Επαναληπτικές ασκήσεις στα αόριστα ολοκληρώματα

A [1.32 α] $\int e^{2x^2+\log x}\,dx = \int e^{2x^2} x\,dx = \frac{1}{4}\int e^{2x^2} 4x\,dx$

$\underset{y=2x^2}{=} \frac{1}{4}\int e^y\,dy = \frac{1}{4}e^y + c = \frac{1}{4}e^{2x^2} + c$

A [1.32 β] $\int e^{k\,\mathrm{Arccos}\,x}\,dx \quad (k\in\mathbb{R})$

$\int e^{k\,\mathrm{Arccos}\,x}\,dx = x e^{k\,\mathrm{Arccos}\,x} - \int x e^{k\,\mathrm{Arccos}\,x}\left(-\frac{k}{\sqrt{1-x^2}}\right)dx$

$\int e^{k\,\mathrm{Arccos}\,x}\frac{x}{\sqrt{1-x^2}}\,dx = -\int e^{k\,\mathrm{Arccos}\,x}\,\frac{1}{2}\left(\frac{-2x}{\sqrt{1-x^2}}\right)dx$

$= -e^{k\,\mathrm{Arccos}\,x}\sqrt{1-x^2} + \int \sqrt{1-x^2}\,e^{k\,\mathrm{Arccos}\,x}\left(-\frac{k}{\sqrt{1-x^2}}\right)dx$

$\Rightarrow \int e^{k\,\mathrm{Arccos}\,x}\,dx = \frac{x-k\sqrt{1-x^2}}{1+k^2}\,e^{k\,\mathrm{Arccos}\,x} + c$

A[1.32γ] $$\int \frac{\sqrt{\alpha^2-x^2}}{x}\,dx \underset{y=\frac{x}{\alpha}\ (\alpha>0)}{=} \alpha\int\frac{\sqrt{1-y^2}}{y}\,dy \underset{y=\sin\vartheta}{=} \alpha\int\frac{\cos^2\vartheta}{\sin\vartheta}\,d\vartheta$$

$$\underset{t=\cos\vartheta}{=} -\alpha\int\frac{t^2}{1-t^2}\,dt = \alpha\int\frac{t^2}{t^2-1}\,dt = \alpha t + \alpha\int\frac{dt}{t^2-1}$$

$$= \alpha t + \frac{\alpha}{2}\log\left|\frac{t-1}{t+1}\right| + c = \alpha\sqrt{1-y^2} + \frac{\alpha}{2}\log\left|\frac{\sqrt{1-y^2}-1}{\sqrt{1-y^2}+1}\right| + c$$

$$= \sqrt{\alpha^2-x^2} + \frac{\alpha}{2}\log\left|\frac{\sqrt{\alpha^2-x^2}-\alpha}{\sqrt{\alpha^2-x^2}+\alpha}\right| + c = \sqrt{\alpha^2-x^2} + \alpha\log\left|\frac{\alpha-\sqrt{\alpha^2-x^2}}{x}\right| + c$$

A[1.32δ] $$\int e^x\,\frac{1+x^2}{(1+x)^2}\,dx = -e^x(1+x^2)\frac{1}{1+x} + \int e^x(1+x^2+2x)\frac{1}{1+x}\,dx$$

$$= -e^x\,\frac{1+x^2}{1+x} + e^x(1+x) - e^x + c = e^x\,\frac{x+x^2-1-x^2}{1+x} + c = e^x\,\frac{x-1}{x+1} + c$$

A[1.34α] $$\int x\, e^{x^2}(1+x^2)\,dx = \frac{1}{2}\, e^{x^2}(1+x^2) - \frac{1}{2}\int e^{x^2}2x\,dx = \frac{1}{2}e^{x^2}x^2 + c$$

A[1.34β] $$\int \frac{\log(\alpha^2+\beta^2x^2)}{x^2}\,dx = -\frac{\log(\alpha^2+\beta^2x^2)}{x} + \int \frac{1}{x}\,\frac{1}{\alpha^2+\beta^2x^2}\,\beta^2 2x\,dx$$

$$= -\frac{1}{x}\log(\alpha^2+\beta^2x^2) + \frac{2\beta}{\alpha}\int\frac{\frac{\beta}{\alpha}dx}{1+(\frac{\beta}{\alpha}x)^2} = -\frac{1}{x}\log(\alpha^2+\beta^2x^2)$$

$$+\frac{2\beta}{\alpha}\operatorname{Arctg}\left(\frac{\beta}{\alpha}x\right) + C \quad (\alpha\neq 0)$$

A [1.35 δ] $\displaystyle\int\frac{x-1}{(x^2+2x+3)^2}dx = \frac{1}{2}\int\frac{2x+2}{(x^2+2x+3)^2}dx - 2\int\frac{dx}{((x+1)^2+2)^2}$

$$\int\frac{dx}{((x+1)^2+2)^2} = \frac{1}{4}\int\frac{dx}{\left(\left(\frac{x+1}{\sqrt{2}}\right)^2+1\right)^2} \underset{y=\frac{x+1}{\sqrt{2}}}{=} \frac{\sqrt{2}}{4}\int\frac{dy}{(y^2+1)^2}$$

$$\int\frac{dy}{(y^2+1)^2} = \int\frac{dy}{1+y^2} - \int\frac{y^2}{(1+y^2)^2}dy$$

$$\int\frac{y^2}{(1+y^2)^2}dy = -y\,\frac{1}{2}\,\frac{1}{1+y^2} + \frac{1}{2}\int\frac{dy}{1+y^2} = -\frac{1}{2}\frac{y}{1+y^2} + \frac{1}{2}\operatorname{Arctan} y + c$$

$$\Rightarrow \int\frac{dy}{(y^2+1)^2} = \frac{1}{2}\operatorname{Arctan} y + \frac{1}{2}\frac{y}{1+y^2} + c$$

$$\Rightarrow \int\frac{x-1}{(x^2+2x+3)^2}dx = -\frac{1}{2}\frac{x+2}{x^2+2x+3} - \frac{\sqrt{2}}{4}\operatorname{Arctg}\left(\frac{x+1}{\sqrt{2}}\right) + c$$

A [1.35 α]
$$\int \frac{x \log x}{(x^2-1)^{3/2}}\,dx = \log x\left(-\frac{1}{\sqrt{x^2-1}}\right) + \int \frac{1}{x}\frac{1}{\sqrt{x^2-1}}\,dx$$

$$\int \frac{1}{x}\frac{1}{\sqrt{x^2-1}}\,dx \underset{x=\frac{1}{\cos\vartheta}}{=} \int \cos\vartheta\,\frac{\cos\vartheta}{\sin\vartheta}\,\frac{1}{\cos^2\vartheta}\sin\vartheta\,d\vartheta = \vartheta + c = \operatorname{Arccos}\frac{1}{x} + c$$

$$dx = +\frac{1}{\cos^2\vartheta}\sin\vartheta\,d\vartheta$$

$$\sqrt{x^2-1} = \frac{\sin\vartheta}{\cos\vartheta}$$

$$\Rightarrow \int \frac{x \log x}{(x^2-1)^{3/2}}\,dx = -\frac{\log x}{\sqrt{x^2-1}} + \operatorname{Arccos}\frac{1}{x} + c$$

A [1.35 η]
$$\int \frac{\log(\cos x)}{\cos^2 x}\,dx = \log(\cos x)\,\mathrm{tg}\,x + \int \mathrm{tg}^2 x\,dx$$

$$\int \mathrm{tg}^2 x\,dx \underset{t=\mathrm{tg}\,x}{=} \int \frac{t^2}{1+t^2}\,dt = t - \int\frac{dt}{1+t^2} = t - \operatorname{Arctg} t + c = \mathrm{tg}\,x - x + c$$

$$dt = (1+t^2)\,dx$$

$$\Rightarrow \int \frac{\log(\cos x)}{\cos^2 x}\,dx = \left(\log(\cos x) + 1\right)\mathrm{tg}\,x - x + c$$

