A Steenrod-Milnor action ordering on Dickson invariants

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Abstract. Let $\mathcal{A}$ be the Steenrod algebra and $D(V)$ the Dickson algebra. An ordering in $D(V)$ is defined according to the Steenrod algebra action. Using this ordering, we prove the following: Let $f \in End_{\mathcal{A}}(D(V))$ be an $\mathcal{A}$-linear degree preserving map. If f is non-zero on the lowest degree, then f is an isomorphism. Moreover, $End_{\mathcal{A}}\left(\overline{D(V)}\right)$ is a local ring, where $\overline{D(V)}$ is its augmentation ideal.

1. Statement of results

It is known that the classical Dickson algebra D_k is a polynomial algebra:

$$D_k \cong \mathbb{F}_p[d_1, ..., d_k]$$

Mùi related D_k (for $p = 2$) with the dual of the Dyer-Lashof algebra calculated by Madsen. Motivated by topological questions regarding the cohomology of an infinite (finite) loop space and influenced by the work of Campbell, Cohen, Peterson and Selick in [**2**] and [**3**] we study the problem under which conditions is an $\mathcal{A}$-endomorphism of $D(V) := (H^*(V))^{GL(k,\mathbb{F}_p)}$ an isomorphism. Here $\mathcal{A}$ stands for the Steenrod algebra.

Firstly, we consider the classical Dickson algebra D_k. Where modifications are needed between the case $p = 2$ and $p > 2$ they are provided. Given a sequence of k non-negative integers $\bar{n} = (n_k, n_1, ..., n_{k-1})$ let

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$d^{\bar{n}} := \prod_1^k d_i^{n_i}$. Our first task is to prove that there exists a unique p-th power Steenrod operation P^{p^m} of smallest degree such that

$$P^{p^m} d^{\bar{n}} \neq 0$$

Moreover, the new element has the property that there exists at least one p-th power of a generator $d_i^{p^{t_i}}$ such that $d_i^{p^{t_i}}$ divides $d^{\bar{n}}$ and $t_i + i - 1 = m$. Applying this property again on $P^{p^m} d^{\bar{n}}$ we get

$$P^{p^{m-1}} P^{p^m} d^{\bar{n}} \neq 0$$

Then we iterate: $P^{p^{t_i}} \ldots P^{p^m} d^{\bar{n}} \neq 0$. We are interested in finding the longest such sequence of Steenrod operations such that $P^{p^{t_{i(l)}}} \ldots P^{p^m} d^{\bar{n}}$ is a non-zero monomial. We call such a sequence a **Steenrod-Milnor** action on $d^{\bar{n}}$ denoted by $P^{\Gamma(\bar{n})}$ (please see definition 5). Now we iterate this procedure on the monomial $P^{\Gamma(\bar{n})} d^{\bar{n}}$ until the resulting monomial is $d_k^{p^q}$ for the smallest q.

Theorem 12 *There exists a sequence of Steenrod-Milnor operations* P^Γ *such that* $P^\Gamma d^{\bar{n}} = \lambda d_k^{p^{l(\bar{n})}}$. *Here* $\lambda \in (\mathbb{F}_p)^*$.

Next, given two monomials $d^{\bar{n}}$ and $d^{\bar{n}'}$ we define an ordering according to their Steenrod-Milnor actions $P^{\Gamma(\bar{n})}$ and $P^{\Gamma(\bar{n}')}$. We call this ordering a Steenrod-Milnor (S-M) ordering (please see definition 13). Using this ordering we prove the following theorem:

Theorem 15 *Let* $f : (D_k)^G \to (D_k)^G$ *be an* $\mathcal{A}$*-linear map of degree* 0 *such that* $f(d_{k-1}) \neq 0$. *Then* f *is a upper triangular map with respect to S-M ordering and hence an isomorphism.*

We note that the last Theorem is not true for the upper triangular ring, please see example 16.

This finishes section 3 which consists of the technical part of this work. We extend the theorem above to the full ring of invariants, $D(V)$, in section 4.

Theorem 19 *Let* $g : D(V) \to D(V)$ *be an* A*-linear map of degree* 0 *such that* $g(M_k) \neq 0$. *Then* g *is an isomorphism. Here* $M_k = \prod_1^k x_i L_k^{p-2}$ *is the element of lowest degree.*

In section 5 we investigate the structure of $End_{\mathcal{U}}(D(V))$. A shorter and elegant proof of the next corollary was suggested by H.-W. Henn.

Corollary 27 $End_{\mathcal{U}}(\overline{D(V)})$ is a local $\mathbb{F}_p$-algebra with dimension n as a vector space over $\mathbb{F}_p$ (i.e. f or $Id - f$ is an isomorphism for any f in $End_{\mathcal{U}}(\overline{D(V)})$). Moreover, if I is the ideal generated by its nilpotent elements, then $End_{\mathcal{U}}(\overline{D(V)})/I \equiv \mathbb{F}_p$.

Finally we apply Theorem 19 to the study of self maps between infinite loop spaces. We obtain an alternative proof of theorem 4.1 page 28 of Campbell, Peterson and Selick:

Theorem [**2**]*Let* $f:\Omega_0^\infty S^\infty \to \Omega_0^\infty S^\infty$ *be an* H*-map which induces an isomorphism on* $H_{2p-3}(\Omega_0^\infty S^\infty;\mathbb{F}_p)$. *If* $p>2$ *suppose in addition that* f *is a loop map or that*

$$f_*(d_{2,2})^* \neq 0$$

for some $\lambda \in (\mathbb{F}_p)^*$. *Then* $f_{(p)}$ *is a homotopy equivalence. Here* $(d_{2,2})^*$ *is the* hom-*dual of the top degree Dickson generator* of D_2 in $R[2]$.

2. Introduction

Let $\mathcal{A}$ be the Steenrod algebra and $\mathcal{U}$ the category of unstable $\mathcal{A}$-modules which is a full subcategory of $\mathbb{F}_p$-graded $\mathcal{A}$-modules and morphisms being $\mathcal{A}$-linear maps of degree 0 that is degree preserving. Let V and W finite dimensional vector spaces over $\mathbb{F}_p$ and $H^*(V)$ the mod p cohomology of its classifying space: $H^*(V) := H^*(BV,\mathbb{F}_p)$. Moreover

$$H^*(V) \cong E(x_1,...,x_k)\otimes P[y_1,...,y_k]$$

where $V^* =< x_1,...,x_k >$, $\beta x_i = y_i$ and $|x_i| = 1$.

A map of unstable $\mathcal{A}$-algebras $f^*: H^*(W) \to H^*(V)$ is determined by its action in degree 1 that is by an element of $Hom(V,W)$ which is isomorphic to $Hom_{\mathcal{K}}(H^*(W), H^*(V))$. Here $\mathcal{K}$ is the category of unstable $\mathcal{A}$-algebras. There is also an isomorphism for $\mathcal{A}$-linear maps:

$$Hom_{\mathcal{U}}(H^*(W), H^*(V)) \cong F_p[Hom(V,W)]$$

It is known that an $\mathcal{A}$-linear map $f^*: H^*(\mathbb{F}_p) \to H^*(\mathbb{F}_p)$ is determined by its direct sum components $H^*(\mathbb{F}_p) \cong \bigoplus_1^{p-1} H_i$ and it is an isomorphism, if it is an isomorphism in degree $2i-1$ for $i=1,...,p-1$. Here $(H_i)^* = \tilde{H}^*(\mathbb{F}_p)$ for $* \equiv 2i$ or $2i-1 \bmod (p-1)$. The algebraic structure of $H^*(V)$ as an $\mathcal{A}$-module has been studied extensively ([**9**]).

