[§2.9] Θεωρήματα μέσης τιμής (του ολοκληρωτικού λογισμού)

Notiztitel 13.05.2012

Θ. [2.72] (Πρώτο Θεώρημα Μέσης Τιμής)

$f, g : [\alpha, \beta] \to \mathbb{R}$ με fg, g ολοκληρώσιμες στο $[\alpha, \beta]$ και

α) $m \le f(x) \le M \quad \forall x \in [\alpha, \beta]$

β) g έχει σταθερό πρόσημο στο $[\alpha, \beta]$

(δηλ. $g(x) \ge 0 \ \forall x \in [\alpha, \beta]$ ή $g(x) \le 0 \ \forall x \in [\alpha, \beta]$)

$$\Rightarrow \quad \exists\, \eta \in [m, M] : \quad \int_\alpha^\beta f(x) g(x)\, dx = \eta \int_\alpha^\beta g(x)\, dx \qquad (1)$$

Αν επιπλέον f συνεχής, τότε

$$\exists\, \xi \in [\alpha, \beta] : \quad \int_\alpha^\beta f(x) g(x)\, dx = f(\xi) \int_\alpha^\beta g(x)\, dx \qquad (2)$$

Απόδειξη:

Χ.β.τ.γ. έστω $g(x) \leq 0 \ \forall x \in [\alpha, \beta]$

$$\underset{\alpha)}{\Longrightarrow} \quad M g(x) \leq f(x) g(x) \leq m g(x) \quad \forall x \in [\alpha, \beta]$$

$$\underset{\Theta.[2.44\,\beta)]}{\Longrightarrow} \quad M \int_\alpha^\beta g(x)dx \leq \int_\alpha^\beta f(x) g(x) dx \leq m \int_\alpha^\beta g(x) dx \qquad (3)$$

Αν $\int_\alpha^\beta g(x) dx = 0 \underset{(3)}{\Longrightarrow} \int_\alpha^\beta f(x) g(x) dx = 0 \Longrightarrow (1), (2)$

Αν $\int_\alpha^\beta g(x)\, dx < 0 \quad \left(\Theta.[2.44\,\alpha)] : \int_\alpha^\beta g(x) dx \leq 0 \right)$

$$\underset{(3)}{\Longrightarrow} \quad m \leq \frac{\int_\alpha^\beta f(x) g(x) dx}{\int_\alpha^\beta g(x) dx} =: \eta \leq M$$

Αν f συνεχής, τότε για $m = \min f$, $M = \max f$ $\exists\, \xi \in [\alpha, \beta]$ $f(\xi) = \eta$ (Θ. ενδιάμεσων τιμών). $\square$

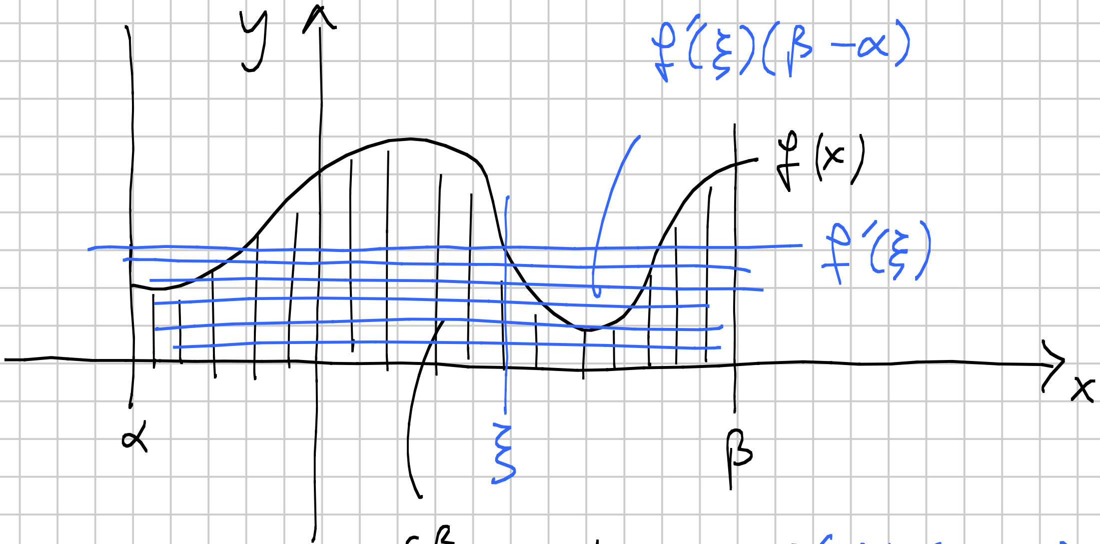
Πόρισμα: $f : [\alpha,\beta] \to \mathbb{R}$ ολοκληρώσιμη, $m \le f(x) \le M \ \forall x \in [\alpha,\beta]$ 18-13