A [1.38 ε] $\int \operatorname{Arctg}\sqrt{1-x^2}\,dx = x\operatorname{Arctg}\sqrt{1-x^2} - \int x \frac{1}{1+1-x^2}\frac{1}{2}\frac{1}{\sqrt{1-x^2}}(-2x)\,dx$

$$= x\operatorname{Arctg}\sqrt{1-x^2} + \int \frac{x^2}{2-x^2}\frac{1}{\sqrt{1-x^2}}\,dx$$

$$\int \frac{x^2}{2-x^2}\frac{1}{\sqrt{1-x^2}}\,dx = -\int \frac{1}{\sqrt{1-x^2}}\,dx + 2\int \frac{1}{2-x^2}\frac{1}{\sqrt{1-x^2}}\,dx$$

$$\int \frac{1}{2-x^2}\frac{1}{\sqrt{1-x^2}}\,dx \underset{x=\sin\vartheta}{=} \int \frac{1}{1+\cos^2\vartheta}\frac{1}{\cos\vartheta}\cos\vartheta\,d\vartheta = \int \frac{1}{1+\cos^2\vartheta}\,d\vartheta =$$

$$\left[t = \operatorname{tg}\vartheta \Rightarrow dt = \frac{1}{\cos^2\vartheta}\,d\vartheta = (1+\operatorname{tg}^2\vartheta)\,d\vartheta = (1+t^2)\,d\vartheta, \right.$$

$$\left. t^2 = \frac{\sin^2\vartheta}{\cos^2\vartheta} = \frac{1}{\cos^2\vartheta} - 1 \Rightarrow \cos^2\vartheta = \frac{1}{1+t^2} \Rightarrow 1+\cos^2\vartheta = \frac{2+t^2}{1+t^2} \right]$$

$$\underset{t=\operatorname{tg}\vartheta}{=} \int \frac{1+t^2}{2+t^2}\frac{1}{1+t^2}\,dt = \int \frac{dt}{2+t^2} = \frac{1}{2}\int \frac{dt}{1+\left(\frac{t}{\sqrt{2}}\right)^2} = \frac{1}{\sqrt{2}}\operatorname{Arctg}\left(\frac{\operatorname{tg}\vartheta}{\sqrt{2}}\right) + c$$

$$= \frac{1}{\sqrt{2}}\operatorname{Arctg}\left(\frac{x}{\sqrt{2}\sqrt{1-x^2}}\right) + c$$

$$\Rightarrow \int \operatorname{Arctg}\sqrt{1-x^2}\,dx = x\operatorname{Arctg}\sqrt{1-x^2} - \operatorname{Arcsin} x + \sqrt{2}\operatorname{Arctg}\left(\frac{x}{\sqrt{2}\sqrt{1-x^2}}\right) + c$$

A [1.38 oz] $\int \operatorname{Arctg} x \, \log(1+x^2)\, dx$

$$\int \log(1+x^2)\,dx = x\log(1+x^2) - \int x \frac{2x}{1+x^2}\,dx$$
$$= x\log(1+x^2) - 2x + 2\operatorname{Arctg} x + c$$

$$\Rightarrow \int \operatorname{Arctg} x \, \log(1+x^2)\,dx = \operatorname{Arctg} x \left(x\log(1+x^2) - 2x + 2\operatorname{Arctg} x\right)$$
$$- \underbrace{\int \frac{x\log(1+x^2)}{1+x^2}\,dx}_{= \frac{1}{4}\log^2(1+x^2)+c} + \underbrace{\int \frac{2x}{1+x^2}\,dx}_{= \log(1+x^2)+c} - \underbrace{\int 2\operatorname{Arctg} x \frac{1}{1+x^2}\,dx}_{= \operatorname{Arctg}^2 x + c}$$

$$\Rightarrow \int \operatorname{Arctg} x \, \log(1+x^2)\,dx = x\operatorname{Arctg} x \, \log(1+x^2) - 2x\operatorname{Arctg} x$$
$$+ \operatorname{Arctg}^2 x - \frac{1}{4}\log^2(1+x^2) + \log(1+x^2) + c$$

A[1.1α] $\int (\sqrt{x}+1)(x-\sqrt{x}+1)\,dx \underset{y=\sqrt{x}}{=} 2\int (y+1)(y^2-y+1)\,y\,dy$ 4-A|7

$dy = \frac{dx}{2\sqrt{x}} = \frac{dx}{2y}$

$= 2\left(\frac{y^5}{5}+\frac{y^2}{2}\right)+c = \frac{2}{5}x^{\frac{5}{2}}+x+c$

A[1.1γ] $\int \frac{2x-\sqrt{\mathrm{Arcsin}\,x}}{\sqrt{1-x^2}}\,dx \underset{y=\mathrm{Arcsin}\,x}{=} \int (2\sin y-\sqrt{y})\,dy$

$dy = \frac{dx}{\sqrt{1-x^2}}$

$= -2\cos y - \frac{2}{3}y^{\frac{3}{2}}+c = -2\sqrt{1-\sin^2 y}-\frac{2}{3}y^{\frac{3}{2}}+c$

$= -2\sqrt{1-x^2}-\frac{2}{3}(\mathrm{Arcsin}\,x)^{\frac{3}{2}}+c$

A[1.1δ] $\int \frac{m}{\sqrt[3]{(\alpha+\beta x)^2}}\,dx = \frac{m}{\beta}\int \frac{dy}{y^{\frac{2}{3}}} = \frac{m}{\beta}\frac{y^{\frac{1}{3}}}{\frac{1}{3}}+c = \frac{3m}{\beta}(\alpha+\beta x)^{\frac{1}{3}}+c$

$y=\alpha+\beta x$

$dy=\beta\,dx \ (\beta\neq 0)$

A[1.1ε] $\int \frac{(1+x)^2}{x(1+x^2)}\,dx = \int \frac{1+x^2+2x}{x(1+x^2)}\,dx = \int \frac{dx}{x}+2\int\frac{dx}{1+x^2} =$

$= \ln|x| + 2\,\mathrm{Arctg}\,x + c$

A [1.2α] $\int \frac{x+2}{2x-1}\,dx = \frac{1}{2}\int \frac{2x+4}{2x-1}\,dx = \frac{1}{2}\int dx + \frac{5}{2}\int \frac{dx}{2x-1} =$

$= \frac{x}{2} + \frac{5}{4}\int \frac{d(2x-1)}{2x-1} = \frac{x}{2} + \frac{5}{4}\ln|2x-1| + c$

A [1.3 ε] $\int \frac{e^x}{e^x+1}\,dx \underset{\substack{y=e^x \\ dy=e^x dx}}{=} \int \frac{dy}{y+1} = \ln|y+1| + c = \ln|e^x+1| + c$

A [1.3 στ] $\int \frac{1+\cos^2 x}{1+\cos(2x)}\,dx = \int \frac{1+\cos^2 x}{2\cos^2 x}\,dx = \frac{1}{2}\left(\int \frac{dx}{\cos^2 x} + \int dx\right) = \frac{1}{2}(\mathrm{tg}\,x + x) + c$

A [1.4 α] $\int \frac{\sqrt{\alpha^2 - x^2}}{x^2}\,dx \underset{\substack{x=\alpha \sin\vartheta,\ \vartheta\in(-\frac{\pi}{2},\frac{\pi}{2}) \\ dx=\alpha\cos\vartheta\, d\vartheta\ (\alpha>0)}}{=} \int \frac{\alpha^2\cos^2\vartheta}{\alpha^2\sin^2\vartheta}\,d\vartheta =$

