Εβδομάδα 3η / Θεωρία / 29.3.12

[§1.7] Ολοκλήρωση τριγωνομετρικών συναρτήσεων

A' $\boxed{\int R(\sin x, \cos x)\,dx}$, $R(y,z)$ ρητή συνάρτηση

$$\boxed{t = \operatorname{tg}\frac{x}{2}} = \frac{\sin\frac{x}{2}}{\cos\frac{x}{2}} \underset{(1)}{=} \frac{\sin x}{2\cos^2\frac{x}{2}} \underset{(2)}{=} \frac{\sin x}{\cos x + 1}, \quad x \in (-\pi, \pi)$$

$$\Leftrightarrow \quad x = 2\operatorname{Arctg} t, \quad t \in \mathbb{R} \quad \Rightarrow \quad \frac{dx}{dt} = \frac{2}{1+t^2},$$

$$t^2 = \operatorname{tg}^2\frac{x}{2} = \frac{\sin^2\frac{x}{2}}{\cos^2\frac{x}{2}} = \frac{1}{\cos^2\frac{x}{2}} - 1 \Leftrightarrow \cos^2\frac{x}{2} = \frac{1}{1+t^2} \underset{(2)}{\Rightarrow} \cos x = \frac{1-t^2}{1+t^2},$$

$$\sin x = 2\sin\frac{x}{2}\cos\frac{x}{2} = 2\operatorname{tg}\frac{x}{2}\cos^2\frac{x}{2} = \frac{2t}{1+t^2}$$

$$\Rightarrow \boxed{\int R(\sin x, \cos x)\,dx = \int R\left(\frac{2t}{1+t^2}, \frac{1-t^2}{1+t^2}\right)\frac{2}{1+t^2}\,dt}$$

$$\left[(1)\ \sin x = 2\sin\frac{x}{2}\cos\frac{x}{2}, \quad (2)\ \cos x = \cos^2\frac{x}{2} - \sin^2\frac{x}{2} = 2\cos^2\frac{x}{2} - 1\right]$$

Ειδικές περιπτώσεις ($\Rightarrow$ απλούστερες αντικαταστάσεις):

1. $R(-\sin x, \cos x) = -R(\sin x, \cos x)$

$t = \cos x \Leftrightarrow x = \operatorname{Arccos} t \Rightarrow dx = -\frac{1}{\sqrt{1-t^2}}dt, \quad \sin x = \sqrt{1-t^2}$

$\Rightarrow \int R(\sin x, \cos x)\,dx = \int R(-\sqrt{1-t^2}, t)\frac{1}{\sqrt{1-t^2}}dt$

2. $R(\sin x, -\cos x) = -R(\sin x, \cos x)$

$t = \sin x \Leftrightarrow x = \operatorname{Arcsin} t \Rightarrow dx = \frac{1}{\sqrt{1-t^2}}dt, \quad \cos x = \sqrt{1-t^2}$

$\Rightarrow \int R(\sin x, \cos x)\,dx = \int R(t, \sqrt{1-t^2})\frac{1}{\sqrt{1-t^2}}dt$

3. $R(-\sin x, -\cos x) = R(\sin x, \cos x)$

$t = \operatorname{tg} x \Leftrightarrow x = \operatorname{Arctg} t \Rightarrow dx = \frac{1}{1+t^2}dt, \quad \cos x = \frac{1}{\sqrt{1+t^2}}, \sin x = \frac{t}{\sqrt{1+t^2}}$

$\Rightarrow \int R(\sin x, \cos x)\,dx = \int R\left(\frac{t}{\sqrt{1+t^2}}, \frac{1}{\sqrt{1+t^2}}\right)\frac{1}{1+t^2}dt$

Β΄ $\boxed{R(\sin x, \cos x) = \sin^m x \cos^n x \ , \ m, n \in \mathbb{Q}}$

α) $m, n \in \mathbb{N} \cup \{0\}$: το $\int R(\sin x, \cos x)\,dx$ υπολογίζεται αναγωγικά και, ειδικότερα, αν m, n άρτιοι : $\sin^2 x = \dfrac{1-\cos 2x}{2}$, $\cos^2 x = \dfrac{1+\cos 2x}{2}$

β) $m, n \in \mathbb{Z}$: m περιττός : $t = \cos x$ (βλ. Α΄, ειδ. περίπτ. 1.)

n — " — : $t = \sin x$ (βλ. — " — 2.)

$m+n$ άρτιος : $t = \tan x$ (βλ. — " — 3.)