$\Rightarrow \exists \eta \in [m, M] : \int_\alpha^\beta f(x)\,dx = \eta(\beta-\alpha)$

Αν επιπλέον f συνεχής, τότε $\exists \xi \in [\alpha,\beta] : \int_\alpha^\beta f(x)\,dx = f(\xi)(\beta-\alpha)$

[Απόδειξη: Εφαρμόζουμε το προηγούμενο Θεώρημα για $g \equiv 1$.]

Παρατ. [2.73]: f παραγ., f' ολοκλ.: $\int_\alpha^\beta f'(x)\,dx \underset{\text{ΘΘΑΛ}}{=} f(\beta)-f(\alpha) \underset{\text{ΘΜΤ}}{=} f'(\xi)(\beta-\alpha)$



$$\int_\alpha^\beta f(x)\,dx = f'(\xi)(\beta-\alpha)$$

Θ. [2.75] (Δεύτερο Θεώρημα Μέσης Τιμής)

$f, g : [\alpha, \beta] \to \mathbb{R}$ με

α) f μονότονη

β) g ολοκληρώσιμη και έχει σταθερό πρόσημο

$$\Rightarrow \quad \exists \; \xi \in [\alpha, \beta] : \int_\alpha^\beta f(x) g(x) dx = f(\alpha) \int_\alpha^\xi g(x) dx + f(\beta) \int_\xi^\beta g(x) dx$$

Απόδειξη:

g ολοκλ. $\Rightarrow$ $G(z) := \int_\alpha^z g(x) dx$, $z \in [\alpha, \beta]$, συνεχής $\Rightarrow$

$\Rightarrow$ $H(z) := \int_\alpha^\beta g(x) dx - G(z) = \int_z^\beta g(x) dx$, $z \in [\alpha, \beta]$, συνεχής

$\Rightarrow$ $F(z) := \int_\alpha^\beta f(x) g(x) dx - f(\alpha) G(z) - f(\beta) H(z)$, $z \in [\alpha, \beta]$, συνεχής

$\left[f \text{ μονότονη} \Rightarrow f \text{ ολοκλ.} \underset{g \text{ ολοκλ.}}{\Rightarrow} fg \text{ ολοκλ.} \Rightarrow \exists \int_\alpha^\beta f(x) g(x) dx \in \mathbb{R} \right]$,

$F(\alpha) = \int_\alpha^\beta f(x) g(x) dx - f(\beta) \int_\alpha^\beta g(x) dx$, $F(\beta) = \int_\alpha^\beta f(x) g(x) dx - f(\alpha) \int_\alpha^\beta g(x) dx$ (1)

Χ.β.τ.γ. Έστω f αύξουσα και $g(x) \geq 0 \ \forall x \in [\alpha, \beta]$

$\Rightarrow f(\alpha) \leq f(x) \leq f(\beta) \ \forall x \in [\alpha, \beta]$

$\Rightarrow f(\alpha) g(x) \leq f(x) g(x) \leq f(\beta) g(x) \ \forall x \in [\alpha, \beta]$

$\Rightarrow f(\alpha) \int_\alpha^\beta g(x)\,dx \leq \int_\alpha^\beta f(x) g(x)\,dx \leq f(\beta) \int_\alpha^\beta g(x)\,dx$

$\underset{(1)}{\Rightarrow} F(\alpha) \leq 0, \ F(\beta) \geq 0 \Rightarrow F(\alpha) F(\beta) \leq 0$

Αν $F(\alpha) = 0 \Leftrightarrow \int_\alpha^\beta f(x) g(x)\,dx = f(\beta) \int_\alpha^\beta g(x)\,dx \underset{\xi := \alpha}{\Leftrightarrow}$ το αποδεικτέο