The general linear group $G := GL(k,\mathbb{F}_p)$ acts on V and hence on $H^*(V)$. The ring of invariants

$$D(V) := (H^*(V))^G$$

called the "Dickson algebra" was described by Dickson for $D_k := (P[y_1,...,y_k])^G$ and Mùi for the general case. Dickson proved that $D_k \cong \mathbb{F}_p[d_{k,1},...,d_{k,k}]$ ([**4**]), is again a polynomial algebra with $|d_{k,i}| = 2(p^k - p^i)$. Let us briefly describe its generators $d_{k,i}$:

$$h_t = \prod_{u\in<y_1,...,y_{t-1}>} (y_t+u) \text{ for } 1\leq t\leq k$$

$$d_{k,i} = h_k^{p-1} d_{k-1,i} + d_{k-1,i-1}^p$$

Mùi ([**8**]) proved that $D(V)$ is a tensor product between $D(V)$ and the $\mathbb{F}_p$-module spanned by the set of elements consisting of the following monomials:

$$M_{k;s_1,\ldots,s_l} L_k^{p-2};\ 0 \le l \le k-1, \text{ and } 0 \le s_1 < \cdots < s_l \le k-1$$

Its algebra structure is determined by the following relations:
a) $(M_{k;s_1,\ldots,s_l} L_k^{p-2})^2 = 0$ for $0 \le l \le k-1$, and $0 \le s_1 < \cdots < s_l \le k-1$.
b) $M_{k;s_1,\ldots,s_l} L_k^{(p-2)} d_{k,k-1}^{m-1} = (-1)^{(k-l)(k-l-1)/2} \prod_{t=1}^{k-l} M_{k;0,\ldots,\widehat{k-s_t},\ldots,k-1} L_k^{p-2}$.
Here $0 \le l \le k-1$, and $0 \le s_1 < \cdots < s_l \le k-1$. Those elements are described as follows:

$$M_{k;s_1,\ldots,s_l} = \frac{1}{(k-l)!} \begin{vmatrix} x_1 & \cdots & x_1 \\ \vdots & & \vdots \\ x_1 & \cdots & x_k \\ y_1^{p^{s_1}} & \cdots & y_k^{p^{s_1}} \\ \vdots & & \vdots \\ y_1^{p^{s_l}} & \cdots & y_k^{p^{s_l}} \end{vmatrix}$$

Here there are $k-l$ rows of x_i's and the s_i-th's powers are completing the rest of the determinant above, where $0 \le s_1 < \cdots < s_l \le k-1$. The row $\left(y_1^{p^i}, \ldots, y_k^{p^i}\right)$ is omitted in the determinant above and $1 \le i \le k-1$.
$|M_{k;s_1,\ldots,s_l}| = k - l + 2(p^{s_1} + \cdots + p^{s_l})$. And $L_k = \prod_1^k h_i$.

From now on we write d_i for $d_{k,i}$.

Since the operation of G on $H^*(V)$ commutes with the action of the Steenrod algebra, $D(V)$ is also a module (in fact an algebra) over $\mathcal{A}$.

3. A Steenrod-Milnor action ordering on Dickson invariants

We shall recall some well known results concerning the action of the Steenrod algebra on Dickson algebra generators.

PROPOSITION 1. [**6**] *(Th. 30, p. 169)*

$$P^{p^t}(d_i^{p^l}) = \begin{cases} d_{i-1}^{p^l}, & \text{if } t = l+i-1 \text{ and } i < k \\ -d_i^{p^l} d_{i-1}^{p^l}, & \text{if } t = l+k-1 \\ 0, & \text{otherwise} \end{cases}.$$

A similar result holds for the generators of $D(V)$.

PROPOSITION 2. [**6**] *(Th. 36, p. 170)1) Let $q>0$. If $q=\sum_{i}^{k-1} a_t p^{t+l}$ such that $p-1\geq a_t\geq a_{t-1}>a_{i-1}=0$. Then*

$$P^q d_k^{p^l} = d_k^{p^l} (-1)^{a_{k-1}} \prod_{i}^{k-1} \binom{a_t}{a_{t-1}} d_t^{p^l(a_t-a_{t-1})}$$

Otherwise, $P^q d_k^{p^l}=0$. If $t=0$, then $d_0\equiv d_k$.

2) Let $q=\sum_{s}^{k-1} a_t p^{t+l}>0$ such that $p-1\geq a_t\geq a_{t-1}\geq a_{\boldsymbol{i}}\geq 0$ and $a_{\boldsymbol{i}}+1\geq a_{\boldsymbol{i}-1}\geq a_t\geq a_{t-1}\geq a_{s-1}=0$. Then

$$P^q d_{\boldsymbol{i}}^{p^l} = d_{\boldsymbol{i}}^{p^l}(-1)^{a_{k-1}} \left(\prod_{i+1}^{k-1}\binom{a_t}{a_{t-1}}\right)\binom{a_{\boldsymbol{i}}+1}{a_{\boldsymbol{i}-1}}\left(\prod_{s}^{i-1}\binom{a_t}{a_{t-1}}\right)\prod_{s}^{k-1} d_t^{p^l(a_t-a_{t-1})}$$

Here $a_{s-1}=0$. Otherwise, $P^q d_i^{p^l}=0$.

REMARK 3. *Please note that the case $a_{\boldsymbol{i}}=0$ and $a_{\boldsymbol{i}-1}=1$ is allowed in the proposition above.*

We shall apply formulas above on a monomial in the Dickson algebra starting with the smaller non-zero p-th power. Let us firstly demonstrate our method.

EXAMPLE 4. *Let $p=2$ and $k=3$. Let*

$$d^{\bar{n}} = d_3^{2+2^2} d_1^{2^3+2^4} d_2^{2^2+2^3}$$

$$\bar{n} = (n_3 = 2^1+2^2, n_1 = 2^3+2^4, n_2 = 2^2+2^3)$$

Let us write n_i in its p-adic form:

$$n_i = n_{i0} + n_{i1}p + \ldots$$

Here $n_{3,0}=1$, $n_{1,0}=3$ and $n_{2,0}=2$. We define

$$m(\bar{n}) = \min\{n_{3,0}+k-1, n_{1,0}+k-3, n_{2,0}+k-2\} = 3$$

$$I(\bar{n}) = \{i \mid m(\bar{n}) = n_{i,0}+i-1\} = \{3,1,2\}$$

and

$$i(\bar{n}) = \max I(\bar{n}) = 3$$

We apply $i(\bar{n})=3$ squaring operations, namely:

$$Sq^{2^{m(\bar{n})}},\ Sq^{2^{m(\bar{n})-1}},\ \text{and } Sq^{2^{m(\bar{n})-2}}$$

