Εβδομάδα 3η / Ασκήσεις / 28.3.12

3-Α/1

Notiztitel 28.03.2012

A[1.19 γ] $\displaystyle\int \frac{dx}{4-x^2-4x} = -\int \frac{dx}{x^2+4x-4} = -\int \frac{dx}{(x+2)^2-8} = \int \frac{dx}{8-(x+2)^2}$

$$= \frac{1}{8}\int \frac{dx}{1-\left(\frac{x}{\sqrt8}+\frac{2}{\sqrt8}\right)^2} = \frac{1}{\sqrt8}\int \frac{d\left(\frac{x}{\sqrt8}+\frac{2}{\sqrt8}\right)}{1-\left(\frac{x}{\sqrt8}+\frac{2}{\sqrt8}\right)^2} = \frac{1}{\sqrt8}\,\mathrm{Arctgh}\left(\frac{x}{\sqrt8}+\frac{2}{\sqrt8}\right) + c$$

$$= \frac{1}{\sqrt8}\,\frac{1}{2}\,\log\left|\frac{1+\frac{x+2}{\sqrt8}}{1-\frac{x+2}{\sqrt8}}\right| + c = \frac{1}{\sqrt8}\,\frac{1}{2}\,\log\left|\frac{x+2+\sqrt8}{\sqrt8-(x+2)}\right| + c$$

$$\left[= -\int \frac{dx}{(x+2)^2-8} = -\int \frac{dx}{(x+2-\sqrt8)(x+2+\sqrt8)} \overset{(*)}{=} -\frac{1}{2\sqrt8}\left(\int \frac{dx}{x+2-\sqrt8} - \int \frac{dx}{x+2+\sqrt8}\right)\right.$$

$$= -\frac{1}{2\sqrt8}\log\left|\frac{x+2-\sqrt8}{x+2+\sqrt8}\right| + c = \frac{1}{2\sqrt8}\log\left|\frac{x+2+\sqrt8}{x+2-\sqrt8}\right| + c = \frac{1}{2\sqrt8}\log\left|\frac{x+2+\sqrt8}{\sqrt8-(x+2)}\right| + c,$$

$(*)$: $\displaystyle\frac{1}{(x-\varrho)(x+\varrho)} \underset{\varrho\neq 0}{=} \frac{A}{x-\varrho} + \frac{B}{x+\varrho} = \frac{1}{2\varrho}\left(\frac{1}{x-\varrho} - \frac{1}{x+\varrho}\right)$ $\left[1 = A(x+\varrho) + B(x-\varrho) \underset{x=\pm\varrho}{\Longrightarrow} A = \frac{1}{2\varrho} = -B\right]$

3-A/2

A [1.19 η] $\displaystyle \int \frac{3x-1}{4x^2-4x+17}\,dx = \frac{1}{4}\int \frac{3x-1}{x^2-x+\frac{17}{4}} = \frac{1}{4}\int \frac{3x-1}{\left(x-\frac{1}{2}\right)^2+4}$

$$= \frac{1}{4}\left[\frac{3}{2}\int \frac{2x-1}{\left(x-\frac{1}{2}\right)^2+4}\,dx + \frac{1}{2}\int \frac{dx}{\left(x-\frac{1}{2}\right)^2+4}\right] = \frac{3}{8}\log\left|\left(x-\frac{1}{2}\right)^2+4\right|$$

$$+ \frac{1}{8}\,\frac{1}{4}\int \frac{dx}{\left(\frac{x}{2}-\frac{1}{4}\right)^2+1} = \dots + \frac{1}{16}\int \frac{d\left(\frac{x}{2}-\frac{1}{4}\right)}{\left(\frac{x}{2}-\frac{1}{4}\right)^2+1} = \frac{3}{8}\log\left|\left(x-\frac{1}{2}\right)^2+4\right|$$

$$+ \frac{1}{16}\,\mathrm{Arctg}\left(\frac{x}{2}-\frac{1}{4}\right)+C = \frac{3}{8}\log\left|4x^2-4x+17\right| + \frac{1}{16}\,\mathrm{Arctg}\left(\frac{2x-1}{4}\right)+C$$

$$\Bigg[\Rightarrow \left(\frac{3}{8}\log\left|4x^2-4x+17\right| + \frac{1}{16}\,\mathrm{Arctg}\left(\frac{2x-1}{4}\right)\right)' = \frac{3}{8}\,\frac{8x-4}{4x^2-4x+17}$$

$$+ \frac{1}{16}\,\frac{1}{1+\frac{(2x-1)^2}{16}}\,\frac{2}{4} = \frac{3x-\frac{3}{2}}{4x^2-4x+17} + \frac{1}{2}\,\frac{1}{4x^2-4x+17} \Bigg]$$

