Εβδομάδα 2η / Ασκήσεις / 21.3.12 2-A/1

Notiztitel 20.03.2012

A [1.2 σε] $\displaystyle\int \frac{2^x}{\sqrt{1-4^x}}\,dx$

$y = 2^x \Rightarrow 4^x = (2^2)^x = 2^{2x} = (2^x)^2 = y^2$ και

$\frac{dy}{dx} = (2^x)' = (e^{x\ln 2})' = e^{x\ln 2}\ln 2 = 2^x \ln 2 > 0 \ \forall x \in \mathbb{R}$

$\Rightarrow \displaystyle\int \frac{2^x}{\sqrt{1-4^x}}\,dx = \frac{1}{\ln 2}\int \frac{2^x \ln 2\, dx}{\sqrt{1-(2^x)^2}} = \frac{1}{\ln 2}\int \frac{dy}{\sqrt{1-y^2}} =$

$= \frac{1}{\ln 2}\,\mathrm{Arcsin}\, y + C = \frac{1}{\ln 2}\,\mathrm{Arcsin}(2^x) + C,$

για $x \in (-\infty, 0)$ $\big(\Leftrightarrow y \in (0,1)\big)$

A [1.3 γ] $\int \cos x \sin 3x \, dx$

$\sin(3x) = \sin(x+2x) = \sin x \cos(2x) + \cos x \sin(2x)$

$\Rightarrow \quad \cos x \sin 3x = \cos x \sin x \cos(2x) + \cos^2 x \sin(2x)$

$= \frac{1}{2} \sin(2x) \cos(2x) + \frac{\cos(2x)+1}{2} \sin(2x)$

$= \sin(2x)\cos(2x) + \frac{1}{2}\sin(2x)$

$= \frac{1}{2}\sin(4x) + \frac{1}{2}\sin(2x)$

$\Rightarrow \int \cos x \sin 3x \, dx = \frac{1}{2}\left(\int \sin(4x)\,dx + \int \sin(2x)\,dx\right) =$

$= \frac{1}{2}\left(\frac{1}{4}\int \sin y \, dy + \frac{1}{2}\int \sin z \, dz\right)$

$y = 4x \Rightarrow dy = 4\,dx$

$z = 2x \Rightarrow dz = 2\,dx$

$= -\frac{1}{4}\left(\frac{1}{2}\cos(4x) + \cos(2x)\right) + C$ für $x \in \mathbb{R}$

12-A13

Πδ [1.24] $\int \frac{dx}{x\sqrt{x^{2\nu}-\alpha^{2\nu}}}$ $(\alpha>0,\ \nu\in\mathbb{N},\ x>\alpha$ ή $x<-\alpha)$

$$\int \frac{dx}{x\,\alpha^{\nu}\sqrt{\left(\frac{x}{\alpha}\right)^{2\nu}-1}} = \int \frac{d\left(\frac{x}{\alpha}\right)}{\left(\frac{x}{\alpha}\right)\alpha^{\nu}\sqrt{\left(\frac{x}{\alpha}\right)^{2\nu}-1}} \underset{y=\frac{x}{\alpha}}{=} \frac{1}{\alpha^{\nu}}\int \frac{dy}{y\sqrt{y^{2\nu}-1}}$$

$$= \frac{1}{\nu\alpha^{\nu}}\int \frac{\nu\, y^{\nu-1}\,dy}{y^{\nu}\sqrt{(y^{\nu})^{2}-1}} \underset{\substack{z=y^{\nu}\\ dz=\nu y^{\nu-1}dy}}{=} \frac{1}{\nu\alpha^{\nu}}\int\frac{dz}{z\sqrt{z^{2}-1}} = \frac{1}{\nu\alpha^{\nu}}\int\frac{dz}{z^{2}\sqrt{1-\frac{1}{z^{2}}}}$$

$$\underset{\substack{u=\frac{1}{z}\\ du=-\frac{1}{z^{2}}dz}}{=} -\frac{1}{\nu\alpha^{\nu}}\int\frac{du}{\sqrt{1-u^{2}}} = -\frac{1}{\nu\alpha^{\nu}}\operatorname{Arcsin} u + c = -\frac{1}{\nu\alpha^{\nu}}\operatorname{Arcsin}\left(\frac{\alpha}{x}\right)^{\nu}+c$$