$= \int \frac{1-\sin^2\vartheta}{\sin^2\vartheta}\,d\vartheta = -\mathrm{ctg}\,\vartheta - \vartheta + c = -\mathrm{ctg}\left(\mathrm{Arcsin}\frac{x}{\alpha}\right) - \mathrm{Arcsin}\frac{x}{\alpha} + c$

$= -\frac{\sqrt{1-\left(\frac{x}{\alpha}\right)^2}}{\frac{x}{\alpha}} - \mathrm{Arcsin}\frac{x}{\alpha} + c = -\frac{\sqrt{\alpha^2-x^2}}{x} - \mathrm{Arcsin}\frac{x}{\alpha} + c$

A [1.4γ] $\displaystyle\int \frac{x^2}{\sqrt{\alpha^2-x^2}}\,dx \underset{\substack{x=\alpha\sin\vartheta\\ dx=\alpha\cos\vartheta\, d\vartheta}}{=} \int \alpha^2\sin^2\vartheta\, d\vartheta = \alpha^2\int \frac{1-\cos(2\vartheta)}{2}\, d\vartheta$

$\displaystyle = \frac{\alpha^2}{2}\left(\vartheta - \frac{1}{2}\sin(2\vartheta)\right)+c = \frac{\alpha^2}{2}\left(\operatorname{Arcsin}\frac{x}{\alpha} - \frac{x}{\alpha}\sqrt{1-\left(\frac{x}{\alpha}\right)^2}\right)+c$

$\displaystyle = \frac{\alpha^2}{2}\operatorname{Arcsin}\frac{x}{\alpha} - \frac{x}{2}\sqrt{\alpha^2-x^2}+c$

A [1.5β] $\displaystyle\int \frac{dx}{x^2\sqrt{\alpha^2+x^2}} \underset{\boxed{\substack{x=\alpha\sinh z\\ dx=\alpha\cosh z\, dz}}}{=} \int \frac{dz}{\alpha^2\sinh^2 z} = -\frac{1}{\alpha^2}\frac{\cosh z}{\sinh z}+c$

$\displaystyle = -\frac{1}{\alpha}\frac{\sqrt{1+\left(\frac{x}{\alpha}\right)^2}}{x}+c = -\frac{1}{\alpha^2}\frac{\sqrt{\alpha^2+x^2}}{x}+c$

A [1.7β] $\displaystyle\int (\cos x-\sin x)\sqrt{\sin x+\cos x}\,dx \underset{\substack{t=\sin x+\cos x\\ dt=(\cos x-\sin x)dx}}{=} \int\sqrt{t}\,dt = \frac{2}{3}t^{\frac{3}{2}}+c$

$\displaystyle = \frac{2}{3}(\sin x+\cos x)^{\frac{3}{2}}+c$

A [1.7 γ] $\displaystyle\int \frac{e^{\alpha \operatorname{Arcsin} x}}{\sqrt{1-x^2}}\,dx \underset{y=\operatorname{Arcsin} x}{=} \int e^{\alpha y}\,dy = \frac{1}{\alpha} e^{\alpha \operatorname{Arcsin} x} + c$

A [1.7 δ] $\displaystyle\int \frac{e^{\alpha \operatorname{Arctg} x}}{1+x^2}\,dx \underset{y=\operatorname{Arctg} x}{=} \int e^{\alpha y}\,dy = \frac{1}{\alpha} e^{\alpha \operatorname{Arctg} x} + c$

A [1.8 α] $\displaystyle\int \operatorname{tg}\frac{1}{x}\,\frac{dx}{x^2} \underset{y=\frac{1}{x}}{=} -\int \operatorname{tg} y\,dy \underset{z=\cos y}{=} \int \frac{dz}{z} =$

$= \ln|z| + c = \ln\left|\cos\left(\frac{1}{x}\right)\right| + c$

A [1.8 β] $\displaystyle\int \frac{e^x}{e^{2x}+1}\,dx \underset{y=e^x}{=} \int \frac{dy}{1+y^2} = \operatorname{Arctg}(e^x) + c$

A [1.8 γ] $\displaystyle\int \left(1-\frac{1}{x^2}\right) e^{x+\frac{1}{x}}\,dx \underset{y=x+\frac{1}{x}}{=} e^{x+\frac{1}{x}} + c$

A [1.8 δ] $\int \frac{\sin x}{\sqrt{\alpha^2+\beta^2-2\alpha\beta\cos x}}\,dx \underset{y=-\cos x}{=} \int \frac{dy}{\sqrt{\alpha^2+\beta^2+2\alpha\beta y}}\,dy$

$\underset{t=\sqrt{\alpha^2+\beta^2+2\alpha\beta y}}{=} \frac{1}{\alpha\beta}\int \frac{t}{t}\,dt = \frac{1}{\alpha\beta}\sqrt{\alpha^2+\beta^2-2\alpha\beta\cos x} + C$

$dt = \frac{\alpha\beta}{t}\,dy$

A [1.10 α] $\int x\cos x\,dx = x\sin x - \int \sin x\,dx = x\sin x + \cos x + C$

A [1.11 α] $\int (\mathrm{Arcsin}\,x)^2\,dx = x(\mathrm{Arcsin}\,x)^2 - \int x\, 2(\mathrm{Arcsin}\,x)\frac{1}{\sqrt{1-x^2}}\,dx$

$= x(\mathrm{Arcsin}\,x)^2 + (\mathrm{Arcsin}\,x)\,2\sqrt{1-x^2} - 2\int \sqrt{1-x^2}\frac{1}{\sqrt{1-x^2}}\,dx$

$= x\,(\mathrm{Arcsin}\,x)^2 + 2\sqrt{1-x^2}\,\mathrm{Arcsin}\,x - 2x + C$

A [1.11 γ] $\int \cos(\log x)\,dx = x\cos(\log x) + \int x \sin(\log x)\frac{1}{x}\,dx$

$= x\cos(\log x) + x\sin(\log x) - \int x\cos(\log x)\frac{1}{x}\,dx$

$= \frac{x}{2}\left(\cos(\log x) + \sin(\log x)\right) + c$

A [1.11 δ] $\int \log\left(\sqrt{1-x}+\sqrt{1+x}\right)dx = x\log\left(\sqrt{1-x}+\sqrt{1+x}\right)$

$- \int x \frac{1}{\sqrt{1-x}+\sqrt{1+x}}\,\frac{1}{2}\left(\frac{1}{\sqrt{1+x}} - \frac{1}{\sqrt{1-x}}\right)dx$

$= -\frac{1}{4}\int \frac{1}{\sqrt{1-x^2}}\left(\sqrt{1-x}-\sqrt{1+x}\right)^2 dx = -\frac{1}{4}\int\frac{1}{\sqrt{1-x^2}}\left(1-x+1+x-2\sqrt{1-x}\sqrt{1+x}\right)dx$

$= -\frac{1}{2}\int\frac{dx}{\sqrt{1-x^2}} + \frac{1}{2}\int dx = -\frac{1}{2}\operatorname{Arcsin} x + \frac{1}{2}x$

$\Rightarrow \int \log\left(\sqrt{1-x}+\sqrt{1+x}\right)dx = x\log\left(\sqrt{1-x}+\sqrt{1+x}\right) + \frac{1}{2}\operatorname{Arcsin} x - \frac{x}{2} + c$