3-A/3

A [1.20 β] (με α=1, β=2) $\int \frac{x^3}{(x-1)^2(x-2)}dx = I$

$(x-1)^2(x-2) = (x^2-2x+1)(x-2) = x^3-2x^2+x-2x^2+4x-2$

$= x^3-4x^2+5x-2 \Rightarrow x^3 = (x-1)^2(x-2)+4x^2-5x+2$

$\Rightarrow I = \int dx + \int \frac{4x^2-5x+2}{(x-1)^2(x-2)}dx = x + A\log|x-1| - B\frac{1}{x-1} + C\log|x-2| + c$

όπου $\frac{4x^2-5x+2}{(x-1)^2(x-2)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x-2}$

$\Rightarrow 4x^2-5x+2 = A(x-1)(x-2) + B(x-2) + C(x-1)^2$

$\underset{x=1,2,0}{\Rightarrow}$ $4-5+2 = -B$, $16-10+2 = C$, $2 = A2-2B+C$

$\Rightarrow B=-1$, $C=8$, $A = \frac{2-C+2B}{2} = \frac{-6-2}{2} = -4$

$\Rightarrow I = x - 4\log|x-1| + \frac{1}{x-1} + 8\log|x-2| + c$

$\left[= x - \frac{1}{1-2}\frac{1}{x-1} + \frac{2-6}{(1-2)^2}\log|x-1| + \frac{2^3}{(1-2)^2}\log|x-2| + c\right]$

3-A/4

A [1.21 g] $\int \frac{x^2+1}{x(x^2+4)}\,dx = I$

$$\frac{x^2+1}{x(x^2+4)} = \frac{A}{x} + \frac{Bx+C}{x^2+4} \Rightarrow x^2+1 = A(x^2+4) + (Bx+C)x$$

$$\Rightarrow \quad 1\cdot x^2 = (A+B)\cdot x^2 \,,\quad 0\cdot x^1 = C\cdot x \,,\quad 1\cdot x^0 = 4A$$

$$\Rightarrow \quad A = \frac{1}{4} \,,\quad B = \frac{3}{4} \,,\quad C = 0$$

$$\Rightarrow \quad I = \frac{1}{4}\left(\int \frac{dx}{x} + 3\int \frac{x}{x^2+4}\,dx\right) = \frac{1}{4}\left(\log|x| + \frac{3}{2}\log|x^2+4|\right) + c$$

$$= \frac{1}{8}\log\left(x^2(x^2+4)^3\right) + c$$

A [1.34 ε] $\int \operatorname{Arcsin}\sqrt{\frac{x}{x+1}}\,dx$

$$= x \operatorname{Arcsin}\sqrt{\frac{x}{x+1}} - \int x \frac{1}{\sqrt{1-\frac{x}{x+1}}}\ \frac{1}{2}\sqrt{\frac{x+1}{x}}\ \frac{1}{(x+1)^2}\,dx$$

$$= x \operatorname{Arcsin}\sqrt{\frac{x}{x+1}} - \frac{1}{2}\int \frac{\sqrt{x}}{x+1}\,dx$$

$$-\frac{1}{2}\int \frac{\sqrt{x}}{x+1}\,dx = -\int \frac{x}{x+1}\ \frac{1}{2\sqrt{x}}\,dx \underset{y=\sqrt{x}}{=} -\int \frac{y^2}{y^2+1}\,dy = -y + \int \frac{dy}{y^2+1}$$

$$= -\sqrt{x} + \operatorname{Arctan}\sqrt{x} + c$$

$$\Rightarrow \int \operatorname{Arcsin}\sqrt{\frac{x}{x+1}} = x \operatorname{Arcsin}\sqrt{\frac{x}{x+1}} - \sqrt{x} + \operatorname{Arctan}\sqrt{x} + c$$

