Εβδομάδα 3η / Ασκήσεις / 25. 11. 11 3A/1

Notiztitel 25.11.2011

$\boxed{E1/1}$ $A = \left\{ x \in \mathbb{R} : \left| \frac{2-x}{x} \right| < \left| \frac{1+x}{1-x} \right| \right\}$

Να βρεθούν τα sup A και inf A

Λύση: $x \neq 0, 1$, $A = \{ x \in \mathbb{R} : |(2-x)(1-x)| < |x(1+x)| \}$

$(2-x)(1-x) \geq 0 \iff (x-2)(x-1) > 0 \underset{x \neq 1}{\iff} x \geq 2 \vee x < 1$

$(2-x)(1-x) < 0 \iff x \in (1, 2)$

$x(1+x) \geq 0 \underset{x \neq 0}{\iff} x > 0 \vee x \leq -1$

$x(1+x) < 0 \iff x \in (-1, 0)$

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$$\Rightarrow A = \{ x \le -1 : (2-x)(1-x) < x(1+x) \}$$

$$\cup \{ x \in (-1,0) : (2-x)(1-x) < -x(1+x) \}$$

$$\cup \{ x \in (0,1) : (2-x)(1-x) < x(1+x) \}$$

$$\cup \{ x \in (1,2) : -(2-x)(1-x) < x(1+x) \}$$

$$\cup \{ x \geq 2 : (2-x)(1-x) < x(1+x) \}$$

$$\Rightarrow A = \emptyset \cup \emptyset \cup \left(\tfrac{1}{2}, 1\right) \cup (1,2) \cup [2,\infty)$$

(1),(2),(3)

$$= \left(\tfrac{1}{2}, 1\right) \cup (1, \infty)$$

$\Rightarrow \inf A = \frac{1}{2}$ και $\sup A \notin \mathbb{R}$ (δεν υπάρχει ή $\sup A$ "=" ∞)

3A/3

(1) $(2-x)(1-x) < x(1+x) \iff 2-3x+x^2 < x+x^2$

$\iff x > \frac{1}{2}$

(2) $(2-x)(1-x) < -x(1+x) \iff 2-3x+x^2 < -x-x^2$

$\iff 2-2x < -2x^2$

$\iff 1-x < -x^2$

$\iff x^2-x+1 < 0$

$\iff \left(x-\frac{1}{2}\right)^2 + \frac{3}{4} < 0$

(3) $-(2-x)(1-x) < x(1+x) \iff (2-x)(1-x) > -x(1+x)$

$\iff \left(x-\frac{1}{2}\right)^2 + \frac{3}{4} > 0$

Εναλλακτικά:

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$$\left|\frac{2-x}{x}\right| < \left|\frac{1+x}{1-x}\right| \underset{x\neq 1}{\iff} |(2-x)(1-x)| < |x(1+x)|$$

$$\iff (2-x)^2(1-x)^2 < x^2(1+x)^2$$

$$\iff (4-4x+x^2)(1-2x+x^2) < x^2(1+x^2+2x)$$

$$\iff 4-4x+x^2-8x+8x^2-2x^3+4x^2-4x^3+\cancel{x^4} < x^2+2x^3+\cancel{x^4}$$

$$\iff 4-12x+12x^2-8x^3<0$$

$$\iff 2x^3-3x^2+3x-1>0$$

$$\iff (2x-1)x^2-(2x-1)x+(2x-1)=(2x-1)\underbrace{(x^2-x+1)}_{=\left(x-\frac{1}{2}\right)^2+\frac{3}{4}}>0$$

$$\iff (2x-1)\left(\left(x-\tfrac{1}{2}\right)^2+\frac{3}{4}\right)>0$$

$$\iff 2x-1>0 \underset{x\neq 1}{\implies} A=\left(\tfrac{1}{2},1\right)\cup(1,\infty) \implies \inf A=\tfrac{1}{2},$$

$$\sup A \text{ "="} \infty$$

$\boxed{E1/2}$ $A \subseteq [m, \infty)$, $\boxed{m > 0}$ [Για $m=0$ η β) δεν ισχύει για $A \subseteq (0,\infty)$]

$$m = \inf A \iff \begin{cases} \alpha) & m \leq x \quad \forall x \in A \\ \beta) & \forall \lambda > 1 \quad \exists\, x \in A : x < \lambda m \end{cases}$$

