THE GENERIC ANISOTROPY OF SIMPLICIAL 1-SPHERES

STAVROS ARGYRIOS PAPADAKIS AND VASILIKI PETROTOU

Abstract. The concept of generic anisotropy of a simplicial sphere was introduced by the authors as a key ingredient for a second proof of McMullen's g-conjecture for simplicial spheres. In the present note we establish the generic anisotropy of all 1-dimensional simplicial spheres over an arbitrary field. We also prove that the determinant of the middle bilinear pairing of the generic Artinian reduction of the Stanley-Reisner ring determines the simplicial 1-sphere.

1. Introduction

The notion of generic anisotropy was introduced by the authors in [7], and used for the study of the Lefschetz properties of the general Artinian reduction of the Stanley-Reisner ring of simplicial spheres. As demonstrated in [2, 7], generic anisotropy implies the Strong Lefschetz property and Adiprasito's Hall-Laman relations and biased pairing property, which were defined in [1].

We recall the definition of generic anisotropy. Assume $d \geq 2$ is an integer and D is a simplicial sphere of dimension $d-1$ with vertex set $\{1, \ldots, m\}$. Assume k_1 is any field and denote by k the field of fractions of the polynomial ring

$$k_1[a_{i,j} : 1 \leq i \leq d,\ 1 \leq j \leq m].$$

We define the polynomial ring $R = k[x_1, \ldots, x_m]$, where all the variables x_i have degree 1. We denote by $I_D \subset R$ the Stanley-Reisner ideal of D and we set $k[D] = R/I_D$. For $i = 1, \ldots, d$, we set

$$f_i = \sum_{j=1}^{m} a_{i,j} x_j,$$

and define $A = k[D]/(f_1, \ldots, f_d)$. Hence, A is the generic Artinian reduction of $k_1[D]$ in the sense of [7, Definition 2.2].

Definition 1.1. ([7, Definition 3.2]) We call D generically anisotropic over k_1, if for all integers j with $1 \leq 2j \leq d$ and all nonzero elements $u \in A_j$ we have $u^2 \neq 0$.

The following is an important conjecture related to anisotropy.

Conjecture 1.2. *(Anisotropy conjecture) If D is a simplicial sphere and k_1 is a field, then D is generically anisotropic over k_1.*

When the characteristic of k_1 is equal to 2 the conjecture is true by [7]. Unfortunately, establishing the conjecture when the characteristic is not equal to 2 seems a harder task. A first step in that direction is the following theorem, which is the main result of the present note and will be proven in Subsection 4.1.

Theorem 1.3. *Assume D is a simplicial sphere of dimension 1 and k_1 is any field. Then D is generically anisotropic over k_1.*

We remark that under the characteristic 2 assumption, Adiprasito and the authors proved, in [2], anisotropy for cycles and pseudomanifolds. The same paper also established, in all characteristics, a weaker version of anisotropy.

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Section 2 contains a specific form of Gauss elimination that we use. Section 3 recalls a well-known general method useful for proving that a polynomial is nonzero.

In Section 5, we study, with D a 1-dimensional simplicial sphere, the middle bilinear pairing of A, and establish that its determinant (with respect to an arbitrary basis) determines the set of facets of D. In Subsection 5.1, we conjecture that the analogous property is true for any odd dimensional simplicial sphere.

1.1. **Notation.** We will use the definitions of [3, Section 5.1] for the notions of simplicial complex, vertices, faces, facets and dimension of a simplicial complex, and the notions of Stanley-Reisner ideal (also known as face ideal) and Stanley-Reisner ring (also known as face ring) of a simplicial complex over a field. If $n \geq 1$ is an integer, a simplicial sphere of dimension n is a simplicial complex D of dimension n such that there exists a geometric realization of D, in the sense of [3, Definition 5.2.8], which is homeomorphic to the sphere S^n. An additional important general reference is [8].

2. A general proposition related to elimination

In this section we describe a specific form of Gauss elimination that is used in the present paper.

Assume R is a commutative ring with unit, and n, m, Z are positive integers with $n < m \leq Z$. Assume that, for $1 \leq j \leq m$, x_j is an element of R and that for $1 \leq i \leq n$ and $1 \leq j \leq Z$, $a_{i,j}$ is an element of R. We denote by M the $n \times Z$ matrix with (i,j)-entry equal to $a_{i,j}$.