A [1.13 α] $\int x^2 e^x \sin x \, dx$

$$\int e^x \sin x \, dx = e^x \sin x - \int e^x \cos x \, dx = e^x(\sin x - \cos x) - \int e^x \sin x \, dx$$

$$= \frac{1}{2} e^x(\sin x - \cos x) + c \quad (1)$$

$$\Rightarrow \int x^2 e^x \sin x \, dx \underset{(1)}{=} \frac{1}{2} x^2 e^x (\sin x - \cos x) - \int x e^x (\sin x - \cos x) dx \ (3)$$

$$\int e^x \cos x \, dx = e^x \cos x + \int e^x \sin x \, dx \underset{(1)}{=} \frac{1}{2} e^x (\sin x + \cos x) + c \ (2)$$

$$\Rightarrow \int e^x (\sin x - \cos x) \, dx \underset{(1),(2)}{=} -e^x \cos x + c$$

$$\Rightarrow \int x e^x (\sin x - \cos x) dx = -x e^x \cos x + \int e^x \cos x \, dx$$

$$\underset{(2)}{=} -x e^x \cos x + \frac{1}{2} e^x (\sin x + \cos x) + c$$

$$\Rightarrow \int x^2 e^x \sin x \, dx \underset{(3)}{=} \frac{1}{2} x^2 e^x (\sin x - \cos x) + x e^x \cos x - \frac{1}{2} e^x (\sin x + \cos x) + c$$

$$\left[= \frac{1}{2} e^x \left((x^2 - 1) \sin x - (x-1)^2 \cos x \right) + c \right]$$

A [1.14α] $\int x \operatorname{Arctg} x \, dx = \frac{x^2}{2} \operatorname{Arctg} x - \int \frac{x^2}{2} \frac{1}{1+x^2} dx$

$= \frac{x^2}{2} \operatorname{Arctg} x - \frac{x}{2} + \frac{1}{2} \operatorname{Arctg} x + c$

A [1.14β] $\int \left(x + \frac{1}{x} + \frac{1}{x^2}\right) \log x \, dx = \left(\frac{x^2}{2} + \log x - \frac{1}{x}\right) \log x$

$- \int \left(\frac{x^2}{2} + \log x - \frac{1}{x}\right) \frac{1}{x} dx = \left(\frac{x^2}{2} + \log x - \frac{1}{x}\right) \log x$

$- \frac{x^2}{4} - \frac{1}{x} - \frac{\log^2 x}{2} + c = \left(\frac{x^2}{2} - \frac{1}{x}\right) \log x - \frac{x^2}{4} - \frac{1}{x} + \frac{\log^2 x}{2} + c$

A [1.14γ] $\int \frac{\operatorname{Arcsin} x}{\sqrt{x+1}} dx = 2\sqrt{x+1} \operatorname{Arcsin} x - 2 \int \frac{\sqrt{x+1}}{\sqrt{1-x^2}} dx$

$= 2\sqrt{x+1} \operatorname{Arcsin} x + 4\sqrt{1-x} + c$

A [1.14ε] $\int \frac{e^{m \operatorname{Arctg} x}}{(1+x^2)^2} dx = \frac{1}{m} e^{m \operatorname{Arctg} x} \frac{1}{1+x^2}$

$+ \frac{1}{m} \int e^{m \operatorname{Arctg} x} \frac{2x}{(1+x^2)^2} dx$

$$\int e^{m\,\mathrm{Arctg}\,x}\frac{x}{(1+x^2)^2}\,dx = \frac{1}{m}e^{m\,\mathrm{Arctg}\,x}\frac{x}{1+x^2}$$

$$-\frac{1}{m}\int e^{m\,\mathrm{Arctg}\,x}\frac{1-x^2}{(1+x^2)^2}\,dx$$

$$= \frac{1}{m}e^{m\,\mathrm{Arctg}\,x}\frac{x}{1+x^2} + \frac{1}{m}\int\frac{e^{m\,\mathrm{Arctg}\,x}}{1+x^2}\,dx - \frac{2}{m}\int\frac{e^{m\,\mathrm{Arctg}\,x}}{(1+x^2)^2}\,dx$$

$$= \frac{1}{m}e^{m\,\mathrm{Arctg}\,x}\left(\frac{x}{1+x^2}+\frac{1}{m}\right) - \frac{2}{m}\int\frac{e^{m\,\mathrm{Arctg}\,x}}{(1+x^2)^2}\,dx$$

$$\Longrightarrow \int\frac{e^{m\,\mathrm{Arctg}\,x}}{(1+x^2)^2}\,dx = \frac{1}{m}e^{m\,\mathrm{Arctg}\,x}\left(\frac{1}{1+x^2}+\frac{2}{m}\left(\frac{x}{1+x^2}+\frac{1}{m}\right)\right)$$

$$-\frac{4}{m^2}\int\frac{e^{m\,\mathrm{Arctg}\,x}}{(1+x^2)^2}\,dx = \frac{1}{m\left(1+\frac{4}{m^2}\right)}e^{m\,\mathrm{Arctg}\,x}\left(\frac{1}{1+x^2}+\frac{2x}{m(1+x^2)}+\frac{2}{m^2}\right)+c$$

$$= \frac{1}{m^2+4}e^{m\,\mathrm{Arctg}\,x}\,\frac{m^2+2xm+2(1+x^2)}{m(1+x^2)} + C$$

A[1.15α] $$\int \frac{x\sin x}{(1+\cos x)^2}dx = x\frac{1}{1+\cos x} - \int\frac{dx}{1+\cos x} = \frac{x}{1+\cos x} - \int\frac{dx}{2\cos^2\frac{x}{2}}$$

$$= \frac{x}{1+\cos x} - \int\frac{d\left(\frac{x}{2}\right)}{\cos^2\frac{x}{2}} = \frac{x}{1+\cos x} - \operatorname{tg}\frac{x}{2} + c$$

A[1.15β] $$\int\frac{x^2e^x}{(x+2)^2}dx = -\frac{x^2e^x}{x+2} + \int\frac{(2x+x^2)e^x}{x+2}dx = -\frac{x^2e^x}{x+2}$$

$$= -\frac{x^2e^x}{x+2} + \int xe^x dx = -\frac{x^2e^x}{x+2} + xe^x - \int e^x dx = e^x\,\frac{x-2}{x+2} + c$$

A[1.19δ] $$\int\frac{dx}{x^2-\alpha^2} = \int\frac{dx}{(x-\alpha)(x+\alpha)} = \frac{1}{2\alpha}\left(\int\frac{dx}{x-\alpha} - \int\frac{dx}{x+\alpha}\right)$$

$$\left[\frac{1}{(x-\alpha)(x+\alpha)} = \frac{A}{x-\alpha} + \frac{B}{x+\alpha} \Rightarrow 1 = A(x+\alpha) + B(x-\alpha) \Rightarrow A=-B,\ A\alpha - B\alpha = 1\right.$$

$$\left.\Rightarrow A = \frac{1}{2\alpha} = -B \quad (\alpha\neq 0)\right] \Rightarrow \int\frac{dx}{x^2-\alpha^2} = \frac{1}{2\alpha}\ln\left|\frac{x-\alpha}{x+\alpha}\right| + c$$

A[1.19ε] $$\int\frac{dx}{\alpha^2-x^2} = -\int\frac{dx}{x^2-\alpha^2} \underset{A[1.19\delta]}{=} \frac{1}{2\alpha}\ln\left|\frac{x+\alpha}{x-\alpha}\right| + c$$

A [1.20 α] $\int \frac{(x-1)^2}{x^2+2x+2}\,dx = \int \frac{x^2-2x+1}{x^2+2x+2}\,dx = x - \int \frac{4x+1}{x^2+2x+2}\,dx$

$= x - 2\int \frac{2x+2}{x^2+2x+2}\,dx + 3\int \frac{dx}{x^2+2x+2} = x - 2\ln|x^2+2x+2| + 3\int \frac{dx}{(x+1)^2+1}$

$= x - 2\ln(x^2+2x+2) + 3\,\mathrm{Arctg}(x+1) + c$

A [1.21 ε] $\int \frac{(x^2-1)^2}{(1+x)(1+x^2)^3}\,dx = \int \frac{(1+x)(x-1)^2}{(1+x^2)^3}\,dx =$

