[§1.6] Ολοκλήρωση άρρητων συναρτήσεων

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A′ $\displaystyle\int R\left(x, \sqrt[\nu]{\frac{\alpha x+\beta}{\gamma x+\delta}}\right) dx$, $R(x,y)$ ρητή, $\nu \in \mathbb{N}$, $\nu \geq 2$,

$\alpha\delta \neq \beta\gamma$, $\dfrac{\alpha x+\beta}{\gamma x+\delta} > 0$ αν ν άρτιος

$$\frac{\alpha x+\beta}{\gamma x+\delta} = t^\nu \iff (\alpha-\gamma t^\nu)x = \delta t^\nu - \beta \iff x = \frac{\delta t^\nu - \beta}{\alpha - \gamma t^\nu}$$

$$\Rightarrow \frac{dx}{dt} = \frac{\nu\delta t^{\nu-1}(\alpha-\gamma t^\nu) + (\delta t^\nu - \beta)\nu\gamma t^{\nu-1}}{(\alpha-\gamma t^\nu)^2}$$

$$= \frac{\nu t^{\nu-1}(\delta\alpha - \beta\gamma)}{(\alpha-\gamma t^\nu)^2}$$

$$\Rightarrow \int R\left(x, \sqrt[\nu]{\frac{\alpha x+\beta}{\gamma x+\delta}}\right) dx = \nu(\delta\alpha-\beta\gamma)\int R\left(\frac{\delta t^\nu-\beta}{\alpha-\gamma t^\nu}, t\right)\frac{t^{\nu-1}}{(\alpha-\gamma t^\nu)^2}\, dt$$

[Β′ $\int R\left(x, \sqrt[\mu]{\frac{\alpha x+\beta}{\gamma x+\delta}}, \sqrt[\nu]{\frac{\alpha x+\beta}{\gamma x+\delta}}, \dots\right) dx$ (R κλπ. όπως στο Α′; $\mu, \nu, \dots \in \mathbb{N}$)

$\frac{\alpha x+\beta}{\gamma x+\delta} = t^{\rho}$, ρ ελάχιστο κοινό πολλαπλάσιο των $\mu, \nu, \dots$]

Α[1.22] α) $\displaystyle\int \frac{\sqrt{x+1}+2}{(x+1)^2-\sqrt{x+1}}\,dx \underset{\substack{x+1=t^2 \\ \Rightarrow dx = 2t\,dt}}{=} \int \frac{t+2}{t^4-t}\,2t\,dt =$

$$= 2\int \frac{t+2}{t^3-1}\,dt = 2\int \frac{t+2}{(t-1)(t^2+t+1)}\,dt = 2\int \frac{A}{t-1}\,dt + 2\int \frac{Bt+C}{t^2+t+1}\,dt$$

$$\frac{t+2}{(t-1)(t^2+t+1)} = \frac{A}{t-1} + \frac{Bt+C}{t^2+t+1} \Leftrightarrow t+2 = A(t^2+t+1) + (Bt+C)(t-1)$$

$t=0$: $2 = A - C$, $t=1$: $3 = 3A \Rightarrow A = 1$, $C = -1$, $t=-1$: $1 = A + 2B + 2 \Rightarrow B = -1$

$$\Rightarrow 2\int \frac{t+2}{t^3-1}\,dt = 2\ln|t-1| - 2\int \frac{t+1}{t^2+t+1}\,dt$$

$$\int \frac{t+1}{t^2+t+1}\,dt = \frac{1}{2}\int \frac{2t+1}{t^2+t+1}\,dt + \frac{1}{2}\int \frac{1}{t^2+t+1}\,dt = \frac{1}{2}\ln|t^2+t+1|$$

$$+\frac{1}{2}\int \frac{1}{(t+\frac{1}{2})^2+\frac{3}{4}}\,dt = \frac{1}{2}\ln|t^2+t+1| + \frac{1}{2}\,\frac{2}{\sqrt{3}}\,\mathrm{Arctg}\,\frac{2(t+\frac{1}{2})}{\sqrt{3}} + c$$

$$\Rightarrow \int \frac{\sqrt{x+1}+2}{(x+1)^2-\sqrt{x+1}}\,dx = \ln\frac{(t-1)^2}{t^2+t+1} - \frac{2}{\sqrt{3}}\,\mathrm{Arctg}\,\frac{2t+1}{\sqrt{3}} + c$$

$$\underset{t=\sqrt{x+1}}{=} \ln\left(\frac{(\sqrt{x+1}-1)^2}{x+\sqrt{x+1}+2}\right) - \frac{2}{\sqrt{3}}\,\mathrm{Arctg}\,\frac{2\sqrt{x+1}+1}{\sqrt{3}} + c$$