A [1.34 η] $\displaystyle\int \frac{\operatorname{Arctg} x}{x^2(1+x^2)}\,dx$

$$\int \frac{dx}{x^2(1+x^2)} = \int \frac{dx}{x^2} - \int \frac{dx}{x^2+1} = -\frac{1}{x} - \operatorname{Arctg} x + c$$

$$\left[\frac{1}{x^2(1+x^2)} = \frac{A}{x} + \frac{B}{x^2} + \frac{\Gamma x + \Delta}{1+x^2} \right.$$

$$\Rightarrow \quad 1 = A(x+x^3) + B(1+x^2) + (\Gamma x + \Delta)x^2$$

$$= (A+\Gamma)x^3 + (B+\Delta)x^2 + Ax + B$$

$$\left. \Rightarrow \quad A = \Gamma = 0, \quad B = -\Delta = 1 \Rightarrow \frac{1}{x^2(1+x^2)} = \frac{1}{x^2} - \frac{1}{x^2+1} \right]$$

$$\Rightarrow \int \frac{\operatorname{Arctg} x}{x^2(1+x^2)}\,dx = -\left(\frac{1}{x} + \operatorname{Arctg} x\right)\operatorname{Arctg} x$$

$$+ \int \left(\frac{1}{x} + \operatorname{Arctg} x\right)\frac{1}{1+x^2}\,dx$$

$$\int \frac{\operatorname{Arctg} x}{1+x^2}\,dx = \frac{1}{2}\int \left(\operatorname{Arctg}^2 x\right)' dx = \frac{1}{2}\operatorname{Arctg}^2 x + C,$$

$$\left[\ \frac{1}{x(1+x^2)} = \frac{A}{x} + \frac{Bx+\Gamma}{1+x^2} \Rightarrow 1 = A + Ax^2 + Bx^2 + \Gamma x\right.$$

$$\left.\Rightarrow A=1,\ B=-1,\ \Gamma=0 \Rightarrow \frac{1}{x(1+x^2)} = \frac{1}{x} - \frac{x}{1+x^2} \Rightarrow\ \right]$$

$$\int \frac{dx}{x(1+x^2)} = \int \frac{dx}{x} - \int \frac{x}{1+x^2}\,dx = \ln|x| - \frac{1}{2}\ln|1+x^2| + C$$

$$\Rightarrow \int \frac{\operatorname{Arctg} x}{x^2(1+x^2)}\,dx = -\frac{\operatorname{Arctg} x}{x} - \frac{1}{2}\operatorname{Arctg}^2 x + \ln\frac{|x|}{\sqrt{1+x^2}} + C$$

A [1.34 δ] $\int x \operatorname{Arcsin} x^2\,dx = \frac{1}{2}\int 2x \operatorname{Arcsin} x^2\,dx =$

$y = x^2$
$dy = 2x\,dx$

$= \frac{1}{2}\int \operatorname{Arcsin} y\,dy = \frac{1}{2}\left(y \operatorname{Arcsin} y - \int \frac{y}{\sqrt{1-y^2}}\,dy\right)$

$= \frac{1}{2}\left(y \operatorname{Arcsin} y + \sqrt{1-y^2}\right) + c$

$= \frac{1}{2}\left(x^2 \operatorname{Arcsin} x^2 + \sqrt{1-x^4}\right) + c$

A [1.34 α] $\int \operatorname{Arctg}(1+\sqrt{x})\,dx = x \operatorname{Arctg}(1+\sqrt{x}) - \int x \frac{1}{1+(1+\sqrt{x})^2}\frac{1}{2\sqrt{x}}\,dx$

$= x \operatorname{Arctg}(1+\sqrt{x}) - \int \frac{(\sqrt{x}+1-1)^2}{1+(1+\sqrt{x})^2}\,\frac{1}{2\sqrt{x}}\,dx$

$\underset{y=1+\sqrt{x}}{=} x \operatorname{Arctg}(1+\sqrt{x}) - \int \frac{(y-1)^2}{1+y^2}\,dy$

$\int \frac{(y-1)^2}{1+y^2}\,dy = \int \frac{y^2+1-2y}{1+y^2}\,dy = y - \int \frac{2y}{1+y^2}\,dy = y - \log|1+y^2| + c$

$\Rightarrow \int \operatorname{Arctg}(1+\sqrt{x})\,dx = x \operatorname{Arctg}(1+\sqrt{x}) - \sqrt{x} + \log|1+(1+\sqrt{x})^2| + c$

