A Steenrod-Milnor action ordering on Dickson invariants

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Abstract. Let \( A \) be the Steenrod algebra and \( D(V) \) the Dickson algebra. An ordering in \( D(V) \) is defined according to the Steenrod algebra action. Using this ordering, we prove the following: Let \( f \in \text{End}_A(D(V)) \) be an \( A \)-linear degree preserving map. If \( f \) is non-zero on the lowest degree, then \( f \) is an isomorphism. Moreover, \( \text{End}_A(D(V)) \) is a local ring, where \( D(V) \) is its augmentation ideal.

1. Statement of results

It is known that the classical Dickson algebra \( D_k \) is a polynomial algebra:

\[
D_k \cong \mathbb{F}_p[d_1, \ldots, d_k]
\]

Mùi related \( D_k \) (for \( p = 2 \)) with the dual of the Dyer-Lashof algebra calculated by Madsen. Motivated by topological questions regarding the cohomology of an infinite (finite) loop space and influenced by the work of Campbell, Cohen, Peterson and Selick in [2] and [3] we study the problem under which conditions is an \( A \)-endomorphism of \( D(V) := (H^*(V))^{GL(k, \mathbb{F}_p)} \) an isomorphism. Here \( A \) stands for the Steenrod algebra.

Firstly, we consider the classical Dickson algebra \( D_k \). Where modifications are needed between the case \( p = 2 \) and \( p > 2 \) they are provided. Given a sequence of \( k \) non-negative integers \( \bar{n} = (n_k, n_1, \ldots, n_{k-1}) \) let

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2000 Mathematics Subject Classification. Primary 55S10, 13A50; Secondary 55S12, 55P47, 13H05.

Key words and phrases. Dickson algebra, Steenrod algebra, Isomorphisms of Dickson algebras, Local rings.

This paper is in final form and no version of it will be submitted for publication elsewhere.
$d^n := \prod_1^k d_i^{n_i}$. Our first task is to prove that there exists a unique $p$-th power Steenrod operation $P^{p^m}$ of smallest degree such that

$$P^{p^m} d^n \neq 0$$

Moreover, the new element has the property that there exists at least one $p$-th power of a generator $d_i^{p^t_i}$ such that $d_i^{p^t_i}$ divides $d^n$ and $t_i + i - 1 = m$. Applying this property again on $P^{p^m} d^n$ we get

$$P^{p^m-1} P^{p^m} d^n \neq 0$$

Then we iterate: $P^{p^t_i} \ldots P^{p^m} d^n \neq 0$. We are interested in finding the longest such sequence of Steenrod operations such that $P_i^{t_i(i)} \ldots P^{p^m} d^n$ is a non-zero monomial. We call such a sequence a Steenrod-Milnor action on $d^n$ denoted by $P^{\Gamma(n)}$ (please see definition 5). Now we iterate this procedure on the monomial $P^{\Gamma(n)} d^n$ until the resulting monomial is $d_k^q$ for the smallest $q$.

**Theorem 12** There exists a sequence of Steenrod-Milnor operations $P^\Gamma$ such that $P^\Gamma d^n = \lambda d_k^{q(n)}$. Here $\lambda \in (\mathbb{F}_p)^*$.

Next, given two monomials $d^n$ and $d^{n'}$ we define an ordering according to their Steenrod-Milnor actions $P^{\Gamma(n)}$ and $P^{\Gamma(n')}$. We call this ordering a Steenrod-Milnor (S-M) ordering (please see definition 13). Using this ordering we prove the following theorem:

**Theorem 15** Let $f : (D_k)^G \to (D_k)^G$ be an $A$-linear map of degree 0 such that $f(d_{k-1}) \neq 0$. Then $f$ is a upper triangular map with respect to S-M ordering and hence an isomorphism.

We note that the last Theorem is not true for the upper triangular ring, please see example 16.

This finishes section 3 which consists of the technical part of this work. We extend the theorem above to the full ring of invariants, $D(V)$, in section 4.

**Theorem 19** Let $g : D(V) \to D(V)$ be an $A$-linear map of degree 0 such that $g(M_k) \neq 0$. Then $g$ is an isomorphism. Here $M_k = \prod_1^k x_i L_k^{p^2}$ is the element of lowest degree.

In section 5 we investigate the structure of $\text{End}_d(D(V))$. A shorter and elegant proof of the next corollary was suggested by H.-W. Henn.

**Corollary 27** $\text{End}_d(D(V))$ is a local $\mathbb{F}_p$-algebra with dimension $n$ as a vector space over $\mathbb{F}_p$ (i.e. $f$ or $Id - f$ is an isomorphism for any $f$ in $\text{End}_d(D(V))$). Moreover, if $I$ is the ideal generated by its nilpotent elements, then $\text{End}_d(D(V))/I \equiv \mathbb{F}_p$. 
Finally we apply Theorem 19 to the study of self maps between infinite loop spaces. We obtain an alternative proof of theorem 4.1 page 28 of Campbell, Peterson and Selick:

**Theorem 2** Let \( f : \Omega^\infty S^\infty \to \Omega^\infty S^\infty \) be an \( H \)-map which induces an isomorphism on \( H_{2p-3}(\Omega^\infty S^\infty, \mathbb{F}_p) \). If \( p > 2 \) suppose in addition that \( f \) is a loop map or that

\[
 f_*(d_{2,2})^* \neq 0
\]

for some \( \lambda \in (\mathbb{F}_p)^* \). Then \( f(p) \) is a homotopy equivalence. Here \((d_{2,2})^*\) is the hom-dual of the top degree Dickson generator of \( D_2 \) in \( R[2] \).

2. Introduction

Let \( \mathcal{A} \) be the Steenrod algebra and \( \mathcal{U} \) the category of unstable \( \mathcal{A} \)-modules which is a full subcategory of \( \mathbb{F}_p \)-graded \( \mathcal{A} \)-modules and morphisms being \( \mathcal{A} \)-linear maps of degree 0 that is degree preserving. Let \( V \) and \( W \) finite dimensional vector spaces over \( \mathbb{F}_p \) and \( H^*(V) \) the mod \( p \) cohomology of its classifying space: \( H^*(V) := H^*(BV, \mathbb{F}_p) \). Moreover

\[
 H^*(V) \cong E(x_1, ..., x_k) \otimes P[y_1, ..., y_k]
\]

where \( V^* = <x_1, ..., x_k> \), \( \beta x_i = y_i \) and \( |x_i| = 1 \).