Αν $F(\beta) = 0 \Leftrightarrow \int_\alpha^\beta f(x) g(x)\,dx = f(\alpha) \int_\alpha^\beta g(x)\,dx \underset{\xi := \beta}{\Leftrightarrow}$ το αποδεικτέο

Αν $F(\alpha) F(\beta) < 0 \underset{\text{Bolzano}}{\Longrightarrow} \exists \ \xi \in (\alpha, \beta) : F(\xi) = 0 \Leftrightarrow$ το αποδεικτέο

$\square$

Εφ. [3., § 2.9] $f, g : [\alpha, \beta] \to \mathbb{R}$ με

α) f μονότονη και παραγωγίσιμη με f' ολοκληρώσιμη

β) g συνεχής

$\Rightarrow \exists\, \xi \in [\alpha, \beta] : \int_\alpha^\beta f(x) g(x)\, dx = f(\alpha) \int_\alpha^\xi g(x)\, dx + f(\beta) \int_\xi^\beta g(x)\, dx$

Απόδειξη:

g συνεχής $\Rightarrow$ g ολοκλ. $\Rightarrow$ $\exists\ G(x) := \int_\alpha^x g(t)\, dt,\ x \in [\alpha, \beta]$,

παραγωγίσιμη με $G'(x) = g(x)$ $\Rightarrow$ G συνεχής $\Rightarrow$ G ολοκλ.

f μονότονη $\Rightarrow$ f ολοκλ. $\underset{g \text{ ολοκλ.}}{\Rightarrow}$ fg ολοκλ.

Επίσης, f', $G' = g$ ολοκλ.

$$\underset{\text{Παραγ. ολοκλ.}}{\Rightarrow} \int_\alpha^\beta f(x) g(x)\, dx = \int_\alpha^\beta f(x) G'(x)\, dx = \underbrace{f(x) G(x) \Big|_\alpha^\beta} - \int_\alpha^\beta f'(x) G(x)\, dx \quad (1)$$

$$= f(\beta) G(\beta) - f(\alpha) \underbrace{G(\alpha)}_{=0} = f(\beta) G(\beta)$$

f μονότονη και παραγωγίσιμη $\Rightarrow$ f' έχει σταθερό πρόσημο,

G συνεχής ($\Rightarrow G([\alpha,\beta]) = [\min G, \max G]$), f', $f'G$ ολοκλ.

$\Rightarrow$ Πρ. ΘΜΤ Ολ. Λογ. (Θ. [2.72]) $\exists\, \xi \in [\alpha,\beta]$: $\displaystyle\int_\alpha^\beta f'(x)\,G(x)\,dx = G(\xi) \underbrace{\int_\alpha^\beta f'(x)\,dx}_{\underset{\text{ΘΘΑΛ}}{=} f(\beta)-f(\alpha)}$

$\underset{(1)}{\Rightarrow}$ $\exists\, \xi \in [\alpha,\beta]$: $\displaystyle\int_\alpha^\beta f(x)\,g(x)\,dx = f(\beta)G(\beta) - G(\xi)\,(f(\beta)-f(\alpha))$

$= f(\alpha)\,G(\xi) + f(\beta)\,(G(\beta) - G(\xi))$

$\displaystyle = f(\alpha)\int_\alpha^\xi g(x)\,dx + f(\beta)\Big(\underbrace{\int_\alpha^\beta g(x)\,dx - \int_\alpha^\xi g(x)\,dx}_{=\int_\xi^\beta g(x)\,dx}\Big)$

$\square$

Θ. [2.76] (Θεώρημα Bonnet)

$f, g : [\alpha, \beta] \to \mathbb{R}$, g ολοκληρώσιμη και σταθερού προσήμου. Τότε

i) f φθίνουσα, θετική $\Rightarrow \exists\, \xi_1 \in [\alpha,\beta] : \int_\alpha^\beta f(x) g(x)\,dx = f(\alpha) \int_\alpha^{\xi_1} g(x)\,dx$

ii) f αύξουσα, θετική $\Rightarrow \exists\, \xi_2 \in [\alpha,\beta] : \int_\alpha^\beta f(x) g(x)\,dx = f(\beta) \int_{\xi_2}^{\beta} g(x)\,dx$

