On the invariants of the tangent cone of a one dimensional local ring

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• Let $(A, \mathfrak{m})$ be a a Noetherian local ring of (krull) dimension d.

(*Typically, A will be the local ring at the origin of and affine algebraic variety V of dimension d defined over* $K = \overline{K}$.)

The tangent cone of A is defined as the Noetherian graded ring

$$G(\mathfrak{m}) = \bigoplus_{n \geq 0} \mathfrak{m}^n / \mathfrak{m}^{n+1}$$

where the multiplication is given in a natural way by

$$\overline{x} \in \mathfrak{m}^r / \mathfrak{m}^{r+1}, \, \overline{y} \in \mathfrak{m}^s / \mathfrak{m}^{s+1} : \, \overline{x} \cdot \overline{y} = \overline{xy} \in \mathfrak{m}^{r+s} / \mathfrak{m}^{r+s+1}$$

Assume $(A, \mathfrak{m})$ is as in the "typical" case:

$$A = K[x_1, \dots, x_n]_{(x_1,\dots,x_n)}/I(V)_{(x_1,\dots,x_n)}$$

where V is an algebraic variety in the affine space $\mathbb{A}_k^n$ of dimension $d = \dim A$ and $I(V)$ is the definition ideal of V.

It is not hard to see that the tangent cone of A may be obtained as

$$G(\mathfrak{m}) \simeq K[x_1, \dots, x_n]/(L(F)_{F \in I(V)})$$

where $L(F)$ denotes the leading form of a polynomial F.

Given a point in $\mathbb{A}_k^n$, it is a zero of the form $L(F)$ if and only if it defines a tangent line at the origin to the hypersurface defined by F.

As a consequence, the set of points (algebraic variety) defined by the ideal $J = (\{L(F)\}_{F \in I(V)})$ is the set of tangent lines at zero to the variety V: that is,

the geometric tangent cone.

The set J_1 of linear forms of this ideal can be described in terms of derivatives, namely:

$$J_1 = (\{dF = \sum_{i=1}^{n} \frac{\partial F}{\partial x_i}(0) \cdot x_i\}_{F \in I(V)})$$

It defines a linear variety that contains the geometric tangent cone, namely

the tangent space.

There is an easy exact sequence of K-vector spaces.

Denoting by $\mathfrak{n}$ the ideal $(x_1, \ldots, x_n)$ we have

$$0 \to J_1/J_1 \cap \mathfrak{n}^2 \to \mathfrak{n}/\mathfrak{n}^2 \to \mathfrak{m}/\mathfrak{m}^2 \to 0$$

So the dimension of the tangent space equals to

$$\dim_K \mathfrak{m}/\mathfrak{m}^2 = \mu(\mathfrak{m})$$

where $\mu(\mathfrak{m})$ denotes de minimal number of generators of $\mathfrak{m}$, that is the (embedding dimension of A).

Now remember that a point in a variety V is non-singular if the dimension of its tangent space equals to the dimension of the variety at the point, that is, the dimension of the local ring of the point.

So the origin is non-singular if the embedding dimension of A equals to its dimension, that is

A is a regular local ring

In terms of the tangent cone one can see that, in general,

$(A, \mathfrak{m})$ is regular if and only if $G_A(\mathfrak{m}) \simeq (A/\mathfrak{m})[x_1, \ldots, x_d]$

but in general its structure is rather complicated.

Even in the ”typical” case, the presentation of the tangent cone as

$$G(\mathfrak{m}) \simeq K[x_1, \ldots, x_n]/(L(F)_{F \in I(V)})$$

does not help too much, because if $I(V) = (F_1, \ldots, F_r)$ it is not true in general that

$$(L(F)_{F \in I(V)}) = (L(F_1), \ldots, L(F_r))$$

The (Krull) dimension of the tangent cone is equal to d, the dimension of A.

• Now, remember that given a positively graded ring S over a field K (or more in general, over an artinian ring) of dimension d, one can consider the Hilbert function of S, which is numerical function of polynomial type of degree $d-1$, the corresponding polynomial of degree $d-1$ being the Hilbert polynomial of S.