$$Sq^{2^{m(\bar{n})}} d^{\bar{n}} = d_3^{2+2^2} d_2^2 d_1^{2^3+2^4} d_2^{2^2+2^3} + d_3^{2+2^2} d_3^{2^3} d_1^{2^4} d_2^{2^2+2^3} + d_3^{2+2^2} d_1^{2^2+2^3+2^4} d_2^{2^3}$$

$$Sq^{2^{m(\bar{n})-1}}\left[d_3^{2+2^2} d_2^2 d_1^{2^3+2^4} d_2^{2^2+2^3} + d_3^{2+2^2} d_3^{2^3} d_1^{2^4} d_2^{2^2+2^3} + d_3^{2+2^2} d_1^{2^2+2^3+2^4} d_2^{2^3}\right] =$$

$$d_3^{2+2^2} d_1^2 d_1^{2^3+2^4} d_2^{2^2+2^3} + d_3^{2+2^2+2^2} d_1^{2^3+2^4} d_2^{2^3}$$

$$d_3^{2+2^2}d_1^2d_1^{2^3+2^4}d_2^{2^2+2^3}+d_3^{2+2^2+2^2}d_1^{2^3+2^4}d_2^{2^3}$$

$$Sq^{2^{m(\bar{n})-2}}\left[d_3^{2+2^2}d_1^2d_1^{2^3+2^4}d_2^{2^2+2^3}+d_3^{2+2^2+2^2}d_1^{2^3+2^4}d_2^{2^3}\right]=d_3^{2^3}d_1^{2^3+2^4}d_2^{2^2+2^3}$$

Finally,

$$Sq^{2^{m(\bar{n})-2}}Sq^{2^{m(\bar{n})-1}}Sq^{2^{m(\bar{n})}}d^{\bar{n}}=d_3^{2^3}d_1^{2^3+2^4}d_2^{2^2+2^3}$$

Let $\bar{n}=(2^3,2^3+2^4,2^2+2^3)$. *Then* $m(\bar{n})=3$, $I(\bar{n})=\{1,2\}$, $i(\bar{n})=2$.

$$Sq^{2^2}Sq^{2^3}d^{\bar{n}}=d_3^{2^2+2^3}d_1^{2^3+2^4}d_2^{2^3}$$

Let $\bar{n}=(2^2+2^3,2^3+2^4,2^3)$. *Then* $m(\bar{n})=3$, $I(\bar{n})=\{1\}$, $i(\bar{n})=1$.

$$Sq^{2^3}d^{\bar{n}}=d_3^{2^2+2^4}d_1^{2^4}d_2^{2^3}$$

Please note that at each step $m(\bar{n})=3$ *and the cardinality of* $I(\bar{n})$ *is reduced by 1.*

We call $Sq^{2^3}Sq^{2^2}Sq^{2^3}Sq^{2^1}Sq^{2^2}Sq^{2^3}$ *a Steenrod-Milnor operation of type* $\bar{n}$ *and denote it by* $Sq^{\Gamma(\bar{n})}$. *Please note that the* n_i*'s have been decreased and* n_k *increased respectively.*

Let $\bar{n}=(2^2+2^4,2^4,2^3)$. *Then* $m(\bar{n})=4$, $I(\bar{n})=\{3,1,2\}$, $i(\bar{n})=3$.

$$Sq^{2^2}Sq^{2^3}Sq^{2^4}d^{\bar{n}}=d_{3,3}^{2^3+2^4}d_{3,1}^{2^4}d_{3,2}^{2^3}$$

Let $\bar{n}=(2^3+2^4,2^4,2^3)$. *Then* $m(\bar{n})=4$, $I(\bar{n})=\{1,2\}$, $i(\bar{n})=2$.

$$Sq^{2^3}Sq^{2^4}d^{\bar{n}}=d_3^{2^5}d_1^{2^4}$$

Let $\bar{n}=(2^5,2^4)$. *Then* $m(\bar{n})=4$, $I(\bar{n})=\{1\}$, $i(\bar{n})=1$.

$$Sq^{2^4}d^{\bar{n}}=d_3^{2^4+2^5}$$

Let $\bar{n}=(2^4)$. *Then* $m(\bar{n})=6$, $I(\bar{n})=\{3\}$, $i(\bar{n})=3$.

$$Sq^{2^4}Sq^{2^5}Sq^{2^6}d^{\bar{n}}=d_3^{2^6}$$

Finally,

$$Sq^{\Gamma\left(2^4+2^5,0,0\right)}Sq^{\Gamma\left(2^2+2^4,2^4,2^3\right)}Sq^{\Gamma\left(2^1+2^2,2^3+2^4,2^2+2^3\right)}d_3^{2+2^2}d_1^{2^3+2^4}d_2^{2^2+2^3}=d_3^{2^6}$$

Definition 5. *Let* $\bar{n}=(n_k,n_1,...,n_{k-1})$ *and* $n_i=\sum_{t=1}^{l(i)}a_{i,t}p^{n_{i,t}}$ *its p-adic expansion with* $\prod a_{i,t}\neq 0$.

a) Let $m(\bar{n}):=\min\{n_{i,0}+i-1 \mid 1\le i\le k\}$.

b) Let $I(\bar{n}):=\{i \mid \ m(\bar{n})=n_{i,0}+i-1\}$.

c) Let $i(\bar{n}):=\max I(\bar{n})$.

d) Let $J(\bar{n}):=\begin{cases}\left(a_{i_1,0},...,a_{i(\bar{n}),0}\right), & \text{if } i(\bar{n})<k\\ \left(a_{i_1,0},...,p-a_{i(\bar{n}),0}\right), & \text{if } i(\bar{n})=k\end{cases}$. *For* $p>2$.

Remark 6. *If* $p=2$, *then* $a_{i,t}=1$ *and* $I(\bar{n})$ *determines* $J(\bar{n})$.

DEFINITION 7. *a) For m and l natural numbers such that $l \leq m$, let $P^{\Gamma(m,l)}$ stand for the Steenrod operation $P^{p^{m-l+1}}P^{p^{m-l+2}}...P^{p^{m}}$.*

b) Given a sequence $\bar{n}$, a triad is defined as above $(m(\bar{n}), I(\bar{n}), J(\bar{n}))$. We define a sequence of Steenrod operations associated with this triad as follows

$$P^{\Gamma(m(\bar{n}),I(\bar{n}),J(\bar{n}))} := \underbrace{P^{\Gamma(m(\bar{n}),i_1)}...P^{\Gamma(m(\bar{n}),i_1)}}_{a_{i_1,0}}...\underbrace{P^{\Gamma(m(\bar{n}),i(\bar{n}))}...P^{\Gamma(m(\bar{n}),i(\bar{n}))}}_{a_{i(\bar{n}),0}(or\ p-a_{i(\bar{n}),0})}$$

*We call this operation, $P^{\Gamma(m(\bar{n}),I(\bar{n}),J(\bar{n}))}$, a **Steenrod-Milnor operation** of type $(\bar{n})$ and denote it by $P^{\Gamma(\bar{n})}$.*

REMARK 8. *If $p = 2$, then $Sq^{\Gamma(\bar{n})} = Sq^{\Gamma(m(\bar{n}),i_1)}...Sq^{\Gamma(m(\bar{n}),i(\bar{n}))}$.*

