Nonlocal problems for superconductivity

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The full Maxwell's equations ($\delta = 1$) and quasi-static Maxwell's equations ($\delta = 0$) for linear materials are considered in a bounded domain Ω in $\mathbb{R}^3$ with a Lipschitz continuous boundary Γ. They can be written as

$$\nabla \times \boldsymbol{H} = \boldsymbol{J} + \delta \partial_t \boldsymbol{D} = \boldsymbol{J} + \delta \varepsilon \partial_t \boldsymbol{E}$$
$$\nabla \times \boldsymbol{E} = -\partial_t \boldsymbol{B} = -\mu \partial_t \boldsymbol{H}.$$

The domain Ω is occupied by a superconductive material. This is a material, which loses all resistivity below a certain temperature T_c. The current density $\boldsymbol{J}$ is supposed to be the sum of a normal and superconducting part, that is $\boldsymbol{J} = \boldsymbol{J}_n + \boldsymbol{J}_s$. Below the critical temperature T_c, the current consists of superconducting electrons and normal electrons. Above the critical temperature only normal electrons occur. The normal density current $\boldsymbol{J}_n$ is required to satisfy Ohm's law $\boldsymbol{J}_n = \sigma \boldsymbol{E}$.
Fritz and Heinz London, postulated in 1935 two equations, in addition to Maxwell's equations, governing the electromagnetic field in a superconductor:

$$\partial_t \boldsymbol{J}_s = \Lambda^{-1} \boldsymbol{E} \qquad \text{and} \qquad \nabla \times \boldsymbol{J}_s = -\Lambda^{-1} \boldsymbol{B},$$

where $\Lambda = \frac{m}{n_s q^2}$, with n_s the number of superelectrons per unit volume, m and q their mass and electric charge respectively. Since $\nabla \cdot \boldsymbol{B} = \nabla \cdot \boldsymbol{H} = 0$, there exists a magnetic vector potential $\boldsymbol{A}$ such that the second London equation can be rewritten in the local form $\boldsymbol{J}_s = -\Lambda^{-1} \boldsymbol{A}$.
The local theory of the brothers London is generalized to nonlocal theories. We consider the nonlocal representation of the superconductive current by Eringen in 1984. This representation identifies the state of the superconductor, at time t, with the field $\boldsymbol{H}(\cdot, t)$ and is given by the linear functional

$$\boldsymbol{J}_s(\boldsymbol{x}, t) = \int_\Omega \sigma_0 \left(|\boldsymbol{x} - \boldsymbol{x}'| \right) (\boldsymbol{x} - \boldsymbol{x}') \times \boldsymbol{H}(\boldsymbol{x}', t) \, \mathrm{d}\boldsymbol{x}', \qquad (\boldsymbol{x}, t) \in \Omega \times [0, T].$$

A straightforward calculation gives $\nabla \times \boldsymbol{J}_s = -\mathcal{K} \star \boldsymbol{H}$, with $\mathcal{K}$ an integral kernel function.
Taking into account the previous considerations, we suggest a parabolic ($\delta = 0$) and a hyperbolic model ($\delta = 1$) for nonlocal superconductivity:

$$\delta \varepsilon \mu \partial_{tt} \boldsymbol{H} + \sigma \mu \partial_t \boldsymbol{H} - \Delta \boldsymbol{H} + \mathcal{K} \star \boldsymbol{H} = \boldsymbol{0}.$$

We develop a time-discrete numerical scheme for the approximation of the solution of the PDEs in suitable function spaces. Moreover, we prove the well-posedness of both problems.