Let us precise this for the tangent cone.

For any $n \geq 0$, we have the function $H_{G_A(\mathfrak{m})}(n)$ defined as

$$H_{G_A(\mathfrak{m})}(n) = \dim_{A/\mathfrak{m}}(\mathfrak{m}^n/\mathfrak{m}^{n+1})$$

There exists a polynomial $P_{G_A(\mathfrak{m})}(x)$ of degree $d-1$ such that for $n \gg 0$

$$H_{G_A(\mathfrak{m})}(n) = P_{G_A(\mathfrak{m})}(n)$$

We may write this polynomial in the binomial form

$$P_{G_A(\mathfrak{m})}(x) = \sum_{i=0}^{d-1} (-1)^i e_i \binom{x+d-i-1}{d-i-1}$$

The leading coefficient e_0 is then the Hilbert multiplicity of $G_A(\mathfrak{m})$: $e(A)$, also named

the Hilbert-Samuel multiplicity of A.

The Hilbert-Samuel multiplicity is one of the most important invariants associated to a local ring and plays an important role in the study of local algebraic singularities.

The multiplicity of a regular local ring is always one. But in fact, under mild conditions (unmixed) the following holds:

$(A, \mathfrak{m})$ is a regular local ring if and only if $e(A) = 1$.

The notion of reduction (introduced by D. G. Northcott and D. Rees) is a very useful tool in order to study asymptotic properties of ideals.

We understand by asymptotic properties those that describe the stable behavior of an ideal by taking high powers.

For instance, the well known Artin-Rees lemma is such a kind of property.

And also the Hilbert-Samuel polynomial and so the multiplicity.

- Let I be an ideal in a A.

We say that $J \subset I$ is a reduction of I if there exists a positive integer $n_0 \geq 0$ such that

$$I^{n+1} = JI^n$$

for any $n \geq n_0$.

The lowest n_0 such that the above holds is called the reduction number of I with respect to J: $r_J(I)$.

All the reductions of I have the same radical and so the same height: $\mathrm{ht}\,(I)$.

Among the reductions of I we may distinguish the minimal ones:

We say that a reduction J of I is minimal if there is no other reduction of I contained in it.

Minimal reductions always exist.

Minimal reductions are generated by very special minimal systems of generators:

A minimal system of generators of a minimal reduction is a family of analytically independent elements.

Moreover, its cardinality is less or equal that the dimension of A.

And if the residue field of A is infinite, all the minimal reductions have the same minimal number of generators.

We call this new invariant of I

$$\text{the analytic spread of } I\text{: } \ell(I)$$

.

• Let us particularize this for the maximal ideal $\mathfrak{m} \subset A$ (we assume that the residue field is infinite).

The exists an ideal $J \subset \mathfrak{m}$ (in fact any minimal reduction of $\mathfrak{m}$) such that

$$\dim A \leq \mathrm{ht}\,(\mathfrak{m}) \leq \mu(J) \leq \dim A$$

and so

$$\mu(J) = \dim A$$

That is, J is a parameter ideal of A.

Moreover: if $a_1, \dots, a_d$ is a minimal system of generators of J then their initial forms satisfy

$$a_1^*, \dots, a_d^* \in \mathfrak{m}/\mathfrak{m}^2 \setminus \{0\} \subset G_A(\mathfrak{m})$$

and they are algebraically independent over $A/\mathfrak{m}$ (in particular, $a_1, \dots, a_d$ is part of a minimal system of generators of $\mathfrak{m}$).

Also, they generate a subalgebra such that

$$(A/\mathfrak{m})[a_1^*, \dots, a_d^*] \hookrightarrow G_A(\mathfrak{m})$$

is a graded finite extension.

That is, the above extension is a graded Noether normalization.

• Given $I \in A$ and $J \subset I$ a minimal reduction, the reduction number of I with respect to J: $r_J(I)$ may depend on J.

So we may consider the lowest of such numbers. We call it the reduction number of I: $r(I)$.

In the case of the maximal ideal we call it

the reduction number of A: $r(A)$.