Εφ [1, § 1.2] α) $\displaystyle\int \frac{dx}{\sqrt{1-x^2}-1} \underbrace{=}_{\substack{x=\sin\vartheta\in(-1,0) \text{ ή } (0,1),\\ \vartheta\in(-\frac{\pi}{2},0) \text{ ή } (0,\frac{\pi}{2}),\\ dx=\cos\vartheta\, d\vartheta,\ \sqrt{1-x^2}=\cos\vartheta}} \int \frac{\cos\vartheta\, d\vartheta}{\cos\vartheta-1}$

$$= \int d\vartheta + \int \frac{d\vartheta}{\cos\vartheta-1}$$

$$= \vartheta + \int \frac{\cos\vartheta+1}{\cos^2\vartheta-1}\, d\vartheta = \operatorname{Arc\,sin} x - \int \frac{\cos\vartheta+1}{\sin^2\vartheta}\, d\vartheta$$

$$= \operatorname{Arc\,sin} x - \int \frac{dx}{x^2} - \int \frac{1}{\sin^2\vartheta}\, d\vartheta = \operatorname{Arc\,sin} x + \frac{1}{x} + \operatorname{ctg}\vartheta + C$$

$$= \operatorname{Arc\,sin} x + \frac{1}{x} + \frac{\sqrt{1-x^2}}{x} + C \quad \text{για } x\in(-1,0) \text{ ή } (0,1)$$

β) $\displaystyle\int \frac{dx}{x\sqrt{1+x^6}} \underbrace{=}_{\substack{y=x^6\\ dy=6x^5dx=6y\frac{1}{x}dx}} \frac{1}{6}\int \frac{1}{y}\frac{1}{\sqrt{1+y}}\, dy \underbrace{=}_{\substack{t=\sqrt{1+y}\\ dt=\frac{1}{2}\frac{1}{\sqrt{1+y}}dy}} \frac{1}{3}\int \frac{dt}{t^2-1}$

$$= -\frac{1}{3}\operatorname{Arctgh} t + c = -\frac{1}{3}\operatorname{Arctgh}\sqrt{1+x^6} + C \quad \text{για } x>0 \text{ ή } x<0$$

[αφού [Νε. I, σελ. 278] $\operatorname{Arctgh} z = \frac{1}{2}\log\frac{z+1}{z-1} = \log\sqrt{\frac{z+1}{z-1}}$, $|z|>1$

$\Longleftrightarrow -\operatorname{Arctgh} z = \log\sqrt{\frac{z-1}{z+1}}$ έχουμε

$$-\operatorname{Arctgh}\sqrt{1+x^6} = \log\sqrt{\frac{\sqrt{1+x^6}-1}{\sqrt{1+x^6}+1}} = \log\sqrt{\frac{(\sqrt{1+x^6}-1)^2}{(1+x^6)-1}} =$$

$$= \log\frac{\sqrt{1+x^6}-1}{|x|^3} \ ]$$

[Εναλλακτικά [Νε. II, σελ. 13, βλ. και Παρατ. 1.23 (ii)]:

$$\int\frac{dx}{x\sqrt{1+x^6}} \underset{x^3=\operatorname{tg}\vartheta,\ \vartheta\in(0,\frac{\pi}{2})\ \text{ή}\ \vartheta\in(-\frac{\pi}{2},0)}{=} \frac{1}{3}\int\frac{1}{\operatorname{tg}\vartheta}\frac{1}{\sqrt{1+\operatorname{tg}^2\vartheta}}\frac{1}{\cos^2\vartheta}\,d\vartheta$$

$$\underbrace{dx^3}_{=3x^2dx} = \frac{1}{\cos^2\vartheta}d\vartheta \Leftrightarrow dx = \frac{1}{3x^2}\frac{1}{\cos^2\vartheta}d\vartheta$$

$$= \frac{1}{3}\int\frac{1}{\operatorname{tg}\vartheta}\frac{1}{\cos\vartheta}d\vartheta = \frac{1}{3}\int\frac{d\vartheta}{\sin\vartheta} \underset{\Pi\delta[1.14]}{=} \log\left|\operatorname{tg}\frac{\vartheta}{2}\right| + c = \log\left|\frac{\sqrt{1+x^6}-1}{x^3}\right| + c$$