A map of unstable \( \mathcal{A} \)-algebras \( f^* : H^*(W) \to H^*(V) \) is determined by its action in degree 1 that is by an element of \( Hom(V, W) \) which is isomorphic to \( Hom_K(H^*(W), H^*(V)) \). Here \( K \) is the category of unstable \( \mathcal{A} \)-algebras. There is also an isomorphism for \( \mathcal{A} \)-linear maps:

\[
 Hom_{\mathcal{U}}(H^*(W), H^*(V)) \cong F_p[Hom(V, W)]
\]

It is known that an \( \mathcal{A} \)-linear map \( f^* : H^*(\mathbb{F}_p) \to H^*(\mathbb{F}_p) \) is determined by its direct sum components \( H^*(\mathbb{F}_p) \cong \bigoplus_{i=1}^{p-1} H_i \) and it is an isomorphism, if it is an isomorphism in degree \( 2i-1 \) for \( i = 1, ..., p-1 \). Here \( (H_i)^* = \tilde{H}^*(\mathbb{F}_p) \) for \( * \equiv 2i \) or \( 2i-1 \mod (p-1) \). The algebraic structure of \( H^*(V) \) as an \( \mathcal{A} \)-module has been studied extensively ([9]).

The general linear group \( G := GL(k, \mathbb{F}_p) \) acts on \( V \) and hence on \( H^*(V) \). The ring of invariants

\[
 D(V) := (H^*(V))^G
\]

called the "Dickson algebra" was described by Dickson for \( D_k := (P[y_1, ..., y_k])^G \) and Mùi for the general case. Dickson proved that \( D_k \cong \mathbb{F}_p[d_{k,1}, ..., d_{k,k}] \) ([4]), is again a polynomial algebra with \( |d_{k,i}| = 2(p^k - p^i) \). Let us briefly describe its generators \( d_{k,i} \):

\[
 h_t = \prod_{u \in <y_1, ..., y_{t-1}>} (y_t + u) \quad \text{for} \ 1 \leq t \leq k
\]
\[ d_{k,i} = h_k^{p-1}d_{k-1,i} + d_{k-1,i-1}^{p} \]

Mühi ([8]) proved that \( D(V) \) is a tensor product between \( D(V) \) and the \( \mathbb{F}_p \)-module spanned by the set of elements consisting of the following monomials:

\[ M_{k; s_1, \ldots, s_l} L_k^{p-2}; \quad 0 \leq l \leq k - 1, \quad \text{and} \quad 0 \leq s_1 < \cdots < s_l \leq k - 1 \]

Its algebra structure is determined by the following relations:

a) \( (M_{k; s_1, \ldots, s_l} L_k^{p-2})^2 = 0 \) for \( 0 \leq l \leq k - 1, \) and \( 0 \leq s_1 < \cdots < s_l \leq k - 1. \)

b) \( M_{k; s_1, \ldots, s_l} L_k^{(p-2)} d_{k,k-1}^{p-1} = (-1)^{(k-l)(k-l-1)/2} \prod_{t=1}^{k-l} M_{k; 0, \ldots, k-s_t, \ldots, k-1} L_k^{p-2}. \)

Here \( 0 \leq l \leq k - 1, \) and \( 0 \leq s_1 < \cdots < s_l \leq k - 1. \) Those elements are described as follows:

\[
M_{k; s_1, \ldots, s_l} = \frac{1}{(k-l)!}
\begin{vmatrix}
x_1 & \cdots & x_1 \\
\vdots & & \vdots \\
x_1 & \cdots & x_k \\
y_1^{p^s_1} & \cdots & y_k^{p^s_1} \\
\vdots & & \vdots \\
y_1^{p^s_l} & \cdots & y_k^{p^s_l}
\end{vmatrix}
\]

Here there are \( k - l \) rows of \( x_i \)'s and the \( s_i \)-th's powers are completing the rest of the determinant above, where \( 0 \leq s_1 < \cdots < s_l \leq k-1. \) The row \( (y_1^{p^s_i}, \ldots, y_k^{p^s_i}) \) is omitted in the determinant above and \( 1 \leq i \leq k-1. \)

\[ |M_{k; s_1, \ldots, s_l}| = k - l + 2(p^{s_1} + \cdots + p^{s_l}). \quad \text{And} \quad L_k = \prod_{i=1}^{k} h_i. \]

From now on we write \( d_i \) for \( d_{k,i}. \)

Since the operation of \( G \) on \( H^*(V) \) commutes with the action of the Steenrod algebra, \( D(V) \) is also a module (in fact an algebra) over \( A. \)

### 3. A Steenrod-Milnor action ordering on Dickson invariants

We shall recall some well known results concerning the action of the Steenrod algebra on Dickson algebra generators.

**Proposition 1.** [6] (Th. 30, p. 169)

\[ P^{p^i}(d_i^{p^j}) = \begin{cases} 
  d_i^{p^j}, & \text{if } t = l + i - 1 \text{ and } i < k \\
  -d_i^{p^j} d_i^{p^j}, & \text{if } t = l + k - 1 \\
  0, & \text{otherwise}
\end{cases} \]

A similar result holds for the generators of \( D(V). \)
PROPOSITION 2. [6] (Th. 36, p. 170) Let \( q > 0 \). If \( q = \sum_{i} a_{i} p_{i}^{j+i} \)
such that \( p - 1 \geq a_{t} \geq a_{t-1} > a_{t-1} = 0 \). Then

\[
P_{i} d_{k}^{\nu} = d_{k}^{\nu} (-1)^{a_{k-1}} \prod_{i}^{k-1} \left( \frac{a_{t}}{a_{t-1}} \right) d_{t}^{\nu (a_{t} - a_{t-1})}
\]

Otherwise, \( P_{i} d_{k}^{\nu} = 0 \). If \( t = 0 \), then \( d_{0} = d_{k} \).

2) Let \( q = \sum_{s} a_{t} p_{s}^{j+s} > 0 \) such that \( p - 1 \geq a_{t} \geq a_{t-1} \geq a_{i} \geq 0 \) and
\( a_{i} + 1 \geq a_{i-1} \geq a_{t} \geq a_{t-1} \geq a_{s-1} = 0 \). Then

\[
P_{i} d_{i}^{\nu} = d_{i}^{\nu} (-1)^{a_{k-1}} \left( \prod_{i}^{k-1} \left( \frac{a_{t}}{a_{t-1}} \right) \right) \left( a_{i} + 1 \right) \left( \prod_{s}^{i-1} \left( \frac{a_{t}}{a_{t-1}} \right) \right) \prod_{s}^{k-1} d_{t}^{\nu (a_{t} - a_{t-1})}
\]

Here \( a_{s-1} = 0 \). Otherwise, \( P_{i} d_{i}^{\nu} = 0 \).

REMARK 3. Please note that the case \( a_{i} = 0 \) and \( a_{i-1} = 1 \) is allowed in the proposition above.

We shall apply formulas above on a monomial in the Dickson algebra starting with the smaller non-zero \( p \)-th power. Let us firstly demonstrate our method.