$= \int \frac{x^3-2x^2+x+x^2-2x+1}{(1+x^2)^3}\,dx = \int \frac{x^3+x}{(1+x^2)^3}\,dx - \int \frac{x^2+1}{(1+x^2)^3}\,dx$

$-\int \frac{2x}{(1+x^2)^3}\,dx + 2\int \frac{dx}{(1+x^2)^3} = -\frac{1}{2}\frac{1}{1+x^2} - \int \frac{dx}{(1+x^2)^2} + \frac{1}{2}\frac{1}{(1+x^2)^2} + 2\int \frac{dx}{(1+x^2)^3}$

$\int \frac{dx}{(1+x^2)^3} = \int \frac{dx}{(1+x^2)^2} - \int \frac{x^2}{(1+x^2)^3}\,dx$

$\int \frac{x^2}{(1+x^2)^3}\,dx = x\,\frac{1}{2}\,\frac{(1+x^2)^{-2}}{-2} - \int \frac{1}{2}\,\frac{(1+x^2)^{-2}}{-2}\,dx = -\frac{x}{4}\frac{1}{(1+x^2)^2} + \frac{1}{4}\int \frac{dx}{(1+x^2)^2}$

$$\Longrightarrow \int \frac{dx}{(1+x^2)^3} = \frac{x}{4}\frac{1}{(1+x^2)^2} + \frac{3}{4}\int\frac{dx}{(1+x^2)^2}$$

$$\int \frac{dx}{(1+x^2)^2} = \int \frac{dx}{1+x^2} - \int \frac{x^2}{(1+x^2)^2} = \operatorname{Arctg} x - \left(x\,\frac{1}{2}\,\frac{(1+x^2)^{-1}}{-1} + \frac{1}{2}\int\frac{dx}{1+x^2}\right)$$

$$= \frac{1}{2}\operatorname{Arctg} x + \frac{x}{2}\frac{1}{1+x^2} + C$$

$$\Longrightarrow \int \frac{dx}{(1+x^2)^3} = \frac{x}{4}\frac{1}{(1+x^2)^2} + \frac{3}{8}\operatorname{Arctg} x + \frac{3}{8}\frac{x}{1+x^2} + C$$

$$\Longrightarrow \int \frac{(x^2-1)^2}{(1+x)(1+x^2)^3} = -\frac{1}{2}\frac{1}{1+x^2} - \frac{1}{2}\operatorname{Arctg} x - \frac{x}{2}\frac{1}{1+x^2} + \frac{1}{2}\frac{1}{(1+x^2)^2}$$

$$+ \frac{x}{2}\frac{1}{(1+x^2)^2} + \frac{3}{4}\operatorname{Arctg} x + \frac{3}{4}\frac{x}{1+x^2} + C$$

$$= \frac{1}{4}\operatorname{Arctg} x + \frac{1}{2}\frac{1+x}{(1+x^2)^2} + \frac{1}{4}\frac{x}{1+x^2} - \frac{1}{2}\frac{1}{1+x^2} + C$$

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A [1.22 β] $\int \sqrt{\frac{1-x}{1+x}}\,\frac{1}{x}\,dx = \int t\,\frac{1+t^2}{t^2-1}\,\frac{4t}{(1+t^2)^2}\,dt = 4\int \frac{t^2}{(t^2-1)(1+t^2)}\,dt$

$t=\sqrt{\frac{1-x}{1+x}}$

$dt = \frac{1}{2t}\,\frac{-1-x-1+x}{(1+x)^2}\,dx = -\frac{1}{t}\,\frac{1}{(1+x)^2}\,dx = \frac{-(1+t^2)^2}{4t}\,dx$

[(*) $t^2 = \frac{1-x}{1+x} \Leftrightarrow -t^2 = \frac{x-1}{1+x} = 1-\frac{2}{1+x} \Leftrightarrow \frac{1+t^2}{2} = \frac{1}{1+x} \Leftrightarrow 1+x = \frac{2}{1+t^2} \Leftrightarrow x = \frac{1-t^2}{1+t^2}$]

$= 4\int \frac{dt}{1+t^2} + 4\int \frac{dt}{(t-1)(t+1)(1+t^2)}\,dt$

$\frac{1}{(t-1)(t+1)(1+t^2)} = \frac{A}{t-1} + \frac{B}{t+1} + \frac{\Gamma t+\Delta}{1+t^2} \Rightarrow 1 = A(t+1)(1+t^2) + B(t-1)(1+t^2)$

$+\Gamma t(t^2-1) + \Delta(t^2-1) \Rightarrow A+B+\Gamma=0,\ A-B+\Delta=0,\ A+B-\Gamma=0,$

$A-B-\Delta=1 \Rightarrow \Gamma=0,\ A+B=0,\ A-B=-\Delta,\ \Delta=-\frac{1}{2},\ A-B=\frac{1}{2},$

$A=\frac{1}{4},\ B=-\frac{1}{4} \Rightarrow \int \frac{dt}{t-1}$

$$\Rightarrow 4\int \frac{dt}{(t-1)(t+1)(1+t^2)} = \int\left(\frac{1}{t-1} - \frac{1}{t+1} - 2\frac{1}{1+t^2}\right)dt$$

$$\Rightarrow 4\int \frac{t^2}{(t^2-1)(1+t^2)}\,dt = 4\,\mathrm{Arctg}\,t + \log|t-1| - \log|t+1| - 2\,\mathrm{Arctg}\,t + c$$

$$= 2\,\mathrm{Arctg}\,t + \log\left|\frac{t-1}{t+1}\right| + c$$

$$\Rightarrow \int \sqrt{\frac{1-x}{1+x}}\,\frac{1}{x}\,dx = 2\,\mathrm{Arctg}\sqrt{\frac{1-x}{1+x}} + \log\left|\frac{\sqrt{\frac{1-x}{1+x}}-1}{\sqrt{\frac{1-x}{1+x}}+1}\right| + c$$

$$= 2\,\mathrm{Arctg}\sqrt{\frac{1-x}{1+x}} + \log\left|\frac{\sqrt{1-x}-\sqrt{1+x}}{\sqrt{1-x}+\sqrt{1+x}}\right| + c$$

A [1.22 δ]

$$\int \sqrt{\frac{1-\sqrt{x}}{1+\sqrt{x}}}\,dx = \int \frac{\sqrt{1-x}}{1+\sqrt{x}}\,dx \underset{\substack{t=\sqrt{x} \\ dt=\frac{1}{2}\frac{1}{t}dx}}{=} \int \frac{\sqrt{1-t^2}}{1+t}\,2t\,dt$$

$$= 2\int \sqrt{1-t^2}\,dt - 2\int \frac{\sqrt{1-t^2}}{1+t}\,dt$$

$$\int \sqrt{1-t^2}\,dt \underset{t=\sin\vartheta}{=} \int \cos^2\vartheta\,d\vartheta = \cos\vartheta\sin\vartheta + \int \sin^2\vartheta\,d\vartheta = \cos\vartheta\sin\vartheta$$

$$+\vartheta - \int \cos^2\vartheta\,d\vartheta = \frac{1}{2}(\cos\vartheta\sin\vartheta+\vartheta)+c = \frac{1}{2}\left(\sqrt{1-t^2}\,t + \operatorname{Arcsin} t\right)+c$$

$$\int \frac{\sqrt{1-t^2}}{1+t}\,dt \underset{t=\sin\vartheta}{=} \int \frac{\cos^2\vartheta}{1+\sin\vartheta}\,d\vartheta = \int \frac{1-\sin^2\vartheta}{1+\sin\vartheta}\,d\vartheta = \int (1-\sin\vartheta)\,d\vartheta$$

$$= \vartheta + \cos\vartheta + c = \vartheta + \sqrt{1-\sin^2\vartheta} + c = \operatorname{Arcsin} t + \sqrt{1-t^2} + c$$

$$\Rightarrow \int \frac{\sqrt{1-t^2}}{1+t}\,2t\,dt = \sqrt{1-t^2}\,t + \operatorname{Arcsin} t - 2\operatorname{Arcsin} t - 2\sqrt{1-t^2} + c$$

$$\Rightarrow \int \sqrt{\frac{1-\sqrt{x}}{1+\sqrt{x}}}\,dx = \sqrt{1-x}\,(\sqrt{x}-2) - \operatorname{Arcsin}\sqrt{x} + c$$

