Some error estimates for the finite volume element method for a parabolic problem

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$$u_t - \Delta u = 0, \text{ in } \Omega, \quad u = 0, \text{ on } \partial\Omega, \quad \text{for } t \geq 0, \quad u(0) = v, \text{ in } \Omega, \tag{1}$$

where Ω is a bounded convex polygonal domain in $\mathbb{R}^2$. We study the spatially semidiscrete finite volume method for (1), where we seek an approximation $\widetilde{u}_h(t) \in S_h$ of $u(t)$, with S_h the piecewise linear functions on a triangulation $\mathcal{T}_h = \{\tau\}$ of Ω, with h denoting the maximum diameter of the triangles $\tau \in \mathcal{T}_h$. The semidiscrete finite volume method is then to find $\widetilde{u}_h(t) \in S_h$, such that

$$\int_{V_z} \widetilde{u}_{h,t}\, dx - \int_{\partial V_z} \nabla \widetilde{u}_h \cdot n\, ds = 0, \forall z \in Z_h^0, \quad \text{for } t > 0, \ \text{ with } \widetilde{u}_h(0) = v_h \in S_h,$$

where v_h is a given approximation of v, Z_h^0 is the set of interior vertices of $\mathcal{T}_h$ and V_z is a finite collection of nonoverlapping control volumes of Ω. We show that

$$\|\widetilde{u}_h(t) - u(t)\| \leq Ch^2 t^{-1} \|v\|, \quad \text{for } t > 0, \tag{2}$$

with $v_h = P_h v$, P_h the L_2 projection on S_h, $|v|_0 = \|v\| = (v, v)^{1/2}$ the norm in $L_2(\Omega)$, under a hypothesis on $\mathcal{T}_h$, which is satisfied for symmetric triangulations. For less restrictive meshes, such as almost symmetric triangulations, we derive (2) with the addition of a logarithmic factor and for piecewise almost symmetric triangulations, we show instead a $O(h^{3/2} t^{-1})$ bound. Further, without any assumption on the mesh we obtain a $O(ht^{-1/2})$ bound. In addition, we give examples of nonsymmetric partitions in two space dimension, that illustrate our theoretical results.