Απόδειξη:

i) Δευτ. ΘΜΤ Ολ. Λογ. (Θ. [2.75]) : $\exists\, \xi \in [\alpha, \beta] : \int_\alpha^\beta f(x)\, g(x)\, dx =$

$$= f(\alpha) \underbrace{\int_\alpha^\xi g(x)\,dx}_{=:G(\xi)} + f(\beta) \underbrace{\int_\xi^\beta g(x)\,dx}_{=G(\beta)-G(\xi)} = \underbrace{\big(f(\alpha) - f(\beta)\big)}_{=:m_1 \geq 0} G(\xi) + \underbrace{f(\beta)}_{=:m_2 > 0} G(\beta) \quad (1)$$

$G : [\alpha, \beta] \to \mathbb{R}$ συνεχής $\Rightarrow G([\alpha,\beta]) = [m, M]$, $m := \min G$, $M := \max G$

$$\Rightarrow G(\xi), G(\beta) \in [m, M] \underset{m_1, m_2 \geq 0}{\Longrightarrow} m_1 G(\xi) \in [m_1 m, m_1 M],\ m_2 G(\beta) \in [m_2 m, m_2 M]$$

$$\Rightarrow m_1 G(\xi) + m_2 G(\beta) \in [(m_1 + m_2) m, (m_1 + m_2) M] \underset{m_1 + m_2 > 0}{\Longleftrightarrow} \frac{m_1 G(\xi) + m_2 G(\beta)}{m_1 + m_2} \in [m, M]$$

$$\Rightarrow \exists\, \xi_1 \in [\alpha,\beta] : \frac{m_1 G(\xi) + m_2 G(\beta)}{m_1+m_2} = G(\xi_1)$$

$$\underset{(1)}{\Rightarrow} \int_\alpha^\beta f(x) g(x)\,dx = (m_1+m_2)\,G(\xi_1) = f(\alpha)\int_\alpha^{\xi_1} g(x)\,dx$$

ii) $\exists\, \xi \in [\alpha,\beta] : \int_\alpha^\beta f(x) g(x)\,dx = f(\alpha)\, G(\xi) + f(\beta)\big(G(\beta)-G(\xi)\big)$

$$= \underbrace{\big(f(\beta)-f(\alpha)\big)}_{=:\, m_1' \geq 0}\underbrace{\big(G(\beta)-G(\xi)\big)}_{=:\, H(\xi)} + \underbrace{f(\alpha)}_{= m_2' > 0}\underbrace{G(\beta)}_{= H(\alpha)} \qquad (3)$$

$H : [\alpha,\beta] \to \mathbb{R}$ συνεχής, $H([\alpha,\beta]) = [m', M'] \Rightarrow \dfrac{m_1' H(\xi) + m_2' H(\alpha)}{m_1'+m_2'} \in [m', M']$

$$\Rightarrow \exists\, \xi_2 \in [\alpha,\beta] : \frac{m_1' H(\xi) + m_2' H(\alpha)}{m_1'+m_2'} = H(\xi_2)$$

$$\underset{(3)}{\Rightarrow} \int_\alpha^\beta f(x) g(x)\,dx = (m_1'+m_2')\,H(\xi_2) = f(\beta)\int_{\xi_2}^{\beta} g(x)\,dx$$

$\square$

Ασκ. [2, §2.9] $\quad \frac{\pi}{6} \le \int_0^{\frac{1}{2}} \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}} \le \frac{\pi}{6}\left(1-\frac{1}{4}k^2\right)^{-\frac{1}{2}}, \quad |k|<2$

Απόδειξη:

$$f(x) = (1-k^2x^2)^{-\frac{1}{2}}, \quad g(x) = (1-x^2)^{-\frac{1}{2}}, \quad x\in\left[0,\tfrac{1}{2}\right]$$