αφού $\operatorname{tg}\frac{\vartheta}{2} = \frac{\sin\frac{\vartheta}{2}}{\cos\frac{\vartheta}{2}} = \frac{\sin\frac{\vartheta}{2}\cos\frac{\vartheta}{2}}{\cos^2\frac{\vartheta}{2}} = \frac{\sin\vartheta}{\cos\vartheta+1} = \frac{\operatorname{tg}\vartheta}{1+\sqrt{\operatorname{tg}^2\vartheta+1}} = \frac{\sqrt{\operatorname{tg}^2\vartheta+1}-1}{\operatorname{tg}\vartheta}$]

A [1.5 γ]

$$\int \frac{x^{2\nu}}{(\alpha^2+x^2)^{\nu+3/2}}\,dx = \int \frac{x^{2\nu}}{(\alpha^2+x^2)^{\nu}\left(\sqrt{\alpha^2+x^2}\right)^3}\,dx$$

$$= \int \frac{x^{2\nu}}{\left(\sqrt{\alpha^2+x^2}\right)^{2\nu+3}}\,dx \underset{\substack{x=\alpha\,\mathrm{tg}\,\vartheta,\ \vartheta\in(-\frac{\pi}{2},\frac{\pi}{2})}}{=} \int \frac{\alpha^{2\nu}\,\mathrm{tg}^{2\nu}\vartheta}{\alpha^{2\nu+3}\frac{1}{\cos^{2\nu+3}\vartheta}}\,\alpha\frac{1}{\cos^2\vartheta}\,d\vartheta$$

$$dx = \alpha\frac{1}{\cos^2\vartheta}\,d\vartheta$$

$$\sqrt{\alpha^2+x^2} \underset{\alpha>0}{=} \alpha\sqrt{1+\mathrm{tg}^2\vartheta} = \frac{\alpha}{\cos\vartheta}$$

$$= \frac{1}{\alpha^2}\int \mathrm{tg}^{2\nu}\vartheta\,\cos^{2\nu+1}\vartheta\,d\vartheta = \frac{1}{\alpha^2}\int \sin^{2\nu}\vartheta\,\cos\vartheta\,d\vartheta$$

$$\underset{\substack{y=\sin\vartheta\\ dy=\cos\vartheta\,d\vartheta}}{=} \frac{1}{\alpha^2}\int y^{2\nu}\,dy = \frac{1}{\alpha^2}\frac{y^{2\nu+1}}{2\nu+1}+C = \frac{1}{\alpha^2}\frac{\sin^{2\nu+1}\vartheta}{2\nu+1}+C$$

$$= \frac{1}{\alpha^2(2\nu+1)}\,\mathrm{tg}^{2\nu+1}\vartheta\,\frac{1}{\frac{1}{\cos^{2\nu+1}\vartheta}}+C = \frac{1}{\alpha^2(2\nu+1)}\frac{\mathrm{tg}^{2\nu+1}\vartheta}{\sqrt{1+\mathrm{tg}^2\vartheta}^{\,2\nu+1}}+C$$

$$= \frac{1}{\alpha^2(2\nu+1)}\left(\frac{\mathrm{tg}\,\vartheta}{\sqrt{1+\mathrm{tg}^2\vartheta}}\right)^{2\nu+1}+C = \frac{1}{\alpha^2(2\nu+1)}\left(\frac{x}{\sqrt{\alpha^2+x^2}}\right)^{2\nu+1}+C \quad \begin{pmatrix}\nu\in\mathbb{N},\\ \alpha>0,\\ x\in\mathbb{R}\end{pmatrix}$$

2-A/7

A [1.5 δ] $\displaystyle\int \frac{dx}{x\sqrt{1+x^4}} \overset{=}{=} \int \frac{1}{2\,tg\vartheta}\,\frac{1}{\cos^2\vartheta}\,\frac{1}{\frac{1}{\cos\vartheta}}\,d\vartheta$