A [1.22] γ) $\int x\sqrt{\frac{x-1}{x+1}}\,dx$ [μετά πιο απλά]

$$\frac{x-1}{x+1} = t^2 \Leftrightarrow x-1 = t^2x+t^2 \Leftrightarrow (1-t^2)x = t^2+1 \Leftrightarrow x = \frac{t^2+1}{1-t^2}$$

$$\Rightarrow dx = \frac{2t(1-t^2)+(t^2+1)2t}{(1-t^2)^2}\,dt \Leftrightarrow dx = \frac{4t}{(1-t^2)^2}\,dt$$

$$\Rightarrow \int x\sqrt{\frac{x-1}{x+1}}\,dx = \int \frac{t^2+1}{1-t^2}\,t\,\frac{4t}{(1-t^2)^2}\,dt = -4\int \frac{(t^2+1)t^2}{(t^2-1)^3}\,dt$$

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$$\frac{(t^2+1)t^2}{(t^2-1)^3} = \frac{(t^2+1)t^2}{(t-1)^3(t+1)^3} = \frac{A}{t-1} + \frac{B}{(t-1)^2} + \frac{C}{(t-1)^3} + \frac{D}{t+1} + \frac{E}{(t+1)^2} + \frac{F}{(t+1)^3}$$

$$\Leftrightarrow \quad (t^2+1)t^2 = A(t-1)^2(t+1)^3 + B(t-1)(t+1)^3 + C(t+1)^3$$
$$+ D(t-1)^3(t+1)^2 + E(t-1)^3(t+1) + F(t-1)^3$$

$t=1: \quad 2 = C\,8 \Leftrightarrow \boxed{C = \frac{1}{4}}$ (1)

$t=-1: \quad 2 = F(-8) \Leftrightarrow \boxed{F = -\frac{1}{4}}$ (2)

$t=2: \quad 20 = A\,27 + B\,27 + C\,27 + D\,9 + E\,3 + F$

$t=-2: \quad 20 = -A\,9 + B\,3 - C - D\,27 + E\,27 - F\,27$

$t=0: \quad 0 = A - B + C - D - E - F$ (6)

$$t^4 + t^2 = (A+D)t^5 + (\dots)t^4 + (\dots)t^3 + (\dots)t^2 + (\dots)t + (\dots)$$

$\Rightarrow A = -D$ (3)

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$(1),(2),(3) \Rightarrow \quad 20 = A\,18 + B\,27 + E\,3 + \frac{26}{4}$

$\Leftrightarrow \quad \frac{27}{2} = A\,18 + B\,27 + E\,3$

$\Leftrightarrow \quad \frac{9}{2} = A\,6 + B\,9 + E, \qquad (4)$

$20 = A\,18 + B\,3 + E\,27 + \frac{26}{4}$

$\Leftrightarrow \quad \frac{27}{2} = A\,18 + B\,3 + E\,27$

$\Leftrightarrow \quad \frac{9}{2} = A\,6 + B + E\,9 \qquad (5)$

$(4),(5) \Rightarrow \quad B\,8 = E\,8 \Leftrightarrow B = E \qquad (7)$

$(6),(3),(7) \Rightarrow \quad 0 = 2A - 2B + \frac{1}{2} \Leftrightarrow A = B - \frac{1}{4} \qquad (8)$

$(5),(8),(7) \Rightarrow \quad \frac{9}{2} = A\,6 + B\,10 = B\,16 - \frac{3}{2} \Leftrightarrow B\,16 = 6 \Leftrightarrow B = \frac{3}{8}$

$\Rightarrow \quad A = \frac{1}{8}, \; B = \frac{3}{8}, \; C = \frac{1}{4}, \; D = -\frac{1}{8}, \; E = \frac{3}{8}, \; F = -\frac{1}{4}$

$$\Rightarrow \quad -4\int \frac{(t^2+1)\,t^2}{(t^2-1)^3}\,dt = -\frac{1}{2}\int\frac{dt}{t-1} - \frac{3}{2}\int\frac{dt}{(t-1)^2} - \int\frac{dt}{(t-1)^3}$$

$$+\frac{1}{2}\int\frac{dt}{t+1} - \frac{3}{2}\int\frac{dt}{(t+1)^2} + \int\frac{dt}{(t+1)^3}$$

$$= -\frac{1}{2}\ln|t-1| + \frac{3}{2}\,\frac{1}{t-1} + \frac{1}{2}\,\frac{1}{(t-1)^2}$$