γ) $m, n \in \mathbb{Q}$: $\displaystyle\int R(\sin x, \cos x)\,dx \underset{t=\sin x}{=} \int t^m (1-t^2)^{\frac{n-1}{2}}\,dt$ (βλ. — " —, 2.)

διωνυμικό ολοκλήρωμα (βλ. [§1.6, ΣΤ΄]) που υπολογίζεται στις περιπτώσεις α) $\frac{n-1}{2} \in \mathbb{Z}$, β) $\frac{m+1}{2} \in \mathbb{Z}$, γ) $\frac{m+n}{2} \in \mathbb{Z}$

[Για αποδείξεις αναγωγικών τύπων, ειδικότερες περιπτώσεις και παραδείγματα, βλ. [Ne. II, σελ. 63-66]]

[§ 1.8] Ολοκλήρωση υπερβολικών συναρτήσεων

$\boxed{\int R(\sinh x, \cosh x)\,dx}$, $R(y,z)$ ρητή συνάρτηση,

$\boxed{t = \tanh \frac{x}{2}} \Leftrightarrow x = 2\,\mathrm{Artanh}\, t \Rightarrow dx = \frac{2}{1-t^2}\,dt$,

$$t = \tanh\frac{x}{2} = \frac{\sinh\frac{x}{2}}{\cosh\frac{x}{2}} \underset{(1)}{=} \frac{\sinh x}{2\cosh^2\frac{x}{2}} \underset{(2)}{=} \frac{\sinh x}{\cosh x + 1} \underset{(3)}{=} \frac{\sqrt{\cosh^2 x - 1}}{\cosh x + 1}$$

$$\Rightarrow \cosh x = \frac{1+t^2}{1-t^2} \Rightarrow \sinh x = \sqrt{\cosh^2 x - 1} = \frac{2t}{1-t^2}$$

$$\Rightarrow \boxed{\int R(\sinh x, \cosh x)\,dx = \int R\left(\frac{2t}{1-t^2}, \frac{1+t^2}{1-t^2}\right)\frac{2}{1-t^2}\,dt}$$

(1) $\sinh(\alpha+\beta) = \sinh\alpha\cosh\beta + \cosh\alpha\sinh\beta \Rightarrow \sinh(2\alpha) = 2\sinh\alpha\cosh\alpha$

(2) $\cosh(\alpha+\beta) = \cosh\alpha\cosh\beta + \sinh\alpha\sinh\beta \Rightarrow \cosh(2\alpha) = \cosh^2\alpha + \sinh^2\alpha \underset{(3)}{=} 2\cosh^2\alpha - 1$

(3) $\cosh^2\alpha - \sinh^2\alpha = 1$

Παρατήρηση: Κατά περίπτωση, αφού $\sinh x = \frac{e^x - e^{-x}}{2}$, $\cosh x = \frac{e^x + e^{-x}}{2}$, η αντικατάσταση $t = e^x$ μπορεί να οδηγεί σε πιο απλά ολοκληρώματα.

[Οι πιο πάνω αντικαταστάσεις (σελ. 1-4) οδηγούν σε συναρτήσεις ρητές ως προς t]

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A [1.27 α] $\displaystyle\int \frac{dx}{\sin x\,(2+\cos x-2\sin x)} = I = \int R(\sin x, \cos x)\,dx$

με $\displaystyle R(y,z) = \frac{1}{y(2+z-2y)}$, $t = \operatorname{tg}\frac{x}{2} \Rightarrow$

$$I = \int \frac{1}{\frac{2t}{1+t^2}\left(2+\frac{1-t^2}{1+t^2}-2\frac{2t}{1+t^2}\right)}\,\frac{2}{1+t^2}\,dt = \int \frac{1+t^2}{t(t^2-4t+3)}\,dt$$

$$= \int \frac{1+t^2}{t((t-2)^2-1)}\,dt = \int \frac{1+t^2}{t(t-3)(t-1)}\,dt$$

$$\frac{1+t^2}{t(t-3)(t-1)} = \frac{A}{t} + \frac{B}{t-1} + \frac{\Gamma}{t-3}$$

$\Leftrightarrow \quad 1+t^2 = A(t^2-4t+3) + B(t^2-3t) + \Gamma(t^2-t)$

$= (A+B+\Gamma)t^2 + (-4A-3B-\Gamma)t + 3A$

$\Rightarrow A = \frac{1}{3}$, $B+\Gamma = \frac{2}{3}$, $3B+\Gamma = -\frac{4}{3} \Rightarrow 2B = -2 \Rightarrow \Gamma = \frac{5}{3}$