$x^2 = tg\vartheta,\ \vartheta\in(0,\frac{\pi}{2})$

$\underbrace{d(x^2)}_{=2x\,dx} = \frac{1}{\cos^2\vartheta}\,d\vartheta \iff dx = \frac{1}{2x}\,\frac{1}{\cos^2\vartheta}\,d\vartheta$

$\sqrt{1+x^4} = \sqrt{1+tg^2\vartheta} = \frac{1}{\cos\vartheta}$

$= \frac{1}{2}\displaystyle\int \frac{d\vartheta}{\sin\vartheta}$

$\underset{\Pi\delta[1.14]}{=} \frac{1}{2}\log\left|tg\frac{\vartheta}{2}\right| + c$

με $tg\frac{\vartheta}{2} = \dfrac{tg\vartheta}{1+\sqrt{1+tg^2\vartheta}} = \dfrac{x^2}{1+\sqrt{1+x^4}}$ $\left(= \dfrac{\sqrt{1+x^4}-1}{x^2}\right)$

$\Longrightarrow \displaystyle\int \frac{dx}{x\sqrt{1+x^4}} = \frac{1}{2}\log\left(\frac{x^2}{1+\sqrt{1+x^4}}\right) + c$ $\left(= \frac{1}{2}\log\left(\frac{\sqrt{1+x^4}-1}{x^2}\right) + c\right)$

για $x>0$ ή $x<0$

A [1.8 ε] $\displaystyle\int \frac{\alpha x^2-\beta}{x^2+(\alpha x^2+\beta)^2}\,dx = \int \frac{1}{1+(\alpha x+\frac{\beta}{x})^2}\left(\alpha-\frac{\beta}{x^2}\right)dx \overset{=}{=}$

$z = \alpha x + \frac{\beta}{x} \iff dz = \left(\alpha - \frac{\beta}{x^2}\right)dx$

$\displaystyle\int \frac{dz}{1+z^2} = \mathrm{Arctan}\, z + c = \mathrm{Arctan}\left(\alpha x + \frac{\beta}{x}\right) + c$

για $x>0$ ή $x<0$ $(\alpha,\beta>0)$

ΠΔ [1.47] $\displaystyle\int \frac{\operatorname{Arcsin} x}{(1-x^2)^{3/2}}\,dx$

$$\int \frac{dx}{(1-x^2)^{3/2}} = \int \frac{dx}{\left(\sqrt{1-x^2}\right)^3} \underset{\substack{x=\sin\vartheta,\\ \vartheta\in(-\frac{\pi}{2},\frac{\pi}{2})}}{=} \int \frac{\cos\vartheta\, d\vartheta}{\cos^3\vartheta} = \int \frac{d\vartheta}{\cos^2\vartheta}$$

$$= \operatorname{tg}\vartheta + c = \frac{\sin\vartheta}{\cos\vartheta} + c = \frac{\sin\vartheta}{\sqrt{1-\sin^2\vartheta}} + c = \frac{x}{\sqrt{1-x^2}} + c \quad (A[1.48])$$

$$\Longrightarrow \int \frac{\operatorname{Arcsin} x}{(1-x^2)^{3/2}}\,dx = \int \underbrace{\operatorname{Arcsin} x}_{=f}\ \underbrace{\frac{1}{(1-x^2)^{3/2}}}_{=g'}\,dx = \underbrace{\operatorname{Arcsin} x}_{=f}\ \underbrace{\frac{x}{\sqrt{1-x^2}}}_{=g}$$

$$- \int \underbrace{\frac{1}{\sqrt{1-x^2}}}_{=f'}\ \underbrace{\frac{x}{\sqrt{1-x^2}}}_{=g}\,dx = \frac{x}{\sqrt{1-x^2}} \operatorname{Arcsin} x - \int \frac{x}{1-x^2}\,dx$$

$$= \frac{x}{\sqrt{1-x^2}} \operatorname{Arcsin} x + \frac{1}{2}\int \frac{-2x}{1-x^2}\,dx = \frac{x}{\sqrt{1-x^2}} \operatorname{Arcsin} x + \frac{1}{2}\log(1-x^2) + c$$