A [1.24 γ] $\displaystyle \int \frac{dx}{x\sqrt{-x^2+5x-6}} \underset{x=\frac{1}{t}}{=} \int t\left(-\frac{1}{t^2}\right)\frac{1}{\sqrt{-\frac{1}{t^2}+\frac{5}{t}-6}}\,dt =$

$$= -\int \frac{dt}{\sqrt{-1+5t-6t^2}} = -\frac{12}{\sqrt{6}}\int \frac{dt}{\sqrt{1-(12t-5)^2}} \underset{y=12t-5}{=} -\frac{1}{\sqrt{6}}\int \frac{dy}{\sqrt{1-y^2}} =$$

$$= -\frac{1}{\sqrt{6}}\,\mathrm{Arc\,sin}\,(12t-5)+c = \frac{1}{\sqrt{6}}\,\mathrm{Arc\,sin}\left(\frac{5x-12}{x}\right)+c$$

[Επαλήθευση: $\displaystyle \frac{1}{\sqrt{6}}\,\frac{1}{\left(1-\left(\frac{5x-12}{x}\right)^2\right)^{\frac{1}{2}}}\,\frac{12}{x^2} = \frac{12}{\sqrt{6}}\,\frac{1}{x}\,\frac{1}{(x^2-25x^2-(12)^2+120x)^{1/2}}$

$$= \frac{1}{x}\,\frac{12}{\sqrt{6}}\,\frac{1}{\sqrt{24}}\,\frac{1}{(-x^2-6+5x)^{1/2}}$$ (βλ. και [Nr. I, A.4.44])]

A [1.24 ε] $\displaystyle \int \frac{dx}{x\sqrt{x^2-2x+3}} \underset{x=\frac{1}{t}}{=} -\int \frac{dt}{\sqrt{3t^2-2t+1}} = -\sqrt{\frac{3}{2}}\int \frac{dt}{\sqrt{\left(\frac{3t-1}{\sqrt{2}}\right)^2+1}}$

$$\underset{y=\frac{3t-1}{\sqrt{2}}}{=} -\frac{1}{\sqrt{3}}\int \frac{dy}{\sqrt{y^2+1}} = -\frac{1}{\sqrt{3}}\log\left(y+\sqrt{y^2+1}\right)+c \quad \text{με } y=\frac{1}{\sqrt{2}}\left(\frac{3}{x}-1\right)$$

[Επαλήθευση: $\displaystyle -\frac{1}{\sqrt{3}}\,\frac{1}{y+\sqrt{y^2+1}}\left(y'+\frac{1}{2}\frac{1}{\sqrt{y^2+1}}\,2yy'\right) =$

$$= -\frac{1}{\sqrt{3}}\,\frac{1}{y+\sqrt{y^2+1}}\left(1+\frac{y}{\sqrt{y^2+1}}\right)y' = -\frac{1}{\sqrt{3}}\,\frac{1}{\sqrt{y^2+1}}\,y' = \frac{1}{x^2}\sqrt{\frac{3}{2}}\,\frac{1}{\sqrt{\frac{1}{2}\left(\frac{3}{x}-1\right)^2+1}}$$

$$\left.= \frac{1}{x}\,\frac{1}{\sqrt{\frac{2}{3}x^2\left(\frac{1}{2}\left(\frac{3}{x}-1\right)^2+1\right)}} = \frac{1}{x}\,\frac{1}{\sqrt{\frac{2}{3}x^2+\frac{1}{3}(3-x)^2}} = \frac{1}{x}\,\frac{1}{\sqrt{\frac{2}{3}x^2+3+\frac{1}{3}x^2-2x}}\right]$$

A [1.24ζ] $\displaystyle\int\sqrt{1-4x-x^2}\,dx = \int\sqrt{5-(x+2)^2}\,dx = \sqrt{5}\int\sqrt{1-\left(\frac{x+2}{\sqrt{5}}\right)^2}\,dx$

$$\underset{y=\frac{x+2}{\sqrt{5}}}{=} 5\int\sqrt{1-y^2}\,dy = 5\left(y\sqrt{1-y^2}-\int y\,\frac{1}{2}\,\frac{1}{\sqrt{1-y^2}}(-2y)\,dy\right)$$

$$= 5\left(y\sqrt{1-y^2}+\int\frac{y^2}{\sqrt{1-y^2}}\,dy\right) = 5\left(y\sqrt{1-y^2}-\int\sqrt{1-y^2}\,dy+\int\frac{dy}{\sqrt{1-y^2}}\right)$$

$$= \frac{5}{2}\left(y\sqrt{1-y^2}+\operatorname{Arcsin} y\right)+c = \frac{5}{2}\left(\frac{x+2}{\sqrt{5}}\sqrt{1-\left(\frac{x+2}{\sqrt{5}}\right)^2}+\operatorname{Arcsin}\frac{x+2}{\sqrt{5}}\right)+c$$

$$= \frac{1}{2}(x+2)\sqrt{1-4x-x^2}+\frac{5}{2}\operatorname{Arcsin}\frac{x+2}{\sqrt{5}}+c$$

A [1.25ε] $\displaystyle\int\frac{dx}{(x-1)\sqrt{x^2-4x+2}} =: I$, $\quad x-1=\frac{1}{t} \Rightarrow x=\frac{t+1}{t}$, $\quad dx = -\frac{1}{t^2}\,dt$

$$\Rightarrow I = -\int t \frac{t}{\sqrt{(t+1)^2 - 4t(t+1) + 2t^2}} \frac{1}{t^2} dt = -\int \frac{dt}{\sqrt{-t^2 - 2t + 1}} = -\int \frac{dt}{\sqrt{2 - (t+1)^2}}$$

$$= -\frac{1}{\sqrt{2}} \int \frac{dt}{\sqrt{1 - \left(\frac{t+1}{\sqrt{2}}\right)^2}} = -\operatorname{Arcsin}\left(\frac{t+1}{\sqrt{2}}\right) + c = -\operatorname{Arcsin}\left(\frac{x}{\sqrt{2}(x-1)}\right) + c$$

$$\left[= \operatorname{Arcsin}\left(\frac{x}{\sqrt{2}(1-x)}\right) + c = \operatorname{Arccos}\left(\frac{x}{\sqrt{2}(x-1)}\right)\left(-\frac{\pi}{2}\right) + c \right]$$

A [1.26 δ] $\displaystyle\int \frac{dx}{\sqrt[4]{1+x^4}} = \int \frac{dx}{(1+x^4)^{\frac{1}{4}}} = \int (1+x^4)^{-\frac{1}{4}} dx = \int x^0 (1+x^4)^{-\frac{1}{4}} dx$