PROPOSITION 9. *a) Let $m = n + k - 1$, then $P^{\Gamma(m,k)}d_k^{p^n} = -d_k^{2p^n}$ ($Sq^{\Gamma(m,k)}d_k^{2^n} = d_k^{2^{n+1}}$).*

b) Let $m = n + k - 1$, then $\underbrace{P^{\Gamma(m,k)}...P^{\Gamma(m,k)}}_{p-1}d_k^{p^n} = -d_k^{p^{n+1}}$.

c) $$P^{p^{m(\bar{n})}}d^{\bar{n}} = \sum_{i_r\in I(\bar{n})-\{k\}} a_{i_r,0}d^{\bar{n}}d_{i_r-1}^{p^{n_{i_r,0}}}d_{i_r}^{-p^{n_{i_r,0}}} + d^{\bar{n}}d_k^{-a_{k,0}p^{n_{k,0}}}P^{p^{m(\bar{n})}}d_k^{a_{k,0}p^{n_{k,0}}}.$$

d) $$P^{\Gamma(m(\bar{n}),i(\bar{n}))}d^{\bar{n}} = \begin{cases} a_{i(\bar{n}),0}d^{\bar{n}}d_k^{p^{n_{i(\bar{n}),0}}}d_{i(\bar{n})}^{-p^{n_{i(\bar{n}),0}}}, & \text{if } k > i(\bar{n}) \\ -a_{k,0}d^{\bar{n}}d_k^{p^{n_{k,0}}}, & \text{if } k = i(\bar{n}) \end{cases}.$$

e) If $k > i(\bar{n})$, then

$$\underbrace{P^{\Gamma(m(\bar{n}),i(\bar{n}))}...P^{\Gamma(m(\bar{n}),i(\bar{n}))}}_{a_{i(\bar{n}),0}}d^{\bar{n}} = (a_{i(\bar{n}),0})!d^{\bar{n}}d_k^{a_{i(\bar{n}),0}p^{n_{i(\bar{n}),0}}}d_{i(\bar{n})}^{-a_{i(\bar{n}),0}p^{n_{i(\bar{n}),0}}}.$$

Otherwise,

$$\underbrace{P^{\Gamma(m(\bar{n}),k)}...P^{\Gamma(m(\bar{n}),k)}}_{p-a_{k,0}}d^{\bar{n}} = (-1)^{p-a_{k,0}}\frac{(p-1)!}{(a_{k,0}-1)!}d^{\bar{n}}d_k^{p^{n_{k,0}+1}}d_k^{-a_{k,0}p^{n_{k,0}}}.$$

PROOF. a) We apply proposition 1:

$$P^{p^t}d_k^{p^l} = 0, \text{ if } t \neq l + k - 1$$

$$P^{p^t}d_i^{p^l} = 0, \text{ if } t \neq l + k - 1 \text{ or } l + i - 1$$

Now the statement follows using Cartan formula.

b) is an application of a).

c) Proposition 1 and Cartan formula implies the statement, since $p^{m(\bar{n})}$ is the least p-th power which provides a non-zero Steenrod operation.

d) We apply proposition 1:

$$P^{\Gamma(n_i+i-1,i)}d_i^{p^{n_i}} = d_k^{p^{n_i}}$$

If $n_i+i-1 > m(\bar{n})$ or $l > i$, then $P^{\Gamma(m(\bar{n}),l)}d_i^{p^{n_i}} = 0$. Now the statement is an application of c).

e) This is a repeated application of d). Two main cases should be considered depending on $i(\bar{n})$. Moreover, the number of times which the operation $P^{\Gamma(m(\bar{n}),i(\bar{n}))}$ has to be applied depends on $a_{i(\bar{n}),0}$. We describe the first step in details. The next steps follow the same pattern.

We first apply $P^{\Gamma(m(\bar{n}),i(\bar{n}))}$. Let $\bar{n}' = \bar{n}+(0,...,0,-p^{n_{i(\bar{n}),0}},0,...,0,p^{n_{i(\bar{n}),0}})$. The next Steenrod-Milnor operation shall be applied depends on $m(\bar{n}')$ and $i(\bar{n}')$.

Let us compare $d^{\bar{n}}$ and $d^{\bar{n}'}$. First case: $i(\bar{n}) < k$, then $n_{i(\bar{n}),0}+k-1 > m(\bar{n})$. Now, if $a_{i(\bar{n}),0} = 1$ and $I(\bar{n}) = \{i(\bar{n})\}$, then $m(\bar{n}') > m(\bar{n})$. In this case our statement follows for $a_{i(\bar{n}),0} = 1$. Otherwise, $m(\bar{n}') = m(\bar{n})$ and $P^{\Gamma(m(\bar{n}),i(\bar{n}))}$ shall be applied again.
Second case: $i(\bar{n}) = k$. Let $a_{k,0} = p-1$, then $m(\bar{n}') = m(\bar{n})+1$. Again our statement follows. Let $a_{k,0} < p-1$, then $m(\bar{n}') = m(\bar{n})$ and $P^{\Gamma(m(\bar{n}),i(\bar{n}))}$ shall be applied again. ■

Let us comment on the statement of last proposition. Let $d^{\bar{n}}$ be a monomial and $d^{\bar{n}'}$ the resulting monomial as in the statement of e) above. If for each index $i_r \in I(\bar{n})$ a suitable Steenrod-Milnor operation is defined, then the smallest p-th components of the n_i are reduced and that of n_k is increased respectively.

The next technical results are needed for the proof of the main Theorem.

Corollary 10. *Given $\bar{n}$, let $n_i = \sum_{t=1}^{l(i)} a_{i,t}p^{n_{i,t}}$.*

a) Let $q = \sum_{i_r\in I(\bar{n})} a_{i_r,0}p^{n_{i_t,0}}$. If $k > i(\bar{n})$, then

$$P^{\Gamma(\bar{n})}d^{\bar{n}} = \lambda d^{\bar{n}} d_k^{(q)} \prod_{i_r\in I(\bar{n})} d_{i_r}^{-a_{i_r,0}p^{n_{i_r,0}}}$$

b) Let $q' = p^{n_{k,0}+1} + \sum_{i_r\in I(\bar{n})-\{k\}} a_{i_r,0}p^{n_{i_t,0}}$. If $k = i(\bar{n})$, then

$$P^{\Gamma(\bar{n})}d^{\bar{n}} = \lambda d^{\bar{n}} d_k^{(q')} \prod_{i_r\in I(\bar{n})} d_{i_r}^{-a_{i_r,0}p^{n_{i_r,0}}}$$

Here $\lambda \in (\mathbb{F}_p)^*$.