Εφ [1, §1.3] α) $\int \frac{3x^2-1}{2x\sqrt{x}}\,dx = \frac{3}{2}\int \sqrt{x}\,dx - \frac{1}{2}\int \frac{dx}{x^{3/2}} =$

$= \frac{3}{2}\,\frac{x^{\frac{3}{2}}}{\frac{3}{2}} - \frac{1}{2}\,\frac{x^{-\frac{1}{2}}}{-\frac{1}{2}} + c = x^{\frac{3}{2}} + x^{-\frac{1}{2}} + c = \frac{x^2+1}{\sqrt{x}} + c$

$\Rightarrow \int \frac{3x^2-1}{2x\sqrt{x}}\,\mathrm{Arctg}\,x\,dx = \frac{x^2+1}{\sqrt{x}}\,\mathrm{Arctg}\,x - \int \frac{x^2+1}{\sqrt{x}}\,\frac{1}{1+x^2}\,dx$

$= \frac{x^2+1}{\sqrt{x}}\,\mathrm{Arctg}\,x - 2\sqrt{x} + c$

β) $\int e^{\sin x}\,\frac{x\cos^3 x - \sin x}{\cos^2 x}\,dx \underset{\substack{y=\sin x\\ dy=\cos x\,dx}}{=} \int e^y\left(\mathrm{Arcsin}\,y - \frac{y}{(1-y^2)^{3/2}}\right)dy$

$= e^y\,\mathrm{Arcsin}\,y - \int e^y\,\frac{1}{(1-y^2)^{1/2}}\,dy - \int e^y\,\frac{y}{(1-y^2)^{3/2}}\,dy$

όπου, αφού $\int \frac{y}{(1-y^2)^{3/2}}\,dy = -\frac{1}{2}\int \frac{-2y}{(1-y^2)^{3/2}}\,dy \underset{z=1-y^2}{=} -\frac{1}{2}\int \frac{dz}{z^{3/2}} =$

$$= -\frac{1}{2}\,\frac{z^{-\frac{1}{2}}}{-\frac{1}{2}} + c = \frac{1}{z^{\frac{1}{2}}} + c = \frac{1}{(1-y^2)^{\frac{1}{2}}} + c,$$

$$\int e^y \frac{y}{(1-y^2)^{3/2}}\,dy = e^y \frac{1}{(1-y^2)^{1/2}} - \int e^y \frac{1}{(1-y^2)^{1/2}}\,dy$$

$$\Longrightarrow \int e^y \left(\operatorname{Arcsin} y - \frac{y}{(1-y^2)^{3/2}}\right) dy = e^y\left(\operatorname{Arcsin} y - \frac{1}{(1-y^2)^{1/2}}\right) + c$$

$$\Longrightarrow \int e^{\sin x}\,\frac{x\cos^3 x - \sin x}{\cos^2 x}\,dx = e^{\sin x}\left(x - \frac{1}{\cos x}\right) + c, \quad x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$$

$$\gamma)\quad \int e^x \frac{1+\sin x}{1+\cos x}\,dx \underset{(*)}{=} \int e^x \left(\left(\operatorname{tg}\frac{x}{2}\right)' + \operatorname{tg}\frac{x}{2}\right) dx =$$

$$= e^x \operatorname{tg}\frac{x}{2} - \int e^x \operatorname{tg}\frac{x}{2}\,dx + \int e^x \operatorname{tg}\frac{x}{2}\,dx = e^x \operatorname{tg}\frac{x}{2} + c$$

$$(*)\quad \frac{1+\sin x}{1+\cos x} = \frac{1 + 2\sin\frac{x}{2}\cos\frac{x}{2}}{2\cos^2\frac{x}{2}} = \frac{1}{2}\,\frac{1}{\cos^2\frac{x}{2}} + \operatorname{tg}\frac{x}{2} = \left(\operatorname{tg}\frac{x}{2}\right)' + \operatorname{tg}\frac{x}{2}$$