$$+\frac{1}{2}\ln|t+1| + \frac{3}{2}\,\frac{1}{t+1} - \frac{1}{2}\,\frac{1}{(t+1)^2} \quad + C$$

$$= \frac{1}{2}\ln\left|\frac{t+1}{t-1}\right| + \frac{1}{2}\left(\frac{3t-2}{(t-1)^2} + \frac{3t+2}{(t+1)^2}\right) + C$$

$$= \frac{1}{2}\ln\left|\frac{t+1}{t-1}\right| + \frac{(3t^2-1)\,t}{(t^2-1)^2} + C$$

$$\Rightarrow \int x\sqrt{\frac{x-1}{x+1}}\,dx \underset{t^2=\frac{x-1}{x+1}}{=} \frac{1}{2}\ln\left|\frac{\sqrt{x-1}+\sqrt{x+1}}{\sqrt{x-1}-\sqrt{x+1}}\right| + \frac{(x+1)^2}{4}\sqrt{\frac{x-1}{x+1}}\;2\,\frac{x-2}{x+1} + c$$

$$= \frac{1}{2}\ln\left|x+\sqrt{x^2-1}\right| + \frac{\sqrt{x^2-1}}{2}(x-2) + C$$

[Εναλλακτικά (πιο απλά): $\int x\sqrt{\frac{x-1}{x+1}}\,dx = \int x\,\frac{x-1}{\sqrt{x^2-1}}\,dx$

$$= \int x\,\frac{x}{\sqrt{x^2-1}}\,dx - \int \frac{x}{\sqrt{x^2-1}}\,dx$$

$$\int \frac{x}{\sqrt{x^2-1}}\,dx = \frac{1}{2}\int \frac{2x}{\sqrt{x^2-1}}\,dx = \sqrt{x^2-1} + c \Rightarrow \int x\frac{x}{\sqrt{x^2-1}}\,dx = x\sqrt{x^2-1} - \int\sqrt{x^2-1}\,dx$$

$$\int\sqrt{x^2-1}\,dx \underset{x=\cosh z,\, z>0}{=} \int \sinh^2 z\,dz = \int \frac{\cosh(2z)-1}{2}\,dz \underset{y=2z}{=} \int \frac{\cosh y - 1}{4}\,dy$$

$$= \frac{\sinh(2z)-2z}{4} + c = \frac{1}{2}\sqrt{\cosh^2 z - 1}\,\cosh z - \frac{1}{2}z + c$$

$$= \frac{1}{2}\,x\sqrt{x^2-1} - \frac{1}{2}\operatorname{Arccosh} x + c = \frac{1}{2}\,x\sqrt{x^2-1} - \frac{1}{2}\log\left|x+\sqrt{x^2-1}\right| + c$$

$$\Rightarrow \int x\sqrt{\frac{x-1}{x+1}}\,dx = \frac{1}{2}\,x\sqrt{x^2-1} + \frac{1}{2}\log\left|x+\sqrt{x^2-1}\right| - \sqrt{x^2-1} + c \;]$$

Γ' $\int R\left(x, \sqrt{\alpha x^2+\beta x+\gamma}\right) dx$, $R(x,y)$ ρητή , $\alpha, \beta, \gamma \in \mathbb{R}$,
$\alpha \neq 0$, $\alpha x^2+\beta x+\gamma > 0$

[Αντικαταστάσεις του Euler :

1. $\alpha > 0$: $\sqrt{\alpha x^2+\beta x+\gamma} = t \pm \sqrt{\alpha}\, x$

2. $\alpha < 0$, $\beta^2 - 4\alpha\gamma > 0$: $\sqrt{\alpha x^2+\beta x+\gamma} = t|x-\rho_1|$, $\alpha\rho_1^2+\beta\rho_1+\gamma = 0$

3. $\alpha < 0$, $\gamma > 0$: $\sqrt{\alpha x^2+\beta x+\gamma} = tx \pm \sqrt{\gamma}$]

Πχ. [1.68]: Αφού $\alpha x^2+\beta x+\gamma = \alpha\left[\left(x+\frac{\beta}{2\alpha}\right)+\frac{4\alpha\gamma-\beta^2}{4\alpha^2}\right]$, αν γίνεται καλύτερα να χρησιμοποιούμε τις αντικαταστάσεις της [§1.2] (Σημ. 1-/18-27)

