A [2.5 i)] Να αποδείξετε με τη βοήθεια της συνθήκης του Riemann, ότι οι παρακάτω συναρτήσεις είναι ολοκληρώσιμες στο $[0,1]$:

α) $f(x) = x$

Λ.: $P = \{ x_k = \frac{k}{\nu} : k = 0, \dots, \nu \}$,

$$U(P,f) - L(P,f) = \sum_{k=1}^{\nu} \left(\sup f|_{[x_{k-1},x_k]} - \inf f|_{[x_{k-1},x_k]} \right)(x_k - x_{k-1})$$

$$= \sum_{k=1}^{\nu} (x_k - x_{k-1})^2 = \sum_{k=1}^{\nu} \frac{1}{\nu^2} = \frac{1}{\nu}$$

Αφού $\forall \varepsilon > 0 \ \exists \nu \in \mathbb{N}: \frac{1}{\nu} < \varepsilon$ (Αρχιμήδεια Ιδιότητα) έχουμε

$\forall \varepsilon > 0 \ \exists P \in \mathcal{P}([0,1])$ (με $\nu > \frac{1}{\varepsilon}$): $U(P,f) - L(P,f) < \varepsilon$

β) $f(x) = x^2$

Λ.: $P = \{ x_k = \frac{k}{\nu} : k = 0, \dots, \nu \}$,

$$U(P,f) - L(P,f) = \sum_{k=1}^{\nu} \left(\sup f|_{[x_{k-1}, x_k]} - \inf f|_{[x_{k-1}, x_k]} \right)(x_k - x_{k-1})$$

$$= \sum_{k=1}^{\nu} \left(x_k^2 - x_{k-1}^2 \right)(x_k - x_{k-1}) = \sum_{k=1}^{\nu} (x_k - x_{k-1})^2 (x_k + x_{k-1}) =$$

$$= \sum_{k=1}^{\nu} \frac{1}{\nu^2} \frac{k+k-1}{\nu} = \frac{1}{\nu^3}\left(2 \sum_{k=1}^{\nu} k - \sum_{k=1}^{\nu} 1 \right) = \frac{1}{\nu^3}\left(2 \frac{\nu(\nu+1)}{2} - \nu \right) = \frac{1}{\nu}$$

$\Longrightarrow$ $\forall \varepsilon > 0 \ \exists P \in \mathcal{P}([0,1]) \ (\mu\varepsilon \ \nu > \frac{1}{\varepsilon}) : \ U(P,f) - L(P,f) < \varepsilon$

βλ. α)

$\boxed{5\text{-}A/3}$

γ) $f(x) = e^x$

Λ.: $P = \{ x_k = \frac{k}{\nu} : k = 0, \dots, \nu \}$,

$$U(P,f) - L(P,f) = \sum_{k=1}^{\nu} \left(\sup f|_{[x_{k-1}, x_k]} - \inf f|_{[x_{k-1}, x_k]} \right)(x_k - x_{k-1})$$

$$= \sum_{k=1}^{\nu} \left(e^{\frac{k}{\nu}} - e^{\frac{k-1}{\nu}} \right) \frac{1}{\nu} \underset{\text{ΘΜΤ}}{=} \sum_{k=1}^{\nu} e^{\xi_k} \frac{1}{\nu^2} \quad \text{με } \xi_k \in \left(\frac{k-1}{\nu}, \frac{k}{\nu} \right) \subset (0,1) \ \forall k = 1, \dots, \nu$$

και αφού $f(x) = e^x$ αύξουσα $e^{\xi_k} < e \quad \forall k = 1, \dots, \nu$

$$\Longrightarrow \quad U(P,f) - L(P,f) < e \sum_{k=1}^{\nu} \frac{1}{\nu^2} = \frac{e}{\nu}$$

Απ' την Αρχιμήδεια Ιδιότητα $\forall \varepsilon > 0 \ \exists \nu \in \mathbb{N} : \frac{e}{\nu} < \varepsilon$ και συνεπώς