EXAMPLE 4. Let \( p = 2 \) and \( k = 3 \). Let

\[
d^{\nu} = d_{2}^{2+2^2} d_{1}^{2+2^4} d_{2}^{2+2^3}
\]

\( \tilde{n} = (n_{3} = 2^1 + 2^2, n_{1} = 2^3 + 2^4, n_{2} = 2^2 + 2^3) \)

Let us write \( n_{i} \) in its \( p \)-adic form:

\( n_{i} = n_{i0} + n_{i1} p + \ldots \)

Here \( n_{30} = 1, n_{10} = 3 \) and \( n_{20} = 2 \). We define

\( m (\tilde{n}) = \min \{ n_{30} + k - 1, n_{10} + k - 3, n_{20} + k - 2 \} = 3 \)

\( I (\tilde{n}) = \{ i | m (\tilde{n}) = n_{i0} + i - 1 \} = \{ 3, 1, 2 \} \)

and

\( i (\tilde{n}) = \max I (\tilde{n}) = 3 \)

We apply \( i (\tilde{n}) = 3 \) squaring operations, namely:

\[
Sq_{m (\tilde{n})} d^{\nu}, Sq_{m (\tilde{n}) - 1}, \text{ and } Sq_{m (\tilde{n}) - 2}
\]

\[
Sq_{m (\tilde{n})} d^{\nu} = d_{3}^{2+2^2} d_{1}^{2+2^4} d_{2}^{2+2^3} + d_{3}^{2+2^2} d_{1}^{2} d_{2}^{2^2} + d_{3}^{2+2^2} d_{1}^{4} d_{2}^{2^3} + d_{3}^{2+2^2} d_{1}^{2} d_{2}^{2^3} d_{2}^{2^4}
\]

\[
Sq_{m (\tilde{n}) - 1} \left[ d_{3}^{2+2^2} d_{1}^{2} d_{2}^{2^2} + d_{3}^{2+2^2} d_{1}^{4} d_{2}^{2^3} + d_{3}^{2+2^2} d_{1}^{2} d_{2}^{2^3} d_{2}^{2^4} \right] =
\]

\[
d_{3}^{2+2^2} d_{1}^{3} d_{2}^{2^2} + d_{3}^{2+2^2} d_{1}^{3} d_{2}^{2^3} + d_{3}^{2+2^2} d_{1}^{3} d_{2}^{2^4}
\]
Finally,\\
\[ S_q^{2m(n) - 2} S_q^{2m(n) - 1} S_q^{2m(n)} d^n = d_3^{23} d_1^{23} d_2^{23} \]

Let \( \bar{n} = (2^3, 2^3 + 2^4, 2^2 + 2^3) \). Then \( m(\bar{n}) = 3, I(\bar{n}) = \{1, 2\} \), \( i(\bar{n}) = 2 \).

\[ S_q^{23} S_q^{23} d^n = d_3^{23} d_1^{23} d_2^{23} \]

Let \( \bar{n} = (2^2 + 2^4, 2^3 + 2^4, 2^3) \). Then \( m(\bar{n}) = 3, I(\bar{n}) = \{1\}, i(\bar{n}) = 1 \).

\[ S_q^{23} d^n = d_3^{23} d_1^{23} d_2^{23} \]

Please note that at each step \( m(\bar{n}) = 3 \) and the cardinality of \( I(\bar{n}) \) is reduced by 1.

We call \( S_q^{23} S_q^{23} S_q^{23} S_q^{23} S_q^{23} \) a Steenrod-Milnor operation of type \( \bar{n} \) and denote it by \( S_q^{\Gamma(\bar{n})} \). Please note that the \( n_i \)'s have been decreased and \( n_k \) increased respectively.

Let \( \bar{n} = (2^2 + 2^4, 2^3, 2^3) \). Then \( m(\bar{n}) = 4, I(\bar{n}) = \{3, 1, 2\}, i(\bar{n}) = 3 \).

\[ S_q^{23} S_q^{23} d^n = d_3^{23} d_1^{23} d_2^{23} \]

Let \( \bar{n} = (2^3 + 2^4, 2^3, 2^3) \). Then \( m(\bar{n}) = 4, I(\bar{n}) = \{1, 2\}, i(\bar{n}) = 2 \).

\[ S_q^{23} d^n = d_3^{23} d_1^{23} \]

Let \( \bar{n} = (2^5, 2^4) \). Then \( m(\bar{n}) = 4, I(\bar{n}) = \{1\}, i(\bar{n}) = 1 \).

\[ S_q^{24} d^n = d_3^{24} \]

Let \( \bar{n} = (2^3) \). Then \( m(\bar{n}) = 6, I(\bar{n}) = \{3\}, i(\bar{n}) = 3 \).

\[ S_q^{24} S_q^{25} d^n = d_3^{26} \]

Finally,
\[ S_q^{\Gamma(2^4 + 2^5, 0, 0)} S_q^{\Gamma(2^2 + 2^4, 2^3)} S_q^{\Gamma(2^2 + 2^3 + 2^4, 2^2 + 2^3)} d_3^{23} d_1^{23} d_2^{23} = d_3^{26} \]

**Definition 5.** Let \( \bar{n} = (n_k, n_1, ..., n_k-1) \) and \( n_i = \sum_{l=1}^{l(i)} a_{i,l} p^{n_l} \) its \( p \)-adic expansion with \( \prod a_{i,l} \neq 0 \).

a) Let \( m(\bar{n}) := \min\{n_{i,0} + i - 1 \mid 1 \leq i \leq k\} \).

b) Let \( I(\bar{n}) := \{i \mid m(\bar{n}) = n_{i,0} + i - 1\} \).

c) Let \( i(\bar{n}) := \max I(\bar{n}) \).

d) Let \( J(\bar{n}) := \left\{ \begin{array}{ll} (a_{i,0}, ..., a_{i(\bar{n}),0}) & , \text{ if } i(\bar{n}) < k \\ (a_{i,0}, ..., p - a_{i(\bar{n}),0}) & , \text{ if } i(\bar{n}) = k \end{array} \right. \). For \( p > 2 \).

**Remark 6.** If \( p = 2 \), then \( a_{i,l} = 1 \) and \( I(\bar{n}) \) determines \( J(\bar{n}) \).
DEFINITION 7. a) For \( m \) and \( l \) natural numbers such that \( l \leq m \), let \( P^{(m,l)} \) stand for the Steenrod operation \( P_{p^{m-l+1}}P_{p^{m-l+2}} \ldots P_{p^m} \).

b) Given a sequence \( \vec{n} \), a triad is defined as above \( (m(\vec{n}), I(\vec{n}), J(\vec{n})) \).

We define a sequence of Steenrod operations associated with this triad as follows

\[ P^{\Gamma(m(\vec{n}), I(\vec{n}), J(\vec{n}))} := \prod_{\alpha_i \neq 0} P^{\Gamma(m(\vec{n}), i)} \prod_{\alpha_i \neq \alpha_{i+1}} P^{\Gamma(m(\vec{n}), i)} \prod_{\alpha_i = \alpha_{i+1}} P^{\Gamma(m(\vec{n}), i)} \]

We call this operation, \( P^{\Gamma(m(\vec{n}), I(\vec{n}), J(\vec{n}))} \), a Steenrod-Milnor operation of type \( \vec{n} \) and denote it by \( P^{\Gamma(\vec{n})} \).

