Εφ. [1., §2.4] $f(x)=|1-x|$, $x\in[0,3]$. Να υπολογίσετε το άθροισμα Riemann $S(P,f,\Xi)$, θεωρώντας κατάλληλες διαμερίσεις P λεπτότητας που τείνει στο μηδέν, και παίρνοντας ως ενδιάμεσα σημεία τα μέσα των τμημάτων των διαμερίσεων.

Λ.: $f(x)=\begin{cases} 1-x, & x\in[0,1] \\ x-1, & x\in(1,3] \end{cases}$,

$P=\left\{x_k=\frac{k}{\nu} : k=0,\dots,\nu\right\}\cup\left\{y_\lambda=1+\frac{2\lambda}{\nu} : \lambda=1,\dots,\nu\right\}$ με $\|P\|=\frac{2}{\nu}\xrightarrow[\nu\to\infty]{}0$,

$\Xi=\left\{\xi_k=\frac{k-1+k}{2\nu}=\frac{k}{\nu}-\frac{1}{2\nu} : k=1,\dots,\nu\right\}$

$\cup\left\{\eta_\lambda=\frac{y_{\lambda-1}+y_\lambda}{2}=1+\frac{2\lambda}{\nu}-\frac{1}{\nu} : \lambda=1,\dots,\nu\right\}$

$$S(P,f,\Xi) = \sum_{k=1}^{\nu} f(\xi_k)(x_k - x_{k-1}) + \sum_{\lambda=1}^{\nu} f(\eta_\lambda)(y_\lambda - y_{\lambda-1})$$

$$= \sum_{k=1}^{\nu} \left(1 - \left(\frac{k}{\nu} - \frac{1}{2\nu}\right)\right)\frac{1}{\nu} + \sum_{\lambda=1}^{\nu}\left(\left(1 + \frac{2\lambda}{\nu} - \frac{1}{\nu}\right) - 1\right)\frac{2}{\nu}$$

$$= \frac{1}{\nu}\sum_{\mu=1}^{\nu}\left(1 - \frac{\mu}{\nu} + \frac{1}{2\nu} + \frac{4\mu}{\nu} - \frac{2}{\nu}\right) = \frac{1}{\nu}\left(\left(1 + \frac{1}{2\nu} - \frac{2}{\nu}\right)\nu + \frac{3}{\nu}\sum_{\mu=1}^{\nu}\mu\right)$$

$$= 1 + \frac{1}{2\nu} - \frac{2}{\nu} + \frac{3}{\nu}\frac{\nu+1}{2} = 1 + \frac{3}{2} + \frac{1}{2\nu} - \frac{2}{\nu} + \frac{3}{2\nu} = \frac{5}{2}$$

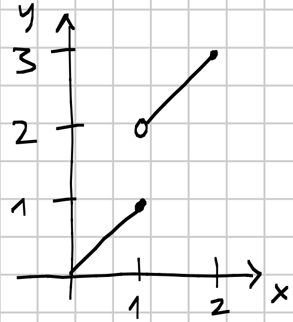
(Αφού η f ως συνεχής είναι και ολοκληρώσιμη και ο αριθμός $A = \frac{5}{2}$ ικανοποιεί την ανισότητα $|S(P,f,\Xi) - A| < \varepsilon$ $\forall \varepsilon > 0$ και για όλες τις πιο πάνω διαμερίσεις P και επιλογές Ξ ενδ. σημ. προκύπτει ότι $\int_0^3 |1-x|\,dx = \frac{5}{2}$.)

Εφ. [2., §2.5] $f(x) = \begin{cases} x, & 0 \le x \le 1 \\ x+1, & 1 < x \le 2 \end{cases}$. Να υπολογίσετε το $L(P,f)$ θεωρώντας κατάλληλες διαμερίσεις του $[0,2]$ λεπτότητας που τείνει στο μηδέν.