The reduction number is a useful invariant of A to study local rings. For instance:

$(A, \mathfrak{m})$ is a regular local ring if and only if $r(A) = 0$.

Founding bounds for the reduction number is an important matter in Commutative Algebra.

• Let $(A, \mathfrak{m})$ be a one dimensional Noetherian local ring (with infinite residue field).

Let:

- $G(\mathfrak{m}) = G_A(\mathfrak{m})$ be the tangent cone of A.
- $b =$ the embedding dimension of $A = \mu(\mathfrak{m})$;
- $r =$ the reduction number of A;
- $e =$ the multiplicity of A;
- x a minimal reduction of $\mathfrak{m}$.

Then we have a graded Noether normalization

$$F(x) = (A/\mathfrak{m})[x^*] \hookrightarrow G(\mathfrak{m}) .$$

where $F(x)$ is the subalgebra generated by $x^* \in \mathfrak{m}/\mathfrak{m}^2$.

In fact, one can see that $F(x)$ is of the form

$$F(x) = \bigoplus_{n\geq 0} \frac{x^n A}{x^n \mathfrak{m}} \hookrightarrow \bigoplus_{n\geq 0} \mathfrak{m}^n/\mathfrak{m}^{n+1}$$

Now,

• since $F(x)$ is a polynomial ring in one variable over a field (and so a principal ideal domain);

• and $G(\mathfrak{m})$ is a finite graded $F(x)$-module,

we have a decomposition of $G(\mathfrak{m})$ as a direct sum of cyclic graded $F(x)$-modules.

Namely, we have that

$$G(\mathfrak{m}) \cong \bigoplus_{i=0}^{s} F(x)(-d_i) \bigoplus_{j=1}^{f} \left(\frac{F(x)}{(x^*)^{c_j} F(x)} \right) (-e_j)$$

for some integers

- $d_0 \leq \cdots \leq d_s$;
- $e_1 \leq \cdots \leq e_f$;
- $c_1, \ldots, c_f$ positive numbers.

• We are going to assume from now on x is a non-zero divisor of A.

In this case this is equivalent to say that A is Cohen-Macaulay.

Then, the reduction number of $\mathfrak{m}$ is independent of the chosen minimal reduction.

Moreover, one can see that the above decomposition can be rewritten as follows:

Proposition (T. Cortadellas and S. Z., 2007)

$$G(\mathfrak{m}) \cong \bigoplus_{i=0}^{r} (F(x)(-i))^{\alpha_i} \bigoplus_{i=1}^{r-1} \bigoplus_{j=1}^{r-i-1} \left(\frac{F(x)}{(x^*)^j F(x)}(-i) \right)^{\alpha_{i,j}},$$

with

- $\alpha_0 = 1$, $\alpha_r \neq 0$ *and*
- $\sum_{i=0}^{r} \alpha_i = e$.

Note that $G(\mathfrak{m})$ is Cohen-Macaulay if and only if it is free as $F(x)$-module.

Thus we have that the Cohen-Macaulay property of the tangent cone is equivalent to $\alpha_{i,j} = 0$ for all i and j.

For instance, this will happen if the reduction number is less or equal than 2.

On the other hand, the set of integers $\{\alpha_i, \alpha_{i,j}\}$ may depend on the chosen minimal reduction (x).

Nevertheless,

Proposition (T. Cortadellas, S. Z., 2007)

The set of integers $\{\alpha_1, \dots, \alpha_r\}$ is independent of the chosen minimal reduction.

Definition

We call the set of integers $\{\alpha_1, \dots, \alpha_r\}$ the invariants of the tangent cone.

The information provided by this set of invariants $\{\alpha_1, \ldots, \alpha_r\}$ (or more in general by the set of integers $\{\alpha_i, \alpha_{i,j}\}$) does not completely recover the structure of $G(\mathfrak{m})$ as a graded algebra.

But from the cohomological point of view it is the same because the local cohomology of $G(\mathfrak{m})$ as a graded $F(x)$-module is the same as its local cohomology as an algebra.