$-\frac{1}{4}$, $\frac{0+1}{4} \notin \mathbb{Z}$, $\frac{0+1}{4} + \left(-\frac{1}{4}\right) = 0 \in \mathbb{Z}$ $\underset{\text{Διων. ολοκλ.}}{\Longrightarrow}$ $\frac{1}{x^4} + 1 = t^4$

$$\Rightarrow x^4 = \frac{1}{t^4 - 1} \Rightarrow 1 + x^4 = \frac{t^4}{t^4 - 1}, \quad x = \frac{1}{(t^4-1)^{\frac{1}{4}}} \Rightarrow dx = -\frac{1}{4} \frac{4t^3}{(t^4-1)^{\frac{5}{4}}} dt$$

$$\Rightarrow \int \frac{dx}{(1+x^4)^{\frac{1}{4}}} = \int \frac{(t^4-1)^{\frac{1}{4}}}{t} (-1) \frac{t^3}{(t^4-1)^{\frac{5}{4}}} dt = -\int \frac{t^2}{t^4 - 1} dt =$$

$$= -\int \frac{dt}{t^2+1} - \int \frac{dt}{t^4-1} = -\frac{1}{2}\int \frac{dt}{t^2+1} - \frac{1}{4}\int \frac{dt}{t-1} + \frac{1}{4}\int\frac{dt}{t+1}$$

$$\left[\frac{1}{t^4-1} = \frac{1}{(t^2-1)(t^2+1)} = \frac{1}{2}\left(\frac{1}{t^2-1} - \frac{1}{t^2+1}\right) = \frac{1}{2}\left(\frac{1}{2}\left(\frac{1}{t-1} - \frac{1}{t+1}\right) - \frac{1}{t^2+1}\right)\right]$$

$$= -\frac{1}{2}\operatorname{Arctg} t - \frac{1}{4}\ln\left|\frac{t-1}{t+1}\right| + c \quad \text{με } t = \frac{(1+x^4)^{\frac{1}{4}}}{x}$$

$$= -\frac{1}{2}\operatorname{Arctg}\left(\frac{(1+x^4)^{\frac{1}{4}}}{x}\right) - \frac{1}{4}\ln\left|\frac{(1+x^4)^{\frac{1}{4}}-x}{(1+x^4)^{\frac{1}{4}}+x}\right| + c$$

A [1.26 ε] $\displaystyle\int \frac{\sqrt[3]{1+\sqrt[4]{x}}}{\sqrt{x}}\,dx = \int x^{-\frac{1}{2}}\left(1+x^{1/4}\right)^{1/3}dx =: I$

$p = \frac{1}{3} \notin \mathbb{Z}$, $\frac{m+1}{n} = \frac{-\frac{1}{2}+1}{\frac{1}{4}} = 2 \in \mathbb{Z} \Rightarrow$ Διωνυμ. Ολοκλ. $\quad 1+x^{\frac{1}{4}} = t^3 \Leftrightarrow t = \left(1+x^{\frac{1}{4}}\right)^{\frac{1}{3}}$,

$$x = (t^3-1)^4 \Rightarrow dx = 4(t^3-1)^3 3t^2\,dt \Rightarrow I := \int \frac{1}{(t^3-1)^2}\, t\, 12t^2 (t^3-1)^3\,dt$$

$$= 12\int (t^3-1)t^3\,dt = 12\frac{t^7}{7} - 12\frac{t^4}{4} + c = \frac{12}{7}\left(1+x^{\frac{1}{4}}\right)^{\frac{7}{3}} - 3\left(1+x^{\frac{1}{4}}\right)^{\frac{4}{3}} + c$$

$$\left[= \left(\frac{12}{7}\left(1+x^{\frac{1}{4}}\right)^2 - 3\left(1+x^{\frac{1}{4}}\right)\right)\left(1+x^{\frac{1}{4}}\right)^{\frac{1}{3}} + c = \frac{3}{7}\left(4\sqrt{x} + x^{\frac{1}{4}} - 3\right)\left(1+x^{\frac{1}{4}}\right)^{\frac{1}{3}} + c\right]$$

A [1.27γ] $\int \frac{dx}{\alpha\cos x + \beta\sin x} =$

$$t = \operatorname{tg}\frac{x}{2} = \frac{\sin\frac{x}{2}}{\cos\frac{x}{2}} = \frac{1}{2}\frac{\sin x}{\cos^2\frac{x}{2}} = \frac{\sin x}{\cos x + 1} = \frac{\sqrt{1-\cos^2 x}}{1+\cos x} \Rightarrow t^2 = \frac{1-\cos x}{1+\cos x}$$

$$= \frac{2}{1+\cos x} - 1 \Rightarrow \frac{t^2+1}{2} = \frac{1}{1+\cos x} \Rightarrow \cos x = \frac{2}{t^2+1} - 1 = \frac{1-t^2}{1+t^2}$$

$$\Rightarrow \sin x = \sqrt{1-\cos^2 x} = \frac{\sqrt{(1+t^2)^2 - (1-t^2)^2}}{1+t^2} = \frac{2t}{1+t^2},$$

$$t = \operatorname{tg}\frac{x}{2} \Rightarrow dt = \frac{1}{2}(1+t^2)\,dx \Rightarrow dx = \frac{2}{1+t^2}\,dt$$

$$\Rightarrow \int \frac{dx}{\alpha\cos x + \beta\sin x} = \int \frac{1+t^2}{\alpha(1-t^2) + \beta 2t}\,\frac{2}{1+t^2}\,dt \underset{\alpha>0}{=} -\frac{2}{\alpha}\int \frac{dt}{t^2 - 1 - \frac{2\beta}{\alpha}t}$$

$$= -\frac{2}{\alpha}\int \frac{dt}{\left(t-\frac{\beta}{\alpha}\right)^2 - \left(1+\left(\frac{\beta}{\alpha}\right)^2\right)} = -\frac{2}{\alpha}\int \frac{dt}{\left(t-\frac{\beta}{\alpha}-\sqrt{1+\left(\frac{\beta}{\alpha}\right)^2}\right)\left(t-\frac{\beta}{\alpha}+\sqrt{1+\left(\frac{\beta}{\alpha}\right)^2}\right)}$$

$$= \frac{1}{\alpha}\frac{1}{\sqrt{1+\left(\frac{\beta}{\alpha}\right)^2}}\left(\int \frac{dt}{t-\frac{\beta}{\alpha}+\sqrt{\ }} - \int \frac{dt}{t-\frac{\beta}{\alpha}-\sqrt{\ }}\right) = \frac{1}{\sqrt{\alpha^2+\beta^2}}\ln\left|\frac{\alpha t - \beta + \sqrt{\alpha^2+\beta^2}}{\alpha t - \beta - \sqrt{\alpha^2+\beta^2}}\right| + C$$

με $t = \operatorname{tg}\frac{x}{2} = \frac{\sin x}{\cos x + 1}$

[Επαλήθευση: $t' = \frac{1}{\cos^2\frac{x}{2}}\,\frac{1}{2} = \frac{1}{\cos x + 1}$

$$\frac{1}{\sqrt{}}\;\frac{\alpha t - \beta - \sqrt{}}{\alpha t - \beta + \sqrt{}}\;\frac{\alpha(\alpha t - \beta - \sqrt{}) - \alpha(\alpha t - \beta + \sqrt{})}{(\alpha t - \beta - \sqrt{})^2}\,t' =$$

$$= -2\alpha\,\frac{1}{(\alpha t - \beta)^2 - (\alpha^2 + \beta^2)}\,t' = 2\,\frac{1}{\alpha(1 - t^2) + 2\beta t}\,t' = \frac{2}{\alpha\left(1 - \frac{s^2}{(c+1)^2}\right) + 2\beta\frac{s}{c+1}}\,\frac{1}{c+1}$$

$$= \frac{2}{\alpha\left(c + 1 - \frac{s^2}{c+1}\right) + 2\beta s} = \frac{2}{\alpha 2c + 2\beta s} \quad \left[\, c + 1 - \frac{s^2}{c+1} = \frac{c^2 + 1 + 2c - s^2}{c+1} = \frac{2c^2 + 2c}{c+1} = 2c \,\right] \Bigg]$$