A [1.34 ζ] $\displaystyle\int \frac{\mathrm{Arc}\sin x}{x^2}\,dx = -\frac{\mathrm{Arc}\sin x}{x} + \int \frac{dx}{x\sqrt{1-x^2}}$

$$\int \frac{dx}{x\sqrt{1-x^2}} \underset{\substack{x=\sin\vartheta,\ \vartheta\in(-\frac{\pi}{2},\frac{\pi}{2}) \\ dx=\cos\vartheta\, d\vartheta}}{=} \int \frac{\cos\vartheta\, d\vartheta}{\sin\vartheta\cos\vartheta} = \int \frac{d\vartheta}{\sin\vartheta}$$

Πδ. [1.14]

$$\left[= \int \frac{d\vartheta}{2\sin\frac{\vartheta}{2}\cos\frac{\vartheta}{2}} = \int \frac{d\vartheta}{2\,\mathrm{tg}\frac{\vartheta}{2}\cos^2\frac{\vartheta}{2}} \underset{y=\mathrm{tg}\frac{\vartheta}{2}}{=} \int \frac{dy}{y} = \log|y|+c\right] = \log\left|\mathrm{tg}\frac{\vartheta}{2}\right| + c$$

με $\displaystyle\mathrm{tg}\frac{\vartheta}{2} = \frac{\sin\frac{\vartheta}{2}}{\cos\frac{\vartheta}{2}} = \frac{\sin\vartheta}{2\cos^2\frac{\vartheta}{2}} = \frac{\sin\vartheta}{\cos\vartheta+1} = \frac{\sin\vartheta}{1+\sqrt{1-\sin^2\vartheta}} = \frac{x}{1+\sqrt{1-x^2}}$

$$\Rightarrow \int \frac{\mathrm{Arc}\sin x}{x^2}\,dx = -\frac{\mathrm{Arc}\sin x}{x} + \log\left|\underbrace{\frac{x}{1+\sqrt{1-x^2}}}\right| + c$$

$$= \frac{x\left(1-\sqrt{1-x^2}\right)}{1-(1-x^2)} = \frac{1-\sqrt{1-x^2}}{x}$$

A [1.35α] $\displaystyle\int\sqrt{e^x-1}\,dx \underset{\substack{y=\sqrt{e^x-1} \\ dy=\frac{1}{2y}(y^2+1)dx}}{=} \int y\,\frac{2y}{y^2+1}\,dy$

$$= 2\int \frac{y^2+1}{y^2+1}\,dy - 2\int\frac{dy}{y^2+1}$$

$$= 2\left(\sqrt{e^x-1} - \mathrm{Arctg}\sqrt{e^x-1}\right) + c$$

A [1.35 γ] $\displaystyle\int \frac{x\cos x}{(x\sin x+\cos x)^2}\,dx$

$$\varphi(x) = x\sin x+\cos x \Rightarrow \varphi'(x) = \sin x + x\cos x - \sin x = x\cos x$$

$$\Rightarrow \int \frac{x\cos x}{(x\sin x+\cos x)^2}\,dx = \int \frac{\varphi'(x)}{\varphi^2(x)}\,dx = -\frac{1}{\varphi(x)} + c = -\frac{1}{x\sin x+\cos x} + c$$

A [1.35 β] $\displaystyle\int \frac{x e^x}{\sqrt{1+e^x}}\,dx$

$$\int \frac{e^x}{\sqrt{1+e^x}}\,dx \underset{y=e^x}{=} \int \frac{dy}{\sqrt{1+y}} = 2\sqrt{1+y} + c = 2\sqrt{1+e^x} + c$$

$$\Rightarrow \int \frac{x e^x}{\sqrt{1+e^x}}\,dx = 2x\sqrt{1+e^x} - 2\int\sqrt{1+e^x}\,dx$$

$$\int\sqrt{1+e^x}\,dx = \int \frac{\sqrt{1+e^x}}{e^x}\,e^x\,dx \underset{y=e^x}{=} \int \frac{\sqrt{1+y}}{y}\,dy \underset{z=\sqrt{1+y}}{=} \int \frac{2z^2}{z^2-1}\,dz$$

$$= 2z + 2\int \frac{dz}{z^2-1} = 2z - \log\left|\frac{z+1}{z-1}\right| + c = 2\sqrt{1+e^x} + \log\frac{\sqrt{1+e^x}-1}{\sqrt{1+e^x}+1} + c$$

$$\Rightarrow \int \frac{x e^x}{\sqrt{1+e^x}}\,dx = (2x-4)\sqrt{1+e^x} + 4\log\left(\sqrt{1+e^x}+1\right) - 2x + c$$