$\forall \varepsilon > 0 \ \exists P \in \mathcal{P}([0,1])$ (με $\nu > \frac{e}{\varepsilon}$): $U(P,f) - L(P,f) < \varepsilon$

A : Να αποδείξετε ότι οι παρακάτω συναρτήσεις ικανοποιούν τη συνθήκη του Riemann

α) $f(x)=|x|,\ x\in[-1,1]$

Λ.: $P=\left\{ x_k=-1+\frac{k}{\nu} : k=0,\dots,\nu \right\} \cup \left\{ y_\lambda=\frac{\lambda}{\nu} : \lambda=1,\dots,\nu \right\}$

$$U(P,f)-L(P,f) = \sum_{k=1}^{\nu}\left(\sup f\big|_{[x_{k-1},x_k]} - \inf f\big|_{[x_{k-1},x_k]}\right)(x_k-x_{k-1})$$

$$+\left(\sup f\big|_{[x_\nu,y_1]} - \inf f\big|_{[x_\nu,y_1]}\right)(y_1-x_\nu)$$

$$+\sum_{\lambda=2}^{\nu}\left(\sup f\big|_{[y_{\lambda-1},y_\lambda]} - \inf f\big|_{[y_{\lambda-1},y_\lambda]}\right)(y_\lambda-y_{\lambda-1})$$

$$=\frac{1}{\nu}\left(\sum_{k=1}^{\nu}\left(-x_{k-1}-(-x_k)\right)+(y_1-x_\nu)+\sum_{\lambda=2}^{\nu}(y_\lambda-y_{\lambda-1})\right)$$

$$= \frac{1}{\nu}\left(\sum_{k=1}^{\nu}\left(1-\frac{k-1}{\nu}-\left(1-\frac{k}{\nu}\right)\right)+\left(\frac{1}{\nu}-0\right)+\sum_{\lambda=2}^{\nu}\left(\frac{\lambda}{\nu}-\frac{\lambda-1}{\nu}\right)\right)$$

$$= \frac{1}{\nu}\left(\sum_{k=1}^{\nu}\left(\frac{k}{\nu}-\frac{k-1}{\nu}\right)+\frac{1}{\nu}+\sum_{\lambda=2}^{\nu}\left(\frac{\lambda}{\nu}-\frac{\lambda-1}{\nu}\right)\right)$$

$$= \frac{1}{\nu}\left(\frac{1}{\nu}+\frac{1}{\nu}\sum_{k=2}^{\nu}1+\frac{1}{\nu}+\frac{1}{\nu}\sum_{\lambda=2}^{\nu}1\right)$$

$$= \frac{1}{\nu^2}\left(2+2\sum_{\mu=2}^{\nu}1\right)=\frac{2}{\nu}$$

Αφού (Αρχιμήδεια Ιδιότητα) $\forall \varepsilon>0\ \exists \nu\in\mathbb{N}: \frac{2}{\nu}<\varepsilon$,

$\forall \varepsilon>0\ \exists\ P\in\mathcal{P}([-1,1])$ (με $\nu>\frac{2}{\varepsilon}$): $U(P,f)-L(P,f)<\varepsilon$

β) $f(x) = \frac{1}{\sqrt{x}}$, $x \in [1,2]$

Λ.: $P = \{ x_k = 1 + \frac{k}{\nu} : k = 0, \dots, \nu \}$

$$U(P,f) - L(P,f) = \sum_{k=1}^{\nu} \left(\sup f|_{[x_{k-1}, x_k]} - \inf f|_{[x_{k-1}, x_k]} \right)(x_k - x_{k-1})$$

$$= \sum_{k=1}^{\nu} \left(\frac{1}{\sqrt{x_{k-1}}} - \frac{1}{\sqrt{x_k}} \right)(x_k - x_{k-1}) = \sum_{k=1}^{\nu} \frac{\sqrt{x_k} - \sqrt{x_{k-1}}}{\sqrt{x_{k-1}}\sqrt{x_k}} (x_k - x_{k-1})$$