A [1.27β] $\displaystyle\int \frac{\cos x}{2 + \sin x}\,dx \underset{y = \sin x}{=} \int \frac{dy}{2 + y} = \ln(2 + \sin x) + c$

A[1.27ε] $\int \frac{dx}{1+\mathrm{tg}\,x} = \int \frac{\cos x}{\cos x+\sin x}\,dx \underset{\substack{t=\mathrm{tg}\,x \\ dt=(1+t^2)dx}}{=} \int \frac{1}{1+t}\,\frac{1}{1+t^2}\,dt$

$\frac{1}{1+t}\,\frac{1}{1+t^2} = \frac{A}{1+t} + \frac{Bt+\Gamma}{1+t^2} \Rightarrow 1 = A(1+t^2) + (Bt+\Gamma)(1+t) \Rightarrow A+B=0,\ B+\Gamma=0,$

$A+\Gamma=1 \Rightarrow B=-A,\ \Gamma=A,\ A=\frac{1}{2} \Rightarrow \frac{1}{1+t}\,\frac{1}{1+t^2} = \frac{1}{2}\left(\frac{1}{1+t} - \frac{t-1}{1+t^2}\right)$

$\Rightarrow \int \frac{1}{1+t}\,\frac{1}{1+t^2}\,dt = \frac{1}{2}\int\frac{dt}{1+t} - \frac{1}{4}\int\frac{2t}{1+t^2}\,dt + \frac{1}{2}\int\frac{dt}{1+t^2}$

$= \frac{1}{2}\ln|t+1| - \frac{1}{4}\ln(1+t^2) + \frac{1}{2}\,\mathrm{Arctg}\,t + c = \frac{1}{2}\ln\left|\frac{t+1}{\sqrt{1+t^2}}\right| + \frac{1}{2}\,\mathrm{Arctg}\,t + c$

$= \frac{1}{2}\ln\left|\frac{\sin x+\cos x}{\cos x\,\frac{1}{\cos x}}\right| + \frac{1}{2}x + c = \frac{1}{2}\left(x + \ln|\sin x+\cos x|\right) + c$

A[1.31α] $\int \sinh^2 x \cosh^3 x\,dx \underset{y=\sinh x}{=} \int y^2(1+y^2)\,dy = \frac{y^3}{3} + \frac{y^5}{5} + c$

$= \frac{\sinh^3 x}{3} + \frac{\sinh^5 x}{5} + c$

A[1.29β] $\int \frac{dx}{\sin^5 x \cos^5 x} = 2^5 \int \frac{dx}{\sin^5(2x)} \underset{y=2x}{=} 2^4 \int \frac{dy}{\sin^5 y} \underset{t=\cos y}{=} -2^4 \int \frac{dt}{(1-t^2)^3}$ 4-A/29

$$\int \frac{dt}{(1-t^2)^3} = \int \frac{dt}{(1-t^2)^2} + \int \frac{t^2}{(1-t^2)^3}\,dt \underset{(1)}{=} \frac{1}{4}\frac{t}{(1-t^2)^2} + \frac{3}{4}\int \frac{dt}{(1-t^2)^2}$$

$$\underset{(2)}{=} \frac{1}{4}\frac{t}{(1-t^2)^2} + \frac{3}{8}\int \frac{dt}{1-t^2} + \frac{3}{8}\frac{t}{1-t^2} \underset{(3)}{=} \frac{1}{4}\frac{t}{(1-t^2)^2} + \frac{3}{8}\frac{t}{1-t^2} + \frac{3}{16}\ln\left|\frac{1+t}{t-1}\right| + c$$

$$\int \frac{t^2}{(1-t^2)^3}\,dt = -\frac{1}{2}\,\frac{1}{(-2)}\,\frac{1}{(1-t^2)^2}\,t - \frac{1}{4}\int \frac{dt}{(1-t^2)^2} \quad (1)$$

$$\int \frac{dt}{(1-t^2)^2} = \int \frac{dt}{1-t^2} + \int \frac{t^2}{(1-t^2)^2}\,dt, \quad \int \frac{t^2}{(1-t^2)^2} = -\frac{1}{2}(-1)\frac{1}{1-t^2}\,t - \frac{1}{2}\int \frac{dt}{1-t^2} \quad (2)$$

$$\int \frac{dt}{1-t^2} = \frac{1}{2}\left(\int \frac{dt}{1-t} + \int \frac{dt}{1+t}\right) = \frac{1}{2}\ln\left|\frac{1+t}{t-1}\right| + c \quad (3)$$

$$\Rightarrow \int \frac{dx}{\sin^5 x \cos^5 x} = -4\,\frac{\cos(2x)}{\sin^4(2x)} - 6\,\frac{\cos(2x)}{\sin^2(2x)} - 3\ln\left|\frac{1+\cos(2x)}{\cos(2x)-1}\right| + c$$

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$$\left[\ \cos(2x) = \cos^2 x - \sin^2 x = 2\cos^2 x - 1 \atop \qquad\qquad\qquad\qquad\qquad = 1 - 2\sin^2 x \right\} \Rightarrow \left|\frac{1+\cos(2x)}{\cos(2x)-1}\right| = \frac{\cos^2 x}{\sin^2 x}$$

$$\Rightarrow \quad -3\log\left|\frac{1+\cos(2x)}{\cos(2x)-1}\right| = -3\log\left(\operatorname{tg}^{-2} x\right) = 6\log|\operatorname{tg} x| + C$$

$$\frac{\cos(2x)}{\sin^2(2x)} = \frac{\cos^2 x - \sin^2 x}{4\sin^2 x\cos^2 x} = \frac{1}{4}\left(\frac{1}{\sin^2 x} - \frac{1}{\cos^2 x}\right) = \frac{1}{4}\left(1 + \operatorname{ctg}^2 x - \operatorname{tg}^2 x - 1\right)$$

$$\frac{\cos(2x)}{\sin^4(2x)} = \frac{\cos^2 x - \sin^2 x}{16\sin^4 x\cos^4 x} = \frac{1}{16}\left(\frac{1}{\sin^4 x\cos^2 x} - \frac{1}{\sin^2 x\cos^4 x}\right)$$

$$= \frac{1}{16}\left(\frac{\cos^4 x + \sin^4 x + 2\cos^2 x\sin^2 x}{\sin^4 x\cos^2 x} - \frac{\cos^4 x + \sin^4 x + 2\cos^2 x\sin^2 x}{\sin^2 x\cos^4 x}\right)$$

$$= \frac{1}{16}\Big(\operatorname{ctg}^4 x\left(1+\operatorname{tg}^2 x\right) + 1 + \operatorname{tg}^2 x + 2\left(1+\operatorname{ctg}^2 x\right)$$

$$\qquad - \left(1+\operatorname{ctg}^2 x\right) - \left(1+\operatorname{ctg}^2 x\right)\operatorname{tg}^4 x - 2\left(1+\operatorname{tg}^2 x\right)\Big)$$

$$= \frac{1}{16}\left(\operatorname{ctg}^4 x - \operatorname{tg}^4 x\right) + \frac{1}{16}\left(\operatorname{ctg}^2 x - \operatorname{tg}^2 x\right) + \frac{1}{16}\underbrace{\operatorname{ctg}^2 x\,\operatorname{tg}^2 x}_{=1}\left(\operatorname{ctg}^2 x - \operatorname{tg}^2 x\right)$$

$$\Rightarrow -4\frac{\cos(2x)}{\sin^4(2x)} - 6\frac{\cos(2x)}{\sin^2(2x)} = \frac{1}{4}\left(\operatorname{tg}^4 x - \operatorname{ctg}^4 x\right) + \frac{1}{2}\left(\operatorname{tg}^2 x - \operatorname{ctg}^2 x\right) + \frac{3}{2}\left(\operatorname{tg}^2 x - \operatorname{ctg}^2 x\right)\ \Big]$$