Assume $b_1, \dots, b_n$ are n integers, with $1 \leq b_i \leq Z$, for all i. We denote by $[b_1, \dots, b_n]$ the determinant of the $n \times n$ matrix, whose i-th column is equal to the b_i-th column of M. For $1 \leq i \leq n$, we set

$$f_i = \sum_{t=1}^{m} a_{i,t} x_t,$$

and denote by $I = (f_1, \dots, f_n)$ the ideal of R generated by the f_i.

Proposition 2.1. *Assume $c_1, \dots, c_{n-1}$ are integers, with $1 \leq c_i \leq Z$ for all i. We have*

$$\sum_{t=1}^{m} [c_1, c_2, \dots, c_{n-1}, t] x_t \in I.$$

Proof. Denote by Q the $n \times (n-1)$ matrix, whose i-th column is the c_i-th column of M. For $1 \leq j \leq n$, we denote by Q_j the determinant of the submatrix of Q obtained by deleting the j-th row of Q. We claim that

$$\sum_{t=1}^{m} [c_1, c_2, \dots, c_{n-1}, t] x_t = \sum_{j=1}^{n} (-1)^{j+n} Q_j f_j.$$

Indeed, on the left hand side the coefficient of x_t is $[c_1, c_2, \dots, c_{n-1}, t]$, while on the right hand side the coefficient is equal to $\sum_{j=1}^{n} (-1)^{j+n} Q_j a_{j,t}$. The two quantities are equal, by developing the determinant $[c_1, c_2, \dots, c_{n-1}, t]$ using the last column. □

3. A general technique for proving a polynomial is nonzero

In this section we discuss a well-known general method which is useful for proving that certain sums of products of bracket polynomials are nonzero. We use it in the proof of Proposition 4.7.

Assume $m \geq 1$, k is a field and $R = k[x_i : 1 \leq i \leq m]$. We denote by $\mathcal{A}_R$ the set of all monomials of R. In other words,

$$\mathcal{A}_R = \{x_1^{a_1} \cdots x_n^{a_m} : a_i \geq 0 \text{ for all } i\}.$$

Following [4, Section 15.2], a monomial order on R is a total order $>$ on $\mathcal{A}_R$ such that if $u_1, u_2, w \in \mathcal{A}_R$ with $u_1 > u_2$ and $w \neq 1$, we then have $wu_1 > wu_2 > u_2$. In addition, by the same reference, the lexicographic order on R with $x_1 > x_2 > \cdots > x_m$ is the total order $>$ on $\mathcal{A}_R$ defined by $x_1^{a_1} \cdots x_m^{a_m} > x_1^{b_1} \cdots x_m^{b_m}$ if and only if $a_i > b_i$ for the first index i such that $a_i \neq b_i$. It is a monomial order on R.

Assume now $>$ is a monomomial order on R. It induces the initial monomial map, $\text{in}_> : R \setminus \{0\} \to \mathcal{A}_R$, defined as follows. Assume $f \in R \setminus \{0\}$. Then, there exist (unique) $s > 0$, $g_1, \dots, g_s \in \mathcal{A}_R$ and $\lambda_1, \dots, \lambda_s \in k \setminus \{0\}$ such that

$$f = \sum_{i=1}^{s} \lambda_i g_i \qquad \text{and} \qquad g_1 > g_2 > g_3 > \cdots > g_s.$$

By definition, $\text{in}_>(f) = g_1$.

Remark 3.1. By the definition of a monomial ordering, we have

$$\text{in}_>(f_1 f_2) = (\text{in}_>(f_1))(\text{in}_>(f_2))$$

for all $f_1, f_2 \in R \setminus \{0\}$.

Moreover, by the definition of a monomial ordering we have the following proposition.

Proposition 3.2. *Assume $f_1, f_2, \dots, f_t \in R \setminus \{0\}$. Assume there exists a with $1 \leq a \leq t$ such that*

$$\text{in}_>(f_a) \ > \ \text{in}_>(f_b)$$

for all b with $1 \leq b \leq t$ and $b \neq a$. Then $\sum_{i=1}^{t} f_i \neq 0$ and $\text{in}_>(\sum_{i=1}^{t} f_i) = \text{in}_>(f_a)$.