$$= \sum_{k=1}^{\nu} \frac{x_k - x_{k-1}}{\sqrt{x_{k-1}}\sqrt{x_k}\left(\sqrt{x_k} + \sqrt{x_{k-1}}\right)} (x_k - x_{k-1})$$

$$= \frac{1}{\nu^2} \sum_{k=1}^{\nu} \frac{1}{\sqrt{x_{k-1}}\sqrt{x_k}\left(\sqrt{x_k} + \sqrt{x_{k-1}}\right)} \leq \frac{1}{\nu^2} \sum_{k=1}^{\nu} \frac{1}{2} = \frac{1}{2\nu}$$

Απ' την Αρχ. Ιδ. $\forall \varepsilon > 0 \ \exists \nu \in \mathbb{N} \ \frac{1}{2\nu} < \varepsilon \Rightarrow$

$\forall \varepsilon > 0 \ \exists P \in \mathcal{P}([1,2])$ (με $\nu > \frac{1}{2\varepsilon}$) : $U(P,f) - L(P,f) < \varepsilon$

γ) $f(x)=\frac{1}{x}$, $x\in[1,2]$

Λ.: $f(x)=\frac{1}{x}$, $x\in[1,2]$

$P=\{x_k=\frac{k}{\nu} : k=0,\dots,\nu\}$,

$$U(P,f)-L(P,f)=\sum_{k=1}^{\nu}\left(\sup f|_{[x_{k-1},x_k]}-\inf f|_{[x_{k-1},x_k]}\right)(x_k-x_{k-1})$$

$$=\sum_{k=1}^{\nu}\left(\frac{1}{x_{k-1}}-\frac{1}{x_k}\right)(x_k-x_{k-1})=\sum_{k=1}^{\nu}\frac{(x_k-x_{k-1})^2}{x_{k-1}x_k}\leq\sum_{k=1}^{\nu}(x_k-x_{k-1})^2$$

$$=\sum_{k=1}^{\nu}\frac{1}{\nu^2}=\frac{1}{\nu}$$

Αρχ. Ιδ.: $\forall\varepsilon>0\ \exists\nu\in\mathbb{N}\ \frac{1}{\nu}<\varepsilon \Rightarrow$

$\forall\varepsilon>0\ \exists P\in\mathcal{P}([1,2])$ (με $\nu>\frac{1}{\varepsilon}$): $U(P,f)-L(P,f)<\varepsilon$

A [2.1 β] Να υπολογίσετε τα $L(P,f)$, $U(P,f)$ αν

$$f(x) = \begin{cases} 1, & 0 \leq x < 1, \\ x+1, & 1 \leq x < 2, \\ 4, & x = 2 \end{cases}$$ και $P = \{0, \frac{1}{2}, 1, \frac{3}{2}, 2\}$

1: $L(P,f) = \sum_{k=1}^{\nu} m_k (x_k - x_{k-1}) = \sum_{k=1}^{4} m_k \frac{1}{2}$,

$U(P,f) = \sum_{k=1}^{\nu} M_k (x_k - x_{k-1}) = \sum_{k=1}^{4} M_k \frac{1}{2}$ με

$m_1 := \inf\{f(x) : x \in [0, \frac{1}{2}]\} = 1$, $M_1 := \sup\{f(x) : x \in [0, \frac{1}{2}]\} = 1$,

$m_2 := \inf\{f(x) : x \in [\frac{1}{2}, 1]\} = 1$, $M_2 := \sup\{f(x) : x \in [\frac{1}{2}, 1]\} = 2$,

$m_3 := \inf\{f(x) : x \in [1, \frac{3}{2}]\} = 2$, $M_3 := \sup\{f(x) : x \in [1, \frac{3}{2}]\} = \frac{5}{2}$,

$m_4 := \inf\{f(x) : x \in [\frac{3}{2}, 2]\} = \frac{5}{2}$, $M_4 := \sup\{f(x) : x \in [\frac{3}{2}, 2]\} = 4$

$\Rightarrow$ $L(P,f) = \frac{1}{2}\left(1 + 1 + 2 + \frac{5}{2}\right) = \frac{13}{4}$, $U(P,f) = \frac{1}{2}\left(1 + 2 + \frac{5}{2} + 4\right) = \frac{19}{4}$.