REMARK 8. If \( p = 2 \), then \( Sq^{\Gamma(\vec{n})} = Sq^{\Gamma(m(\vec{n}), i)} \ldots Sq^{\Gamma(m(\vec{n}), i)} \).

PROPOSITION 9. a) Let \( m = n + k - 1 \), then \( P^{\Gamma(m,k)} d_{p^n} = -d_{p^n} \)

\[ (S_{p^n}^{\Gamma(m,k)} d_{p^n} = d_{p^n+1}^n) \]

b) Let \( m = n + k - 1 \), then \( P^{\Gamma(m,k)} \ldots P^{\Gamma(m,k)} d_{p^n} = -d_{p^n+1} \)

c) \( P^{\Gamma(m,n)} d_{p^n} = \sum_{i_r \in \{1(\vec{n})\}} a_{i_r,0} d_{p^{n-i_r}} d_{p^{i_r-1}} - d_{p^{n-i_r}} d_{p^{n-i_r}} \ldots \)

d) \( P^{\Gamma(m(\vec{n}), i(\vec{n}))} d_{p^n} = \left\{ \begin{array}{ll}
    a_{i(\vec{n}),0} d_{p^{n-i(\vec{n})}} d_{p^{n-i(\vec{n})}} & \text{if } k > i(\vec{n}) \\
    -a_{i(\vec{n}),0} d_{p^{n-i(\vec{n})}} & \text{if } k = i(\vec{n})
\end{array} \right. \)

e) If \( k > i(\vec{n}) \), then

\[ \prod_{\alpha_i \neq 0} P^{\Gamma(m(\vec{n}), i(\vec{n}))} d_{p^n} = (a_{i(\vec{n}),0})! d_{p^{n-i(\vec{n})}} d_{p^{(n-i(\vec{n}))}} \]

Otherwise,

\[ \prod_{\alpha_i \neq 0} P^{\Gamma(m(\vec{n}), i(\vec{n}))} d_{p^n} = (-1)^{p-a_{i(\vec{n}),0}} (p-1)! d_{p^{n-i(\vec{n})}} d_{p^{(n-i(\vec{n}})} \ldots d_{p^{n-i(\vec{n})}} \]

PROOF. a) We apply proposition 1:

\[ P^{\mu} d_{p^l} = 0, \text{ if } t \neq l + k - 1 \]

\[ P^{\mu} d_{l+1} = 0, \text{ if } t \neq l + k - 1 \text{ or } l + i - 1 \]

Now the statement follows using Cartan formula.

b) is an application of a).

c) Proposition 1 and Cartan formula implies the statement, since \( p^{\mu(\vec{n})} \) is the least \( p \)-th power which provides a non-zero Steenrod operation.

d) We apply proposition 1:

\[ P^{\Gamma(n_i+i-1,i)} d_{p^{n_i}} = d_{p^{n_i}} \]

If \( n_i + i - 1 > m(\vec{n}) \) or \( l > i \), then \( P^{\Gamma(m(\vec{n}), l)} d_{p^{n_i}} = 0 \). Now the statement is an application of c).
e) This is a repeated application of d). Two main cases should be considered depending on \(i(\vec{n})\). Moreover, the number of times which the operation \(P^{(m(\vec{n}),i(\vec{n}))}\) has to be applied depends on \(a_{i(\vec{n}),0}\). We describe the first step in details. The next steps follow the same pattern.

We first apply \(P^{(m(\vec{n}),i(\vec{n}))}\). Let \(\vec{n}' = \vec{n} + (0, \ldots, 0, -p^{a_{i(\vec{n}),0}}, 0, \ldots, 0, p^{a_{i(\vec{n}),0}})\). The next Steenrod-Milnor operation shall be applied depends on \(m(\vec{n}')\) and \(i(\vec{n}')\).

Let us compare \(d^n\) and \(d'^n\). First case: \(i(\vec{n}) < k\), then \(n_{i(\vec{n}),0} + k - 1 > m(\vec{n})\). Now, if \(a_{i(\vec{n}),0} = 1\) and \(I(\vec{n}) = \{i(\vec{n})\}\), then \(m(\vec{n}') > m(\vec{n})\). In this case our statement follows for \(a_{i(\vec{n}),0} = 1\). Otherwise, \(m(\vec{n}') = m(\vec{n})\) and \(P^{(m(\vec{n}),i(\vec{n}))}\) shall be applied again.

Second case: \(i(\vec{n}) = k\). Let \(a_{k,0} = p - 1\), then \(m(\vec{n}') = m(\vec{n}) + 1\). Again our statement follows. Let \(a_{k,0} < p - 1\), then \(m(\vec{n}') = m(\vec{n})\) and \(P^{(m(\vec{n}),i(\vec{n}))}\) shall be applied again.

Let us comment on the statement of last proposition. Let \(d^n\) be a monomial and \(d'^n\) the resulting monomial as in the statement of e) above. If for each index \(i_r \in I(\vec{n})\) a suitable Steenrod-Milnor operation is defined, then the smallest \(p\)-th components of the \(n_i\) are reduced and that of \(n_k\) is increased respectively.

The next technical results are needed for the proof of the main Theorem.

**Corollary 10.** Given \(\vec{n}\), let \(n_i = \sum_{t=1}^{l(i)} a_{i,t}p^{n_{i,t}}\).

a) Let \(q = \sum_{i_r \in I(\vec{n})} a_{i_r,0}p^{n_{i_r,0}}\). If \(k > i(\vec{n})\), then

\[P^{(\vec{n})}d^n = \lambda d^n d_k^{(q)} \prod_{i_r \in I(\vec{n})} d_{i_r}^{-a_{i_r,0}p^{n_{i_r,0}}}\]

b) Let \(q' = p^{n_{k,0}+1} + \sum_{i_r \in I(\vec{n}) - \{k\}} a_{i_r,0}p^{n_{i_r,0}}\). If \(k = i(\vec{n})\), then

\[P^{(\vec{n})}d^n = \lambda d^n d_k^{(q')} \prod_{i_r \in I(\vec{n})} d_{i_r}^{-a_{i_r,0}p^{n_{i_r,0}}}\]

Here \(\lambda \in (\mathbb{F}_p)^*\).

