Cofree coalgebras and the components of the Dyer-Lashof algebra

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Algebras

Let K be a commutative ring with a unit.
An augmented graded K-algebra is a graded K-module A equipped with graded k-molules $\phi : A \otimes A \to A$, $\eta : K \to A$ and a morphism of algebras $\epsilon : A \to K$ such that the following diagrams are commutative

$$\begin{tikzcd}
A \otimes A \otimes A \arrow[r, "1\otimes\phi"] \arrow[d, "\phi\otimes 1"'] & A \otimes A \arrow[d, "\phi"] \\
A \otimes A \arrow[r, "\phi"] & A
\end{tikzcd}$$

$$\begin{tikzcd}
K \otimes A \arrow[rr, "\eta\otimes 1"] \arrow[dr, "\cong"'] & & A \otimes A \arrow[dl, "\phi"] \\
& A &
\end{tikzcd}
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A \otimes K \arrow[rr, "1\otimes\eta"] \arrow[dr, "\cong"'] & & A \otimes A \arrow[dl, "\phi"] \\
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If the next diagram is commutative, then our algebra A is anti-commutative.

$$\begin{tikzcd} A \otimes A \arrow[d, "T"'] \arrow[dr, "\phi"] & \\ & A \\ A \otimes A \arrow[ur, "\phi"'] & \end{tikzcd}$$

Free algebra

Let K be a commutative ring with a unit.
Let M be graded K-module M.

The *free algebra generated by* M is an algebra VM together with a linear map $i : M \to VM$ such that for each K-algebra A and linear map $g : M \to A$ there is a unique algebra map

$$g' : VM \to A$$

such that $g'i = g$.

Coalgebras

Let K be a commutative ring with a unit.

A K-coalgebra is a graded module M equipped with morphisms of graded K-modules (a comultiplication) $\Delta : M \to M \otimes M$ and (a counit) $\epsilon : M \to K$ such that the diagrams are commutative.

$$\begin{tikzcd} M \otimes M \otimes M & M \otimes M \arrow[l, "1\otimes\Delta"'] \\ M \otimes M \arrow[u, "\Delta\otimes 1"] & M \arrow[l, "\Delta"'] \arrow[u, "\Delta"'] \end{tikzcd}$$

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Cofree coalgebra

One should think of Δ as giving each element m of M a "K-decomposition" into a sum of pairs of elements of M.

The *cofree coalgebra generated by* M is a coalgebra UM together with a linear map $\pi : UM \to M$ such that for each coalgebra C and linear map $f : C \to M$ there is a unique coalgebra map

$$f' : C \to UM$$

such that $\pi f' = f$.

If $\pi : UM \to M$ is the cofree K-coalgebra over M, then UM must represent all possible decompositions of M, i.e. for each element m of M and each Δ_M such that $\Delta_M(m) = \sum m_{(1)} \otimes m_{(2)}$, there must be elements $[m]$ and $[m_{(i)}]$ of UM such that $\pi[m] = m$, $\pi[m_{(i)}] = m_{(i)}$, and $\Delta_{UM}[m] = \sum [m_{(1)}] \otimes [m_{(2)}]$ in $UM \otimes UM$.

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Hopf algebra

A Hopf algebra over K is a graded K-module A equipped with morthisms of graded k-molules

$$\phi : A \to A \otimes A, \ \eta : K \to A$$

$$\Delta : A \to A \otimes A, \ \epsilon : A \to K$$

such that

- (A, ϕ, η) is an algebra over K with augmentation ϵ,
- (A, Δ, ϵ) is a coalgebra over K with augmentation η,

$$\begin{tikzcd}
A \otimes A \arrow[r, "\phi"] \arrow[d, "\Delta\otimes\Delta"'] & A \arrow[r, "\Delta"] & A \otimes A \\
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The Steenrod algebra $\mathcal{A}$

For a given prime number p, *the Steenrod algebra* $\mathcal{A}$ is the graded Hopf algebra over the field $F \cong Z/pZ$ of p elements, consisting of cohomology operations for $mod - p$ cohomology generated by:

$$P^i \colon H^n(X; F) \to H^{n+2i(p-1)}(X; F);$$

$$\beta \colon H^n(X; F) \to H^{n+1}(X; F) \text{ for } p > 2.$$

β and P^i are additive homomorphisms such that:

- P^0 is the identity, $\beta^2 = 0$.
- $P^i x = x^p$ for $|x| = 2i$ and $P^i x = 0$ for $|x| < 2i$. Moreover,

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- Cartan Formula: $P^n(x \smile y) = \sum_{i+j=n}(P^i x) \smile (P^j y)$ and
- Adem relations:

$$P^a P^b = \sum_i (-1)^{a+i} \binom{(p-1)(b-i)-1}{a-pi} P^{a+b-i} P^i$$

for $a < pb$.