For instance, this is the case for the Castelnuovo-Mumford regularity of $G(\mathfrak{m})$: it is easily seen that:

$$\operatorname{reg}(G(\mathfrak{m})) = r$$

In fact, more in general, the minimal free-resolution of $G(\mathfrak{m})$ as a $F(x)$-module may be easily recovered as:

Proposition

$$0 \longrightarrow \bigoplus_{i=1}^{r-1} F(x)(-i)^{\beta_{1,i}} \longrightarrow \bigoplus_{i=0}^{r-1} F(-i)^{\beta_{0,i}} \longrightarrow G(\mathfrak{m}) \longrightarrow 0$$

- $\beta_{0,i} = \alpha_i + \sum_{j=1}^{r-1} \alpha_{i,j}$
- $\beta_{1,i} = \sum_{k+l=i} \alpha_{k,l}$

In order to compute the invariants of the tangent cone let us define new auxiliary numbers $f_{k,l}$ as

Definition

$$f_{k,l} := \lambda \left(\frac{\mathfrak{m}^k \cap (\mathfrak{m}^{k+l+1} : x^l)}{\mathfrak{m}^{k+1}} \right)$$

One has that

- $f_{i,j} = 0$ if $(i,j) \notin \{(k,l) \mid 1 \leq k \leq r-1 \text{ and } 1 \leq l \leq r-l\}$.
- $f_{r-1,1} = 0$.

Then, it holds:

Proposition

(1) *For* $1 \leq i \leq r-1$, $\alpha_i = \mu(\mathfrak{m}^i) - f_{i,r-i} - \mu(\mathfrak{m}^{i-1}) + f_{i-1,r-i+1}$.

(2) $\alpha_r = \mu(\mathfrak{m}^r) - \mu(\mathfrak{m}^{r-1})$.

(3) $f_{k,l} = \sum_{(i,j)\in\Lambda} \alpha_{i,j}$, *where*

$$\Lambda = \{(i,j) : 1 \leq i \leq k,\ k-i+1 \leq j \leq k-i+l\}$$

(In fact, the formula for computing the numbers $f_{k,l}$ is by means of an invertible matrix that allows to compute the numbers $\alpha_{i,j}$ in terms of the $f_{k,l}$.)

Now we want to make explicit all the computations for the case of numerical semigroup rings.

- Recall that a numerical semigroup S is a subset of $\mathbb{N}$ that is closed under addition, contains the zero element and has finite complement.

- A numerical semigroup S is always finitely generated, and has an unique minimal system of generators: $n_1 < \cdots < n_{b(S)}$.

- n_1 is the lowest integer belonging to S and it is called the multiplicity of S: e(S).

- An ideal of S is a nonempty subset of S such $I + S \subset I$.

- Given an ideal I, we denote by $Ap(I)$ the Apery set of I (with respect to $e(S)$): that is the set of smallest elements in I in each residue class module $e(S)$. Obviously, it has cardinality $e(S)$.

- ......

There is a large list of notions from Commutative Algebra that can be translated (additively) to numerical semigroup (and viceversa) that we will use freely.

- Let $S = \langle n_1, \ldots, n_b \rangle$ be a numerical semigroup minimally generated by $0 < e = e(S) = n_1 < \cdots < n_b = n_{b(S)}$.
- Denote by $M = S \setminus \{0\}$ the maximal ideal of S and let r the reduction number.
- For any ideal $I \subset S$ denote by $Ap(I) = \{\omega_0, \ldots, \omega_{e-1}\}$ the Apery set of I with respect to e.

Then, we may consider for any $0 \leq n \leq r$ the apery set of nM:

$$Ap(nM)$$

We may put all these values in a table that we call the Apery table of S:

$\mathrm{Ap}(S)$	$\omega_{0,0}$	$\omega_{0,1}$	$\cdots$	$\omega_{0,i}$	$\cdots$	$\omega_{0,e-1}$
$\mathrm{Ap}(M)$	$\omega_{1,0}$	$\omega_{1,1}$	$\cdots$	$\omega_{1,i}$	$\cdots$	$\omega_{1,e-1}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\mathrm{Ap}(nM)$	$\omega_{n,0}$	$\omega_{n,1}$	$\cdots$	$\omega_{n,i}$	$\cdots$	$\omega_{n,e-1}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\mathrm{Ap}(rM)$	$\omega_{r,0}$	$\omega_{r,1}$	$\cdots$	$\omega_{r,i}$	$\cdots$	$\omega_{r,e-1}$

To a numerical semigroup S we may associate the corresponding numerical semigroup ring. Given an (infinite) field K,

- let $(A, \mathfrak{m}) = K[[S]] = K[[t^{n_1}, \ldots, t^{n_{b(S)}}]] \subset K[[t]]$.