$\Rightarrow I = \frac{1}{3}\ln|t| - \ln|t-1| + \frac{5}{3}\ln|t-3| + c$, $t = \operatorname{tg}\frac{x}{2}$

A [1.27 δ] $\int \frac{\sin^2 x \cos x}{\sin x + \cos x} dx = I = \int R(\sin x, \cos x)\, dx$

$R(y,z) = \frac{y^2 z}{y+z} = \frac{(-y)^2(-z)}{(-y)+(-z)} = R(-y,-z)$, $t = \operatorname{tg} x$

$\Rightarrow \quad I = \int \frac{t^2}{(1+t^2)^{\frac{3}{2}}} \frac{\sqrt{1+t^2}}{t+1} \frac{1}{1+t^2} dt = \int \frac{t^2}{(1+t^2)^2(t+1)} dt$

$= A \ln|t+1| + \int \frac{Bt+\Gamma}{1+t^2} dt + \int \frac{\Delta t + E}{(1+t^2)^2} dt$

όπου $\frac{t^2}{(1+t^2)^2(t+1)} = \frac{A}{t+1} + \frac{Bt+\Gamma}{1+t^2} + \frac{\Delta t + E}{(1+t^2)^2}$

$\Rightarrow \quad t^2 = A(1+t^2)^2 + (Bt+\Gamma)(1+t^2)(t+1) + (\Delta t + E)(t+1)$

$\Rightarrow \quad 0 \cdot t^4 = (A+B)t^4, \quad 0 \cdot t^3 = (\Gamma + B)t^3, \quad 0 \cdot t = (\underbrace{\Gamma + B}_{=0} + E + \Delta)t,$

$0 \cdot t^0 = (A + \Gamma + E)t^0$ [ή $t=0$: $0 = A + \Gamma + E$,] $t=-1$: $1 = A4$

$$\Rightarrow \quad A = \frac{1}{4} = \Gamma, \quad B = -\frac{1}{4}, \quad E = -\frac{1}{2}, \quad \Delta = \frac{1}{2}$$

$$\Rightarrow \quad I = \frac{1}{4} \ln|t+1| - \frac{1}{4} \int \frac{t-1}{1+t^2} dt + \frac{1}{2} \int \frac{t-1}{(1+t^2)^2} dt$$

$$\int \frac{t-1}{1+t^2} dt = \frac{1}{2} \int \frac{2t}{1+t^2} dt - \int \frac{dt}{1+t^2} = \frac{1}{2} \ln|1+t^2| - \operatorname{Arctg} t + c$$

$$\int \frac{t-1}{(1+t^2)^2} dt = \frac{1}{2} \int \frac{2t}{(1+t^2)^2} dt - \int \frac{dt}{(1+t^2)^2} = -\frac{1}{2} \frac{1}{1+t^2} - \int \frac{dt}{(1+t^2)^2}$$

$$\int \frac{dt}{(1+t^2)^2} = \int \frac{1+t^2-t^2}{(1+t^2)^2} dt = \int \frac{1}{1+t^2} dt + \frac{1}{2} \int t \left(-\frac{2t}{(1+t^2)^2}\right) dt$$

$$= \operatorname{Arctg} t + \frac{1}{2} \left(t \frac{1}{1+t^2} - \int \frac{1}{1+t^2} dt \right)$$

$$= \frac{1}{2} \operatorname{Arctg} t + \frac{1}{2} \frac{t}{1+t^2} + c$$

$$\Rightarrow \quad \int \frac{t-1}{(1+t^2)^2} dt = -\frac{1}{2} \frac{t+1}{1+t^2} - \frac{1}{2} \operatorname{Arctg} t + c$$

$$\Rightarrow \quad I = \frac{1}{4} \ln|t+1| - \frac{1}{8} \ln|1+t^2| + \frac{1}{4} \operatorname{Arctg} t - \frac{1}{4} \frac{t+1}{1+t^2} - \frac{1}{4} \operatorname{Arctg} t + c$$

$$= \frac{1}{4} \left(\ln \left| \frac{t+1}{\sqrt{1+t^2}} \right| - \frac{t+1}{1+t^2} \right) + c$$

$$\underset{t = \operatorname{tg} x}{=} \frac{1}{4} \left(\ln \left| \frac{\operatorname{tg} x + 1}{\sqrt{1+\operatorname{tg}^2 x}} \right| - \frac{\operatorname{tg} x + 1}{1+\operatorname{tg}^2 x} \right) + c$$

$$= \frac{1}{4} \left(\ln |\sin x + \cos x| - \cos x (\sin x + \cos x) \right) + c$$

αφού $\quad 1 + \operatorname{tg}^2 x = 1 + \frac{\sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x}$.