Proof. This is an application of proposition 9 e). ■

Lemma 11. *Given $\bar{n}$, let $\bar{n}(0) = \bar{n}$ and $\bar{n}(1)$ such that $d^{\bar{n}(1)} := P^{\Gamma(\bar{n})}d^{\bar{n}}$.*

i) Then $m(\bar{n}(0)) < m(\bar{n}(1))$.

ii) Let $\sum \bar{n} := \sum\limits_{t<k} \sum\limits_{s} a_{t,s}$, *where* $n_i = \sum\limits_{t=1}^{l(i)} a_{i,t} p^{n_{i,t}}$ *and* $\prod a_{i,t} \neq 0$.

$$I\left(\bar{n}\left(0\right)\right) \neq \{k\} \Longrightarrow \sum \bar{n}\left(0\right) > \sum \bar{n}\left(1\right)$$

Proof. i) Let $\bar{n}\left(0\right) = \left(n_i\left(0\right) | i = 1, ..., k\right)$, $I\left(\bar{n}\left(0\right)\right) = \{i_1, ..., i_t\}$ and $J(\bar{n}\left(0\right)) = (a_1, ..., a_t)$. According to last corollary, $n_i\left(0\right) = n_i\left(1\right)$, if $i \notin I\left(\bar{n}\left(0\right)\right)$ and

$$n_i\left(1\right) = \begin{cases} n_i\left(0\right) - a_{i,0} p^{n_{i,0}}, \text{ if } i \in I\left(\bar{n}\left(0\right)\right) - \{k\} \\ n_k\left(0\right) + \sum\limits_{i_r \in I(\bar{n})} a_{i_r,0} p^{n_{i_t,0}}, \text{ if } k > i\left(\bar{n}\left(0\right)\right) \\ n_k\left(0\right) - a_t p^{n_{k,0}} + p^{n_{k,0}+1} + \sum\limits_{i_r \in I(\bar{n}) - \{k\}} a_{i_r,0} p^{n_{i_t,0}}, \text{ if } k = i\left(\bar{n}\left(0\right)\right) \end{cases}$$

Thus

$$m(\bar{n}\left(1\right)) = \begin{cases} \min\left\{n_{i_t,0} + k - 1, n_{k,0} + k - 1, n_{i,1} + i - 1\right\}, \text{ if } k > i\left(\bar{n}\left(0\right)\right) \\ m(\bar{n}\left(0\right)) + 1, \text{ if } k = i\left(\bar{n}\left(0\right)\right) \end{cases}$$

and the claim follows.

ii) If $I\left(\bar{n}\left(0\right)\right) = \{k\}$, then $\sum \bar{n}\left(0\right) = \sum \bar{n}\left(1\right)$. This follows from formulas above. Otherwise, $\sum \bar{n}\left(0\right) > \sum \bar{n}\left(1\right)$. ∎

Theorem 12. *There exists a sequence of Steenrod-Milnor operations* P^{Γ} *such that* $P^{\Gamma} d^{\bar{n}} = \lambda d_k^{p^{l(\bar{n})}}$. *Here* $\lambda \in (\mathbb{F}_p)^*$.

Proof. We shall describe an algorithm which constructs the required sequence. This algorithm depends heavily on last corollary and lemma.

Step 0. Let $P^{\Gamma} = P^0$.

Step 1. Given $d^{\bar{n}}$ define $I(\bar{n})$, $J(\bar{n})$ and $i\left(\bar{n}\right)$ as in Definition 5 b). Define $P^{\Gamma} := P^{\Gamma(\bar{n})} P^{\Gamma}$.

Step 2. Let $q = \sum\limits_{k > i_r \in I(\bar{n})} a_{i_r,0} p^{n_{i_t,0}}$ and $q' = p^{n_{k,0}+1} + \sum\limits_{k > i_r \in I(\bar{n})} a_{i_r,0} p^{n_{i_t,0}}$. Define

$$d^{\bar{n}} := \begin{cases} \lambda d^{\bar{n}} d_k^{(q)} \prod\limits_{i_r \in I(\bar{n})} d_{i_r}^{-a_{i_r,0} p^{n_{i_r,0}}} \text{ or} \\ \lambda d^{\bar{n}} d_k^{(q')} \prod\limits_{i_r \in I(\bar{n})} d_{i_r}^{-a_{i_r,0} p^{n_{i_r,0}}} \end{cases}$$

given by corollary above. If $n_i < k$ for some $i > 0$ or $n_k \neq p^{l(\bar{n})}$ for some positive integer $l(\bar{n})$, then proceed to step 1. Otherwise, the required sequence is P^{Γ}. Because of last lemma, the procedure above terminates after a finite number of steps. ∎

P^{Γ} as in the last Theorem is a repeated S-M operation. We define an ordering between monomials in D_k according to S-M operations called a **Steenrod-Milnor action ordering** and write S-M ordering.

Definition 13. *For two sequences $\bar{n}$ and $\bar{n}'$ let $n_i = \sum_{t=0}^{l(i)} a_{i,t} p^{n_{i,t}}$ and $n_i' = \sum_{t=0}^{l'(i)} a_{i,t}' p^{n_{i,t}'}$ be their p-adic decompositions respectively. We require $\prod_{i,t} a_{i,t} \prod_{i,t} a_{i,t}' \neq 0$. We define an ordering on monomials of D_k, $d^{\bar{n}} < d^{\bar{n}'}$, if one of the conditions is satisfied referring to the definition 5.*

i) $m(\bar{n}) < m(\bar{n}')$.

ii) Let $m(\bar{n}) = m(\bar{n}')$, $I(\bar{n}) = \{i_1, ..., i_t\}$, $I(\bar{n}') = \{i_1', ..., i_{t'}'\}$, $J(\bar{n}) = (a_1, ..., a_t)$ and $J(\bar{n}') = (a_1', ..., a_{t'}')$. There exists a t_0 with $0 \le t_0 \le t-1$ such that t_0 is maximal with respect to the following condition: $i_{t-s} = i_{t'-s}'$ and $a_{t-s} = a_{t'-s}'$ for $0 \le s \le t_0 - 1$ and either $i_{t-t_0} > i_{t'-t_0}'$ or $i_{t-t_0} = i_{t'-t_0}'$ and $a_{t-t_0} > a_{t'-t_0}'$.

For simplicity we write $\bar{n} < \bar{n}'$ instead of $d^{\bar{n}} < d^{\bar{n}'}$.

Proposition 14. *Let $\bar{n} < \bar{n}'$, then there exists a sequence of Steenrod operations P^{Γ} depending on $\bar{n}$ such that $P^{\Gamma} d^{\bar{n}} \neq 0$ and $P^{\Gamma} d^{\bar{n}'} = 0$.*

Proof. Without lost of generality we suppose that either condition i) or ii) in the last definition is satisfied.
Condition i). Let $m(\bar{n}) < m(\bar{n}')$, then $P^{p^{m(\bar{n})}} d^{\bar{n}} \neq 0$ and $P^{p^{m(\bar{n})}} d^{\bar{n}'} = 0$.

Condition ii). Let $m = m(\bar{n}) = m(\bar{n}')$ and there exists a t_0 with $0 \le t_0 \le t-1$ such that $i_{t-t_0} = i_{t'-t_0}'$ and $a_{t-t_0} > a_{t'-t_0}'$. Let

$$d^{\bar{n}(1)} = \lambda(1) P^{\Gamma\left(m, \left(i_{t-t_0+1}, \dots, i_t\right), \left(a_{t-t_0+1}, \dots, a_t\right)\right)} d^{\bar{n}}$$

and $d^{\bar{n}'(1)} = \lambda'(1) P^{\Gamma\left(m, \left(i_{t-t_0+1}, \dots, i_t\right), \left(a_{t-t_0+1}, \dots, a_t\right)\right)} d^{\bar{n}'}$. Because of our assumption $\bar{n}(1) - \bar{n} = \bar{n}'(1) - \bar{n}'$. Here negative integers are allowed.