A[1.10] γ) $\int \mathrm{Arctg}\, x\, dx = x\, \mathrm{Arctg}\, x - \int \frac{x}{1+x^2} dx = x\, \mathrm{Arctg}\, x - \frac{1}{2}\ln(1+x^2) + c$

δ) $\int \frac{x}{\cos^2 x} dx = x\, \mathrm{tg}\, x - \int \mathrm{tg}\, x\, dx = x\, \mathrm{tg}\, x - \int \frac{\sin x}{\cos x} dx =$

$= x\, \mathrm{tg}\, x + \ln|\cos x| + c$

A[1.14 oz] $\int \frac{x\, \mathrm{Arctg}\, x}{\sqrt{1+x^2}} dx$:

$$\int \frac{x}{\sqrt{1+x^2}} dx = \frac{1}{2}\int \frac{2x}{\sqrt{1+x^2}} dx \underset{y=1+x^2}{=} \frac{1}{2}\int \frac{dy}{\sqrt{y}} = \sqrt{y} + c = \sqrt{1+x^2} + c$$

$$\Rightarrow \int \frac{x\, \mathrm{Arctg}\, x}{\sqrt{1+x^2}} dx = \sqrt{1+x^2}\, \mathrm{Arctg}\, x - \int \sqrt{1+x^2} \frac{1}{1+x^2} dx$$

$$= \sqrt{1+x^2}\, \mathrm{Arctg}\, x - \int \frac{1}{\sqrt{1+x^2}} dx = \sqrt{1+x^2}\, \mathrm{Arctg}\, x - \underbrace{\mathrm{Arcsinh}\, x}_{= \log\left(x + \sqrt{x^2+1}\right)} + c$$

[Nz.I, str. 275]

A [1.15γ] $\displaystyle\int \frac{x^2}{(x\sin x+\cos x)^2}\,dx$:

$$\varphi(x) = x\sin x+\cos x \implies \varphi'(x) = \sin x + x\cos x - \sin x = x\cos x$$

$$\implies \left(\frac{1}{\varphi(x)}\right)' = -\frac{\varphi'(x)}{\varphi^2(x)} = -\frac{x\cos x}{(x\sin x+\cos x)^2}$$

$$\implies \int \frac{x^2}{(x\sin x+\cos x)^2}\,dx = \int \frac{-x}{\cos x}\left(\frac{1}{\varphi(x)}\right)' dx$$

$$= \frac{-x}{\cos x}\,\frac{1}{\varphi(x)} + \int \frac{\cancel{\cos x + x\sin x}}{\cos^2 x}\,\frac{1}{\cancel{x\sin x+\cos x}}\,dx$$

$$= -\frac{x}{\cos x}\,\frac{1}{x\sin x+\cos x} + \operatorname{tg} x + C$$

$$= \frac{1}{\cos x}\left(\sin x - \frac{x}{\cos x + x\sin x}\right) + C = \frac{1}{\cos x}\,\frac{\sin x\cos x + x\sin^2 x - x}{\cos x + x\sin x} + C$$

$$= \frac{\sin x - x\cos x}{\cos x + x\sin x} + C$$

A [1.16] α) $\int x \log\left(\frac{1-x}{1+x}\right)dx = \frac{1}{2}x^2 \log\left(\frac{1-x}{1+x}\right) - \frac{1}{2}\int x^2 \frac{1+x}{1-x}\left(\frac{1-x}{1+x}\right)' dx$

όπου $\left(\frac{1-x}{1+x}\right)' = \frac{-(1+x)-(1-x)}{(1+x)^2} = \frac{-2}{(1+x)^2} \Longrightarrow$

$$\Longrightarrow -\frac{1}{2}\int x^2 \frac{1+x}{1-x}\left(\frac{1-x}{1+x}\right)' dx = \int \frac{x^2}{1-x^2}dx = -\int \frac{1-x^2}{1-x^2}dx + \int \frac{dx}{1-x^2}$$

$$= -x + \text{Arctgh}\, x + c \text{ για } x \in (-1,1)$$

$$\Longrightarrow \int x \log\left(\frac{1-x}{1+x}\right)dx = \frac{1}{2}x^2 \log\left(\frac{1-x}{1+x}\right) - x + \underbrace{\text{Arctgh}}_{=\frac{1}{2}\log\left(\frac{1+x}{1-x}\right)} x + c$$

$$= \frac{1}{2}(x^2-1)\log\left(\frac{1-x}{1+x}\right) - x + c, \quad x \in (-1,1)$$