Corollary 3.3. *Assume $f_1, f_2, \dots, f_t \in R \setminus \{0\}$ satisfy*

$$\text{in}_>(f_i) \ \neq \ \text{in}_>(f_j)$$

for all $1 \leq i, j \leq t$ with $i \neq j$. Then $\sum_{i=1}^{t} f_i \neq 0$.

Proof. Since $\text{in}_>(f_i) \neq \text{in}_>(f_j)$ for all $1 \leq i, j \leq t$ with $i \neq j$, there exists a unique integer a such that $1 \leq a \leq t$ and $\text{in}_>(f_a) > \text{in}_>(f_b)$ for all b with $1 \leq b \leq t$ and $b \neq a$. The result follows by Proposition 3.2. □

4. Anisotropy in dimension 1

Assume k_1 is an arbitrary field, $m \geq 3$ and D is the m-gon, considered as a 1-dimensional simplicial complex with vertex set $\{1, \dots, m\}$ and consecutive vertices $1, 2, \dots, m$. We denote by S_{sp} the polynomial ring

$$S_{sp} = k_1[a_{i,j} : 1 \leq i \leq 2, \ 1 \leq j \leq m]$$

and by k the field of fractions of S_{sp}.

We define the polynomial ring $R = k[x_1, \dots, x_m]$. We denote by $I_D \subset R$ the Stanley-Reisner ideal of D, and set $k[D] = R/I_D$. For $1 \leq i \leq 2$, we set

$$f_i = \sum_{j=1}^{m} a_{i,j} x_j,$$

and define $A = k[D]/(f_1, f_2)$. Therefore, A is the generic Artinian reduction of $k_1[D]$ in the sense of [7, Definition 2.2]. We denote by $\pi : R \to A$ the natural quotient k-algebra homomorphism.

If $m \geq 4$ we have

$$I_D = (x_1 x_j : 3 \leq j \leq m-1) + (x_i x_j : 2 \leq i \leq m-2, i+2 \leq j \leq m),$$

while if $m = 3$, we have $I_D = (x_1 x_2 x_3)$.

For $1 \leq i, j \leq m$ we set $x_{(i,j)} = x_i x_j \in R$, and denote by $[i, j]$ the determinant of the 2×2 matrix with (i, j)-entry equal to $a_{i,j}$.

Proposition 4.1. *Assume*

$$\sigma_1 = (b, d_1), \quad \sigma_2 = (b, d_2),$$

are two ordered facets of D having a common element. We then have

$$[b, d_1]\pi(x_{\sigma_1}) = -[b, d_2]\pi(x_{\sigma_2}). \tag{1}$$

and

$$[b, d_2]\pi(x_b^2) = -[d_1, d_2]\pi(x_{\sigma_1}). \tag{2}$$

Proof. Assume $z = b$ or $z = d_2$. By Proposition 2.1, we have that

$$\sum_{j=1}^{m} [z, j]\pi(x_j) = 0.$$

Hence,

$$\sum_{j=1}^{m} [z, j]\pi(x_j x_b) = 0.$$

We have $[z, z] = 0$ and that σ_1, σ_2 are the only facets of D which contain $\{b\}$. If $z = b$, the only terms of the last sum that are nonzero are for $j = d_1$ and $j = d_2$, which proves Equation (1). If $z = d_2$, the only terms of the last sum that are nonzero are for $j = b$ and $j = d_1$, which proves Equation (2). □

Remark 4.2. It is interesting to compare Proposition 4.1 with the results obtained by Lee in [6, Section 6].

It is clear that Proposition 4.1 implies that

$$[1, 2]\pi(x_1 x_2) = [i, i+1]\pi(x_i x_{i+1}) = [m, 1]\pi(x_m x_1)$$

for all $1 \le i \le m-1$. Since, by [3, Section 5], the vector space A_2 is 1-dimensional, it follows that $\pi(x_1 x_2) \neq 0$. Hence $\pi(x_1 x_2)$ is a basis of the 1-dimensional vector space A_2, and we denote by $\deg : A_2 \to k$ the unique k-linear isomorphism such that $(\deg \circ \pi)([1, 2]x_1 x_2) = 1$.

Moreover, we define the symmetric bilinear map $\rho : A_1 \times A_1 \to k$ by

$$\rho(u, v) = \deg(uv)$$

for all $u, v \in A_1$.

4.1. **The proof of Theorem 1.3.** It is clear that in order to prove Theorem 1.3 it is enough to prove that $\rho(u, u) \neq 0$ whenever $0 \neq u \in A_1$. The following proposition is an important step in the direction of explicitly computing $\rho(u, u)$.