- And for p odd

$$P^a \beta P^b = \sum_i (-1)^{a+i} \binom{(p-1)(b-i)}{a-pi} \beta P^{a+b-i} P^i +$$

$$\sum_i (-1)^{a+i+1} \binom{(p-1)(b-i)-1}{a-pi-1} P^{a+b-i} \beta P^i$$

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Unstable $\mathcal{A}$-modules

An $\mathcal{A}$-module M is called unstable, if $\beta^k P^i m = 0$ for $2i + k > |m|$ and $m \in M$.

The module $\Sigma^n(\mathcal{A}/ < P^I, I$ admissible and $e(I) < n + 1 >)$ is called the free unstable cyclic $\mathcal{A}$-module on one generator of degree n and is denoted by $F(n)$.

A free unstable $\mathcal{A}$-module is the direct sum of free unstable cyclic $\mathcal{A}$-modules.

The category of unstable modules was defined by Massey and Peterson and is denoted by $\mathcal{U}$.

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Let M be an unstable $\mathcal{A}$-modules and TM its tensor algebra.

Let VM be the quotient of TM by the ideal generated by

$$x \otimes y - (-1)^{|x||y|} y \otimes x \text{ and } P^{|x|/2}x - x^p.$$

VM is called the free unstable $\mathcal{A}$-algebra generated by M.

If M is a free unstable $\mathcal{A}$-module, then VM is called the completely free unstable $\mathcal{A}$-algebra generated by M.

Serre, $p = 2$: $H^*(K(Z/2Z, n); Z/2Z) \equiv VF(n)$.

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Unstable opposite $\mathcal{A}$-modules

Let $\mathcal{MA}$ be the category of connected unstable opposite $\mathcal{A}$-modules.

An unstable opposite $\mathcal{A}$-module consists of a positively graded F-module M

and a graded module map $\mathcal{A}^i \otimes M_n \to M_{n-2(p-1)i}$ with the property

$$P^k m = 0, \text{ if } |m| < 2pk; \text{ and } \beta P^k m = 0, \text{ if } |m| = 2pk + 1.$$

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Unstable $\mathcal{A}$-coalgebras

Let $\mathcal{CA}$ be the category of unstable coalgebras i.e.

an object is both an unstable opposite $\mathcal{A}$-module and a connected co-commutative F-coalgebra where these structures are compatible in the following sense.

The comultiplication map in $\mathcal{CA}$ is an unstable $\mathcal{A}$-module map and the p-th root map $\xi : M_{pk} \to M_k$, dual to the p-th power map, satisfies

$$\xi(m) = P^k(m).$$

For example $H_*(X, F)$ is an object in $\mathcal{CA}$ for X a connected space and the co multiplication is induced by the diagonal.

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Cofree Unstable $\mathcal{A}$-coalgebras

For a connected unstable $\mathcal{A}$-module M, the cofree unstable $\mathcal{A}$-coalgebra generated by M, UM, has the following universal property:

UM comes with an $\mathcal{A}$-module map $i : UM \to M$ and if C is an unstable $\mathcal{A}$-coalgebra and $f : C \to M$ an $\mathcal{A}$-module map, there exists a unique $\mathcal{A}$-coalgebra map

$$\bar{f} : C \to UM$$

such that $f = i\bar{f}$.

If M is of finite type, then UM **is dual to the free unstable $\mathcal{A}$-algebra VM^* generated by the dual $\mathcal{A}$-module M^*.**

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The Dyer-Lashof algebra $\mathcal{R}$

For a given prime number p, the Dyer-Lashof algebra $\mathcal{R}$ is the graded Hopf algebra over the field F of p elements, consisting of homology operations for $mod - p$ homology on infinite loop spaces QX generated by:

$$Q^i : H_n(QX; F) \to H_{n+2i(p-1)}(QX; F);$$

$$\beta Q^i : H_n(QX; F) \to H_{n-1+2i(p-1)}(QX; F) \text{ for } p > 2.$$

- βQ^i and Q^i are additive homomorphisms such that:
- $Q^i x = x^p$ for $|x| = 2i$ and $Q^i x = 0$ for $|x| > 2i$.