A [1.24] α) $\displaystyle\int \frac{dx}{1+\sqrt{x^2+2x+2}} = \int \frac{dx}{1+\sqrt{(x+1)^2+1}} \underset{y=x+1}{=} \int \frac{dy}{1+\sqrt{y^2+1}}$

$$\underset{y=\sinh z}{=} \int \frac{\cosh z}{1+\cosh z}\,dz = z - \int \frac{dz}{1+\cosh z} = z - \frac{1}{2}\int \frac{dz}{\cosh^2\left(\frac{z}{2}\right)} = z - \tanh\left(\frac{z}{2}\right) + c$$

$$= z - \frac{\sinh\left(\frac{z}{2}\right)\cosh\left(\frac{z}{2}\right)}{\cosh^2\left(\frac{z}{2}\right)} + C = z - \frac{\sinh z}{2\cosh^2\left(\frac{z}{2}\right)} + C$$

$$= z - \frac{\sinh z}{1+\cosh z} + C = z - \frac{\sinh z}{1+\sqrt{1+\sinh^2 z}} + C$$

$$= \operatorname{Arcsinh}(x+1) - \frac{x+1}{1+\sqrt{1+(x+1)^2}} + C$$

$$= \log\left(x+1+\sqrt{(x+1)^2+1}\right) - \frac{x+1}{1+\sqrt{(x+1)^2+1}} + C$$

$$= \log\left(x+1+\sqrt{x^2+2x+2}\right) - \frac{x+1}{1+\sqrt{x^2+2x+2}} + C$$

[Επαλήθευση : $\frac{1}{x+1+\sqrt{\ }}\left(1+\frac{x+1}{\sqrt{\ }}\right) - \frac{1+\sqrt{\ } - (x+1)\frac{x+1}{\sqrt{\ }}}{\left(1+\sqrt{\ }\right)^2}$

$$= \frac{1}{\sqrt{\ }} - \frac{\sqrt{\ } + \cancel{(x+1)^2} + 1 - \cancel{(x+1)^2}}{\left(1+\sqrt{\ }\right)^2\sqrt{\ }} = \frac{1+\sqrt{\ }}{\left(1+\sqrt{\ }\right)\sqrt{\ }} - \frac{1}{\left(1+\sqrt{\ }\right)\sqrt{\ }} = \frac{1}{1+\sqrt{\ }}$$]

$$\text{Δ}' \quad \int \frac{P_\nu(x)}{\sqrt{\alpha x^2+\beta x+\gamma}}\,dx \, , \quad P_\nu \text{ πολυώνυμο βαθμού } \nu \, (\in \mathbb{N})$$ 3-1/10

$$\int \frac{P_\nu(x)}{\sqrt{\alpha x^2+\beta x+\gamma}}\,dx = \underbrace{Q_{\nu-1}(x)}_{\text{πολυώνυμο βαθμού } \nu-1} \sqrt{\alpha x^2+\beta x+\gamma} + k \int \frac{dx}{\sqrt{\alpha x^2+\beta x+\gamma}}$$

$$\Rightarrow \frac{P_\nu(x)}{\sqrt{\alpha x^2+\beta x+\gamma}} = Q'_{\nu-1}(x)\sqrt{\alpha x^2+\beta x+\gamma} + Q_{\nu-1}(x)\,\frac{1}{2}\,\frac{2\alpha x+\beta}{\sqrt{\alpha x^2+\beta x+\gamma}}$$

$$+ k \frac{1}{\sqrt{\alpha x^2+\beta x+\gamma}}$$

$$\Rightarrow \underbrace{P_\nu(x)}_{\text{πολυώνυμο βαθμού } \nu} = \underbrace{Q'_{\nu-1}(x)\left(\alpha x^2+\beta x+\gamma\right) + \frac{1}{2}(2\alpha x+\beta)\,Q_{\nu-1}(x) + k}_{\text{πολυώνυμο βαθμού } \nu}$$

$\Rightarrow$ $\nu+1$ εξισώσεις για τον προσδιορισμό $\nu+1$ αγνώστων, των ν συντελεστών του $Q_{\nu-1}$, και του k

A [1.25] α) $\int \frac{x^2}{\sqrt{1-2x-x^2}}\,dx = (\alpha x+\beta)\sqrt{1-2x-x^2} + k\int \frac{dx}{\sqrt{1-2x-x^2}}$ 3-1/11

$$\Rightarrow \alpha\sqrt{1-2x-x^2} + (\alpha x+\beta)\,\frac{1}{2}\,\frac{-2x-2}{\sqrt{1-2x-x^2}} + k\,\frac{1}{\sqrt{1-2x-x^2}} = \frac{x^2}{\sqrt{1-2x-x^2}}$$