$\underset{\substack{\text{Πρ. ΘΜΤ Ολ. Λογ.}\\ (\text{Θ.}[2.72])}}{\Longrightarrow} \quad \exists\ \xi\in\left[0,\tfrac{1}{2}\right] : \int_0^{\frac{1}{2}} f(x)g(x)\,dx = (1-k^2\xi^2)^{-\frac{1}{2}}\int_0^{\frac{1}{2}}\frac{1}{\sqrt{1-x^2}}dx$

$$= (1-k^2\xi^2)^{-\frac{1}{2}} \operatorname{Arcsin} x\Big|_0^{\frac{1}{2}} = (1-k^2\xi^2)^{-\frac{1}{2}}\left(\operatorname{Arcsin}\tfrac{1}{2} - \operatorname{Arcsin} 0\right)$$

$$= (1-k^2\xi^2)^{-\frac{1}{2}}\frac{\pi}{6} \in \left[\frac{\pi}{6}, \left(1-\left(\frac{k}{2}\right)^2\right)^{-\frac{1}{2}}\frac{\pi}{6}\right]$$

A [2.44] α) $\exists\ \xi_1, \xi_2 \in [0,1]$: $\int_0^1 \frac{\sin(\pi x)}{x^2+1}\,dx = \frac{2}{\pi(\xi_1^2+1)} = \frac{\pi}{4}\sin(\pi\xi_2)$

β) $\frac{1}{\pi} \leq \int_0^1 \frac{\sin(\pi x)}{x^2+1}\,dx \leq \frac{2}{\pi}$

Απόδειξη :

$f(x) = \frac{1}{x^2+1}$ συνεχής, $g(x) = \sin(\pi x) \geq 0\ \forall x \in [0,1]$ συνεχής

$\underset{\Theta.[2.72]}{\Longrightarrow} \exists\ \xi_1 \in [0,1] : \int_0^1 \frac{\sin(\pi x)}{x^2+1}\,dx = \frac{1}{\xi_1^2+1}\int_0^1 \sin(\pi x)\,dx$

και $\int_0^1 \sin(\pi x)\,dx = \frac{1}{\pi}\int_0^1 \sin(\underbrace{\pi x}_{=\varphi(x)})\,\pi\,dx = \frac{1}{\pi}\int_0^{\pi}\sin x\,dx = -\frac{1}{\pi}\cos \pi x\Big|_0^{\pi}$

$= -\frac{1}{\pi}(\cos\pi - \cos 0) = -\frac{1}{\pi}(-1-1) = \frac{2}{\pi}$

$\Longrightarrow \exists\ \xi_1 \in [0,1] : \int_0^1 \frac{\sin(\pi x)}{x^2+1}\,dx = \frac{2}{\pi}\frac{1}{\xi_1^2+1} \in \left[\frac{1}{\pi}, \frac{2}{\pi}\right]$ (δηλ. β))

$f(x) = \sin(\pi x)$ συνεχής, $g(x) = \frac{1}{x^2+1} \geq 0\ \forall x \in \mathbb{R}$ συνεχής

$\underset{\Theta.[2.72]}{\Longrightarrow} \exists\ \xi_2 \in [0,1] : \int_0^1 \frac{\sin(\pi x)}{x^2+1}\,dx = \sin(\pi\xi_2)\underbrace{\int_0^1 \frac{dx}{x^2+1}}_{= \operatorname{Arctg} x|_0^1 = \frac{\pi}{4}} = \frac{\pi}{4}\sin(\pi\xi_2)$

A [2.48] $f_{\alpha,\beta}:\mathbb{R}\to\mathbb{R}$, $f_{\alpha,\beta}(x)=e^{-\alpha x}\sin^2\beta x$, $\alpha>0$.

α) Να υπολογίσετε το αόριστο ολοκλήρωμα της $f_{\alpha,\beta}$

β) Να αποδείξετε: $\exists\, \eta\in[e^{-\frac{1}{2}}, e^4]$: $\int_0^2 e^{2x^2} f_{2,\pi}(x)\,dx=\eta$

Λύση:

α) $\beta=0$: $f_{\alpha,0}(x)=0 \Rightarrow \int f_{\alpha,0}(x)\,dx=c$

$\beta\neq0$: $\sin^2\beta x = 1-\cos^2\beta x = 1-\frac{\cos(2\beta x)+1}{2}=\frac{1}{2}\left(1-\cos(2\beta x)\right)$

$\left[\cos(2\alpha)=\cos^2\alpha-\sin^2\alpha = 2\cos^2\alpha-1 \Rightarrow \cos^2\alpha=\frac{\cos(2\alpha)+1}{2}\right]$