Proposition 4.3. *For $1 \le i \le m-1$, we have*

$$(\deg \circ \pi)(x_i x_{i+1}) = \frac{1}{[i, i+1]}.$$

Moreover, we have

$$(\deg \circ \pi)(x_1 x_m) = \frac{1}{[m, 1]}, \quad (\deg \circ \pi)(x_1^2) = -\frac{[m, 2]}{[m, 1][1, 2]}, \quad (\deg \circ \pi)(x_m^2) = -\frac{[m-1, 1]}{[m-1, m][m, 1]}$$

and

$$(\deg \circ \pi)(x_i^2) = -\frac{[i-1, i+1]}{[i-1, i][i, i+1]}$$

for $2 \le i \le m-1$.

Proof. All equalities follow immediately from Proposition 4.1. □

The following proposition allows us to find a suitable base for A_1.

Proposition 4.4. *We have $\dim_k A_1 = m-2$. If $\mathcal{S}$ is any subset of $\{1, \ldots, m\}$ of cardinality $m-2$, then the set $\{\pi(x_i) : i \in \mathcal{S}\}$ is a k-basis of A_1.*

Proof. We denote by M the $2 \times m$ matrix with (i,j)-entry equal to $a_{i,j}$. The determinant of every 2×2 submatrix of M is a nonzero element of the field k. Since $A = k[D]/(f_1, f_2)$ and I_D is a homogeneous ideal with generators of degrees ≥ 2, the result follows. □

For $1 \leq i \leq m-2$, we set $e_i = \pi(x_{i+1})$. By Proposition 4.4, the finite sequence

$$e_1, e_2, \dots, e_{m-2}$$

is an ordered basis of A_1.

We define a second basis of A_1 by using the Gram-Schmidt orthogonalization. We set $\tilde{e}_1 = e_1$, and inductively define

$$\tilde{e}_i = e_i + \frac{[1,i]}{[1,i+1]}\tilde{e}_{i-1},$$

for $2 \leq i \leq m-2$.

Proposition 4.5. *For all $1 \leq i \leq m-2$, we have*

$$\tilde{e}_i = \sum_{t=2}^{i+1} \frac{[1,t]}{[1,i+1]}\pi(x_t). \tag{3}$$

Proof. We prove Equation (3) using induction on i. For $i = 1$ it is true by the definition of $\tilde{e}_1$. Assume $1 \leq i \leq m-3$ and that Equation (3) is true for the value i. We have

$$\begin{aligned}\tilde{e}_{i+1} = e_{i+1} + \frac{[1,i+1]}{[1,i+2]}\tilde{e}_i &= \pi(x_{i+2}) + \frac{[1,i+1]}{[1,i+2]}\Big(\sum_{t=2}^{i+1}\frac{[1,t]}{[1,i+1]}\pi(x_t)\Big)\\ &= \pi(x_{i+2}) + \sum_{t=2}^{i+1}\frac{[1,t]}{[1,i+2]}\pi(x_t) = \sum_{t=2}^{i+2}\frac{[1,t]}{[1,i+2]}\pi(x_t).\end{aligned}$$

□

Proposition 4.6. *For all $1 \leq i \leq m-2$, we have*

$$\rho(\tilde{e}_i, \tilde{e}_i) = -\frac{[1,i+2]}{[1,i+1][i+1,i+2]}.$$

Moreover, if $1 \leq j \leq m-2$ and $j \neq i$, we have

$$\rho(\tilde{e}_i, \tilde{e}_j) = 0.$$

Proof. Assume $1 \leq i \leq m-2$. We set $u = \sum_{t=2}^{i+1}[1,t]\pi(x_t)$. By Proposition 2.1,

$$\sum_{t=2}^{m}[1,t]\pi(x_t) = 0.$$

Hence, if $1 \leq r \leq i$, taking into account that $\pi(x_r x_t) = 0$ when $r+2 \leq t \leq m$, we get

$$u\,\pi(x_r) = 0. \tag{4}$$

Assume $1 \leq j < i$. Using Proposition 4.5, Equation (4) implies that $\rho(\tilde{e}_i, \tilde{e}_j) = 0$. Moreover, Equation (4) also implies that