A [2.1 γ] $f(x) = 4 - x$, $x \in [1,3]$. $P = \{ 1 + \frac{2k}{\nu} : k = 0, \dots, \nu \}$

Υπολογίστε τα $L(P,f)$, $U(P,f)$.

Λ: $$L(P,f) = \sum_{k=1}^{\nu} m_k (x_k - x_{k-1}) = \sum_{k=1}^{\nu} \left(4 - \left(1 + \frac{2k}{\nu}\right)\right) \frac{2}{\nu} =$$

$$= \frac{2}{\nu} \sum_{k=1}^{\nu} \left(3 - \frac{2k}{\nu}\right) = \frac{2}{\nu}\left(3\nu - \frac{2}{\nu}\sum_{k=1}^{\nu} k\right) = 6 - \frac{4}{\nu^2}\frac{(\nu+1)\nu}{2}$$

$$= 6 - 2\frac{\nu+1}{\nu} = 4 - \frac{2}{\nu}$$

$$U(P,f) = \sum_{k=1}^{\nu} M_k (x_k - x_{k-1}) = \sum_{k=1}^{\nu} \left(4 - \left(1 + \frac{2(k-1)}{\nu}\right)\right) \frac{2}{\nu} =$$

$$= \frac{2}{\nu} \sum_{k=1}^{\nu} \left(3 - \frac{2k}{\nu} + \frac{2}{\nu}\right) = \frac{2}{\nu}\sum_{k=1}^{\nu}\left(3 - \frac{2k}{\nu}\right) + \frac{2}{\nu}\sum_{k=1}^{\nu}\frac{2}{\nu}$$

$$= 4 - \frac{2}{\nu} + \frac{4}{\nu} = L(P,f) + \frac{4}{\nu} = 4 + \frac{2}{\nu}$$

A [2.1 δ] $f(x)=x$, $x\in[1,2]$, $P_\nu=\left\{1+\frac{k}{\nu} : k=0,\dots,\nu\right\}$,

$Q_\nu=\left\{2^{\frac{k}{\nu}} : k=0,\dots,\nu\right\}$. Βρείτε τα $L(P_\nu,f)$, $U(Q_\nu,f)$

Λ: $$L(P_\nu,f)=\sum_{k=1}^{\nu} m_k\frac{1}{\nu}=\frac{1}{\nu}\sum_{k=1}^{\nu}\left(1+\frac{k-1}{\nu}\right)=1+\frac{1}{\nu^2}\sum_{k=1}^{\nu}(k-1)=$$

$$=1+\frac{1}{\nu^2}\,\frac{\nu(\nu-1)}{2}=1+\frac{\nu-1}{2\nu}=\frac{3\nu-1}{2\nu}$$

$$U(Q_\nu,f)=\sum_{k=1}^{\nu} M_k\left(2^{\frac{k}{\nu}}-2^{\frac{k-1}{\nu}}\right)=\sum_{k=1}^{\nu}2^{\frac{k}{\nu}}\left(2^{\frac{k}{\nu}}-2^{\frac{k-1}{\nu}}\right)$$

$$=\sum_{k=1}^{\nu}2^{\frac{2k}{\nu}}\left(1-\frac{1}{\sqrt[\nu]{2}}\right)=\left(1-\frac{1}{\sqrt[\nu]{2}}\right)\sum_{k=1}^{\nu}\left(\sqrt[\nu]{4}\right)^k$$