A [1.27 σz] $\displaystyle\int \frac{\sin x}{(1-\cos x)^2} dx = I = \int R(\sin x, \cos x)\, dx \;,\; R(y,z) = \frac{y}{(1-z)^2}$

$$\Rightarrow I \underset{t=\cos x}{=} - \int \frac{\sqrt{1-t^2}}{(1-t)^2} \frac{1}{\sqrt{1-t^2}} dt = - \int \frac{dt}{(1-t)^2} \underset{y=1-t}{=} -\frac{1}{y} + c = \frac{1}{\cos x - 1} + c$$

A [1.28 γ] $\int \frac{dx}{1+\sin^2 x} = I$, $R(y,z) = \frac{1}{1+y^2} = R(-y,-z)$

$t = \operatorname{tg} x \Rightarrow I = \int \frac{1}{1+\frac{t^2}{1+t^2}} \frac{1}{1+t^2} dt = \int \frac{dt}{1+2t^2} = \frac{1}{\sqrt{2}} \int \frac{d(\sqrt{2}t)}{1+(\sqrt{2}t)^2}$

$= \frac{1}{\sqrt{2}} \operatorname{Arctg}(\sqrt{2}t) + C = \frac{1}{\sqrt{2}} \operatorname{Arctg}(\sqrt{2}\operatorname{tg} x) + C$

A [1.28 δ] $\int \sqrt{1+\sin x}\, dx = I$

$t = \sin x \Leftrightarrow x = \operatorname{Arcsin} t \Rightarrow dx = \frac{1}{\sqrt{1-t^2}} dt \Rightarrow I = \int \sqrt{\frac{1+t}{1-t^2}} dt$

$= \int \frac{dt}{\sqrt{1-t}} \underset{y=\sqrt{1-t}}{=} \int \frac{-2y\, dy}{y} = -2y + C = -2\sqrt{1-t} + C = -2\sqrt{1-\sin x} + C$

$dy = \frac{-1}{2\sqrt{1-t}} dt = -\frac{1}{2y} dt$

$\Big[= -2\sqrt{1 - 2\sin\frac{x}{2}\cos\frac{x}{2}} + C = -2\sqrt{\cos^2\frac{x}{2} + \sin^2\frac{x}{2} - 2\sin\frac{x}{2}\cos\frac{x}{2}} + C$

$= -2\left|\cos\frac{x}{2} - \sin\frac{x}{2}\right| + C \underset{x\in(-\frac{\pi}{2},\frac{\pi}{2})}{=} 2\left(\sin\frac{x}{2} - \cos\frac{x}{2}\right) + C \Big]$

A [1.29α] $\displaystyle\int \frac{dx}{\sqrt[4]{\sin^3 x \cos^5 x}} = \int \sin^{-\frac{3}{4}} x \cos^{-\frac{5}{4}} x \, dx = I$

$t = \sin x \iff x = \operatorname{Arcsin} t \Rightarrow dx = \frac{1}{\sqrt{1-t^2}} dt, \quad \cos x = \sqrt{1-\sin^2 x} = \sqrt{1-t^2}$

$\Rightarrow \quad I = \int t^{-\frac{3}{4}} (1-t^2)^{-\frac{5}{8}} (1-t^2)^{-\frac{1}{2}} dt = \int t^{-\frac{3}{4}} (1-t^2)^{-\frac{9}{8}} dt$

$= \int t^m (\alpha + \beta t^n)^p dx$ με $m = -\frac{3}{4}$, $n = 2$, $p = -\frac{9}{8}$, $\alpha = 1$, $\beta = -1$

$\Rightarrow \frac{m+1}{n} + p = \frac{1}{8} - \frac{9}{8} = -1 \in \mathbb{Z} \Rightarrow y^8 = \frac{1}{t^2} - 1 \iff t = \frac{1}{\sqrt{y^8+1}} = (y^8+1)^{-\frac{1}{2}}$

$\Rightarrow \quad t^{-\frac{3}{4}} = (y^8+1)^{\frac{3}{8}}, \quad 1-t^2 = 1 - \frac{1}{y^8+1} = \frac{y^8}{y^8+1} \Rightarrow (1-t^2)^{-\frac{9}{8}} = \frac{y^{-9}}{(y^8+1)^{-\frac{9}{8}}},$

