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The homomorphisms between the Dickson-Mùi algebras as modules over the Steenrod algebra. (English summary)
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The Dickson-Mùi algebra $D(V)$ of an elementary abelian p-group V consists of all invariants in the mod p cohomology $H^*(V)$ with respect to the action of the general linear group $\mathrm{GL}(V)$. It is a module over the Steenrod algebra $\mathbb{A}$. As an $\mathbb{A}$-module $D(V)$ splits as the direct sum of its augmentation ideal $\overline{D}(V)$ (the reduced Dickson-Mùi algebra of V) and $\mathbb{F}_p$, given by the unit of $D(V)$. The author determines an explicit basis of the vector space of all $\mathbb{A}$-module homomorphisms between $\overline{D}(V)$ and $\overline{D}(W)$. The basis elements are given as compositions of transfers and restrictions passing through intermediate reduced Dickson-Mùi algebras $D(U)$, where U is an elementary abelian p-group with $\mathrm{rank}(U) \le \min\{\mathrm{rank}(V), \mathrm{rank}(W)\}$.

Furthermore, for $V = W$ the author determines the algebra of all $\mathbb{A}$-module endomorphisms of $D(V)$ (resp. $\overline{D}(V)$); it is isomorphic to an explicit quotient of a polynomial algebra on one indeterminate. As a consequence it is shown that the reduced Dickson-Mùi algebras are atomic in the sense that if an $\mathbb{A}$-module endomorphism of the algebra is nonzero on the least positive degree generator, then it is an automorphism. This implies in particular that the reduced Dickson-Mùi algebra is an indecomposable $\mathbb{A}$-module. Similar results also hold for the odd characteristic usual Dickson algebras given as the invariants of the action of $\mathrm{GL}(V)$ on the symmetric algebra of the dual of V, again considered as an $\mathbb{A}$-module.

In the case of the algebra of automorphisms of $D(V)$ (resp. $\overline{D}(V)$) similar results have been found in an unpublished paper of N. E. Kechagias ["A Steenrod-Milnor action ordering on Dickson invariants", submitted, see http://www.math.uoi.gr/~nondas_k/Steen-Mil.pdf].

The author uses as a major tool the injectivity of H^*V as an object in the category $\mathcal{U}$ of unstable $\mathbb{A}$-modules, as well as the Adams-Gunawardena-Miller Theorem, which states that the canonical homomorphism $\mathbb{F}_p[\mathrm{Hom}(V,W)] \to \mathrm{Hom}_{\mathcal{U}}(H^*W, H^*V)$ is an isomorphism. Using these tools a bit more systematically leads to a positive answer to Conjecture 1.5 of the paper. In fact, injectivity of H^*V implies that for any subgroups $G \subset \mathrm{GL}(V)$ and $H \subset \mathrm{GL}(W)$ there are canonical isomorphisms

$$\mathrm{Hom}_{\mathcal{U}}((H^*W)^H, (H^*V)^G) = \mathrm{Hom}_{\mathcal{U}}((H^*W)^H, H^*V)^G \cong (\mathrm{Hom}_{\mathcal{U}}(H^*W, H^*V)_H)^G$$

(the latter object denoting the G-invariants of the H-coinvariants!), and then the Adams-Gunawardena-Miller Theorem gives

$$\mathrm{Hom}_{\mathcal{U}}((H^*W)^H, (H^*V)^G) \cong (\mathbb{F}_p[\mathrm{Hom}(V,W)]_H)^G \cong \mathbb{F}_p[H\backslash\mathrm{Hom}(V,W)/G],$$

i.e. precisely Conjecture 1.5. If $V = W$ and $H = G$ then composition on the left-hand side corresponds to the product in the Hecke algebra of double cosets on the right-hand side, thus giving an answer to another question posed by the author in the context of Conjecture 1.5. *Hans-Werner Henn*

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Note: This list reflects references listed in the original paper as accurately as possible with no attempt to correct errors.