$$\Rightarrow \alpha(1-2x-x^2) - (x+1)(\alpha x+\beta) + k = x^2$$

$$\Leftrightarrow \alpha - 2\alpha x - \alpha x^2 - \alpha x^2 - \beta x - \alpha x - \beta + k = x^2$$

$$\Rightarrow -2\alpha = 1, \quad -3\alpha - \beta = 0, \quad \alpha - \beta + k = 0$$

$$\Rightarrow \alpha = -\frac{1}{2}, \quad \beta = \frac{3}{2}, \quad k = 2$$

und $\int \frac{dx}{\sqrt{1-2x-x^2}} = \int \frac{dx}{\sqrt{2-(x+1)^2}} = \frac{1}{\sqrt{2}}\int \frac{dx}{\sqrt{1-\left(\frac{x+1}{\sqrt{2}}\right)^2}} \underset{y=\frac{x+1}{\sqrt{2}}}{=} \int \frac{dy}{\sqrt{1-y^2}} = \mathrm{Arcsin}\left(\frac{x+1}{\sqrt{2}}\right) + c$

$$\Rightarrow \int \frac{x^2}{\sqrt{1-2x-x^2}}\,dx = \frac{3-x}{2}\sqrt{1-2x-x^2} + 2\,\mathrm{Arcsin}\left(\frac{x+1}{\sqrt{2}}\right) + c$$

A [1.22 γ] $\int x\sqrt{\frac{x-1}{x+1}}\,dx = \int x\sqrt{\frac{(x-1)^2}{x^2-1}}\,dx = \int \frac{x(x-1)}{\sqrt{x^2-1}}\,dx$

$$\stackrel{!}{=} (\alpha x+\beta)\sqrt{x^2-1} + k\int\frac{dx}{\sqrt{x^2-1}}$$

$$\Rightarrow \frac{x^2-x}{\sqrt{x^2-1}} \stackrel{!}{=} \alpha\sqrt{x^2-1} + (\alpha x+\beta)\frac{x}{\sqrt{x^2-1}} + k\frac{1}{\sqrt{x^2-1}}$$

$$\Rightarrow x^2-x \stackrel{!}{=} \alpha x^2-\alpha + \alpha x^2 + \beta x + k$$

$$\Rightarrow 2\alpha = 1,\ \beta=-1,\ k=\alpha$$

$$\Rightarrow \int x\sqrt{\frac{x-1}{x+1}}\,dx = \left(\frac{x}{2}-1\right)\sqrt{x^2-1} + \frac{1}{2}\int\frac{dx}{\sqrt{x^2-1}}$$

όπου $\int\frac{dx}{\sqrt{x^2-1}} = \log\left|x+\sqrt{x^2-1}\right| + c$

(βλ. και Σημ. 3-1/3-7)

$3-1/13$

[Ε′ α) $\displaystyle\int \frac{P_\nu(x)}{(x-\rho)^m\sqrt{\alpha x^2+\beta x+\gamma}}\,dx$: αντικατάσταση $x-\rho=\frac{1}{t}$

β) $\displaystyle\int \frac{P_\nu(x)}{(x-\rho_1)(x-\rho_2)\sqrt{\alpha x^2+\beta x+\gamma}}\,dx$: $\displaystyle\frac{1}{(x-\rho_1)(x-\rho_2)}=\frac{1}{\rho_1-\rho_2}\left(\frac{1}{x-\rho_1}-\frac{1}{x-\rho_2}\right)$]

A[1.25γ]

$$\int \frac{dx}{(x-1)\sqrt{x^2+x+1}} = -\int \frac{1}{\frac{1}{t}\sqrt{\left(\frac{t+1}{t}\right)^2+\frac{t+1}{t}+1}}\,\frac{1}{t^2}dt$$

$$\boxed{\begin{aligned} x-1&=\frac{1}{t}\\ x&=1+\frac{1}{t}=\frac{t+1}{t}\\ dx&=-\frac{1}{t^2}\,dt\end{aligned}}$$

$$= -\int \frac{1}{\sqrt{(t+1)^2+t(t+1)+t^2}}\,dt = -\int \frac{dt}{\sqrt{3t^2+3t+1}}$$

$$= -\frac{1}{\sqrt{3}}\int \frac{dt}{\sqrt{\left(t+\frac{1}{2}\right)^2+\frac{1}{12}}} = -\frac{1}{\sqrt{3}}\int \frac{d\left(\sqrt{12}\left(t+\frac{1}{2}\right)\right)}{\sqrt{\left(\sqrt{12}\left(t+\frac{1}{2}\right)\right)^2+1}} = -\frac{1}{\sqrt{3}}\int\frac{dy}{\sqrt{y^2+1}}$$