"$\Rightarrow$": Αφού $m = \inf A$ το m είναι κάτω φράγμα $\iff$ α)

και (βλ. Θ. [1.11]) $\forall \varepsilon > 0 \quad \exists\, x \in A : \quad x < m + \varepsilon = m\underbrace{\left(1 + \frac{\varepsilon}{m}\right)}_{=:\,\lambda > 1}$

άρα $\forall \lambda > 1 \quad \exists \left(\varepsilon > 0 : \underbrace{\lambda = 1 + \frac{\varepsilon}{m}}_{\iff \varepsilon = m(\lambda - 1)} \text{ και}\right) x \in A : \quad x < m\lambda$

"$\Leftarrow$": α) $\iff$ m κάτω φράγμα

Έστω τυχαίο $\varepsilon > 0$. Τότε $\lambda = 1 + \frac{\varepsilon}{m} > 1$

και $\exists\, x \in A : x < \lambda m = m + \varepsilon$. Αφού αυτό ισχύει $\forall \varepsilon > 0$

ακολουθεί ότι $m = \inf A$ από το Θ. [1.11]

$\boxed{E1/3}$ A μη κενό και φραγμένο $\Rightarrow$ $\exists \sup A, \inf A \in \mathbb{R}$

$\forall \alpha, \beta \in A : |\alpha - \beta| < 1 \quad (1) \Rightarrow \sup A - \inf A \leq 1$

Απόδειξη:

$$A - A = \{ x \in \mathbb{R} : x = \alpha - \beta, \ \alpha, \beta \in A \} \quad (2)$$

$$(1) \overset{(2)}{\Longleftrightarrow} \forall x \in A - A : |x| < 1 \Leftrightarrow x \in (-1, 1)$$

$$\Leftrightarrow A - A \subseteq (-1, 1)$$

$$\Rightarrow \underbrace{\sup (A - A)}_{= \sup A - \inf A} \leq 1$$

βλ. Άσκηση $\boxed{6}$ στις σημειώσεις 1Α

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$\boxed{1}$ $$\alpha_\nu = \frac{1}{\nu}\Big(\underbrace{1+\frac{1}{2}+\frac{1}{3}+\dots+\frac{1}{\nu}}_{=\sum_{k=1}^{\nu}\frac{1}{k} =: A}\Big), \quad \nu\in\mathbb{N}$$

είναι γνησίως φθίνουσα

<u>Λύση</u>:

$$\alpha_{\nu+1}-\alpha_\nu = \frac{1}{\nu+1}\left(A+\frac{1}{\nu+1}\right) - \frac{1}{\nu}A = \frac{1}{(\nu+1)^2} + A\left(\frac{1}{\nu+1}-\frac{1}{\nu}\right)$$

$$= \frac{1}{(\nu+1)^2} - A\frac{1}{\nu(\nu+1)} = \frac{1}{\nu+1}\left(\frac{1}{\nu+1}-\frac{1}{\nu}A\right) = \frac{1}{\nu+1}\,\frac{\nu-A(\nu+1)}{(\nu+1)\nu}$$

και $\nu-(\nu+1)A = \nu-(\nu+1)-(\nu+1)(A-1) = -1-(\nu+1)(A-1)$

$$\Rightarrow \alpha_{\nu+1}-\alpha_\nu < 0.$$

$\boxed{2}$ $\alpha_\nu = \sqrt[\nu]{x^\nu + y^\nu}$ με $0 < x < y$, $\nu \in \mathbb{N}$ 3A/8

$\Rightarrow$ $\alpha_\nu \to y$

<u>Λύση</u>: $0 < x < y \Rightarrow 0 < x^\nu < y^\nu \Rightarrow y^\nu < x^\nu + y^\nu < 2y^\nu$

Αξ. Διατ.

$\Rightarrow \underbrace{y}_{=\beta_\nu} < \alpha_\nu < \underbrace{\sqrt[\nu]{2}\, y}_{=\gamma_\nu = \sqrt[\nu]{2}\beta_\nu}$ και $\lim \beta_\nu = y$,

$\lim \gamma_\nu \underset{\text{άλγεβρα ορίων}}{=} \underbrace{\lim \sqrt[\nu]{2}}_{=1} \underbrace{\lim \beta_\nu}_{=y}$

$\underset{\text{Θεώρημα Ισοσυγκλ.}}{\Rightarrow}$ $\lim \alpha_\nu = y$.