$$\begin{aligned}\deg(u^2) &= \deg\big(u\,(\sum_{t=2}^{i+1}[1,t]\pi(x_t))\big) = \deg\big([1,i+1]u\pi(x_{i+1})\big)\\ &= \deg\big([1,i+1][1,i]\pi(x_i x_{i+1}) + [1,i+1]^2\pi(x_{i+1}^2)\big)\\ &= [1,i+1]\Big(\frac{[1,i]}{[i,i+1]} - \frac{[1,i+1][i,i+2]}{[i,i+1][i+1,i+2]}\Big)\\ &= [1,i+1]\frac{[1,i][i+1,i+2] - [1,i+1][i,i+2]}{[i,i+1][i+1,i+2]}\\ &= -[1,i+1]\frac{[1,i+2][i,i+1]}{[i,i+1][i+1,i+2]} = -[1,i+1]\frac{[1,i+2]}{[i+1,i+2]},\end{aligned}$$

where we used Proposition 4.3 and the well-known Plücker identity

$$[a,b][c,d]-[a,c][b,d]+[a,d][b,c]=0.$$

Since $\tilde{e}_i = u/[1,i+1]$, this proves the formula for $\rho(\tilde{e}_i,\tilde{e}_i)$. □

We set

$$L=\prod_{s=2}^{m-1}([1,s][s,s+1])$$

and, for $1\le t\le m-2$, define $L_t = L/([1,t+1][t+1,t+2]) \in S_{sp}$.

Using Proposition 4.6, it is clear that to prove Theorem 1.3 it is enough to prove that, if $d_t\in k$ satisfy

$$\sum_{t=1}^{m-2} d_t^2 \frac{[1,t+2]}{[1,t+1][t+1,t+2]}=0,$$

we then have $d_t=0$ for all $1\le t\le m-2$. By clearing denominators, it is enough to prove the following proposition.

Proposition 4.7. *Assume $d_1,\dots,d_{m-2}\in S_{sp}$ satisfy*

(5) $$\sum_{t=1}^{m-2} d_t^2[1,t+2]L_t=0.$$

Then, we have $d_t=0$, for all $1\le t\le m-2$.

Proof. We give to the polynomial ring S_{sp} the lexicographic ordering $>$ with

$$a_{1,1}>a_{1,2}>\cdots>a_{1,m}>a_{2,1}>a_{2,2}>\cdots>a_{2,m}.$$

Using Corollary 3.3, it is enough to prove that if i,j have the properties $1\le i<j\le m-2$, $d_i\neq 0$ and $d_j\neq 0$, we then have

(6) $$\mathrm{in}_>(d_i^2[1,i+2]L_i)\neq \mathrm{in}_>(d_j^2[1,j+2]L_j).$$

Using the definitions of L_i and L_j and Remark 3.1, we have

$$\mathrm{in}_>(d_i^2[1,i+2]L_i)=(\mathrm{in}_>(d_i))^2\cdot(a_{1,1})^{m-2}\cdot\prod_{s=1}^{i}a_{1,s}\cdot\prod_{s=i+2}^{m-1}a_{1,s}\cdot Q_i,$$

and

$$\mathrm{in}_>(d_j^2[1,j+2]L_j)=(\mathrm{in}_>(d_j))^2\cdot(a_{1,1})^{m-2}\cdot\prod_{s=1}^{j}a_{1,s}\cdot\prod_{s=j+2}^{m-1}a_{1,s}\cdot Q_j,$$

where Q_i and Q_j are monomials in the variables $a_{2,1},\dots,a_{2,m}$. Therefore, the variable $a_{1,j+1}$ appears in the monomial $\mathrm{in}_>(d_i^2[1,i+2]L_i)$ with an odd power, and in the monomial $\mathrm{in}_>(d_j^2[1,j+2]L_j)$ with an even power. Hence, Inequality (6) is true. □

Example 4.8. Assume $m=6$. Equation (5) becomes

$$d_1^2[1,3]L_1+d_2^2[1,4]L_2+d_3^2[1,5]L_3+d_4^2[1,6]L_4=0,$$

where

$$L_1=\frac{L}{[1,2][2,3]},\quad L_2=\frac{L}{[1,3][3,4]},\quad L_3=\frac{L}{[1,4][4,5]},\quad L_4=\frac{L}{[1,5][5,6]}$$

and

$$L=[1,2][1,3][1,4][1,5][2,3][3,4][4,5][5,6].$$

5. The determinant of ρ

We keep using the notations of Section 4. Moreover, we denote by N_m the $(m-2)\times(m-2)$ symmetric matrix, with (i,j)-entry equal to $\rho(\tilde{e}_i, \tilde{e}_j)$. We call N_m the matrix of ρ with respect to the ordered basis $\tilde{e}_1, \tilde{e}_2, \dots, \tilde{e}_{m-2}$.