$$=\left(1-\frac{1}{\sqrt[\nu]{2}}\right)\sqrt[\nu]{4}\sum_{k=1}^{\nu}\left(\sqrt[\nu]{4}\right)^{k-1}=\left(\sqrt[\nu]{4}-\sqrt[\nu]{2}\right)\frac{\left(\sqrt[\nu]{4}\right)^{\nu}-1}{\sqrt[\nu]{4}-1}$$

$$=3\,\frac{\sqrt[\nu]{4}-\sqrt[\nu]{2}}{\sqrt[\nu]{4}-1}$$

[Εφ. 1, §2.2] $f:[\alpha,\beta]\to\mathbb{R}$ φραγμένη, $P\in\mathcal{P}(I) \Longrightarrow L(P,f)=-U(P,-f)$

Απόδειξη:

<u>Γενικά</u>: Έστω $\emptyset\neq A\subset\mathbb{R}$ φραγμένο και $-A:=\{-x: x\in A\}$.

$\Longrightarrow \emptyset\neq -A\subset\mathbb{R}$ φραγμένο και $\inf(-A)=-\sup A$, $\sup(-A)=-\inf A$:

$\forall x\in A: \quad \inf A\le x\le \sup A \iff -\inf A\ge -x\ge -\sup A$

$$\Longrightarrow \quad -\sup A\le \inf(-A)\le \sup(-A)\le -\inf A \qquad (1)$$

και, αφού $A=-(-A)=\{-(-x): x\in A\}$, θέτοντας όπου A το $-A$,

$$-\sup(-A)\le \inf A\le \sup A\le -\inf(-A)$$

$$\iff \quad \sup(-A)\ge -\inf A\ge -\sup A\ge \inf(-A) \qquad (2)$$

$$\underset{(1),(2)}{\Longrightarrow} \quad \sup(-A)=-\inf A\ , \quad -\sup A=\inf(-A)$$

<u>Επίσης</u>: $g: \emptyset \neq B \to \mathbb{R}$ φραγμένη $:\Longleftrightarrow \emptyset \neq g(B) \subset \mathbb{R}$ φραγμένο

και για $-g: B \to \mathbb{R}$, $(-g)(x) := -g(x)\ \forall x \in B$: $(-g)(B) = -g(B)$:

$(-g)(B) = \{(-g)(x) : x \in B\} = \{-g(x) : x \in B\} = \{-g(x) : g(x) \in g(B)\}$

$= -g(B)$

<u>Συνεπώς</u>:

Έστω $P = \{\alpha = x_0, x_1, \dots, x_{\nu-1}, x_\nu = \beta\}$, $\nu \in \mathbb{N}$.

$f: [\alpha, \beta] \to \mathbb{R}$ φραγμένη $\Longleftrightarrow -f: [\alpha, \beta] \to \mathbb{R}$ φραγμένη

$$\Rightarrow \exists\ U(P, -f) = \sum_{k=1}^{\nu} \sup(-f)([x_{k-1}, x_k])\,(x_k - x_{k-1})$$

$$= -\sum_{k=1}^{\nu} \inf f([x_{k-1}, x_k])\,(x_k - x_{k-1}) = -L(P, f)$$

[θέτοντας $-f$ όπου f και αφού $-(-f) = f$ ισχύει και: $U(P, f) = -L(P, -f)$]

A [2.4 α] $\quad f, g : [\alpha, \beta] \to \mathbb{R}$ φραγμένες, $f(x) \leq g(x) \ \forall x \in [\alpha, \beta]$

$\Rightarrow \quad L_f \leq L_g \quad , \quad U_f \leq U_g$

$\underset{f, g \text{ ολοκληρώσιμες}}{\Rightarrow} \quad \int_\alpha^\beta f(x)\,dx \leq \int_\alpha^\beta g(x)\,dx$

Απόδειξη:

Γενικά: $\quad f, g : A \to \mathbb{R}$ φραγμένες, $f(x) \leq g(x) \ \forall x \in A \Rightarrow$

$\inf f \ (:= \inf f(A)) \leq f(x) \leq g(x) \leq \sup g \ (:= \sup g(A)) \quad \forall x \in A$

$\Rightarrow \inf f \leq \inf g \quad , \quad \sup f \leq \sup g$

Συνεπώς:

Έστω $P = \{\alpha = x_0, x_1, \dots, x_{\nu-1}, x_\nu = \beta\} \in \mathcal{P}(I), \ \nu \in \mathbb{N}$.

$$L(P, f) = \sum_{k=1}^{\nu} \underbrace{\inf f|_{[x_{k-1}, x_k]}}_{\leq \inf g|_{[x_{k-1}, x_k]}} (x_k - x_{k-1}) \leq \sum_{k=1}^{\nu} \inf g|_{[x_{k-1}, x_k]} (x_k - x_{k-1}) = L(P, g)$$

$$\Rightarrow L(P,f) \leq \sup\{L(P,g) : P \in \mathcal{P}(I)\} =: L_g$$

$$\Rightarrow L_f := \sup\{L(P,f) : P \in \mathcal{P}(I)\} \leq L_g$$

και, αντίστοιχα,

$$U(P,f) = \sum_{k=1}^{\nu} \sup f\big|_{[x_{k-1},x_k]}(x_k - x_{k-1}) \leq \sum_{k=1}^{\nu} \sup g\big|_{[x_{k-1},x_k]}(x_k - x_{k-1}) = U(P,g)$$

$$\Rightarrow U_f := \inf\{U(P,f) : P \in \mathcal{P}(I)\} \leq U(P,g)$$

$$\Rightarrow U_f \leq \inf\{U(P,g) : P \in \mathcal{P}(I)\} = U_g$$

Αν οι f, g είναι ολοκληρώσιμες στο $[\alpha,\beta]$, τότε

$$\int_\alpha^\beta f(x)\,dx = L_f = U_f \leq L_g = U_g = \int_\alpha^\beta g(x)\,dx$$

Εφ [3., §2.4] $f : [\alpha, \beta] \to \mathbb{R}$ φραγμένη και ολοκληρώσιμη στο $[\alpha,\beta]$

$\Longrightarrow$ $\forall\, P \in \mathcal{P}([\alpha,\beta])$ $\forall$ επιλογή Ξ ενδιάμεσων σημείων της P:

$$\left| \int_\alpha^\beta f(x)\,dx - S(P,f,\Xi) \right| \leq U(P,f) - L(P,f)$$

Απόδειξη:

Αφού η f είναι ολοκληρώσιμη στο $[\alpha,\beta]$, ισχύει $A := \int_\alpha^\beta f(x)\,dx = L_f = U_f$ (1)

Αφού η f είναι φραγμένη, ισχύει $L(P,f) \leq S(P,f,\Xi) \leq U(P,f)$ (2)

(βλ. Παρατ. β), Σημ. 5-1/3)

Εξ ορισμού (Ορ. [2.6]), $L(P,f) \leq L_f$, $U_f \leq U(P,f)$ (3)

$$\Longrightarrow \quad A - S(P,f,\Xi) \underset{(2)}{\leq} A - L(P,f) \underset{(1)}{=} U_f - L(P,f) \underset{(3)}{\leq} U(P,f) - L(P,f)$$

$$S(P,f,\Xi) - A \underset{(2)}{\leq} U(P,f) - A \underset{(1)}{=} U(P,f) - L_f \underset{(3)}{\leq} U(P,f) - L(P,f)$$

απ' οπού προκύπτει $\left| A - S(P,f,\Xi) \right| \leq U(P,f) - L(P,f)$ $\square$

A [2.6 β] $f(x) = 2x^2 - 4x$, $x \in [0,1]$. Να υπολογίσετε το $S(P, f, \Xi)$ χωρίζοντας το $[0,1]$ σε ν ίσα τμήματα και θεωρώντας ως ενδιάμεσα σημεία τα μέσα των τμημάτων της διαμέρισης.