$$= -\frac{1}{\sqrt{3}}\log\left(y+\sqrt{y^2+1}\right)+c = -\frac{1}{\sqrt{3}}\log\left(\sqrt{12}\left(t+\frac{1}{2}+\sqrt{\left(t+\frac{1}{2}\right)^2+\frac{1}{12}}\right)\right)+c$$

$$= -\frac{1}{\sqrt{3}}\log\left(t+\frac{1}{2}+\frac{t}{\sqrt{3}}\sqrt{\left(\frac{t+1}{t}\right)^2+\frac{t+1}{t}+1}\right)+c = -\frac{1}{\sqrt{3}}\log\left(\frac{1}{x-1}+\frac{1}{2}\right.$$

$$= -\frac{1}{\sqrt{3}}\log\left(\frac{1}{x-1}+\frac{1}{2}+\frac{1}{\sqrt{3}(x-1)}\sqrt{x^2+x+1}\right)+c$$

A [1.25 ζ]

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$$\int \frac{dx}{(1-x^2)\sqrt{1+x^2}} = \int \frac{dx}{(1-x)(1+x)\sqrt{1+x^2}} = \frac{1}{2}\int \frac{dx}{(x+1)\sqrt{1+x^2}} + \frac{1}{2}\int \frac{dx}{(1-x)\sqrt{1+x^2}}$$

$$\int \frac{dx}{(x+1)\sqrt{1+x^2}} \underset{\boxed{\begin{array}{l} x+1=\frac{1}{t}>0 \\ x=\frac{1}{t}-1=\frac{1-t}{t} \\ dx=-\frac{1}{t^2}dt \end{array}}}{=} -\int \frac{1}{t\sqrt{1+\frac{(1-t)^2}{t^2}}}\,dt = -\int \frac{dt}{\sqrt{t^2+(1-t)^2}} = -\int \frac{dt}{\sqrt{2t^2+1-2t}}$$

$$= -\frac{1}{\sqrt{2}}\int \frac{dt}{\sqrt{t^2-t+\frac{1}{2}}} = -\frac{1}{\sqrt{2}}\int \frac{dt}{\sqrt{\left(t-\frac{1}{2}\right)^2+\frac{1}{4}}} = -\frac{1}{\sqrt{2}}\int \frac{d(2t-1)}{\sqrt{(2t-1)^2+1}} =$$

$$= -\frac{1}{\sqrt{2}} \log\left(2t-1+\sqrt{(2t-1)^2+1}\right)+c = -\frac{1}{\sqrt{2}}\log\left(\frac{1-x}{1+x}+\sqrt{\left(\frac{1-x}{1+x}\right)^2+1}\right)+c$$

$$\int \frac{dx}{(1-x)\sqrt{1+x^2}} \underset{\boxed{\begin{array}{l} 1-x=\frac{1}{t}>0 \\ x=1-\frac{1}{t}=\frac{t-1}{t} \\ dx=\frac{1}{t^2}dt \end{array}}}{=} \int \frac{dt}{t\sqrt{1+\left(\frac{t-1}{t}\right)^2}} = \int \frac{dt}{\sqrt{t^2+(t-1)^2}} \underset{\uparrow}{=} \frac{1}{\sqrt{2}}\log\left(2t-1+\sqrt{(2t-1)^2+1}\right)+c$$

βλ. προηγούμενο

$$= \frac{1}{\sqrt{2}}\log\left(\frac{1+x}{1-x}+\sqrt{\left(\frac{1+x}{1-x}\right)^2+1}\right)+c$$

$$\Rightarrow \int \frac{dx}{(1-x^2)\sqrt{1+x^2}} = \frac{1}{2\sqrt{2}}\log\left(\frac{y+\sqrt{y^2+1}}{\frac{1}{y}+\sqrt{\left(\frac{1}{y}\right)^2+1}}\right)+c \quad \text{με } y=\frac{1+x}{1-x}, \quad x\in(-1,1)$$

$$= \frac{1}{2\sqrt{2}}\log\left(\left(y+\sqrt{y^2+1}\right)\left(\sqrt{\left(\frac{1}{y}\right)^2+1}-\frac{1}{y}\right)\right)+c$$