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- Cartan Formula: $Q^n(x \otimes y) = \sum_{i+j=n} (Q^i x) \otimes (Q^j y)$ and
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$$Q^a Q^b = \sum_i (-1)^{a+i} \binom{(p-1)(i-b)}{pi-a} Q^{a+b-i} Q^i$$

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- And for p odd

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Infinite loop spaces

Y is an infinite loop space, if there exists a sequence of spaces $\{Y_0, Y_1, ...\}$ such that $Y = Y_0$ and $Y_i \simeq \Omega Y_{i+1}$.

Examples:

- $QX = \varinjlim \Omega^n \Sigma^n X$,
- $Z \times BO = Z \times \varinjlim BO_n$ real K-theory,
- $Z \times BU = Z \times \varinjlim BU_n$ complex K-theory.

Theorem (Dyer-Lashof)

$H_*(QX, F)$ is the free commutative algebra generated by $\left(Q^I x\text{, such that } Q^I \in \mathcal{R},\ e(I) \leq |x|\text{, and } x \in H_*(X)\right)$ modulo the ideal generated by

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Problem: i) Classify all compact n-manifolds up to **diffeomorphism.**
ii) Classify all compact n-manifolds up to **cobordism.**

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The cobordism group of n-dimensional unoriented manifolds, is isomorphic to the stable homotopy group:

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The Dyer-Lashof Hopf algebra can be decomposed as coalgebras over the opposite Steenrod algebra with respect to length:

$$\mathcal{R} = \bigoplus_{k \geq 0} R[k].$$

Problem:

- How far is each $R[k]$ from being cofree?
- How about its dual $(R[k])^*$?
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$(R[k])^*$ is related with the Dickson algebra D_k.

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The Dickson algebra

Theorem (Dickson)

The classical Dickson algebra is a polynomial algebra

$$P[y_1, \cdots, y_k]^{GL_k} = P\left[d_{k,1}, \cdots, d_{k,k-1}, d_{k,k}\right].$$

$\left|d_{k,i}\right| = 2\left(p^k - p^{k-i}\right)$, $[\left|d_{k,i}\right| = 2^k - 2^{k-i}]$.

Definition

The extended Dickson algebra, p odd, is given by:

$$H^*\left(BV^k\right)^{GL_k} \cong \left(E\left(x_1, ..., x_k\right) \otimes P\left[y_1, ..., y_k\right]\right)^{GL_k}$$

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Theorem (Mui)

The extended Dickson algebra is described as follows

$$ED_k := (E(x_1, \cdots, x_k) \otimes P[y_1, \cdots, y_k])^{GL_k}.$$

It is a tensor product of the polynomial algebra $P[y_1, \cdots, y_k]^{GL_k}$ *and the* $\mathbb{Z}/p\mathbb{Z}$*-module spanned by the set of elements consisting of the following polynomials:*

$$M_{k;s_1,\ldots,s_m}(L_k)^{p-2};\ 1 \leq m \leq k,\ \textit{and}\ 0 \leq s_1 < \cdots < s_m \leq k-1.$$

There are relations among the generators.

The Dickson algebra

Definition

Let SED_k be the subalgebra of ED_k generated by:

$$d_{k;s+1},\ M_{k;s}(L_k)^{p-2} \text{ and } M_{k;s_1,s_2}(L_k)^{p-2}.$$

Here $0 \leq s \leq k-1$. $0 \leq s_1 < s_2 \leq k-1$.

The Dickson algebra and the Dyer-Lashof algebra

Theorem (Madsen $p = 2$, May p odd)

$R[k]^*$ is generated by $\{\omega_{k,i+1} = \left(Q^{I_{k,i+1}}\right)^*, \tau_{k;i} = \left(Q^{J_{k;i}}\right)^*$, and $\sigma_{k;s,i} = \left(Q^{K_{k;s,i}}\right)^* \mid 0 \leq i \leq k-1$, and $0 \leq s < i\}$, $[\{\omega_{k,i+1} \mid 0 \leq i \leq k-1\}$, for $p = 2]$, modulo certain relations.