β) $\int \sin\sqrt[3]{x}\, dx = \int \sin(\underbrace{x^{\frac{1}{3}}}_{=y})dx$

$= y \Rightarrow dy = \frac{1}{3} x^{-\frac{2}{3}} dx$

$= \frac{1}{3}\frac{1}{y^2} dx$

$\Longleftrightarrow dx = 3y^2 dy$

$= 3\int y^2 \sin y\, dy$

$$= 3\left(-y^2\cos y + 2\int y\cos y\,dy\right)$$

$$= -3y^2\cos y + 6\left(y\sin y - \int \sin y\,dy\right)$$

$$= -3y^2\cos y + 6y\sin y + 6\cos y + c$$

$$= \left(6 - 3x^{\frac{2}{3}}\right)\cos\left(x^{\frac{1}{3}}\right) + 6x^{\frac{1}{3}}\sin\left(x^{\frac{1}{3}}\right) + c$$

γ) $$\int \operatorname{Arccos}\sqrt{\frac{x}{x+1}}\,dx = x\operatorname{Arccos}\sqrt{\frac{x}{x+1}} + \int x\frac{1}{\sqrt{1-\frac{x}{x+1}}}\,\frac{1}{2}\sqrt{\frac{x+1}{x}}\,\frac{1}{(x+1)^2}\,dx$$

$$= x\operatorname{Arccos}\sqrt{\frac{x}{x+1}} + \frac{1}{2}\int\frac{\sqrt{x}}{x+1}\,dx + c$$

και $$\int\frac{\sqrt{x}}{x+1}\,dx \underset{\substack{y=\sqrt{x}\\ dy=\frac{1}{2y}dx}}{=} 2\int\frac{y^2}{y^2+1}\,dy = 2y - 2\int\frac{dy}{1+y^2} = 2\sqrt{x} - 2\operatorname{Arctg}\sqrt{x} + c$$

$$\Rightarrow \int\operatorname{Arccos}\sqrt{\frac{x}{x+1}}\,dx = x\operatorname{Arccos}\sqrt{\frac{x}{x+1}} + \sqrt{x} - \operatorname{Arctg}\sqrt{x} + c$$

$$\delta)\ \int x^2\sqrt{4-x^2}\,dx \underset{\substack{x=2\sin\vartheta,\ \vartheta\in(-\frac{\pi}{2},\frac{\pi}{2})\\ dx=2\cos\vartheta\, d\vartheta}}{=} \int 4\sin^2\vartheta\, 2\cos\vartheta\, 2\cos\vartheta\, d\vartheta$$

$$= 4\int \sin^2(2\vartheta)\, d\vartheta \underset{y=2\vartheta}{=} 2\int \sin^2 y\, dy = 2\left(-\sin y\cos y + \int \cos^2 y\, dy\right)$$

$$= -2\sin y\cos y + 2y - 2\int \sin^2 y\, dy = -\sin y\cos y + y + c$$

$$= -\sin(2\vartheta)\cos(2\vartheta) + 2\vartheta + c = -2\sin\vartheta\cos\vartheta\left(\cos^2\vartheta - \sin^2\vartheta\right) + 2\vartheta + c$$

$$= -2\sin\vartheta\left(1-\sin^2\vartheta\right)^{\frac{1}{2}}\left(1-2\sin^2\vartheta\right) + 2\vartheta + c$$

$$= -x\left(1-\frac{x^2}{4}\right)^{\frac{1}{2}}\left(1-2\frac{x^2}{4}\right) + 2\operatorname{Arcsin}\frac{x}{2} + c$$

$$= -\frac{1}{4}x\left(4-x^2\right)^{\frac{1}{2}}\left(2-x^2\right) + 2\operatorname{Arcsin}\frac{x}{2} + c$$