspace \(W_n\) where \(\lambda_{s,t} = o(m^{-3})\), we obtain

\[
\beta \geq \frac{2}{n_1 + 1} \sin^2 \left( \frac{\pi}{n_1 + 1} \right) \left( \max_{(s,t) \in T_q} \lambda_{s,t} \right)^{-1}
\]
either

\[
\beta \geq \frac{2}{n_2 + 1} \sin^2 \left( \frac{\pi}{n_2 + 1} \right) \left( \max_{(s,t) \in \hat{T}_q} \lambda_{s,t} \right)^{-1}
\]
Therefore, since \( \max_{(s, t) \in \mathcal{T}_g} \lambda_{s, t} = o(m^{-3}) \) either \( \max_{(s, t) \in \mathcal{T}_g} \lambda_{s, t} = o(m^{-3}) \) and \( \frac{2}{m^2 + 1} \sin^2 \left( \frac{\pi}{m^2 + 1} \right) \geq \frac{c}{m^3} \) for some positive \( c \) independent of \( m \), we deduce that
\[
\beta \cdot o(m^{-3}) \geq \frac{c}{m^3}
\]
which is a contradiction. \( \bullet \)

### 2.2. Negative results: the circulant case.

We begin with the main negative conjecture.

**Conjecture 2.2.** Let \( f \) be equivalent to \( p_k(x, y) = (2 - 2 \cos(x))^k + (2 - 2 \cos(y))^k \) with \( k \geq 1 \) and let \( \alpha \) be a fixed positive number independent of \( n \). Then for any sequence \( \{P_n\} \) with \( P_n \) two level circulant and such that
\[
\lambda_{\min}(P_n^{-1}T_n(f)) \geq \alpha \tag{7}
\]
uniformly with respect to \( n \), we have

(a) : the maximal eigenvalue of \( P_n^{-1}T_n(f) \) diverges to infinity (in other words \( \{T_n(f)\} \) does not possess spectrally equivalent preconditioners in the two level circulant algebra);

(b) : if \( \Sigma(X) \) denotes the complete set of the eigenvalues of \( X \), then the number
\[
\#\{\lambda(n) \in \Sigma(P_n^{-1}T_n(f)) : \lambda(n) \rightarrow n \rightarrow \infty \}
\]
tends to infinity as \( n \) tends to infinity (in other words \( \{T_n(f)\} \) does not possess essentially spectrally equivalent preconditioners in the two level circulant algebra).

As observed for Conjecture 2.1, we can reduce the analysis to \( f = p_k \).