Λ.: $P = \left\{ x_k = \frac{k}{\nu} : k = 0, \dots, \nu \right\}$, $\Xi = \left\{ \xi_k = \frac{k-1+k}{2\nu} = \frac{k}{\nu} - \frac{1}{2\nu} : k = 1, \dots, \nu \right\}$

$$S(P, f, \Xi) = \sum_{k=1}^{\nu} f(\xi_k)(x_k - x_{k-1}) = \sum_{k=1}^{\nu} f\left(\frac{k}{\nu} - \frac{1}{2\nu}\right) \frac{1}{\nu} =$$

$$= \frac{1}{\nu} \sum_{k=1}^{\nu} \left(2\left(\frac{k}{\nu} - \frac{1}{2\nu}\right)^2 - 4\left(\frac{k}{\nu} - \frac{1}{2\nu}\right) \right) = \frac{1}{\nu}\left(2 \sum_{k=1}^{\nu}\left(\frac{k^2}{\nu^2} - \frac{k}{\nu^2} + \frac{1}{4\nu^2}\right) - 4 \sum_{k=1}^{\nu}\left(\frac{k}{\nu} - \frac{1}{2\nu}\right)\right)$$

$$= \frac{1}{\nu}\left(\frac{2}{\nu^2}\left(\sum_{k=1}^{\nu} k^2 - \sum_{k=1}^{\nu} k \right) + \frac{1}{2\nu} - \frac{4}{\nu} \sum_{k=1}^{\nu} k + 2 \right)$$

$$= \frac{1}{\nu}\left(\frac{2}{\nu^2}\left(\frac{\nu(\nu+1)(2\nu+1)}{6} - \frac{\nu(\nu+1)}{2} \right) + \frac{1}{2\nu} - \frac{4}{\nu} \frac{\nu(\nu+1)}{2} + 2 \right)$$

(*)

$$= \frac{(\nu+1)(2\nu+1)}{3\nu^2} - \frac{\nu+1}{\nu^2} + \frac{1}{2\nu^2} - \frac{2(\nu+1)}{\nu} + \frac{2}{\nu}$$

$$= \frac{2}{3} + \frac{3\nu+1}{3\nu^2} - \frac{\nu+1}{\nu^2} + \frac{1}{2\nu^2} - 2 = -\frac{4}{3} - \frac{2}{3\nu^2} + \frac{1}{2\nu^2} = -\frac{4}{3} - \frac{1}{6\nu^2}$$

$(*)$ $\sum_{k=1}^{\nu} k^2 = \frac{\nu(\nu+1)(2\nu+1)}{6}$: $\nu=1$: $1 = \frac{1\cdot 2\cdot 3}{6}$ ισχύει.

Έστω ότι ισχύει για $\nu \in \mathbb{N}$, τότε $\sum_{k=1}^{\nu+1} k^2 = \sum_{k=1}^{\nu} k^2 + (\nu+1)^2 = \frac{\nu(\nu+1)(2\nu+1)}{6} + (\nu+1)^2$

$$= \frac{\nu(\nu+1)(2\nu+1) + 6(\nu+1)^2}{6} = \frac{(\nu+1)(\nu(2\nu+1) + 6(\nu+1))}{6} =$$

$$= \frac{(\nu+1)(\nu(2\nu+3) + 4\nu + 6)}{6} = \frac{(\nu+1)(\nu+2)(2\nu+3)}{6} = \frac{(\nu+1)((\nu+1)+1)(2(\nu+1)+1)}{6}$$

δηλ. τότε ισχύει και για $\nu+1$ $\underset{\text{επαγωγή}}{\Longrightarrow}$ $(*)$ ισχύει $\forall \nu \in \mathbb{N}$.