**Proof.** This is an application of proposition 9 e). ■

**Lemma 11.** Given \(\vec{n}\), let \(\vec{n}(0) = \vec{n}\) and \(\vec{n}(1)\) such that \(d^{\vec{n}(1)} := P^{(\vec{n})}d^n\).

i) Then \(m(\vec{n}(0)) < m(\vec{n}(1))\).
ii) Let \( \sum \bar{n} := \sum_{t<k} \sum_s a_{t,s} \), where \( n_i = \sum_{t=1}^{\ell(i)} a_{i,t} p^{n_{i,t}} \) and \( \prod a_{i,t} \neq 0 \).

\[
I (\bar{n}(0)) \neq \{k\} \implies \sum \bar{n}(0) > \sum \bar{n}(1)
\]

**Proof.** i) Let \( \bar{n}(0) = (n_i(0) | i = 1, \ldots, k) \), \( I (\bar{n}(0)) = \{i_1, \ldots, i_t\} \) and \( J(\bar{n}(0)) = (a_1, \ldots, a_t) \). According to last corollary, \( n_i(0) = n_i(1) \), if \( i \notin I (\bar{n}(0)) \) and

\[
n_i(1) = \begin{cases} 
n_i(0) - a_{i,0} p^{n_{i,0}}, & \text{if } i \in I (\bar{n}(0)) - \{k\} \\
n_k(0) + \sum_{i_r \in I(\bar{n})} a_{i_r,0} p^{n_{i_r,0}}, & \text{if } k > i(\bar{n}(0)) \\
n_k(0) - a_{i,0} p^{n_{i,0}} + p^{n_{i,0}+1} + \sum_{i_r \in I(\bar{n}) - \{k\}} a_{i_r,0} p^{n_{i_r,0}}, & \text{if } k = i(\bar{n}(0))
\end{cases}
\]

Thus

\[
m(\bar{n}(1)) = \begin{cases} 
\min\{n_{i,t} + k - 1, n_{k,0} + k - 1, n_{i,1} + i - 1\}, & \text{if } k > i(\bar{n}(0)) \\
m(\bar{n}(0)) + 1, & \text{if } k = i(\bar{n}(0))
\end{cases}
\]

and the claim follows.

ii) If \( I (\bar{n}(0)) = \{k\} \), then \( \sum \bar{n}(0) = \sum \bar{n}(1) \). This follows from formulas above. Otherwise, \( \sum \bar{n}(0) > \sum \bar{n}(1) \).

**Theorem 12.** There exists a sequence of Steenrod-Milnor operations \( P^\lambda \) such that \( P^\lambda d^n = \lambda d^{p^{\ell(n)}} \). Here \( \lambda \in \mathbb{F}_p^\ast \).

**Proof.** We shall describe an algorithm which constructs the required sequence. This algorithm depends heavily on last corollary and lemma.

**Step 0.** Let \( P^0 = P^0 \).

**Step 1.** Given \( d^0 \) define \( I(\bar{n}), J(\bar{n}) \) and \( i(\bar{n}) \) as in Definition 5 b). Define \( P^1 := P^{I(\bar{n})} P^0 \).

**Step 2.** Let \( q = \sum_{k > i_r \in I(\bar{n})} a_{i_r,0} p^{n_{i_r,0}} \) and \( q' = p^{n_{i,0}+1} + \sum_{k > i_r \in I(\bar{n})} a_{i_r,0} p^{n_{i_r,0}} \).

Define

\[
d^{d^n} := \begin{cases} 
\lambda d^n d_k^{(q)} \prod_{i_r \in I(\bar{n})} d_{i_r}^{-a_{i_r,0} p^{n_{i_r,0}}} & \text{or} \\
\lambda d^n d_k^{(q)} \prod_{i_r \in I(\bar{n})} d_{i_r}^{-a_{i_r,0} p^{n_{i_r,0}}}
\end{cases}
\]

given by corollary above. If \( n_i < k \) for some \( i > 0 \) or \( n_k = p^{l(\bar{n})} \) for some positive integer \( l(\bar{n}) \), then proceed to step 1. Otherwise, the required sequence is \( P^\lambda \). Because of last lemma, the procedure above terminates after a finite number of steps.

\( P^\lambda \) as in the last Theorem is a repeated S-M operation. We define an ordering between monomials in \( D_k \) according to S-M operations called a **Steenrod-Milnor action ordering** and write S-M ordering.
Definition 13. For two sequences \( \bar{n} \) and \( \bar{n}' \) let
\[
n_i = \sum_{t=0}^{l(i)} a_{i,t} p^{n_{i,t}}
\]
and
\[
n_i' = \sum_{t=0}^{l(i)} a_{i,t}' p^{n_{i,t}'}
\]
be their \( p \)-adic decompositions respectively. We require
\[
\prod_{i,t} a_{i,t} \prod_{i,t} a_{i,t}' \neq 0.
\]
We define an ordering on monomials of \( D_k \), \( d^n < d^{n'} \),
if one of the conditions is satisfied referring to the definition 5.

i) \( m(\bar{n}) < m(\bar{n}') \).

ii) Let \( m(\bar{n}) = m(\bar{n}') \), \( I(\bar{n}) = \{i_1, \ldots, i_t\} \), \( I(\bar{n}') = \{i'_1, \ldots, i'_t\} \), \( J(\bar{n}) = (a_1, \ldots, a_t) \) and \( J(\bar{n}') = (a'_1, \ldots, a'_t) \). There exists a \( t_0 \) with \( 0 \leq t_0 \leq t-1 \)
such that \( t_0 \) is maximal with respect to the following condition:
\( i_{t-s} = i'_{t-s} \) and \( a_{t-s} = a'_{t-s} \) for \( 0 \leq s \leq t_0 - 1 \)
and either \( i_{t-t_0} > i'_{t-t_0} \) or \( i_{t-t_0} = i'_{t-t_0} \) and \( a_{t-t_0} > a'_{t-t_0} \).

For simplicity we write \( \bar{n} < \bar{n}' \) instead of \( d^n < d^{n'} \).

Proposition 14. Let \( \bar{n} < \bar{n}' \), then there exists a sequence of Steen-
rod operations \( P^r \) depending on \( \bar{n} \) such that \( P^r d^n \neq 0 \) and \( P^r d^{n'} = 0 \).

Proof. Without loss of generality we suppose that either condition
i) or ii) in the last definition is satisfied.

Condition i). Let \( m(\bar{n}) < m(\bar{n}') \), then \( P^{m(\bar{n})} d^n \neq 0 \) and \( P^{m(\bar{n})} d^{n'} = 0 \).

Condition ii). Let \( m = m(\bar{n}) = m(\bar{n}') \) and there exists a \( t_0 \) with
\( 0 \leq t_0 \leq t-1 \) such that \( i_{t-t_0} = i'_{t-t_0} \) and \( a_{t-t_0} > a'_{t-t_0} \). Let
\[
d^n(1) = \lambda(1) P^{(m, (i_{t-t_0+1}, \ldots, i_t), (a_{t-t_0+1}, \ldots, a_t) )} d^n
\]
and
\[
d^{n'}(1) = \lambda'(1) P^{(m, (i_{t-t_0+1}, \ldots, i_t), (a_{t-t_0+1}, \ldots, a_t) )} d^{n'}.
\]
Because of our assumption \( \bar{n} (1) = \bar{n}' (1) = \bar{n} \). Here negative integers are allowed.