$dt = -\frac{1}{2}(y^8+1)^{-\frac{3}{2}} 8 y^7 dy = -4 (y^8+1)^{-\frac{3}{2}} y^7 dy$

$\Rightarrow I = -4 \int (y^8+1)^{\frac{3}{8} + \frac{9}{8} - \frac{3}{2}} \frac{1}{y^2} dy = -4 \int \frac{dy}{y^2} = 4 \frac{1}{y} + c = 4 \left(\frac{t^2}{1-t^2} \right)^{\frac{1}{8}} + c$

$= 4 \left(\frac{\sin^2 x}{\cos^2 x} \right)^{\frac{1}{8}} = 4 \operatorname{tg}^{\frac{1}{4}} x + c = 4 \sqrt[4]{\operatorname{tg} x} + c$.

Άσκηση: $\int \frac{dx}{\sin^\nu x} = I_\nu$, $\nu \in \mathbb{N}$

$$I_\nu = \int \sin^{-\nu} x \, dx = \int \sin^{-\nu-1} x \, \sin x \, dx$$

$$= -\sin^{-\nu-1} x \cos x + (-\nu-1) \int \sin^{-\nu-2} x \cos^2 x \, dx$$

$$= -\sin^{-\nu-1} x \cos x + (-\nu-1) I_{\nu+2} - (-\nu-1) I_\nu$$

$$\Rightarrow (\nu+1) I_{\nu+2} = \nu I_\nu - \frac{\cos x}{\sin^{\nu+1} x}$$

$$\Leftrightarrow I_{\nu+2} = \frac{\nu}{\nu+1} I_\nu - \frac{\cos x}{(\nu+1)\sin^{\nu+1} x}, \quad \nu \in \mathbb{N}$$

και $I_1 = \int \frac{dx}{\sin x} = \int \frac{dx}{2 \sin\frac{x}{2} \cos\frac{x}{2}} \underset{y = \operatorname{tg}\frac{x}{2}}{=} \int \frac{1}{\operatorname{tg}\frac{x}{2}} \frac{1}{2\cos^2\frac{x}{2}} dx$

$$dy = \frac{1}{2\cos^2\frac{x}{2}} dx$$

$$= \int \frac{dy}{y} = \log|y| + c = \log\left|\operatorname{tg}\frac{x}{2}\right| + c \quad (\text{βλ. ΠΔ. [1.14], [1.13]}),$$

$$I_2 = \int \frac{dx}{\sin^2 x} = -\operatorname{ctg} x + c \quad \left[\operatorname{ctg}' x = \left(\frac{\cos x}{\sin x}\right)' = \frac{-\sin^2 x - \cos^2 x}{\sin^2 x} = -\frac{1}{\sin^2 x}\right]$$

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A [1.31 β] $\displaystyle\int \frac{dx}{\sinh x \cosh x} = \int \frac{dx}{\operatorname{tgh} x \cosh^2 x} = \int \frac{d(\operatorname{tgh} x)}{\operatorname{tgh} x} =$

$= \log|\operatorname{tgh} x| + c$

$\displaystyle\left[\frac{d}{dx} \operatorname{tgh} x = \operatorname{tgh}' x = \left(\frac{\sinh x}{\cosh x}\right)' = \frac{\cosh^2 x - \sinh^2 x}{\cosh^2 x} = \frac{1}{\cosh^2 x} \right]$

A [1.34 γ] $\displaystyle\int x \operatorname{tg}^2 x \, dx$

$$\int \operatorname{tg}^2 x \, dx = \int \frac{\sin^2 x}{\cos^2 x} dx \underbrace{=}_{\substack{t = \operatorname{tg} x \\ x = \operatorname{Arctg} t \Rightarrow dx = \frac{dt}{1+t^2}}} \int \frac{t^2}{1+t^2} dt = t - \int \frac{dt}{1+t^2} = t - \operatorname{Arctg} t + c$$

$= \operatorname{tg} x - x + c$

$$\Rightarrow \quad \int x \operatorname{tg}^2 x \, dx = x(\operatorname{tg} x - x) - \int (\operatorname{tg} x - x) dx$$

$$= x \operatorname{tg} x - x^2 + \frac{x^2}{2} - \int \operatorname{tg} x \, dx$$

$$\int \operatorname{tg} x \, dx = \int \frac{\sin x}{\cos x} dx = -\int \frac{-\sin x}{\cos x} dx = -\log|\cos x| + c$$

$$\Rightarrow \quad \int x \operatorname{tg}^2 x \, dx = x \operatorname{tg} x - \frac{x^2}{2} + \log|\cos x| + c$$