Theorem (Mui $p = 2$, Kechagias p odd)

Let $T_k : SED_k \to R[k]^*$ be given by $T_k\left(d_{k,i+1}\right) = \omega_{k,i+1}$, $T_k\left(M_{k;i}L_k^{p-2}\right) = \tau_{k,i}$, and $T_k\left(M_{k;s,i}L_k^{p-2}\right) = \sigma_{k;s,i}$. Then T_k is a Steenrod algebra isomorphism.

For $p = 2$, $R[k]^*$ is a polynomial algebra and it is isomorphic with the classical Dickson algebra as Steenrod algebras.

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The Peterson conjecture

The Peterson conjecture is about the **global structure** of the classical Dickson algebra as an unstable algebra over the Steenrod algebra.

This conjecture was solved by Pengelley, Peterson and Williams for $p = 2$ in:

Pengelley, D. J., Peterson, F. P. and Williams, F., "A global structure theorem for the mod 2 Dickson algebras, and unstable cyclic modules over the Steenrod and Kudo-Araki-May algebras", *Math. Proc. Cambridge Philos. Soc.*, **129**, 2000, no. 2, 263–275.

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They proved that the classical Dickson algebra D_k is a **free unstable algebra on a certain cyclic module**, modulo four additional relations.

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The Peterson conjecture and SED_k

- We define an unstable $\mathcal{A}$-module $\mathcal{M}(\mu, u)$
- and from it an unstable $\mathcal{A}$-algebra $\mathcal{Q}(\mu, u)$.
- Finally an isomorphism between $\mathcal{Q}(\mu, u)$ and SED_n will be defined.

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The unstable $\mathcal{A}$-module $\mathcal{M}(\mu, u)$

Definition

The module $\mathcal{M}(\mu, u)$ has two generators μ and u of degrees $2\left(p^n - p^{n-1} - p^{n-2}\right)$ and $2\left(p^n - p^{n-1}\right)$ respectively and relations:

$$P^{p^k}\mu = 0 = P^{p^l}u, \tag{1}$$

$$\text{for } -1 \leq k \leq n-4,\ k = n-2 \text{ and } -1 \leq l \leq n-3;$$

$$P^{p^{n-3}}P^{p^{n-3}}\mu = 0 = P^{p^{n-2}}P^{p^{n-2}}u; \tag{2}$$

$$P^{p^{n-3}}P^{p^{n-2}}P^{p^{n-3}}\mu = 0 = P^{p^{n-2}}P^{p^{n-2}}P^{p^{n-3}}\mu; \tag{3}$$

$$P^{p^{n-1}}P^{p^{n-3}}\mu = P^{p^{n-3}}P^{p^{n-1}}\mu \text{ and } P^{p^{n-2}}P^{p^{n-1}}u = 2P^{p^{n-1}}P^{p^{n-2}}u. \tag{4}$$

The generators are related as follows:

$$P^{(-1,\cdots,n-2)}P^{(-1,\cdots,n-3)}\mu = P^{(0,\cdots,n-2)}u. \tag{5}$$

The unstable $\mathcal{A}$-algebra $\mathcal{Q}(\mu, u)$

Definition

Let $\mathcal{Q}(\mu, u)$ be the free unstable $\mathcal{A}$-algebra on the module $\mathcal{M}(\mu, u)$ subject to the following relations:

$$\mu^2 = 0 \text{ and } \quad P^{p^{n-1}} u = (p-1)u^2. \tag{6}$$

The generators are related as follows:

$$P^{p^{n-1}} \mu = (p-2)\mu u \text{ and} \tag{7}$$

$$P^{p^{n-1}} P^{p^{n-2}} P^{p^{n-3}} \mu = \tag{1}$$

$$-P^{p^{n-2}} P^{p^{n-3}} \mu u + \mu P^{p^{n-3}} P^{p^{n-2}} u - P^{p^{n-3}} \mu P^{p^{n-2}} u. \tag{2}$$

The main Theorem

Theorem

The algebra $\mathcal{Q}(\mu, u)$ *is isomorphic as an* $\mathcal{A}$*-algebra to* SED_n.

Corollary

$R[n]$ *is isomorphic to a subcoalgebra of a cofree unstable coalgebra on two cogenerators.*

Corollary

Let $1 \leq n$, *then* $Hom_{\mathcal{CA}}(R[n], R[n]) \cong F_p$.

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