Now $I(\bar{n}(1)) = \{i_1, ..., i_{t-t_0}\}$ and $I(\bar{n}'(1)) = \left\{i_1', ..., i_{t'-t_0}' = i_{t-t_0}\right\}$ with $a_{t-t_0} > a_{t'-t_0}'$. Let

$$d^{\bar{n}(2)} = \lambda(2) \underbrace{P^{\Gamma\left(m, i_{t-t_0}\right)} \dots P^{\Gamma\left(m, i_{t-t_0}\right)}}_{a_{t'-t_0}'} d^{\bar{n}(1)}$$

and

$$d^{\bar{n}'(2)} = \lambda'(2) \underbrace{P^{\Gamma\left(m, i_{t-t_0}\right)} \dots P^{\Gamma\left(m, i_{t-t_0}\right)}}_{a_{t'-t_0}'} d^{\bar{n}'(1)}$$

Now $I(\bar{n}(2)) = \{i_1, ..., i_{t-t_0}\}$ and $I(\bar{n}'(2)) = \left\{i_1', ..., i_{t'-t_0-1}'\right\}$ with $i(\bar{n}(2)) = i_{t-t_0} > i(\bar{n}'(2)) = i_{t'-t_0-1}'$. Thus

$$P^{\Gamma\left(m, i_{t-t_0}\right)} d^{\bar{n}'(2)} = 0$$

■

Now we are ready to proceed to our main Theorem.

THEOREM 15. *Let $f : D_k \to D_k$ be an $\mathcal{A}$-linear map of degree* 0 *such that $f(d_{k-1}) \neq 0$. Then f is a upper triangular map with respect to S-M ordering and hence an isomorphism.*

PROOF. By hypothesis and proposition 1, $f(d_i) = \lambda d_i$ for $i = 1, ..., k$ after applying a suitable Steenrod operation.

Let $d^{\bar{n}} \in D_k$ and $(d^{\bar{n}_{(1)}}, ..., d^{\bar{n}_{l(\bar{n})}})$ the increasing sequence of elements of degree $|d^{\bar{n}}|$. Let $f(d^{\bar{n}}) = \sum_{t=1}^{l(\bar{n})} a_t d^{\bar{n}_{(t)}}$. Claim: If $\bar{n}_{(t_0)} = \bar{n}$, then $a_t \equiv 0 \bmod p$ for $t < t_0$. Following our last proposition $P^{\Gamma} d^{\bar{n}'} = 0$, if $\bar{n} < \bar{n}'$. We use induction on t for $t < t_0$. Using Theorem 12, proposition 14 and definition of elements P^{Γ}, there exists P^{Γ_1} such that $P^{\Gamma_1} d^{\bar{n}_{(1)}} = \lambda d_k^{p^{l(\bar{n}_{(1)})}}$ and $P^{\Gamma_1} d^{\bar{n}_{(i)}} = 0$ for $i > 1$. Then $P^{\Gamma_1} f(d^{\bar{n}}) = P^{\Gamma_1} \sum_{t=1}^{l(\bar{n})} a_t d^{\bar{n}_{(t)}}$ implies $a_1 \equiv 0 \bmod p$. By induction $P^{\Gamma_i} f(d^{\bar{n}}) = P^{\Gamma_i} \sum_{t=i}^{l(\bar{n})} a_t d^{\bar{n}_{(t)}}$ implies $a_i \equiv 0 \bmod p$ for $i < t_0$. Using proposition 1 and the fact that $f(d_k) \neq 0$, we get $f(d_k^{p^l}) = \lambda' d_k^{p^l}$. The last observation implies $a_{t_0} \neq 0 \bmod p$. Hence f is a upper triangular map. ∎

EXAMPLE 16. *Let $p = 2$ and $H_2 = P[y_1, y_2]^{U_2}$ the ring of upper triangular invariants which is a polynomial algebra on $h_1 = y_1$ and $h_2 = y_2^2 + y_2 y_1$. Let $f : H_2 \to H_2$ be an $\mathcal{A}$-linear map such that $f(h_1) = h_1$. Since $Sq^1 h_1 = h_1^2 \neq h_2$, $f(h_2)$ can be defined independently of h_1: $f(h_2) = a h_2 + b h_1^2$ with $a, b \in \mathbb{F}_2$. Even if $f(h_2) = h_1^2$, f is not an isomorphism: $f(d_{2,1}) = f(h_2 + h_1^2) = 0 = f(d_{2,0}) = f(Sq^1 d_{2,1})$.*

4. The exterior part of the Dickson algebra

Next we extend the previous results to the full Dickson algebra.

LEMMA 17. *i) Let $M_{s_1,...,s_l} L_k^{p-2} d^{\bar{n}}$ be a monomial in $D(V)$ and $P^{B(s_1,...,s_l)} := \beta \underbrace{P^{p^0} \beta} ... \underbrace{P^{p^{k-l-2}} ... P^{p^0} \beta} \underbrace{P^{p^{k-l-1}} ... P^{p^{s_1}}} ... \underbrace{P^{p^{k-2}} ... P^{p^{s_l}}}$. Then*

$$P^{B(s_1,...,s_l)} M_{s_1,...,s_l} L_k^{p-2} d^{\bar{n}} = (-1)^{(k-l-1)!} d_k d^{\bar{n}}$$

ii) Let $M_{s_1,...,s_l} L_k^{p-2} d^{\bar{n}}$ and $M_{s'_1,...,s'_{l'}} L_k^{p-2} d^{\bar{n}'}$ be monomials such that $s_{l-t} < s'_{l'-t}$ and t is minimal with this property, then

$$P^{B(s_1,...,s_l)} M_{s'_1,...,s'_{l'}} L_k^{p-2} d^{\bar{n}'} = 0$$

PROOF. Let us recall that $P^{p^{s_l}}(M_{s_1,\dots,s_l}L_k^{p-2}) = M_{s_1,\dots,s_l-1,s_l+1}L_k^{p-2}$ for $s_l < k-1$ and $P^{p^{s_l}}d_t^{a_t p^{n_t}} \neq 0$ if and only if $n_t = s_l - t + 1$ for $0 \leq t \leq s_l + 1$. If $0 = s_l$, we apply the Bockstein operation β. Thus $P^{p^{k-2}}...P^{p^{s_l}}M_{s_1,\dots,s_l}L_k^{p-2}d^{\bar{n}} = \sum\limits_{0}^{k-1-s_l} M_{s_1,\dots,s_{l-1},s_l+t_l}L_k^{p-2}f_{t_l}$. Here f_{t_l} is a polynomial in D_k.

Let $P^E = \underbrace{P^{p^{k-l-1}}...P^{p^{s_1}}}...\underbrace{P^{p^{k-2}}...P^{p^{s_l}}}$. Iterating the last formula we obtain:

$$P^E M_{s_1,\dots,s_l}L_k^{p-2}d^{\bar{n}} = \sum_{q=1}^{l}\sum_{0}^{s_{q+1}+t_{q+1}-s_q} M_{s_1+t_1,\dots,s_{l-1}+t_{l-1},s_l+t_l}L_k^{p-2}f_{t_1,\dots,t_l}$$

Here $s_{l+1} = 0$ and $t_{l+1} = k-1$.