This is the (complete, local) numerical semigroup ring defined by S.

- The maximal ideal is $\mathfrak{m} = (t^{n_1}, \ldots, t^{n_{b(S)}})$.

It has dimension 1, it is Cohen-Macaulay, the multiplicity e is equal to $e(S) = n_1$, the embedding dimension b is equal $b(S)$, and the ideal (t^e) is a minimal reduction.

Our goal is to describe the structure of the tangent cone of A with respect to the minimal reduction (t^e) ”in terms” of Apery table of S.

Our computations are based on the following results essentially proved by V. Barucci and R. Fröberg (2006). They allow to ”read” from the Apery table the structure of the tangent cone.

- Let us denote $W = K[[t^e]] \subset A$.

Proposition

Let I be an ideal of S and $\mathfrak{I}$ the ideal of A generated by $\{t^n\}_{n\in I}$. If $Ap(I) = \{\omega_0, \ldots, \omega_{e-1}\}$ is the Apery set of I with respect to e, then $\mathfrak{I}$ is a free W-module generated by $t^{\omega_0}, \ldots, t^{\omega_{e-1}}$.

Proposition

For each $n \geq 0$

$$\mathfrak{m}^n = Wt^{\omega_{n,0}} \oplus \cdots \oplus Wt^{\omega_{n,e-1}},$$

with $\omega_{n+1,i} = \omega_{n,i} + e \cdot \epsilon$ and $\epsilon \in \{0, 1\}$.

Consider the following example:

- $S = \langle 10, 11, 19 \rangle$.
- $A = k[[t^{10}, t^{11}, t^{19}]]$.
- $x = t^{10}$ the minimal reduction.

By using the NumericalSgps package of GAP we may compute the reduction number of M (which is 8) and also the Apery sets of the ideals nM, for $n \leq 8$.

The following is the Apery table in this case:

A concrete example

$Ap(S)$	0	11	22	33	44	55	66	57	38	19
$Ap(M)$	10	11	22	33	44	55	66	57	38	19
$Ap(2M)$	20	21	22	33	44	55	66	57	38	29
$Ap(3M)$	30	31	32	33	44	55	66	57	48	39
$Ap(4M)$	40	41	42	43	44	55	66	67	58	49
$Ap(5M)$	50	51	52	53	54	55	66	77	68	59
$Ap(6M)$	60	61	62	63	64	65	66	77	78	69
$Ap(7M)$	70	71	72	73	74	75	76	77	88	79
$Ap(8M)$	80	81	82	83	84	85	86	87	88	89

Now, we may "read" on the Apery table the set of invariants $\{\alpha_i, \alpha_{i,j}\}$.

For that we will observe the shape of the "stairs" defined by the numbers in the columns of the table and their increasing.

In particular, we get the degrees of the free summands of the tangent cone (and so the (α_i)'s) in the following way:

$Ap(S)$	0	11	22	33	44	55	66	57	38	19
$Ap(M)$	10	11	22	33	44	55	66	57	38	19
$Ap(2M)$	20	21	22	33	44	55	66	57	38	29
$Ap(3M)$	30	31	32	33	44	55	66	57	48	39
$Ap(4M)$	40	41	42	43	44	55	66	67	58	49
$Ap(5M)$	50	51	52	53	54	55	66	77	68	59
$Ap(6M)$	60	61	62	63	64	65	66	77	78	69
$Ap(7M)$	70	71	72	73	74	75	76	77	88	79
$Ap(8M)$	80	81	82	83	84	85	86	87	88	89