6-A/B

Λύση:

$$P = \left\{ x_k = \frac{k}{\nu} : k = 0, \dots, \nu \right\} \cup \left\{ y_\lambda = 1 + \frac{\lambda}{\nu} : \lambda = 1, \dots, \nu \right\} \text{ με } \|P\| = \frac{1}{\nu} \xrightarrow[\nu \to \infty]{} 0$$



$$\inf f|_{[x_{k-1}, x_k]} = f(x_{k-1}) = x_{k-1} = \frac{k-1}{\nu}, \quad k = 1, \dots, \nu,$$

$$\inf f|_{[y_{\lambda-1}, y_\lambda]} = f(y_{\lambda-1}) = y_{\lambda-1} + 1 = 2 + \frac{\lambda-1}{\nu}, \quad \lambda = 2, \dots, \nu,$$

$$\inf f|_{[x_\nu, y_1]} = \inf f|_{[1, 1+\frac{1}{\nu}]} = \inf \left(\{f(1)\} \cup f\left(\left(1, 1+\tfrac{1}{\nu}\right]\right) \right)$$

$$= \inf \left(\{1\} \cup \left(2, 2+\tfrac{1}{\nu}\right) \right) = 1$$

$$L(P,f) = \sum_{k=1}^{\nu} \inf f|_{[x_{k-1}, x_k]} (x_k - x_{k-1}) + \inf f|_{[x_\nu, y_1]} (y_1 - x_\nu)$$

$$+ \sum_{\lambda=2}^{\nu} \inf f|_{[y_{\lambda-1}, y_\lambda]} (y_\lambda - y_{\lambda-1}) = \sum_{k=1}^{\nu} \frac{k-1}{\nu} \frac{1}{\nu} + \frac{1}{\nu} + \sum_{\lambda=2}^{\nu} \left(2 + \frac{\lambda-1}{\nu}\right) \frac{1}{\nu} =$$

$$= \frac{1}{\nu} + \frac{1}{\nu} \sum_{\mu=2}^{\nu} \left(\frac{\mu-1}{\nu} + 2 + \frac{\mu-1}{\nu} \right) = \frac{1}{\nu} + \frac{2(\nu-1)}{\nu} + \frac{2}{\nu^2} \sum_{\mu=2}^{\nu} \mu - 1 =$$

$$= 2 - \frac{1}{\nu} + \frac{2}{\nu^2} \sum_{\mu=1}^{\nu-1} \mu = 2 - \frac{1}{\nu} + \frac{2}{\nu^2} \frac{(\nu-1)\nu}{2} = 2 - \frac{1}{\nu} + 1 - \frac{1}{\nu} = 3 - \frac{2}{\nu}$$

Εφ. [1., §2.6] $f : [\alpha,\beta] \to \mathbb{R}$ ολοκληρώσιμη στο $[\alpha,\beta]$ με $\inf f > 0$

$\Rightarrow \frac{1}{f}, \sqrt{f}$ ολοκληρώσιμες στο $[\alpha,\beta]$

Απόδειξη:

f ολοκληρώσιμη $\Rightarrow$ f φραγμένη $\Leftrightarrow \exists\, m := \inf f,\ M := \sup f \in \mathbb{R}$

$$\underset{m>0}{\Longrightarrow} \quad \sup \frac{1}{f} = \frac{1}{m}, \quad \inf \frac{1}{f} = \frac{1}{M}, \quad \sup \sqrt{f} = \sqrt{M}, \quad \inf \sqrt{f} = \sqrt{m} \qquad (*)$$

[$\forall x \in [\alpha,\beta] : 0 < m \leq f(x) \leq M \Leftrightarrow \frac{1}{m} \geq \frac{1}{f(x)} \geq \frac{1}{M}, \quad \sqrt{m} \leq \sqrt{f(x)} \leq \sqrt{M}$ (1),

$$0 < m = \inf f \Rightarrow \forall \varepsilon > 0 \ \exists x \in [\alpha,\beta] : f(x) < m + m^2\varepsilon \Rightarrow \frac{1}{f(x)} > \frac{1}{m+m^2\varepsilon}$$

$$= \frac{1}{m} - \frac{m^2\varepsilon}{m(m+\varepsilon)} > \frac{1}{m} - \varepsilon \ \underset{(1)}{\Longrightarrow} \ \sup \frac{1}{f} = \frac{1}{m},$$

$$0 < M = \sup f \Rightarrow \forall \varepsilon \in \left(0, \frac{1}{M}\right) \exists x \in [\alpha,\beta] : f(x) > M - \frac{M^2}{2}\varepsilon > \frac{M}{2} > 0$$

$$\Rightarrow \frac{1}{f(x)} < \frac{1}{M - \frac{M^2}{2}\varepsilon} = \frac{1}{M} + \frac{M^2}{2}\frac{\varepsilon}{M(M-\frac{M^2}{2}\varepsilon)} = \frac{1}{M} + \varepsilon \ \underset{(1)}{\Longrightarrow} \ \inf \frac{1}{f} = \frac{1}{M}$$