A [1.35 3] $\displaystyle\int x \log(1+x^3)\,dx = \frac{1}{2}x^2\log(1+x^3) - \frac{1}{2}\int x^2\frac{3x^2}{1+x^3}\,dx$ 3-A/M

$$\frac{x^4}{1+x^3} = \frac{x(x^3+1)-x}{1+x^3} = x - \frac{x}{1+x^3} = x - \frac{x}{(x+1)(x^2-x+1)} =$$

$$= x - \frac{x}{(x+1)\left(\left(x-\frac{1}{2}\right)^2+\frac{3}{4}\right)} = x - \frac{A}{x+1} - \frac{Bx+\Gamma}{x^2-x+1}$$

με $x = A(x^2-x+1) + (Bx+\Gamma)(x+1) \Rightarrow A+B=0,\ A+\Gamma=0, -A+B+\Gamma=1$

$$\Rightarrow B=\Gamma=-A=\frac{1}{3} \Rightarrow \frac{x^4}{1+x^3} = x + \frac{1}{3}\frac{1}{x+1} - \frac{1}{3}\frac{x+1}{x^2-x+1} =$$

$$= x + \frac{1}{3}\frac{1}{x+1} - \frac{1}{6}\frac{2x-1}{x^2-x+1} - \frac{1}{2}\frac{1}{x^2-x+1}$$

$$\Rightarrow -\frac{3}{2}\int\frac{x^4}{1+x^3}\,dx = -\frac{3}{2}\frac{x^2}{2} - \frac{1}{2}\log|x+1| + \frac{1}{4}\log|x^2-x+1|$$

$$+\frac{3}{4}\int\frac{dx}{\left(x-\frac{1}{2}\right)^2+\frac{3}{4}} = - \dots - \dots + \dots + \int\frac{dx}{\left(\frac{2}{\sqrt{3}}\left(x-\frac{1}{2}\right)\right)^2+1}$$

$$\Rightarrow -\frac{3}{2}\int \frac{x^4}{1+x^3}\,dx = -\ldots-\ldots+\ldots+\frac{\sqrt{3}}{2}\,\mathrm{Arctg}\left(\frac{2}{\sqrt{3}}\left(x-\frac{1}{2}\right)\right)+c$$

$$\Rightarrow \int x\log(1+x^3)\,dx = \frac{x^2}{2}\log(1+x^3) - \frac{3}{4}x^2 - \frac{1}{2}\log|x+1|$$

$$+\frac{1}{4}\log|x^2-x+1| + \frac{\sqrt{3}}{2}\,\mathrm{Arctg}\left(\frac{2}{\sqrt{3}}\left(x-\frac{1}{2}\right)\right)+c$$

A [1.35 ε] (Σεπτ. 2010) $\displaystyle\int \frac{1}{x\sqrt{x}}\log\frac{1}{1-x}\,dx = \int x^{-\frac{3}{2}}\log\frac{1}{1-x}\,dx$

$$= -2x^{-\frac{1}{2}}\log\frac{1}{1-x} + 2\int \frac{1}{\sqrt{x}}(1-x)\left(-\frac{1}{(1-x)^2}(-1)\right)dx$$

$$= -\frac{2}{\sqrt{x}}\log\frac{1}{1-x} + 2\int\frac{1}{\sqrt{x}}\,\frac{1}{1-x}\,dx \underset{(*)}{=} -\frac{2}{\sqrt{x}}\log\frac{1}{1-x} + 2\log\left|\frac{\sqrt{x}+1}{\sqrt{x}-1}\right| + c$$

$$(*):\int\frac{1}{\sqrt{x}}\,\frac{1}{1-x}\,dx \underset{y=\sqrt{x}}{=} 2\int\frac{1}{1-y^2}\,dy = 2\int\frac{1}{(1-y)(1+y)}\,dy = -\int\left(\frac{1}{y-1}-\frac{1}{y+1}\right)dy$$

$$= -\left(\log|\sqrt{x}-1| - \log|\sqrt{x}+1|\right)+c = \log\left|\frac{\sqrt{x}+1}{\sqrt{x}-1}\right|+c$$