Now \( I(\bar{n} (1)) = \{i_1, \ldots, i_{i-t_0}\} \) and \( I(\bar{n}' (1)) = \{i'_1, \ldots, i'_{t-t_0} = i_{t-t_0}\} \)
with \( a_{t-t_0} > a'_{t-t_0} \). Let
\[
d^n(2) = \lambda(2) P^{(m, i_{t-t_0}) \ldots P^{(m, i_{t-t_0})} d^n(1)}
\]
and
\[
d^{n'}(2) = \lambda'(2) P^{(m, i_{t-t_0}) \ldots P^{(m, i_{t-t_0})} d^{n'}(1)}
\]
Now \( I(\bar{n} (2)) = \{i_1, \ldots, i_{t-t_0}\} \) and \( I(\bar{n}' (2)) = \{i'_1, \ldots, i'_{t-t_0-1} = i_{t-t_0-1}\} \) with
\( i(\bar{n} (2)) = i_{t-t_0} > i(\bar{n}' (2)) = i'_{t-t_0-1} \). Thus
\[
P^{(m, i_{t-t_0})} d^{n'}(2) = 0.
\]
Now we are ready to proceed to our main Theorem.

**Theorem 15.** Let \( f : D_k \rightarrow D_k \) be an \( \mathcal{A} \)-linear map of degree 0 such that \( f(d_{k-1}) \neq 0 \). Then \( f \) is a upper triangular map with respect to S-M ordering and hence an isomorphism.

**Proof.** By hypothesis and proposition 1, \( f(d_i) = \lambda d_i \) for \( i = 1, \ldots, k \) after applying a suitable Steenrod operation.

Let \( d^n \in D_k \) and \( (d^{n(1)}, \ldots, d^{n(i)}) \) the increasing sequence of elements of degree \( |d^n| \). Let \( f(d^n) = \sum_{t=1}^{l(n)} a_t d^{n(i)} \). Claim: If \( \bar{n}(t_0) = \bar{n} \), then \( a_t \equiv 0 \mod p \) for \( t < t_0 \). Following our last proposition \( P^n \) \( d^r = 0 \), if \( \bar{n} < \bar{n}' \). We use induction on \( t \) for \( t < t_0 \). Using Theorem 12, proposition 14 and definition of elements \( P^n \), there exists \( P^{n_1} \) such that \( P^{n_1}d^{n(i)} = \lambda a_k^{l(n)}(d_{n(i)}) \) and \( P^{n_1}d^{n(i)} = 0 \) for \( i > 1 \). Then

\[
P^{n_1}f(d^n) = P^{n_1} \sum_{t=1}^{l(n)} a_t d^{n(i)} \text{ implies } a_1 \equiv 0 \mod p.
\]

By induction \( P^{n_1}f(d^n) = P^{n_1} \sum_{t=1}^{l(n)} a_t d^{n(i)} \) implies \( a_t \equiv 0 \mod p \) for \( i < t_0 \). Using proposition 1 and the fact that \( f(d_k) \neq 0 \), we get \( f(d^n_k) = \lambda a_k^{l(n)}d_k^{l(n)} \). The last observation implies \( a_t_0 \neq 0 \mod p \). Hence \( f \) is a upper triangular map.

**Example 16.** Let \( p = 2 \) and \( H_2 = P[y_1, y_2] \) the ring of upper triangular invariants which is a polynomial algebra on \( h_1 = y_1 \) and \( h_2 = y_2^2 + y_2y_1 \). Let \( f : H_2 \rightarrow H_2 \) be an \( \mathcal{A} \)-linear map such that \( f(h_1) = h_1 \). Since \( Sq^1h_1 = h_2^2 \neq h_2 \), \( f(h_2) \) can be defined independently of \( h_1 \):

\[
f(h_2) = ah_2 + bh_1^2 \quad \text{with } a, b \in \mathbb{F}_2.
\]

Even if \( f(h_2) = h_2^2 \), \( f \) is not an isomorphism: \( f(d_{2,1}) = f(h_2 + h_1^2) = 0 = f(d_{2,0}) = f(Sq^1d_{2,1}) \).

4. The exterior part of the Dickson algebra

Next we extend the previous results to the full Dickson algebra.

**Lemma 17.** i) Let \( M_{s_1, \ldots, s_t} L_k^{p-2}d^n \) be a monomial in \( D(V) \) and \( P^{B(s_1, \ldots, s_t)} := \prod_{k=1}^{p-2} \prod_{k=1}^{p-2} P^{V_{s_1}} \prod_{k=1}^{p-2} P^{V_{s_t}} \). Then

\[
P^{B(s_1, \ldots, s_t)} M_{s_1, \ldots, s_t} L_k^{p-2}d^n = (-1)^{(k-1)}! d_k d^n
\]

ii) Let \( M_{s_1, \ldots, s_t} L_k^{p-2}d^n \) and \( M_{s'_1, \ldots, s'_t} L_k^{p-2}d'^n \) be monomials such that \( s_{l-t} < s'_{l-t} \) and \( t \) is minimal with this property, then

\[
P^{B(s_1, \ldots, s_t)} M_{s'_1, \ldots, s'_t} L_k^{p-2}d'^n = 0
\]
Proof. Let us recall that $P^p \beta (M_{s_1,\ldots,s_l} L_k^{p-2}) = M_{s_1,\ldots,s_{l-1},s_l+1} L_k^{p-2}$ for $s_l < k - 1$ and $P^p \beta (M_{s_1,\ldots,s_l} L_k^{p-2}) = 0$ if and only if $s_l - t + 1$ for $0 \leq t \leq s_l + 1$. If $0 = s_t$, we apply the Bockstein operation $\beta$. Thus $P^p \beta \ldots P^p \beta M_{s_1,\ldots,s_l} L_k^{p-2} f_t^\bar{n} = \sum_{0 \leq t \leq s_l + 1} M_{s_1,\ldots,s_{l-1},s_l+t} L_k^{p-2} f_t$. Here $f_t$ is a polynomial in $D_k$.

Let $\mathcal{P} = \{P^p \beta \ldots P^p \beta, P^p \ldots P^p \beta\}$. Iterating the last formula we obtain:

$$P^{\mathcal{P}} M_{s_1,\ldots,s_l} L_k^{p-2} f_t^\bar{n} = \sum_{\min k \leq l} s_{q+1} t_{q+1} s_q M_{s_1+t_1,\ldots,s_{l-1}+t_l} L_k^{p-2} L_k^{p-2} f_{t_1,\ldots,t_l}$$

Here $s_{l+1} = 0$ and $t_{l+1} = k - 1$.