Let us suppose that $s_1 + t_1 < k - l$. Let $P^{\Delta} = P^{p^{k-l-2}}...P^{p^0}\beta$ and $A = M_{s_1+t_1,\dots,s_{l-1}+t_{l-1},s_l+t_l}L_k^{p-2}f_{t_1,\dots,t_l}$. There are $s_1 + t_1 - 1 \leq k - l - 2$ positions to be filled by powers of y's using Steenrod operations: $\beta\underbrace{P^{p^0}\beta}...\underbrace{P^{p^{k-l-2}}...P^{p^0}\beta}$. Since there are $k-l$ β's in this sequence and only $s_1+t_1-1 \leq k-l-2$ positions, it is obvious that $P^{\Delta}A = 0$. Now suppose that $s_1+t_1 = k-l$ and one operation P^{p^q} of P^{Δ} is not applied on A. Then it will be less positions than the number of remaining β's. In that case $\beta\underbrace{P^{p^0}\beta}...\underbrace{P^{p^{k-l-2}}...P^{p^0}\beta}M_{s_1+t_1,\dots,s_{l-1}+t_{l-1},s_l+t_l}L_k^{p-2}f_{t_1,\dots,t_l} = 0$. The claim follows. ■

COROLLARY 18. *Let $M_{s_1,\dots,s_l}L_k^{p-2}d^{\bar{n}} \in D(V)$. There exists a sequence of S-M operations such that*

$$P^{\Gamma}P^{B(s_1,\dots,s_l)}M_{s_1,\dots,s_l}L_k^{p-2}d^{\bar{n}} = \lambda d_k^{p^q}$$

and q is minimal with this property.

Here $\lambda \in (\mathbb{Z}/p\mathbb{Z})^*$.

Now we are ready to proceed to our main Theorem.

THEOREM 19. *Let $g : D(V) \to D(V)$ be an $\mathcal{A}$-linear map degree 0 such that $g(M_k) \neq 0$. Then g is an isomorphism. Here $M_k = \prod\limits_{1}^{k} x_i L_k^{p-2}$.*

PROOF. Let $g : D(V) \to D(V)$ such that $g(M_k) = \lambda M_k$. Then $g(d_k) = \lambda' d_k$ with $\lambda' \neq 0 \bmod p$ and $g(d_{k-1}) = \lambda' d_{k-1}$. Please recall that $\beta P^1\beta...P^{p^{k-2}}...P^p P^1\beta M_k = d_k$. Thus g is an isomorphism in D_k. ■

We recall that an $\mathcal{A}$-module is indecomposable, if it is not a nontrivial direct sum.

DEFINITION 20. *Let $\overline{D(V)}$ denote the augmentation ideal of $D(V)$.*

Corollary 21. *$\overline{D(V)}$ is not directly decomposable as an $\mathcal{A}$-module.*

Proof. Assume $\overline{D(V)} = \bigoplus_{i \in I} D(V)_i$ such that $D(V)_i \neq 0$. If $d(i)$ and $d(j)$ are monomials in $D(V)_i$ and $D(V)_i$ respectively, then there exist P^{Γ} and $P^{\Gamma'}$ such that $a_i P^{\Gamma} d(i) = d_k^{p^l} = b_j P^{\Gamma} d(j)$. ■

5. The structure of $End_{\mathcal{U}}(D(V))$

It is known that if an R-module M is directly indecomposable and of finite length, then $End(M_R)$ is a local ring and its non-invertible elements are precisely its nilpotent elements. According to last corollary $\overline{D(V)}$ is directly indecomposable. It is also known that the set $\left\{ d_{k-1}^{(p^l-1)/(p-1)} | l \geq 1 \right\}$ is linearly independent ([**5**]). Thus $\overline{D(V)}$ is not of finite length. But we shall prove that $End_{\mathcal{U}}\left(\overline{D(V)}\right)$ is a local ring.

We follow the approach suggested by H.-W. Henn. We shall omit technical details which will appear in [**7**].

We state a remarkable theorem due to Adams, Gunawardena and Miller (Th. 1.6, p. 437 [**1**]), we apply it for $s = 0$, $t = 0$, $M = \mathbb{F}_p$ and $U = V$.

Theorem 22. *$\mathbb{F}_p[End(V)] \cong End_{\mathcal{U}}(H^*(V))$. Here $\mathbb{F}_p[End(V)]$ is the monoid algebra of the monoid under composition $End(V)$.*

Using the Theorem above one can reduce the problem to a linear algebra one.

Theorem 23. *$Hom_{\mathcal{U}}(D(V), H^*(V)) \cong \mathbb{F}_p[G \backslash End(V)]$.*

Proof. We view $End(V)$ as a monoid with respect to composition of linear maps. It admits a left and a right action by itself. Let $\mathbb{F}_p[End(V)]$ be the associated monoid algebra. Let $\mathbb{F}_p[G \setminus End(V)]$ be the vector space on the set of orbits $G \setminus End(V)$. The monoid $End(V)$ acts on the right of $GL(V) \setminus End(V)$ by $\overline{f} \cdot h = \overline{fh}$. Because of the right action above $\mathbb{F}_p[G \setminus End(V)]$ becomes an $\mathbb{F}_p[End(V)]$-module.

Let $f : D(V) \to H^*(V)$ and $g : H^*(V) \to H^*(V)$, then $gf : D(V) \to H^*(V)$ and $Hom_{\mathcal{U}}(D(V), H^*(V))$ becomes a left $End_{\mathcal{U}}(H^*(V))$-module. Moreover, by the AGM-theorem, it becomes a right $\mathbb{F}_p[End(V)]$-module.

We recall that $H^*(V)$ is an unstable $\mathcal{A}$-module. Thus given $f \in Hom_{\mathcal{U}}(D(V), H^*(V))$ an $\bar{f} \in Hom_{\mathcal{U}}(H^*(V), H^*(V))$ is induced. Applying the theorem above, $\bar{f}$ is identified with a $\varphi : V \to V$ such that φ is G-invariant because f is. The isomorphism follows. ■

Proposition 24. *$End_{\mathcal{U}}\left(D\left(V\right)\right)\cong End_{End_{\mathcal{U}}\left(H^{*}(V)\right)}\left(Hom_{\mathcal{U}}\left(D\left(V\right),H^{*}(V)\right)\right)$.*

Proof. Let

$$\Phi : End_{\mathcal{U}}\left(D\left(V\right)\right)\rightarrow End_{End_{\mathcal{U}}\left(H^{*}(V)\right)}\left(Hom_{\mathcal{U}}\left(D\left(V\right),H^{*}(V)\right)\right)$$

given by $\Phi\left(f\right)\left(h\right)=hf$ for $f\in End_{\mathcal{U}}\left(D\left(V\right)\right)$ and $h\in Hom_{\mathcal{U}}\left(D\left(V\right),H^{*}(V)\right)$. Moreover, $k\Phi\left(f\right)\left(h\right)=\Phi\left(f\right)\left(kh\right)$ for $k\in End_{\mathcal{U}}\left(H^{*}(V)\right)$.

Φ is 1-1: Let $\Phi\left(f\right)=\Phi\left(f'\right)$, then $\Phi\left(f\right)\left(i\right)=\Phi\left(f'\right)\left(i\right)$ for $i: D\left(V\right)\hookrightarrow H^{*}(V)$. Thus $f=f'$.