#### Proof of Conjecture 2.2: item (a) in the specific case where \( k = 1 \).

Let \( \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{N(n)} \) be the eigenvalues of \( P_n^{-1}T_n(p_1) \) with \( P_n \) being a two level circulant matrix. For the sake of notational simplicity we assume that \( n_1 \sim n_2 \sim m \) i.e. the partial dimensions are proportional. By contradiction we suppose that \( \{P_n\} \) and \( \{T_n(p_1)\} \) are spectrally equivalent that is there exists positive constants \( \alpha \) and \( \beta \) independent of \( n \) such that
\[
\lambda_{N(n)} \geq \alpha \quad \text{and} \quad \lambda_1 \leq \beta.
\]
We consider the eigenvectors of the two level circulant algebra \( f_s^{(n_1)} \otimes f_t^{(n_2)} \), \( (s, t) \in \{1, \ldots, n_1\} \times \{1, \ldots, n_2\} \) and by making use of the relation (2) we look for a contradiction. Setting \( v = f_s^{(n_1)} \otimes f_t^{(n_2)} \) and calling \( \lambda_{s, t} \) the corresponding eigenvalue of \( P_n \), we have
\[
\beta \geq \lambda_1 \geq \frac{v^H T_n(p_1) v}{v^H P_n v} = \frac{v^H C_n(p_1) v + v^H \tilde{T}_n(p_1) v}{v^H P_n v} = \frac{\tilde{p}_{s, t} + v^H \tilde{T}_n(p_1) v}{\lambda_{s, t}},
\]
where
\[ \tilde{T}_n(p_1) = (J_{n_1}^{(1-n_1)} + J_{n_1}^{(n_1-1)}) \otimes I_{n_2} + I_{n_1} \otimes (J_{n_2}^{(1-n_2)} + J_{n_2}^{(n_2-1)}) \]
and therefore
\[ v^H \tilde{T}_n(p_1)v = \frac{1}{n_1} 2 \cos \left( \frac{2\pi (s-1)}{n_1} \right) + \frac{1}{n_2} 2 \cos \left( \frac{2\pi (t-1)}{n_2} \right). \]
Consequently we have
\[ \lambda_{s,t} \geq \frac{1}{\beta} \left( p_{s,t}^c + \frac{1}{n_1} 2 \cos \left( \frac{2\pi (s-1)}{n_1} \right) + \frac{1}{n_2} 2 \cos \left( \frac{2\pi (t-1)}{n_2} \right) \right) \]
and finally, by using simple asymptotical expansions and the explicit expression of \( p_{s,t}^c \), and by recalling that \( n_1 \sim n_2 \sim m \), we deduce that there exists an absolute constant \( c \) independent of \( m \) such that
\[ \lambda_{s,t} \geq \frac{c}{\beta m} \]
On the other hand, from the explicit knowledge of the eigenvalues of \( T_n(p_1) \) and by recalling that \( n_1 \sim n_2 \sim m \), we infer that \( T_n(f) \) has \( g(m) \) eigenvalues \( \lambda(m) \) going to zero asymptotically faster than \( m^{-1} \) i.e. \( \lambda(m) = o(m^{-1}) \) where \( g(m) \to \infty \) with the constraint that \( g(m) = o(m^{1/2}) \). Finally we deduced a contradiction since at least \( g(m) \) eigenvalues of the preconditioned matrix collapse to zero as \( m \) tends to infinity and this cannot happen under the assumption (7).

Remark 2.1. Both the proofs given here for the two level case work unchanged in an arbitrary number \( d \geq 2 \) of levels with \( k = 1 \) (circulant class) and \( k = 2 \) (\( \tau \) algebra). The extension of the proof of Conjecture 2.2, item (a) to the case of a general \( k > 1 \) and with a different position of the zero seems to be easier than the corresponding generalization of Conjecture 2.1 since in the latter we essentially use the nonnegative definiteness of the low rank correction \( H_n(P_k) \) while in the case of a generic \( k \) the correction \( H_n(P_k) \) is no longer definite and indeed has positive and negative eigenvalues. This essentially means that the proof of item (b) can be a difficult task in general while the generalization of the proof of item (a) should be more straightforward having in mind the scheme of the proof in Conjecture 2.2 with \( k = 1 \). However for both the cases there is a philosophical reason for which these negative results should be generalizable: indeed we have proved these negative results for the easiest cases where the degree of ill-conditioning is the lowest possible among polynomial generating functions. Therefore we expect that a worsening of the condition number should lead to an even more negative result.

3. Conclusions

From this note and from [15, 18], the following message can be read: the \( d \) level case with \( d \geq 2 \) is dramatically different from the scalar Toeplitz case in terms of preconditioning using fast transforms algebras. More precisely in the \( d \) level case with \( d \geq 2 \) it is impossible (except for rare exceptions) to find superlinear and/or (essentially) spectrally equivalent preconditioners. Therefore the only techniques for which the optimality can be proved are those based on multilevel band Toeplitz preconditioning and approximation theory techniques [12, 8, 16] (see also [9] for a new preconditioning scheme). However it should be mentioned that the problem of solving optimally those multilevel banded systems is still a difficult problem that
has been solved (both theoretically and practically) only in some cases with the help of a multigrid strategy [6, 20, 19]. In conclusion, the positive message of this note is the invitation for researchers working in this area to pay more attention to the optimality analysis of multigrid strategies for multilevel banded Toeplitz structures.

References

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