Let us suppose that $s_1 + t_1 < k - l$. Let $P^{\Delta} = P^{p^k-1} \ldots P^{p^0}$ and $A = M_{s_1+t_1,\ldots,s_{l-1}+t_{l-1},s_l+t_l} L_k^{p-2} f_{t_1,\ldots,t_l}$. There are $s_1 + t_1 - 1 \leq k - l - 2$ positions to be filled by powers of $y$‘s using Steenrod operations: $\beta \{P^p \beta \ldots P^p \beta, P^p \ldots P^p \beta\}$. Since there are $k - 1$ ‘s in this sequence and only $s_1 + t_1 - 1 \leq k - l - 2$ positions, it is obvious that $P^{\Delta} A = 0$. Now suppose that $s_1 + t_1 = k - l$ and one operation $P^{p^0}$ of $P^{\Delta}$ is not applied on $A$. Then it will be less positions than the number of remaining ‘s. In that case $\beta \{P^p \beta \ldots P^p \beta, P^p \ldots P^p \beta\} M_{s_1+t_1,\ldots,s_{l-1}+t_{l-1},s_l+t_l} L_k^{p-2} f_{t_1,\ldots,t_l} = 0$. The claim follows.

Corollary 18. Let $M_{s_1,\ldots,s_l} L_k^{p-2} d_t^\bar{n} \in D(V)$. There exists a sequence of $S\cdot M$ operations such that

$$P^{\mathcal{S}} M_{s_1,\ldots,s_l} L_k^{p-2} d_t^\bar{n} = \lambda d_t^\bar{n}$$

and $q$ is minimal with this property.

Here $\lambda \in (\mathbb{Z}/p\mathbb{Z})^*$. Now we are ready to proceed to our main Theorem.

Theorem 19. Let $g : D(V) \rightarrow D(V)$ be an $A$-linear map degree 0 such that $g(M_k) \neq 0$. Then $g$ is an isomorphism. Here $M_k = \prod_{1}^{k} x_i L_k^{p-2}$.

Proof. Let $g : D(V) \rightarrow D(V)$ such that $g(M_k) = \lambda M_k$. Then $g(d_k) = \lambda' d_k$ with $\lambda' \neq 0 \mod p$ and $g(d_{k-1}) = \lambda' d_{k-1}$. Please recall that $\beta P^1 \beta \ldots P^{l-2} \beta M_k = d_k$. Thus $g$ is an isomorphism in $D_k$.

We recall that an $A$-module is indecomposable, if it is not a non-trivial direct sum.

Definition 20. Let $\overline{D(V)}$ denote the augmentation ideal of $D(V)$.
Corollary 21. \( \overline{D(V)} \) is not directly decomposable as an \( \mathcal{A} \)-module.

Proof. Assume \( \overline{D(V)} = \bigoplus_{i \in I} D(V)_i \) such that \( D(V)_i \neq 0 \). If \( d(i) \) and \( d(j) \) are monomials in \( D(V)_i \) and \( D(V)_i \), respectively, then there exist \( P^r \) and \( P^{r'} \) such that \( a_i P^r d(i) = d_0^j = b_j P^{r'} d(j) \).

5. The structure of \( \text{End}_U(D(V)) \)

It is known that if an \( R \)-module \( M \) is directly indecomposable and of finite length, then \( \text{End}(M_R) \) is a local ring and its non-invertible elements are precisely its nilpotent elements. According to last corollary \( \overline{D(V)} \) is directly indecomposable. It is also known that the set \( \left\{ d_{k-1}^{(p'-1)/(p-1)} | l \geq 1 \right\} \) is linearly independent ([5]). Thus \( \overline{D(V)} \) is not of finite length. But we shall prove that \( \text{End}_U \left( \overline{D(V)} \right) \) is a local ring.

We follow the approach suggested by H.-W. Henn. We shall omit technical details which will appear in [7].

We state a remarkable theorem due to Adams, Gunawardena and Miller (Th. 1.6, p. 437 [1]), we apply it for \( s = 0, t = 0, M = \mathbb{F}_p \) and \( U = V \).

Theorem 22. \( \mathbb{F}_p[\text{End}(V)] \cong \text{End}_U(H^*(V)) \). Here \( \mathbb{F}_p[\text{End}(V)] \) is the monoid algebra of the monoid under composition \( \text{End}(V) \).

Using the Theorem above one can reduce the problem to a linear algebra one.

Theorem 23. \( \text{Hom}_U \left( D(V), H^*(V) \right) \cong \mathbb{F}_p[G \setminus \text{End}(V)] \).

Proof. We view \( \text{End}(V) \) as a monoid with respect to composition of linear maps. It admits a left and a right action by itself. Let \( \mathbb{F}_p[\text{End}(V)] \) be the associated monoid algebra. Let \( \mathbb{F}_p[G \setminus \text{End}(V)] \) be the vector space on the set of orbits \( G \setminus \text{End}(V) \). The monoid \( \text{End}(V) \) acts on the right of \( GL(V) \setminus \text{End}(V) \) by \( f \cdot h = f \overline{h} \). Because of the right action above \( \mathbb{F}_p[G \setminus \text{End}(V)] \) becomes an \( \mathbb{F}_p[\text{End}(V)] \)-module.

Let \( f : D(V) \to H^*(V) \) and \( g : H^*(V) \to H^*(V) \), then \( gf : D(V) \to H^*(V) \) and \( \text{Hom}_U(D(V), H^*(V)) \) becomes a left \( \text{End}_U(H^*(V)) \)-module. Moreover, by the AGM-theorem, it becomes a right \( \mathbb{F}_p[\text{End}(V)] \)-module.

We recall that \( H^*(V) \) is an unstable \( \mathcal{A} \)-module. Thus given \( f \in \text{Hom}_U(D(V), H^*(V)) \) an \( f \in \text{Hom}_U(H^*(V), H^*(V)) \) is induced. Applying the theorem above, \( \tilde{f} \) is identified with \( \varphi : V \to V \) such that \( \varphi \) is \( G \)-invariant because \( f \) is. The isomorphism follows. ■
 Proposition 24. \( \text{End}_U(D(V)) \cong \text{End}_{\text{End}_U(H^*(V))} (\text{Hom}_U(D(V), H^*(V))). \)

Proof. Let

\[ \Phi : \text{End}_U(D(V)) \to \text{End}_{\text{End}_U(H^*(V))} (\text{Hom}_U(D(V), H^*(V))) \]

given by \( \Phi(f)(h) = hf \) for \( f \in \text{End}_U(D(V)) \) and \( h \in \text{Hom}_U(D(V), H^*(V)). \) Moreover, \( k\Phi(f)(h) = \Phi(f)(kh) \) for \( k \in \text{End}_U(H^*(V)). \)

\( \Phi \) is 1-1: Let \( \Phi(f) = \Phi(f') \), then \( \Phi(f)(i) = \Phi(f')(i) \) for \( i : D(V) \to H^*(V) \). Thus \( f = f' \).