Φ is onto: $\forall h\in Hom_{\mathcal{U}}\left(D\left(V\right),H^{*}(V)\right)$, $\exists\overline{h}\in End_{\mathcal{U}}\left(H^{*}(V)\right)$ such that $h=\overline{h}i$ because $H^{*}(V)$ is injective.
Let $\Psi\in End_{End_{\mathcal{U}}\left(H^{*}(V)\right)}\left(Hom_{\mathcal{U}}\left(D\left(V\right),H^{*}(V)\right)\right)$, then $\overline{h}\Psi\left(i\right)=\Psi\left(\overline{h}i\right)=\Psi\left(h\right)$. Each $g\in G$ defines a map $g\in End_{\mathcal{U}}\left(H^{*}(V)\right)$. For such a map $g\Psi\left(i\right)=\Psi\left(gi\right)=\Psi\left(i\right)$, thus $\Psi\left(i\right)\in End_{\mathcal{U}}\left(D\left(V\right)\right)$. Let $f_{\Psi}=\Psi\left(i\right)$, then $\Phi\left(f_{\Psi}\right)\left(h\right)=hf_{\Psi}=\overline{h}if_{\Psi}=\overline{h}f_{\Psi}=\Psi\left(h\right)$. ■

Theorem 25. *$End_{\mathcal{U}}\left(D\left(V\right)\right)\cong End_{\mathbb{F}_p[End(V)]}\left(\mathbb{F}_p[G\diagdown End\left(V\right)]\right)$.*

The next proposition provides all technical details for the conclusion of this section, namely $End_{\mathcal{U}}\left(\overline{D\left(V\right)}\right)$ is a local ring along with its structure.

The $\mathcal{A}$-submodule of $D\left(V\right)$ consisting of degree 0 elements is isomorphic with $\mathbb{F}_p$. Thus $End_{\mathcal{U}}(D\left(V\right))\cong\mathbb{F}_p\oplus End_{\mathcal{U}}(\overline{D\left(V\right)})$. Although the subspace of the zero orbit in $\mathbb{F}_p[G\diagdown End\left(V\right)]$ is a direct summand, it is not a direct summand as an $\mathbb{F}_p[End\left(V\right)]$-module but it can be decomposed as follows. Let $\mathbb{F}_p[\widehat{G\diagdown End}\left(V\right)]$ be the vector space on

$$\left\{\overline{f}\mid f\in End\left(V\right),f\neq 0_V\right\}$$

We identify the neutral element $\sum\limits_{f\neq 0_V}0\overline{f}$ with the zero orbit $\overline{0}$. It becomes an $\mathbb{F}_p[End\left(V\right)]$-module.

Theorem 26. [**7**]*Let $R:=End_{\mathbb{F}_p[End(V)]}\left(\mathbb{F}_p[\widehat{G\diagdown End}\left(V\right)]\right)$, then R has dimension n as a vector space over $\mathbb{F}_p$. There is a set of generators $\{\psi_i|1\leq i\leq n\}$ such that:*
1) $\psi_0=0$ is the neutral element and $\psi_n=1$ is the identity map;
2) Let $n>l\geq k>0$, then $\psi_k\psi_l=\psi_l\psi_k=\psi_{\max(0,k+l-n)}$.

Because of the isomorphism $End_{\mathcal{U}}(\overline{D\left(V\right)})\cong R$, the next corollary follows.

Corollary 27. *$End_{\mathcal{U}}(\overline{D\left(V\right)})$ is a local $\mathbb{F}_p$-algebra with dimension n as a vector space over $\mathbb{F}_p$ (i.e. f or $Id-f$ is an isomorphism for* *any f in $End_{\mathcal{U}}(\overline{D\,(V)})$). Moreover, if I is the ideal generated by its nilpotent elements, then $End_{\mathcal{U}}(\overline{D\,(V)})/I \equiv \mathbb{F}_p$.*

6. An application

We close this work by applying our result in the mod p homology of QS^0. Firstly, we recall the isomorphism between the hom-dual of the Dyer-Lashof algebra and the Dickson algebra.

Proposition 28. *a) Let $SD\,(V)$ be the subalgebra of $D\,(V)$ generated by $\{d_i, M_{s_1}L_k^{p-2}, M_{s_1',s_2'}L_k^{p-2}\}$ where $1 \le i \le k$, $0 \le s_1 \le k-1$ and $0 \le s_1' < s_2' \le k-1$. If $f : SD\,(V) \to SD\,(V)$ satisfies*

$$f(M_{k-2,k-1}L_k^{p-2}) = \lambda M_{k-2,k-1}L_k^{p-2} \neq 0$$

then f is an isomorphism.

b) Let $I[k]$ be the ideal of $SD\,(V)$ generated by

$$\{d_k, M_{s_1}L_k^{p-2}, M_{s_1',k-1}\}$$

then the induced map f which satisfies $f(M_{k-2,k-1}L_k^{p-2}) = \lambda M_{k-2,k-1}L_k^{p-2}$ is also an isomorphism.

Proposition b) above is a reformulation of Theorem 4.1 in [**2**].

Let $R =< Q^{(I,J)} | I = (i_1, ..., i_n), J = (\varepsilon_1, ..., \varepsilon_n) >$ be the Dyer-Lashof algebra, then $H_*(Q_0S^0; \mathbb{F}_p)$ is the free commutative algebra generated by $\Phi(R)$ subject to the following relation $Q^{(I,J)} \approx (Q^{(I',J')})^p$ if $I = (i_1, I')$, $J = (0, J')$ and $exc(Q^{(I,J)}) = 0$. Here $\Phi : R \to H_*(Q_0S^0; \mathbb{F}_p)$ is the $\mathcal{A}_*$-module map given by $\Phi(Q^{(I,J)}) = Q^{(I,J)}[1] * [-p^{l(I)}]$, $[1]$ is a generator of $\tilde{H}_0(S^0; \mathbb{F}_p)$, $[r] = [1]^r$ and $l(I)$ is the length of I. Thus there exists an $\mathcal{A}_*$-module isomorphism between the generators of $H_*(Q_0S^0; \mathbb{F}_p)$ and the quotient R/Q_0R where $Q_0R = \{Q^{(I,J)} | exc(I, J) = 0\}$. It is known that $R[k]^* \cong SD\,(V)$ as Steenrod algebras and $(R/Q_0R)[k]^* \cong I[k]$ as Steenrod modules. Here $R = \bigoplus R[k]$. Now the following Theorem is a consequence of last corollary.

Theorem 29. [**2**]*Let $f : \Omega_0^\infty S^\infty \to \Omega_0^\infty S^\infty$ be an H-map which induces an isomorphism on $H_{2p-3}(\Omega_0^\infty S^\infty; \mathbb{F}_p)$. If $p > 2$ suppose in addition that f is a loop map or that*

$$f_*(d_{2,0})^* \neq 0$$

for some $\lambda \in (\mathbb{F}_p)^*$. Then $f_{(p)}$ is a homotopy equivalence. Here $(d_{2,0})^*$ is the hom-dual *of the top degree Dickson generator of D_2 in $R[2]$.*

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