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[Επαλήθευση: $\frac{1}{2\sqrt{2}}\left[\log\left(\left(y+\sqrt{y^2+1}\right)\left(\sqrt{\left(\frac{1}{y}\right)^2+1}-\frac{1}{y}\right)\right)\right]' =$

$$= \frac{1}{2\sqrt{2}} \frac{\left(y' + \frac{yy'}{\sqrt{y^2+1}}\right)\left(\sqrt{y^{-2}+1}-y^{-1}\right) + \left(y+\sqrt{y^2+1}\right)\left(\frac{-y^{-3}y'}{\sqrt{y^{-2}+1}} + y^{-2}y'\right)}{\left(y+\sqrt{y^2+1}\right)\left(\sqrt{y^{-2}+1}-y^{-1}\right)}$$

$$= \frac{y'}{2\sqrt{2}}\left(\frac{1}{\sqrt{y^2+1}} + y^{-2}\frac{1}{\sqrt{y^{-2}+1}}\right) = \frac{y'}{2\sqrt{2}}\frac{1}{\sqrt{y^2+1}}\left(1+\frac{1}{y}\right)$$

όπου $y' = \left(\frac{1+x}{1-x}\right)' = \frac{1-x+(1+x)}{(1-x)^2} = \frac{2}{(1-x)^2}$,

$1+\frac{1}{y} = 1+\frac{1-x}{1+x} = \frac{1+x+1-x}{1+x} = \frac{2}{1+x}$,

$\sqrt{y^2+1} = \sqrt{\left(\frac{1+x}{1-x}\right)^2+1} = \frac{\sqrt{(1+x)^2+(1-x)^2}}{1-x} = \frac{\sqrt{2}\sqrt{1+x^2}}{1-x}$

και άρα

$$\frac{1}{2\sqrt{2}}\left[\log(\dots)\right]' = \frac{y'}{2\sqrt{2}}\frac{1}{\sqrt{y^2+1}}\left(1+\frac{1}{y}\right) = \frac{1}{2\sqrt{2}}\frac{2}{(1-x)^2}\frac{1-x}{\sqrt{2}\sqrt{1+x^2}}\frac{2}{1+x}$$

$$= \frac{1}{(1-x^2)\sqrt{1+x^2}}$$]

ΣΤ' (Διωνυμικά ολοκληρώματα)

$\int x^m (\alpha+\beta x^n)^p dx$, $\alpha, \beta \neq 0$, $m, n, p \in \mathbb{Q}$ $\left(p \vee \frac{m+1}{n} \vee \frac{m+1}{n}+p \in \mathbb{Z}\right)$
(Συνθήκες Chebyshev)

1. $p \in \mathbb{Z}$: $t^r = x$, r ελάχιστο κοινό πολλαπλάσιο των παρονομαστών των m και n

2. $\frac{m+1}{n} \in \mathbb{Z}$: $\alpha+\beta x^n = t^s$, s ο παρονομαστής του p

3. $\frac{m+1}{n}+p \in \mathbb{Z}$: $\alpha x^{-n}+\beta = t^s$, —— " ——

A.[1.26] γ) $\int \sqrt{x(1+x^3)}\, dx \Rightarrow p=\frac{1}{2}=m$, $n=3 \Rightarrow x^{-3}+1=t^2$

$\Leftrightarrow \quad x^3 = \frac{1}{t^2-1} \Leftrightarrow x = \frac{1}{(t^2-1)^{\frac{1}{3}}} \Rightarrow dx = -\frac{1}{3}\frac{2t}{(t^2-1)^{\frac{4}{3}}}dt$

$\Rightarrow \int \sqrt{x(1+x^3)}\, dx = \int \frac{1}{(t^2-1)^{\frac{1}{6}}} \frac{t}{(t^2-1)^{\frac{1}{2}}} \left(-\frac{1}{3}\right) \frac{2t}{(t^2-1)^{\frac{4}{3}}} dt = -\frac{2}{3}\int \frac{t^2}{(t^2-1)^2}dt$

3-1/17

$$\frac{t^2}{(t^2-1)^2} = \frac{A}{t-1} + \frac{B}{(t-1)^2} + \frac{C}{t+1} + \frac{D}{(t+1)^2}$$

$$\Longleftrightarrow \quad t^2 = A(t-1)(t+1)^2 + B(t+1)^2 + C(t-1)^2(t+1) + D(t-1)^2$$

$$\Longrightarrow \quad t=1: \quad 1 = B4 \Rightarrow B = \frac{1}{4}$$

$$t=-1: \quad 1 = D4 \Rightarrow D = \frac{1}{4}$$

$$t=0: \quad 0 = -A + B + C + D \Longleftrightarrow A - C = \frac{1}{2}$$

$$t=2: \quad 4 = A9 + B9 + C3 + D \Longleftrightarrow \frac{1}{2} = A3 + C$$

$$\Longrightarrow A = \frac{1}{4} \Longrightarrow C = -\frac{1}{4}$$

$$\Longrightarrow \frac{t^2}{(t^2-1)^2} = \frac{1}{4}\left(\frac{1}{t-1} + \frac{1}{(t-1)^2} - \frac{1}{t+1} + \frac{1}{(t+1)^2}\right)$$