$\boxed{3}$ ανισότητα Bernoulli :

$$\boxed{(1+x)^{\nu} \geq 1+\nu x \quad \forall \nu \in \mathbb{N}\cup\{0\} \quad \forall x \geq -1}$$

Απόδειξη : $[\nu = 0 : 1 \geq 1]$ $\nu = 1 : \ 1+x \geq 1+x$

$\nu \to \nu+1$:
$$(1+x)^{\nu+1} = (1+x)^{\nu}(1+x) \geq (1+\nu x)(1+x)$$
$$= 1+\nu x + x + \nu x^2$$
$$\geq 1+(\nu+1)x$$

$\boxed{4}$ $|\alpha_\nu - \ell| \leq K|\beta_\nu| \quad \forall \nu \in \mathbb{N}$, $\beta_\nu \to 0 \Rightarrow \alpha_\nu \to \ell$;

$\forall \varepsilon > 0 \ \exists \nu_0 \in \mathbb{N} \ \forall \nu > \nu_0 : |\beta_\nu - 0| < \varepsilon \Rightarrow |\alpha_\nu - \ell| \leq K\varepsilon$, δηλ.

έστω τυχαίο $\varepsilon > 0$, τότε για $\varepsilon' = \frac{\varepsilon}{K}$ $\exists \nu_0 \in \mathbb{N} \ \forall \nu > \nu_0 \ |\alpha_\nu - \ell| \leq K\varepsilon' = \varepsilon$

$\boxed{5}$ Υπολογίστε τα όρια των ακολουθιών

$$\frac{7^\nu+2^\nu}{4\cdot 7^\nu+5^\nu}, \quad \frac{\nu \sin(\nu!)}{\nu^2+1}, \quad \sqrt[\nu]{2+\frac{1}{\nu}}$$

$\nu!$ ["ν παραγοντικό"] $:= \nu(\nu-1)\cdots 1$

$$\frac{7^\nu+2^\nu}{4\cdot 7^\nu+5^\nu} = \frac{\underbrace{1^\nu}_{=1} + \underbrace{\left(\frac{2}{7}\right)^\nu}_{\to 0}}{4+\underbrace{\left(\frac{5}{7}\right)^\nu}_{\to 0}} = \frac{1}{4}$$

$$\boxed{|\alpha|<1 \Rightarrow \alpha^\nu \to 0}$$

$$0 < \left|\frac{\nu \sin(\nu!)}{\nu^2+1}\right| \leq \frac{\nu}{\nu^2+1} < \frac{\nu}{\nu^2} = \frac{1}{\nu}$$

$\Rightarrow$ θ. Ισοσυγκλ. $\left|\frac{\nu \sin(\nu!)}{\nu^2+1}\right| \to 0 \iff \frac{\nu \sin(\nu!)}{\nu^2+1} \to 0$

(α_ν) <u>μηδενική ακολουθία</u> $:\Longleftrightarrow \alpha_\nu \to 0 \Longleftrightarrow |\alpha_\nu| \to 0$

$$\alpha_\nu \to 0 \Longleftrightarrow \forall \varepsilon>0\ \exists \nu_0 \in \mathbb{N}\ \forall \nu>\nu_0\ \underbrace{|\alpha_\nu|}_{=||\alpha_\nu||} \to 0 \Longleftrightarrow |\alpha_\nu| \to 0$$

$$\alpha_\nu = x \in \mathbb{R} \Rightarrow \alpha_\nu \to x \Longleftrightarrow \forall \varepsilon>0\ \exists \nu_0 \in \mathbb{N}\ \forall \nu>\nu_0\ \underbrace{|\alpha_\nu - x|}_{\substack{=0 \\ \text{για } \alpha_\nu = x}} < \varepsilon$$

$$\sqrt[\nu]{2+\frac{1}{\nu}} \in \Big(\underbrace{\sqrt[\nu]{2}}_{\to 1}, \underbrace{\sqrt[\nu]{3}}_{\to 1}\Big) \underset{\text{Θ. Ισοσυγκλ.}}{\Longrightarrow} \sqrt[\nu]{2+\frac{1}{\nu}} \to 1$$

$$\boxed{\alpha > 0 \Rightarrow \sqrt[\nu]{\alpha} \to 1}$$