$\Rightarrow \int f_{\alpha,\beta}(x)\,dx = \frac{1}{2}\left(\int e^{-\alpha x}dx - \int e^{-\alpha x}\cos(2\beta x)\,dx\right)=$

$=\frac{1}{-2\alpha}e^{-\alpha x}-\frac{1}{2}\int e^{-\alpha x}\cos(2\beta x)\,dx$

και

$$\int e^{-\alpha x}\cos(2\beta x)\,dx = \frac{1}{-\alpha}e^{-\alpha x}\cos(2\beta x) + \frac{1}{\alpha}\int e^{-\alpha x}\left(-\sin(2\beta x)\,2\beta\,dx\right.$$

$$= \frac{1}{-\alpha}e^{-\alpha x}\cos(2\beta x) - \frac{2\beta}{\alpha}\left(\frac{1}{-\alpha}e^{-\alpha x}\sin(2\beta x) + \frac{1}{\alpha}\int e^{-\alpha x}\cos(2\beta x)2\beta dx\right)$$

$$= \frac{1}{-\alpha}e^{-\alpha x}\cos(2\beta x) + \frac{2\beta}{\alpha^2}e^{-\alpha x}\sin(2\beta x) - \frac{4\beta^2}{\alpha^2}\int e^{-\alpha x}\cos(2\beta x)dx$$

$$\Rightarrow \int e^{-\alpha x}\cos(2\beta x)\,dx = \frac{1}{1+\frac{4\beta^2}{\alpha^2}}e^{-\alpha x}\left(\frac{1}{-\alpha}\cos(2\beta x) + \frac{2\beta}{\alpha^2}\sin(2\beta x)\right) + c$$

$$= \frac{e^{-\alpha x}}{\alpha^2+4\beta^2}\left(2\beta\sin(2\beta x) - \alpha\cos(2\beta x)\right) + c$$

$$\Rightarrow \int f_{\alpha,\beta}(x)\,dx = \frac{1}{2}\left(\frac{\alpha\cos(2\beta x) - 2\beta\sin(2\beta x)}{\alpha^2+4\beta^2} - \frac{1}{\alpha}\right)e^{-\alpha x} + c$$

β) $$\int_0^2 e^{2x^2} f_{2,\pi}(x)\,dx = \int_0^2 e^{2x^2-2x}\sin^2(\pi x) = \eta\int_0^2 \sin^2(\pi x)\,dx$$

με $\eta \in \left[\min_{x\in[0,2]} e^{2x^2-2x}, \max_{x\in[0,2]} e^{2x^2-2x}\right]$ (Πρώτο Θ.Μ.Τ., Θ [2.72])

$$g(x) := e^{2x^2-2x} \Rightarrow g'(x) = g(x)(4x-2) = 0 \text{ για } x = \tfrac{1}{2}$$

και $g''(x) = g(x)\left((4x-2)^2+4\right) = 4g(x)\left(4x^2-4x+2\right) =$

$= 16\,g(x)\left(x^2-x+\frac{1}{2}\right) = 16\,g(x)\left(\left(x-\frac{1}{2}\right)^2+\frac{1}{4}\right) > 0 \quad \forall x \in \mathbb{R}$

$\Rightarrow \quad \min_{x\in\mathbb{R}} g(x) = g\left(\frac{1}{2}\right) = e^{-\frac{1}{2}} = \min_{x\in[0,2]} g(x)$,

$\max_{x\in[0,2]} g(x) = \max\{g(0), g(2)\} = \max\{1, e^4\} = e^4$

$\Rightarrow \quad \eta \in \left[e^{-\frac{1}{2}}, e^4\right]$

Απ' την άλλη $\quad \int_0^2 \sin^2(\pi x)\,dx \underset{y=\pi x}{=} \frac{1}{\pi}\int_0^{2\pi} \sin^2 y\,dy$

και αφού $\quad \int \sin^2 y\,dy = -\sin y\cos y + \int (1-\sin^2 y)\,dy =$

$= \frac{1}{2}\left(y - \sin y\cos y\right) + c \quad \Rightarrow \quad \int_0^{2\pi} \sin^2 y\,dy = \pi$

και άρα $\quad \int_0^2 e^{2x^2} f_{2,\pi}(x)\,dx = \eta \in \left[e^{-\frac{1}{2}}, e^4\right]$