$$\Longrightarrow \int \frac{t^2}{(t^2-1)^2}\,dt = \frac{1}{4}\left(\ln|t-1| - \ln|t+1| + \frac{(t-1)^{-1}}{-1} + \frac{(t+1)^{-1}}{-1}\right) + c$$

$$= \frac{1}{4}\left(\ln\left|\frac{t-1}{t+1}\right| - \frac{1}{t-1} - \frac{1}{t+1}\right) + c$$

$$\Longrightarrow \int \sqrt{x(1+x)}\,dx = \frac{1}{6}\ln\left|\frac{t+1}{t-1}\right| + \frac{1}{3}\frac{t}{t^2-1} + c, \quad t = \sqrt{1+x^{-3}}$$

A. [1.26] α) $\int \sqrt[3]{x(1-x^2)}\, dx$ 3-1/18

Διωνυμικό ολοκλ. $\int x^m(\alpha+\beta x^n)^p dx$ με $m=\frac{1}{3}=p$, $n=2$, $\alpha=1$, $\beta=-1$.

Αφού $\frac{m+1}{n}+p=\frac{2}{3}+\frac{1}{3}=1\in\mathbb{Z}$, θέτουμε $\frac{1}{x^2}-1=t^3 \Leftrightarrow x^2=\frac{1}{t^3+1}$

$\underset{x>0}{\Longrightarrow} x=\frac{1}{\sqrt{t^3+1}}$, $1-x^2=1-\frac{1}{t^3+1}=\frac{t^3}{t^3+1}$, $dx=-\frac{1}{2}\frac{3t^2}{(\sqrt{t^3+1})^3}dt$

$$\Rightarrow \int\sqrt[3]{x(1-x^2)}\,dx=\int\frac{1}{(t^3+1)^{\frac{1}{6}}}\frac{t}{(t^3+1)^{\frac{1}{3}}}\left(-\frac{3}{2}\right)\frac{t^2}{(t^3+1)^{\frac{3}{2}}}dt=$$

$$=-\frac{3}{2}\int\frac{t^3}{(t^3+1)^2}dt=-\frac{1}{2}\int t\frac{3t^2}{(t^3+1)^2}dt=-\frac{1}{2}\left(-\frac{t}{t^3+1}+\int\frac{dt}{t^3+1}\right)$$

$$t^3+1=(t+1)(t^2-t+1)=(t+1)\left(\left(t-\frac{1}{2}\right)^2+\frac{3}{4}\right)\Rightarrow\frac{1}{t^3+1}=\frac{A}{t+1}+\frac{Bt+\Gamma}{t^2-t+1}$$

$$\Rightarrow 1=A(t^2-t+1)+(Bt+\Gamma)(t+1)=(A+B)t^2+(-A+B+\Gamma)t+(A+\Gamma)$$

$$\Rightarrow B=-A,\ \Gamma=2A,\ 3A=1\Rightarrow\int\frac{dt}{t^3+1}=\frac{1}{3}\int\frac{dt}{t+1}-\frac{1}{3}\int\frac{t-2}{t^2-t+1}dt$$

όπου $\int \frac{t-2}{t^2-t+1}\,dt = \frac{1}{2}\int \frac{2t-4}{t^2-t+1}\,dt = \frac{1}{2}\int \frac{2t-1}{t^2-t+1}\,dt - \frac{3}{2}\int \frac{dt}{(t-\frac{1}{2})^2+\frac{3}{4}}$

$= \frac{1}{2}\ln|t^2-t+1| - 2\int \frac{dt}{(\frac{2}{\sqrt{3}}t-\frac{1}{\sqrt{3}})^2+1} = \frac{1}{2}\ln|t^2-t+1| - \sqrt{3}\,\mathrm{Arctg}\,\frac{2t-1}{\sqrt{3}} + c$

$\Rightarrow \int (x(1-x^2))^{\frac{1}{3}}\,dx = \frac{1}{2}\frac{t}{t^3+1} - \frac{1}{6}\ln|t+1| + \frac{1}{12}\ln|t^2-t+1|$

$- \frac{\sqrt{3}}{6}\,\mathrm{Arctg}\,\frac{2t-1}{\sqrt{3}} + c \ , \quad t = \left(\frac{1-x^2}{x^2}\right)^{\frac{1}{3}}$