$$0 < m = \inf f \implies \forall \varepsilon > 0 \ \exists x \in [\alpha, \beta] : f(x) < m + 2\sqrt{m}\,\varepsilon$$

$$\implies \sqrt{f(x)} < \sqrt{m + 2\sqrt{m}\,\varepsilon} = \sqrt{m} + \frac{2\sqrt{m}\,\varepsilon}{\sqrt{m + 2\sqrt{m}\,\varepsilon} + \sqrt{m}} < \sqrt{m} + \varepsilon \underset{(1)}{\implies} \inf \sqrt{f} = \sqrt{m}$$

$$0 < M = \sup f \implies \forall \varepsilon \in \left(0, \frac{\sqrt{M}}{2\sqrt{2}}\right) \ \exists x \in [\alpha, \beta] : f(x) > M - \sqrt{2M\varepsilon} > \frac{M}{2} > 0$$

$$\implies \sqrt{f(x)} > \sqrt{M - \sqrt{2M\varepsilon}} = \sqrt{M} - \frac{\sqrt{2M\varepsilon}}{\sqrt{M - \sqrt{2M\varepsilon}} + \sqrt{M}} > \sqrt{M} - \frac{\sqrt{2M}\,\varepsilon}{2\sqrt{M - \sqrt{2M\varepsilon}}} > \sqrt{M} - \varepsilon \underset{(1)}{\implies} \sup \sqrt{f} = \sqrt{M}$$

Συνεπώς, οι $g := \frac{1}{f}$, $h := \sqrt{f}$ είναι φραγμένες. Οι g, f είναι ολοκληρώσιμες στο $[\alpha, \beta]$ $\underset{\text{Συνθ. Riemann (B')}}{\iff}$ $\forall \varepsilon > 0 \ \exists P \in \mathcal{P}([\alpha, \beta]) : U(P, g) - L(P, g),\ U(P, h) - L(P, h) < \varepsilon$

Έστω $\varepsilon > 0$. f ολοκληρώσιμη στο $[\alpha, \beta]$

$\underset{\text{Συνθ. Riemann (B')}}{\implies}$ $\exists\ P = \{x_0, \dots, x_\nu\} \in \mathcal{P}([\alpha, \beta])$:

$$U(P, f) - L(P, f) < \varepsilon \min\{m^2, 2\sqrt{m}\} \leq \varepsilon m^2, \varepsilon 2\sqrt{m}$$

Θέτουμε $\forall k=1,\dots,\nu$: $m_k := \inf f|_{[x_{k-1},x_k]}$, $M_k := \sup f|_{[x_{k-1},x_k]}$,

$m_k' := \inf g|_{[x_{k-1},x_k]}$, $M_k' := \sup g|_{[x_{k-1},x_k]}$,

$m_k'' := \inf h|_{[x_{k-1},x_k]}$, $M_k'' := \sup h|_{[x_{k-1},x_k]}$

Από την (*) έχουμε $m_k' = \frac{1}{M_k}$, $M_k' = \frac{1}{m_k}$, $m_k'' = \sqrt{m_k}$, $M_k'' = \sqrt{M_k}$

$$\Rightarrow U(P,g) - L(P,g) = \sum_{k=1}^{\nu} (M_k' - m_k')(x_k - x_{k-1}) = \sum_{k=1}^{\nu} \left(\frac{1}{m_k} - \frac{1}{M_k}\right)(x_k - x_{k-1})$$

$$= \sum_{k=1}^{\nu} \frac{M_k - m_k}{m_k M_k}(x_k - x_{k-1}) \leq \frac{1}{m^2} \sum_{k=1}^{\nu} (M_k - m_k)(x_k - x_{k-1}) = \frac{1}{m^2}\big(U(P,f) - L(P,f)\big) < \varepsilon$$

και $$U(P,h) - L(P,h) = \sum_{k=1}^{\nu} (M_k'' - m_k'')(x_k - x_{k-1}) = \sum_{k=1}^{\nu} \left(\sqrt{M_k} - \sqrt{m_k}\right)(x_k - x_{k-1})$$

$$= \sum_{k=1}^{\nu} \frac{M_k - m_k}{\sqrt{M_k} + \sqrt{m_k}}(x_k - x_{k-1}) \leq \frac{1}{2\sqrt{m}} \sum_{k=1}^{\nu} (M_k - m_k)(x_k - x_{k-1}) = \frac{1}{2\sqrt{m}}\big(U(P,f) - L(P,f)\big) < \varepsilon$$