So there free direct summand of degrees:

$$d_1 = 1, d_2 = 2, d_3 = 3, d_4 = 4$$
$$d_5 = 5, d_6 = 6, d_7 = 7, d_8 = 8, d_9 = 1$$

and the free part is given by

$$\alpha_1 = 2, \alpha_2 = 1, \alpha_3 = 1, \alpha_4 = 1$$
$$\alpha_5 = 1, \alpha_6 = 1, \alpha_7 = 1, \alpha_8 = 1$$

Next, we deal with the direct summands of the torsion of the tangent cone, that is, the $(\alpha_{i,j})$'s:

$Ap(S)$	0	11	22	33	44	55	66	57	38	19
$Ap(M)$	10	11	22	33	44	55	66	57	38	19
$Ap(2M)$	20	21	22	33	44	55	66	57	38	29
$Ap(3M)$	30	31	32	33	44	55	66	57	48	39
$Ap(4M)$	40	41	42	43	44	55	66	67	58	49
$Ap(5M)$	50	51	52	53	54	55	66	77	68	59
$Ap(6M)$	60	61	62	63	64	65	66	77	78	69
$Ap(7M)$	70	71	72	73	74	75	76	77	88	79
$Ap(8M)$	80	81	82	83	84	85	86	87	88	89

So we get

$$\alpha_{3,2} = 1, \alpha_{2,5} = 1$$

and the structure of the tangent cone is

$$G(\mathfrak{m}) \simeq F \oplus (F(-1))^2 \oplus F(-2) \oplus F(-3) \oplus F(-4) \oplus F(-5) \oplus$$
$$F(-6) \oplus F(-7) \oplus F(-8) \oplus (F/(x^*)^2 F)(-3) \oplus (F/(x^*)^5 F)(-2)$$

where F is $F(x)$.

- Let $S = \langle 10, 17, 22, 28 \rangle$.

We use GAP to compute the reduction number (which is 4) and the Apery table of S:

$\mathrm{Ap}(S)$	0	51	22	73	34	45	56	17	28	39
$\mathrm{Ap}(M)$	10	51	22	73	34	45	56	17	28	39
$\mathrm{Ap}(2M)$	20	51	32	73	34	45	56	27	38	39
$\mathrm{Ap}(3M)$	30	51	42	73	44	55	56	37	48	49
$\mathrm{Ap}(4M)$	40	61	52	73	54	65	66	47	58	59

In this case the tangent cone of $A = k[[t^{10}, t^{17}, t^{22}, t^{28}]]$ is Cohen-Macaulay, as it may be seen from the Apery table.

In fact, if we "read" the (α_i)'s we see that there are not "true landings":

$\mathrm{Ap}(S)$	0	51	22	73	34	45	56	17	28	39
$\mathrm{Ap}(M)$	10	51	22	73	34	45	56	17	28	39
$\mathrm{Ap}(2M)$	20	51	32	73	34	45	56	27	38	39
$\mathrm{Ap}(3M)$	30	51	42	73	44	55	56	37	48	49
$\mathrm{Ap}(4M)$	40	61	52	73	54	65	66	47	58	59

So we have the following degrees:

$$d_1 = 3, d_2 = 1, d_3 = 4, d_2 = 4$$
$$d_5 = 2, d_6 = 3, d_7 = 1, d_8 = 1, d_9 = 2$$

and the free part is given by

$$\alpha_1 = 3, \alpha_2 = 3, \alpha_3 = 2, \alpha_4 = 1$$

- Let $S = \langle 4, 11, 29 \rangle$.

The Apery table is now:

$\mathrm{Ap}(S)$	0	29	22	11
$\mathrm{Ap}(M)$	4	29	22	11
$\mathrm{Ap}(2M)$	8	33	22	15
$\mathrm{Ap}(3M)$	12	33	26	19

The tangent cone is Buchsbaum in this case, as it may be seen from the Apery table:

$\mathrm{Ap}(S)$	0	29	22	11
$\mathrm{Ap}(M)$	4	29	22	11
$\mathrm{Ap}(2M)$	8	33	22	15
$\mathrm{Ap}(3M)$	12	33	26	19

There exists a single torsion factor of order 1 of degree 2.