\( \Phi \) is onto: \( \forall h \in \text{Hom}_U(D(V), H^*(V)), \exists \tilde{h} \in \text{End}_U(H^*(V)) \) such that \( h = \tilde{h}i \) because \( H^*(V) \) is injective.

Let \( \Psi \in \text{End}_{\text{End}_U(H^*(V))} (\text{Hom}_U(D(V), H^*(V))) \), then \( \tilde{h}\Psi(i) = \Psi(\tilde{h}i) = \Psi(h) \). Each \( g \in G \) defines a map \( g\Psi(i) = \Psi(gi) = \Psi(i) \), thus \( \Psi(i) \in \text{End}_U(D(V)) \). Let \( f_\Psi = \Psi(i) \), then \( \Phi(f_\Psi)(h) = hf_\Psi = \tilde{h}if_\Psi = \tilde{h}f_\Psi = \Psi(h) \). \( \blacksquare \)

Theorem 25. \( \text{End}_U(D(V)) \cong \text{End}_{\mathbb{F}_p[End(V)]} (\mathbb{F}_p[G \smash \text{End}(V)]). \)

The next proposition provides all technical details for the conclusion of this section, namely \( \text{End}_U(D(V)) \) is a local ring along with its structure.

The \( \mathcal{A} \)-submodule of \( D(V) \) consisting of degree 0 elements is isomorphic with \( \mathbb{F}_p \). Thus \( \text{End}_U(D(V)) \cong \mathbb{F}_p \oplus \text{End}_U(D(V)) \). Although the subspace of the zero orbit in \( \mathbb{F}_p[G \smash \text{End}(V)] \) is a direct summand, it is not a direct summand as an \( \mathbb{F}_p[\text{End}(V)] \)-module but it can be decomposed as follows. Let \( \mathbb{F}_p[G \smash \text{End}(V)] \) be the vector space on

\[ \{ \overline{f} \mid f \in \text{End}(V), f \neq 0_V \} \]

We identify the neutral element \( \sum_{f \neq 0_V} 0 \overline{f} \) with the zero orbit \( 0 \). It becomes an \( \mathbb{F}_p[\text{End}(V)] \)-module.

Theorem 26. [7]Let \( R := \text{End}_{\mathbb{F}_p[End(V)]} \left( \mathbb{F}_p[G \smash \text{End}(V)] \right) \), then \( R \) has dimension \( n \) as a vector space over \( \mathbb{F}_p \). There is a set of generators \( \{ \psi_i \mid 1 \leq i \leq n \} \) such that:

1) \( \psi_0 = 0 \) is the neutral element and \( \psi_n = 1 \) is the identity map;
2) Let \( n > l \geq k > 0 \), then \( \psi_k \psi_l = \psi_l \psi_k = \psi_{\max(0,k+l-n)} \).

Because of the isomorphism \( \text{End}_U(D(V)) \cong R \), the next corollary follows.

Corollary 27. \( \text{End}_U(D(V)) \) is a local \( \mathbb{F}_p \)-algebra with dimension \( n \) as a vector space over \( \mathbb{F}_p \) (i.e. \( f \) or \( \text{Id} - f \) is an isomorphism for
any \( f \) in \( \text{End}_U(\overline{D(V)}) \). Moreover, if \( I \) is the ideal generated by its nilpotent elements, then \( \text{End}_U(\overline{D(V)})/I \equiv \mathbb{F}_p \).

6. An application

We close this work by applying our result in the mod \( p \) homology of \( QS^0 \). Firstly, we recall the isomorphism between the hom-dual of the Dyer-Lashof algebra and the Dickson algebra.

**Proposition 28.** a) Let \( SD(V) \) be the subalgebra of \( D(V) \) generated by
\[
\{d_i, M_s L_k^{p-2}, M_{s_1,s_2} L_k^{p-2}\}
\]
where \( 1 \leq i \leq k \), \( 0 \leq s_1 \leq k - 1 \) and \( 0 \leq s'_1 < s'_2 \leq k - 1 \). If \( f : SD(V) \to SD(V) \) satisfies
\[
f(M_{k-2,k-1}L_k^{p-2}) = \lambda M_{k-2,k-1}L_k^{p-2} \neq 0
\]
then \( f \) is an isomorphism.

b) Let \( I[k] \) be the ideal of \( SD(V) \) generated by
\[
\{d_k, M_s L_k^{p-2}, M_{s_1,k-1}\}
\]
then the induced map \( f \) which satisfies \( f(M_{k-2,k-1}L_k^{p-2}) = \lambda M_{k-2,k-1}L_k^{p-2} \)
is also an isomorphism.

Proposition b) above is a reformulation of Theorem 4.1 in [2].

Let \( R = < Q^{(I,J)} | I = (i_1, ..., i_n), J = (\varepsilon_1, ..., \varepsilon_n) > \) be the Dyer-Lashof algebra, then \( H_*(Q_0 S^0; \mathbb{F}_p) \) is the free commutative algebra generated by \( \Phi(R) \) subject to the following relation \( Q^{(I,J)} \approx (Q^{(I',J')})^p \) if \( I = (i_1, I'), J = (0, J') \) and \( exc(Q^{(I,J)}) = 0. \) Here \( \Phi : R \to H_*(Q_0 S^0; \mathbb{F}_p) \) is the \( \mathcal{A}_* \)-module map given by \( \Phi(Q^{(I,J)}) = Q^{(I,J)}[1] * [-p^{(I)}], \) \( [1] \) is a generator of \( \tilde{H}_0(S^0; \mathbb{F}_p) \), \( [r] = [1]^r \) and \( l(I) \) is the length of \( I \). Thus there exists an \( \mathcal{A}_* \)-module isomorphism between the generators of \( H_*(Q_0 S^0; \mathbb{F}_p) \) and the quotient \( R/Q_0 R \) where \( Q_0 R = \{Q^{(I,J)}|exc(I,J) = 0\} \). It is known that \( R[k]^* \cong SD(V) \) as Steenrod algebras and \( (R/Q_0 R)[k]^* \cong I[k] \) as Steenrod modules. Here \( R = \bigoplus R[k] \). Now the following Theorem is a consequence of last corollary.

**Theorem 29.** [2] Let \( f : \Omega_0^\infty S^\infty \to \Omega_0^\infty S^\infty \) be an \( H \)-map which induces an isomorphism on \( H_{2p-3}(\Omega_0^\infty S^\infty; \mathbb{F}_p) \). If \( p > 2 \) suppose in addition that \( f \) is a loop map or that
\[
f_*(d_{2,0})^* \neq 0
\]
for some \( \lambda \in (\mathbb{F}_p)^* \). Then \( f_{(p)} \) is a homotopy equivalence. Here \( (d_{2,0})^* \) is the hom-dual of the top degree Dickson generator of \( D_2 \) in \( R[2] \).
6.1. Acknowledgement. We would like to express our profound thanks to H.-W. Henn, P. May and L. Schwartz.

References


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