Proposition 5.1. *We have*

$$\det(N_m) = (-1)^m \frac{[1,m]}{\prod_{i=1}^{m-1}[i,i+1]}.$$

Proof. The result follows immediately from Proposition 4.6. □

Remark 5.2. Assume $1 \le c < d \le m$. It is well-known that $[c,d]$ is an irreducible element of S_{sp}. Hence, there exists an induced valuation map

$$\mathrm{val}_{[c,d]} : k \setminus \{0\} \to \mathbb{Z}.$$

Recall that if $f, g \in S_{sp} \setminus \{0\}$, then $\mathrm{val}_{[c,d]}(f)$ is the largest integer s such that $[c,d]^s$ divides f in S_{sp}, and

$$\mathrm{val}_{[c,d]}(f/g) = \mathrm{val}_{[c,d]}(f) - \mathrm{val}_{[c,d]}(g).$$

Remark 5.3. Assume that h is any ordered basis of A_1. We denote by H the matrix of ρ with respect to h. By the basic theory of bilinear forms, there exists an invertible matrix P with entries in k such that

$$H = P^t N_m P.$$

As a consequence, using Proposition 5.1,

$$\det(H) = (-1)^m (\det P)^2 \frac{[1,m]}{\prod_{i=1}^{m-1}[i,i+1]}.$$

Taking into account Remark 5.2, we conclude that we can recover the simplicial complex D from $\det(H)$, since the set of facets of D is exactly the set of ordered pairs (c,d) such that $1 \le c < d \le m$ and $\mathrm{val}_{[c,d]}(\det(H))$ is an odd integer.

5.1. The Odd Multiplicity conjecture.

Assume k_1 is an arbitrary field and E is a simplicial sphere of odd dimension $d-1 \ge 1$ and vertex set $\{1,\dots,m\}$. Assume $e = (e_1,\dots,e_d)$ is an ordered facet of E. We denote by k the field of fractions of the polynomial ring

$$S_{sp} = k_1[a_{i,j} : 1 \le i \le d,\ 1 \le j \le m].$$

We define the polynomial ring $R = k[x_1,\dots,x_m]$. We denote by $I_E \subset R$ the Stanley-Reisner ideal of E, and set $k[E] = R/I_E$. For $1 \le i \le d$, we set

$$f_i = \sum_{j=1}^{m} a_{i,j} x_j,$$

and define $A = k[E]/(f_1,\dots,f_d)$. Consequently, A is the generic Artinian reduction of $k_1[E]$ in the sense of [7, Definition 2.2]. We denote by $\pi : R \to A$ the natural quotient k-algebra homomorphism.

For a sequence $c = (c_1,\dots,c_d)$ of distinct integers with $1 \le c_i \le m$ for all i, we denote by $[c]$ the determinant of the $d \times d$ matrix with (i,j)-entry equal to a_{i,c_j}. It is well-known that $[c]$ is an irreducible element of the polynomial ring S_{sp}.

Arguing as in Section 4, there exists a unique k-linear isomorphism $\deg : A_d \to k$ such that $(\deg \circ \pi)([e]x_e) = 1$, where $x_e = \prod_{i=1}^{d} x_{e_i}$. We define the symmetric bilinear map $\rho : A_{d/2} \times A_{d/2} \to k$ by

$$\rho(u,v) = \deg(uv)$$

for all $u, v \in A_{d/2}$. We assume that h is an ordered basis of $A_{d/2}$ and denote by H the matrix of ρ with respect to the basis h.

Remark 5.3 motivates the following conjecture.

Conjecture 5.4. *(Odd Multiplicity conjecture) Assume* $c = (c_1, \ldots, c_d)$ *is a sequence of integers with* $1 \leq c_1 < c_2 < \cdots < c_d \leq m$*. We then have that* $\mathrm{val}_{[c]}(\det(H))$ *is odd if and only if* c *is a facet of* E*.*

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